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TAUTNESS AND APPLICATIONS OF THE ALEXANDER-SPANIER COHOMOLOGY OF K-TYPES

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Abstract

The aim of the present work is centered around the tautness property for the two K-types of Alexander-Spanier cohomology given by the authors. A version of the continuity property is proved, and some applications are given.

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0 Introduction

It is well-known that in the Alexander-Spanier cohomology theory [17], [18] or in the isomorphic theory of Čech [9], if the coefficient group G is topological then either the theory does not take into account the topology on G [9], [18], or considers only the case when G is compact to obtain a compact cohomology [5], [8]. Continuous cohomology naturally arises when the coefficient group of a cohomology theory is topological [6],[7],[11]. The partially continuous Alexander-Spanier cohomology theory [14] can be considered as a variant of the continuous cohomology of a space with two topologies in the sense of Bott-Haefliger [15]; also it is isomorphic to the continuous cohomology of a simplicial space defined by Brown-Szczarba [6].

The idea of K-groups [1],[2] where K is a locally-finite simplicial complex, is used to introduce the K-types of Alexander-Spanier cohomology with coefficients in a pair (G, G') of topological abelian groups [3],[4]; namely, K-Alexander-Spanier and partially continuous K-Alexander-Spanier cohomologies $\bar{H}^*_K$, $\tilde{H}^*_K$. It is proved that these K-types satisfied the seven Eilenberg-Steenrod axioms [9]; the excision axiom for the second K-type is verified for compact Hausdorff spaces when (G, G') are absolutely retract. Therefore the uniqueness theorem of the cohomology theory on the category of compact polyhedral pairs [9], asserts that our Alexander-Spanier K-types over a pair of absolute retract coefficient abelian groups are naturally isomorphic.

The present work is centered around the tautness property for the Alexander-Spanier K-types cohomology. Roughly speaking, we prove that the K-Alexander-Spanier cohomology of a closed subset in a paracompact space is isomorphic to the direct limit of the K-Alexander-Spanier cohomology of its neighborhoods; and that the partially continuous K-Alexander-Spanier cohomology of a neighborhood retract closed subspace of a Hausdorff space is isomorphic to the direct limit of the partially continuous K-Alexander-Spanier cohomology of its neighborhoods. Also a version of the continuity property is proved. Moreover, we study some application of the K-type cohomologies.

1 Alexander-Spanier Cohomology of K Types

Here we mention the notations which will be used throughout the present work [3],[4].

For an object (X, A) of the category Q of the pairs of topological spaces and their continuous maps, denote by $\Omega(X, A)[\tilde{\Omega}(X, A)]$ the set of the pairs $\bar{\alpha} = (\alpha, \alpha')$, where α is an open covering of X and α' is a subcollection of α covering $A[\alpha' = \alpha \cap A]$; it is directed with respect to the refinement relation $\bar{\alpha} < \bar{\beta}$, i.e. $\alpha < \beta$ and $\alpha' < \beta'$ [9]. Denote by $C^{q(\tau)}(\tilde{X})$ the group of the functions $\varphi^\tau : \tilde{X}^{q(\tau)+1} \to G$, where τ is a simplex in K, $q(\tau) = q + \dim \tau$, $q \geq 0$, and $\tilde{X}$ denotes either a space X or $\alpha \in \Omega(X)$. Let $C^{q(\tau)}(\tilde{X})$ be the subgroup of the direct product $\prod_{\tau \in K} C^{q(\tau)}(\tilde{X})$ consisting of such $\varphi = \{\varphi^\tau\}$ for which the condition (k) is satisfied, which states that there is a cofinite subset $\check{\tau}(\varphi)$ of K, i.e. $K - \check{\tau}(\varphi)$ is finite, such that $(\varphi^\tau)^{-1}(G') = \tilde{X}^{q(\tau)+1}$,

$\forall\tau \in \check{\tau}(\varphi)$. The coboundary $\delta^q : C^q(\tilde{X}) \to C^{q+1}(\tilde{X})$ is

$$(\delta^q\varphi)^\tau = \sum_{i=1}^{q(\tau)+1} (-1)^i \varphi^\tau p_i^{(q(\tau)+1)} + (-1)^{q(\tau)+1} \sum_{\sigma\in \mathrm{st}(\tau)} [\sigma : \tau]\varphi^\sigma \ ,$$

where $\mathrm{st}(\tau) = \{\sigma \in K : \tau \text{ is } (\dim\sigma - 1)\text{–face of } \sigma\}$, $p_i^{(\tau)} : X^{\tau+1} \to X^\tau$ is the projection defined by: if $\hat{t}_i$ is the τ-tuple consisting of $t = (x_0, \ldots, x_\tau) \in X^{\tau+1}$ with x_i omitted, then $p_i^{(\tau)}(t) = \hat{t}_i$, $0 \le i \le \tau$. The cohomology groups of the cochain complex $C^{\neq}(X) = \{C^q(X), \delta^q\}$ is, in general, uninteresting, as shown in the following theorem [3].

