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AN EQUIVALENCE OF A SYSTEM OF PARTIAL INTEGRAL EQUATIONS TO A SYSTEM OF FREDHOLM'S INTEGRAL EQUATIONS

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Abstract

Two systems of partial integral equations are considered. Under some natural conditions the equivalence of these equations, corresponding to the systems of second kind of Fredholm's integral equations, is proved.

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A number of problems in Quantum mechanics, Field theory, Solid state physics and problems of stability of rotor are connected to the study of systems of partial integral equations [1,2,3,4].

Let us consider the following system of non homogeneous partial integral equations

$$f_i(x,y)=\sum_{j=1}^{n}\int_a^b K_{ij}^{(1)}(x,t)f_j(t,y)dt+ \tag{1}$$

$$+\sum_{j=1}^{n}\int_a^b K_{ij}^{(2)}(y,t)f_j(x,t)dt+g_i(x,y), \quad i=\overline{1,n},$$

and the corresponding system of homogeneous partial integral equations

$$f_i(x,y)=\sum_{j=1}^{n}\int_a^b K_{ij}^{(1)}(x,t)f_j(t,y)dt+ \tag{2}$$

$$+\sum_{j=1}^{n}\int_a^b K_{ij}^{(2)}(y,t)f_j(x,t)dt, \quad i=\overline{1,n}.$$

Here the kernels $K_{ij}^{(1)}, K_{ij}^{(2)}$ and the function $g_i \quad i=\overline{1,n}$, are given as continuous functions defined on $[a,b]^2$, with values in C,where C is a complex plan and $f_i, \quad i=\overline{1,n}$, are unknown continuous functions defined on $[a,b]^2$.

The problem of the equivalence of partial integral equations to Fredholm's integral equations was studied in [2-7]. In [2] the following partial integral equation was considered

$$f(x,y)=\int_a^b K^{(1)}(x,y;t)f(t,y)dt+\int_a^b K^{(2)}(x,y;t)f(x,t)dt+g(x,y), \tag{3}$$

and it was shown that equation (3) is equivalent to several different integral equations. These integral equations depend on the way they were obtained and have non simple kernels.

In [5] a more general partial integral equation than (3) with additional total integral terms was considered, and by using another method some integral equations were obtained. In this case the kernels of the integral equations also depend on the way they were obtained and have a non simple form. In [5] it has also been proved the existence of solutions under some additional conditions on kernels. In work [3] equation (3) was investigated in the case when the functions K_1 and K_2 do not depend on $x, y\in[a,b]$, respectively. It was proven that equation (3) is equivalent to a unique integral equation with simple kernel which does not depend on the way it was obtained.

In the present work systems of partial integral equations (1),(11) and (2),(12) are considered, and it will be shown that under some natural conditions these systems of partial integral equations are equivalent to corresponding systems of the second kind of Fredholm's integral equations with quite simple kernels. As a consequence we get a solvability theorem for these systems of partial integral equations.

Let $R_{ij}^{(l)}$ be the resolvent corresponding to the kernel $K_{ij}^{(l)}$, i.e.

$$R_{ij}^{(l)}(x,t) = \frac{D_{ij}^{(l)}(x,t)}{\Delta_l}, l = 1,2,$$

where

$$D_{ij}^{(l)}(x,t) = K_{ij}^{(l)}(x,t) + \sum_{s_1=1}^{n} \int_a^b \begin{vmatrix} K_{ij}^{(l)}(x,t) & K_{is_1}^{(l)}(x,t_1) \\ K_{s_1j}^{(l)}(t_1,t) & K_{s_1s_1}^{(l)}(t_1,t_1) \end{vmatrix} dt + \tag{4}$$

$$+\frac{1}{2!}\sum_{s_1,s_2=1}^{n} \int_a^b \int_a^b \begin{vmatrix} K_{ij}^{(l)}(x,t) & K_{is_1}^{(l)}(x,t_1) & K_{is_2}^{(l)}(x,t_2) \\ K_{s_1j}^{(l)}(t_1,t) & K_{s_1s_1}^{(l)}(t_1,t_1) & K_{s_1s_2}^{(l)}(t_1,t_2) \\ K_{s_2j}^{(l)}(t_2,t) & K_{s_2s_1}^{(l)}(t_2,t_1) & K_{s_2s_2}^{(l)}(t_2,t_2) \end{vmatrix} dt_1dt_2 + \ldots,$$