Theorem 1.1. If $\dim K = 0$, then $H^q(C^{\neq}(X)) \cong G^{*K}$ (the subgroup of $G^K = \prod_{\tau\in K} G^\tau$, $G^\tau = G$, consisting of those elements having all but a finite number of their τ-coordinates in G'), and $H^q(C^{\neq}(X)) = 0$, when $q \neq 0$.

To pass to more interesting cohomology groups, the topology of the space X will be used to define that $\varphi \in C^q(X)$ is said to be K-locally zero on $M \subseteq X$ if there is $\alpha \in \Omega_X(M)$ (the set of external covering of M by open subsets of X) such that φ vanishes on $\alpha \cap M$, i.e. each φ^τ vanishes on $(\alpha \cap M)^{q(\tau)+1}$, where $\alpha^\tau = \cup\{u_\alpha^\tau : u_\alpha \in \alpha\}$. The subgroups of $C^q(X)$ consisting of those elements which are K-locally zero on X, A respectively are denoted by $C_0^q(X)$, $C^q(X, A)$. The K-Alexander-Spanier cohomology of (X, A) over (G, G'), denoted by $\bar{H}_K^*(X, A)$, is the cohomology of the quotient cochain complex $\bar{C}_K^{\neq}(X, A) = C^{\neq}(X, A)/C_0^{\neq}(X)$. If $f : (X, A) \to (Y, B)$ is in Q, $\bar{\beta} \in \Omega(Y, B)$ and $\bar{\alpha} = f^{-1}(\bar{\beta})$, then f defines a cochain map $\bar{f}^{\neq} : \bar{C}_K^{\neq}(Y, B) \to \bar{C}_K^{\neq}(X, A)$, where $\check{\tau}(f^q\varphi) = \check{\tau}(\varphi)$ for each $\varphi \in C^q(Y)$. In turn, $\bar{f}^{\neq}$ induces the homomorphism $\bar{f}^* : \bar{H}_K^*(Y, B) \to \bar{H}_K^*(X, A)$.

On the other hand, for $\bar{\alpha} \in \Omega(X, A)$, denote by $C_{\bar{\alpha}}^q$. The subgroup of $C_{\bar{\alpha}}^q = C^q(\alpha)$ consisting of those φ which vanishes on $\alpha' \cap A$. Then we obtain a direct system $\{C_{\bar{\alpha}}^{\neq}\}_{\Omega(X,A)}$ such that any map $f \in Q$ constitutes a map $F : \{C_{\bar{\beta}}^{\neq}\}_{\Omega(Y,B)} \to \{C_{\bar{\alpha}}^{\neq}\}_{\Omega(X,A)}$ [9]; its limit is F^∞.

Theorem 1.2. The K-Alexander-Spanier cohomology functor $\{\bar{H}_K^*, \bar{f}^*\}$ is naturally isomorphic to the functor $\{\varinjlim\{H^*(C_{\bar{\alpha}}^{\neq})\}_{\Omega(X,A)}, F^{\infty *}\}$ [4].

In the previous part, the topology on (G, G) plays no role; to pass to the second cohomology of K-type we characterize an element $\varphi \in C^q(X)$ to be K-partially continuous if it is continuous on some $\alpha \in \Omega(X)$, i.e. $\varphi^\tau|\alpha^{q(\tau)+1}$ are continuous functions. Let $L^q(X)$ be the group of all such elements, and $M_K^{\neq}(X) = L^{\neq}(X)/C_0^{\neq}(X)$. The subgroup of C_α^q, where $\alpha \in \Omega(X)$, consisting of the K-continuous elements φ, i.e. φ^τ are continuous, is denoted by M_α^q. Let $i : A \hookrightarrow X$, define $M_K^{\neq}(X, A)$ to be the mapping cone of $i^{\neq} : M_K^{\neq}(X) \to M_K^{\neq}(A)$, [13],[18], assuming that $M_K^q(X, A) = M_K^q(X) \oplus M_K^{q-1}(A)$, and the coboundary is $\Delta^q(\varphi, \psi) = (-\delta^q\varphi, i^q\varphi + \delta^{q-1}\psi)$. The cohomology of $M_K^{\neq}(X, A)$ is the partially continuous K-Alexander-Spanier cohomology of (X, A) over the topological pair (G, G') of coefficient groups; it is denoted by $\tilde{H}_K^*(X, A)$.

On the other hand, if $\bar{\alpha} \in \tilde{\Omega}(X, A)$, then i defines a cochain map $i_\alpha^{\neq} : M_\alpha^{\neq} \to M_{\alpha'}^{\neq}$; its mapping cone is denoted by $M_{\bar{\alpha}}^{\neq}$.

Theorem 1.3. For a pair $(X,A)\in Q$ with A is closed, $M_K^{\neq}(X,A)$ is naturally isomorphic to $\varinjlim\{M_{\bar\alpha}^{\neq}\}_{\tilde\Omega(X,A)}$ [4].

Theorem 1.4 For a discrete space, and $q\geq 0$, $\tilde H_K^q(X)\simeq \bar H_K^q(X)$.