$$i,j = \overline{1,n}, l = 1,2,$$

is the corresponding Fredholm's minor and

$$\Delta_l = 1 + \textstyle\sum_{s_1=1}^{n} \int_a^b K_{s_1s_1}^{(l)}(t_1,t_1)dt_1 + \tag{5}$$

$$+\frac{1}{2!}\sum_{s_1,s_2=1}^{n} \int_a^b \int_a^b \begin{vmatrix} K_{s_1s_1}^{(l)}(t_1,t_1) & K_{s_1s_2}^{(l)}(t_1,t_2) \\ K_{s_2s_1}^{(l)}(t_2,t_1) & K_{s_2s_2}^{(l)}(t_2,t_2) \end{vmatrix} dt_1dt_2 + \ldots,$$

is the corresponding Fredholm's determinant[8].

Theorem 1 *Let $\Delta_l \neq 0$,l=1,2. Then the systems of partial integral equations (1) and (2) are equivalent to the following systems of the second type of Fredholm's integral equations, respectively,*

$$f_i(x,y) = G_i(x,y) + \sum_{j,p=1}^{n} \int_a^b \int_a^b R_{ij}^{(1)}(x,t)R_{ip}^{(2)}(y,s)f_p(t,s)dtds,\ i = \overline{1,n},$$

and

$$f_i(x,y) = \sum_{j,p=1}^{n} \int_a^b \int_a^b R_{ij}^{(1)}(x,t)R_{ip}^{(2)}(y,s)f_p(t,s)dtds,\ i = \overline{1,n},$$

where

$$G_i(x,y) = g_i(x,y) + \sum_{j=1}^{n} \int_a^b R_{ij}^{(2)}(y,t)g_j(x,t)dt + \sum_{p=1}^{n} \int_a^b R_{ip}^{(1)}(x,t)g_p(t,y)dt +$$

$$+\sum_{j,p=1}^{n} \int_a^b \int_a^b R_{ip}^{(1)}(x,t)R_{ij}^{(2)}(y,s)g_i(t,s)dtds$$

Corollary. 1 *Assume that the conditions of the theorem are fulfilled. Then the system of non homogeneous partial integral equations (1) has a non trivial solution if and only if the system of homogeneous partial integral equations (2) has only a trivial solution.*

Proof of the theorem. Fixing the variable x, $x \in [a,b]$, and having introduced the following continuous functions

$$f_i(y) = f_i(x,y); \hat{g}_i(y) = \sum_{j=1}^{n} \int_a^b K_{ij}^{(1)}(x,t) f_j(t,y)dt + g_i(x,y), \tag{6}$$

$$y \in [a,b], i = \overline{1,n},$$

we get that the system of partial integral equations (1) is equivalent to the following system of the second kind of Fredholm's integral equations

$$f_i(y) = \hat{g}_i(y) + \sum_{j=1}^{n} \int_a^b K_{ij}^{(2)}(y,t) f_j(t)dt, \quad i = \overline{1,n}.$$

Let Σ denote the set consisting of n identical copies of $[a,b]$, i.e.,

$$\Sigma = \cup_{i=1}^{n} [a,b]_i, \ [a,b]_i = [a,b].$$

We shall define a measure σ on the subsets s of the set Σ according to the formula

$$\sigma(s) = \sigma_1(s) + \sigma_2(s) + \ldots + \sigma_n(s),$$

where $\sigma_i(s) = \mu(s \cap [a,b]_i)$, $i = \overline{1,n}$, and μ is the Lesbegues measure on [a,b].

We introduce the following functions on Σ and on $\Sigma \times \Sigma$:

$$f(Y) = f_i(y), Y = y \in [a,b]_i \, , g(Y) = g_i(y), Y = y \in [a,b]_i; \tag{7}$$

$$K_2(Y,T) = K_{ij}^{(2)}(y,t), Y = y \in [a,b]_i, T = t \in [a,b]_j, \ i,j = \overline{1,n}. \tag{8}$$

Then we obtain the following second type of Fredholm's integral equation:

$$f(Y) = g(Y) + \int_\Sigma K_2(Y,T) f(T) d\sigma(T). \tag{9}$$

According to the Fredholm theorem equation (9) has a unique solution if and only if the determinant $\Delta(K_2)$ corresponding to the kernel K_2 is not equal to zero. In this case the unique solution of equation (9) is represented in the following form:

$$f(Y) = g(Y) + \int_\Sigma R(Y,T;K_2) g(T) d\sigma(T),$$

where $R(Y,T,K_2)$ is the resolvent corresponding to the kernel K_2, that is,

$$R(Y,T;K_2) = \frac{D(Y,T;K_2)}{\Delta(K_2)}, \tag{10}$$

where

$$D(Y,T;K_2)=K_2(Y,T)+\int_\Sigma\begin{vmatrix}K_2(Y,T) & K_2(Y,T_1)\\ K_2(T_1,T) & K_2(T_1,T_1)\end{vmatrix}dT+$$

$$+\frac{1}{2!}\int_\Sigma\int_\Sigma\begin{vmatrix}K_2(Y,T) & K_2(Y,T_1) & K_2(Y,T_2)\\ K_2(T_1,T) & K_2(T_1,T_1) & K_2(T_1,T_2)\\ K_2(T_2,T) & K_2(T_2,T_1) & K_2(T_2,T_2)\end{vmatrix}dT_1dT_2+\ldots,$$

is the Fredholm's minor and

$$\Delta(K_2)=1+\int_\Sigma K_2(T_1,T_1)dT+$$

$$+\frac{1}{2!}\int_\Sigma\int_\Sigma\begin{vmatrix}K_2(T_1,T_1) & K_2(T_1,T_2)\\ K_2(T_2,T_1) & K_2(T_2,T_2)\end{vmatrix}dT_1dT_2+\ldots,$$

is the Fredholm's determinant.

Transforming the integrals over Σ to the integrals over [a,b] we get that $\Delta(K_2)=\Delta_2$. By the conditions of the theorem, $\Delta_2\neq 0$. Therefore, for all $g(Y)$ equation (7) has a unique solution and this solution is represented by formula (10). Taking into account notations (6),(7) and (8), and calculating the minor $D(Y,T;K_2)$ and determinant $\Delta(K_2)$ we shall obtain from (10) the following system of partial integral equations, which is equivalent to system (1):

$$f_i(x,y)=g_i(x,y)+\sum_{j=1}^n\int_a^b R_{ij}^{(2)}(y,t)g_j(x,t)dt+$$

$$+\sum_{j=1}^n\int_a^b K_{ij}^{(1)}(x,t)f_j(t,y)dt+\sum_{j,p=1}^n\int_a^b\int_a^b R_{ij}^{(2)}(y,s)f_p(t,s)dtds,\, i=\overline{1,n}$$

Further, fixing the variable y, $y\in[a,b]$ in this system of equations and having introduced the functions $f_i(x)=f_i(x,y)$ and

$$g_i(x)=g_i(x,y)+\sum_{j=1}^n\int_a^b R_{ij}^{(2)}(y,t)g_j(x,t)dt+$$

$$+\sum_{j,p=1}^n\int_a^b\int_a^b R_{ij}^{(2)}(y,s)f_p(t,s)dtds,\quad i=\overline{1,n},$$

we get the following system of the second kind of Fredholm's integral equations

$$f_i(x)=g_i(x)+\sum_{j=1}^n\int_a^b K_{ij}^{(1)}(x,t)f_j(t)dt,\quad i=\overline{1,n}.$$

Using similar arguments as the above we obtain that the system of partial integral equations (1) is equivalent to the following system of total integral equations

$$f_i(x,y)=g_i(x,y)+\sum_{j=1}^n\int_a^b R_{ij}^{(2)}(y,t)g_j(x,t)dt+$$

$$+\sum_{p=1}^n\int_a^b R_{ip}^{(1)}(x,t)g_p(t,y)dt+$$

$$+\sum_{j,p=1}^{n}\int_a^b\int_a^b R_{ip}^{(1)}(x,t)R_{ij}^{(2)}(y,s)g_i(t,s)dtds+$$

$$+\sum_{j,l=1}^{n}\int_a^b\int_a^b R_{ij}^{(2)}(y,s)\{K_{jr}^{(1)}(x,s)+\sum_{p=1}^{n}\int_a^b R_{ip}^{(1)}(x,t)K_{jr}^{(1)}(t,s)dt\}f_r(t,s)dtds,$$

$$i=\overline{1,n}$$

The following resolvent relations

$$R_{jp}^{(1)}(x,t)=K_{ip}^{(1)}(x,t)+\sum_{r=1}^{n}\int_a^b R_{ir}^{(1)}(x,t_1)K_{ir}^{(1)}(t_1,t)dt_1,$$

$$j,p=\overline{1,n},$$

hold. Therefore using these relations we obtain the part of the proof, concerning non homogeneous equation (1). For the rest of the proof we repeat the process from the first part of the proof having put $g_i=0,\quad i=\overline{1,n}$. So, we show that the homogeneous system of partial integral equations is equivalent to the homogeneous system of the second kind of Fredholm's integral equations.

Now we shall consider the following system of partial integral equations which is more general than (1):

(11)
$$f_i(x,y)=\sum_{j=1}^{n}\int_a^b K_{ij}^{(1)}(x,y;t)f_j(t,y)dt+$$
$$+\sum_{j=1}^{n}\int_a^b K_{ij}^{(2)}(x,y;t)f_j(x,t)dt+g_i(x,y),$$

(12)
$$f_i(x,y)=\sum_{j=1}^{n}\int_a^b K_{ij}^{(1)}(x,y;t)f_j(t,y)dt+$$
$$+\sum_{j=1}^{n}\int_a^b K_{ij}^{(2)}(x,y;t)f_j(x,t)dt\quad i=\overline{1,n}.$$

Here the kernels $K_{ij}^{(1)}, i,j=\overline{1,n}$ and $K_{ij}^{(2)}, i,j=\overline{1,n}$ are continuous functions defined on $[a,b]^3$ and $g_i, i=\overline{1,n}$ are continuous functions on $[a,b]^2$.

Let

(13)
$$R_{ij}^{(1)}(x,y,t)=\frac{D_{ij}^{(1)}(x,y,t)}{\Delta_1(y)},$$

where $D_{ij}^{(1)}(x,y,t)$ and $\Delta_1(y)$ are the Fredholm's minor and determinant corresponding to the kernels $K_{ij}^{(1)}(x,y;t), i,j=\overline{1,n}$ and which are defined by the similar formulas of (4) and (5), and let

$$R_{ij}^{(2)}(x,y,t)=\frac{D_{ij}^{(2)}(x,y,t)}{\Delta_2(x)},$$

where $D_{ij}^{(2)}(x,y,t)$ and $\Delta_2(x)$ are the Fredholm's minor and determinant corresponding to the kernels $K_{ij}^{(2)}(x,y;t), i,j=\overline{1,n}$ which are defined by the similar formulas of (4) and (5).

Theorem 2 *Suppose that for all $x,y\in[a,b]$ the inequalities: $\Delta_2(x)\neq 0$ and $\Delta_2(x)\neq 0$ are satisfied. Then the systems of partial integral equations (11) and (12) are equivalent to the following systems of total integral equations*

$$f_i(x,y)=\sum_{j,p=1}^{n}\int_a^b\int_a^b[R_{ij}^{(1)}(x,y,t)K_{ip}^{(2)}(y,s,t)+$$

$$+R_{ij}^{(2)}(x,y,t)K_{ip}^{(2)}(t,y,s)]f(t,s)dtds+ \tag{14}$$

$$+g_i(x,y)+\sum_{j=1}^{n}\int_a^b R_{ij}^{(1)}(x,y,t)g_j(x,t)dt+\sum_{p=1}^{n}\int_a^b R_{ip}^{(2)}(x,y,t)g_p(t,y)dt$$

and

$$f_i(x,y)=\sum_{j,p=1}^{n}\int_a^b\int_a^b[R_{ij}^{(1)}(x,y,t)K_{ip}^{(2)}(y,s,t)+ \tag{15}$$

$$+R_{ij}^{(2)}(x,y,t)K_{ip}^{(2)}(t,y,s)]f(t,s)dtds$$

Corollary. 2 *Assume that the conditions of the theorem are fulfilled. Then the system of non homogeneous partial integral equations (11) has a non trivial solution for each $g_i, i=\overline{1,n}$ if and only if the system of homogeneous partial integral equations (12) has only a trivial solution.*

Proof Let $f=(f_1,f_2,...,f_n)$and $g=(g_1,g_2,...,g_n)$ be the vector functions defined on $[a,b]^2$ with values in C^n, where C^n is an n-dimensional complex space. We define the operators K_1 and K_2 acting in the Banach space $C_{([a,b]^2,C^n)}$ of the continuous functions defined on $[a,b]^2$ with values in C^n, according to the following formulas:

$$(K_1f)_i(x,y)=\sum_{j=1}^{n}\int_a^b K_{ij}^{(1)}(x,y;t)f_j(t,y)dt$$

$$(K_2f)_i(x,y)=\sum_{j=1}^{n}\int_a^b K_{ij}^{(2)}(x,y;t)f_j(x,t)dt$$

Now the system of partial integral equations (11) can be written in the following operator form

$$f=K_1f+K_2f+g \tag{16}$$

According to the condition of theorem 2, $\Delta_2(y)\neq 0$. Therefore by the Fredholm's theorem the operator $I-K_1$ is invertible and its inverse operator $(I-K_1)^{-1}$ has the following form

$$(I-K_1)^{-1}=I+R_1, \tag{17}$$

where R_1 is the integral operator in the space $C_{([a,b]^2,R^n)}$ given by the following formula

$$(R_1f)_i(x,y)=\sum_{j=1}^{n}\int_a^b R_{ij}^{(1)}(x,y;t)f_j(t,y)dt$$

Using equality (17) we get from (16)

$$f=(I-K_1)^{-1}(K_2f+g)=(I+R_1)(K_2f+g)= \tag{18}$$

$$=K_2f+R_1K_2f+g+R_1g$$

Under the conditions of the theorem the inequality $\Delta_2(x)\neq 0$ holds for all $x\in[a,b]$. Reasoning as above, we get that the operator $(I-K_2)^{-1}$ exists and is represented as follows, $(I-K_2)^{-1}=I+R_2$, where R_2 is the operator in $C_{([a,b]^2,C^n)}$ given by the following formula

$$(R_2f)(x,y)=\sum_{j=1}^{n}\int_a^b R_{ij}^{(2)}(x,y;t)f_j(x,t)dt$$

Using similar arguments as above we obtain from (16)

$$f=K_1f+R_2K_1f+g+R_2g \tag{19}$$

From (18) we obtain

$$K_1f=K_1K_2f+K_1R_1K_2f+K_1g+K_1R_1g \tag{20}$$

For the resolvent R_1 of the operator K_1 the following Fredholm relation is valid,

$$K_1R_1=R_1K_1=R_1-K_1$$

Putting the expression for R_1K_1 to (20) we conclude that

$$K_1f=R_1K_2f+R_1g.$$

Substituting the expression for K_1f to (19) we get the following system of total integral equations

$$f=R_1K_2f+R_1g+R_2K_1f+g+R_2g,$$

or

$$f=(R_1K_2+R_2K_1)f+g+(R_1+R_2)g,$$

which is the same as (14).

We proved the part of theorem 2 concerning non homogeneous equation (11). For the proof of the part concerning homogeneous equation (12) we repeat the process from the first part of the proof putting $g_i=0, i=\overline{1,n}$.

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