Proof. Since $X^{q(\tau)+1}$ admits a discrete topology, it follows that each τ-coordinate φ^τ of $\varphi\in C_K^q(X)$ is continuous [16]. Then φ is K-partially continuous with respect to any $\alpha\in\Omega(X)$. Therefore $L^q(X)=C_K^q(X)$ and $M_K^{\neq}(X)=\bar C_K^{\neq}(X)$.

2 Tautness and Continuity Properties

This article is devoted to study the tautness property for both Alexander-Spanier cohomology of K-types. One of its applications is the continuity property.

The star of a subset A in a space X with respect to $\alpha\in\Omega(X)$ is

$$\mathrm{st}(A,\alpha)=\cup\{U_\alpha\in\alpha : U_d\cap A\neq\emptyset\}$$

The star of α is

$$\alpha^*=\{\mathrm{st}(U_\alpha,\alpha) : u_\alpha\in\alpha\}$$

Definition 2.1 Let α, $\beta\in\Omega(X)$, then β is a star-refinement of α, written $\alpha <^* \beta$, if $\alpha<\beta^*$.

Denote by $\mathcal{N}(A)$ the collections of neighborhoods $\{N\}$ of A in X; it is directed downward by inclusion. If $N_1<N_2$, then the inclusion $\pi_{N_1N_2}: N_2\hookrightarrow N_1$ induces the homomorphisms $\bar\pi^*_{N_1N_2}:\bar H_K^q(N_1)\to \bar H_K^q(N_2)$. Also $i_N : A\hookrightarrow N$ induces $\bar i_N^* : \bar H_K^q(N)\to\bar H_K^q(A)$, and they define a homomorphism

$$I^\infty : \varinjlim\{\bar H_K^q(N),\ \bar\pi^*_{N_1N_2}\}_{\mathcal{N}(A)}\to \bar H_K^q(A)\ .$$

Theorem 2.1 (Tautness). A closed subspace of a paracompact space is a taut subspace relative to the K-Alexander-Spanier cohomology, i.e. I^∞ is an isomorphism for each q and any pair (G,G') of coefficient groups.

Proof. (1) I^∞ is an epimorphism. Actually let $h\in\bar H_K^q(A)$ with representative $\bar\varphi\in\bar C_K^q(A)$, written $h=[\bar\varphi]$. Let $\varphi\in C^q(A)$ such that $\varphi\in\bar\varphi$. Then there is $\alpha=\{u_\alpha=\nu_\alpha\cap A:\nu_\alpha\subseteq X$ is open$\}\in\Omega(A)$ such that

$$(\delta^q\varphi)^\tau|\alpha^{q(\tau)+2}=0 \tag{2.1}$$

Since A is closed, it follows that $\beta=\{\nu_\alpha\}\cup\{X-A\}\in\Omega(X)$. The paracompactness of X is equivalent to the existence of such $\gamma\in\Omega(X)$ that $\beta<^*\gamma$ [21], and a neighborhood N of A and an extension $f:N\to A$ (not necessarily continuous) of the identity map id_A of A, i.e. $fi_N=\mathrm{id}_A$, such that $f(u_\gamma\cap N)\subseteq \mathrm{st}(u_\gamma,\gamma)$ for each $u_\gamma\in\gamma$ [18]. One can show that f defines a cochain map $f^{\neq}: C^{\neq}(A)\to C^{\neq}(N)$ by $(f^q\varphi)^\tau=\varphi^\tau f^{(q(\tau)+1)}$ with $\check\tau(f^q\varphi)=\check\tau(\varphi)$, where

$f^{(\tau)}: N^\tau \to A^\tau$ given by $f(x_0,\dots x_{\tau-1}) = (f(x_0),\dots,f(x_{\tau-1}))$. The relation $\beta < \gamma^*$ yields that for each $u_\gamma \in \gamma$ there is $u_\beta \in \beta$ such that $f(u_\gamma \cap N) \subseteq \mathrm{st}(u_\gamma,\gamma) \subseteq u_\beta$. Because $f(N) = A$, then $f(u_\gamma \cap N) \subseteq u_\beta \cap A \subseteq u_\alpha$ for some $u_\alpha \in \alpha$. By using (2.1), we get $(\delta^q f^q \varphi)^\tau|(\gamma \cap N)^{q(\tau)+2} = 0$, i.e. $\delta^q(f^q\varphi) \in C_0^{q+1}(N)$. Then $f^q\varphi$ represents a cocycle $\overline{f^q\varphi} \in \bar{C}_K^q(N)$ which, in turn, defines $h_N \in \bar{H}_K^q(N)$, i.e. $h_N = [\overline{f^q\varphi}]$. Let $t \in A^{q(\tau)+1}$, then

$$(i_N^q(f^q\varphi))^\tau(t) = \varphi^\tau f^{(q(\tau)+1)} i_N^{(q(\tau)+1)}(t) = \varphi^\tau(t) \ ,$$

and therefore $\bar{i}_N^* h_N = [\overline{(f i_N)^q \varphi}] = [\bar{\varphi}] = h$.

(2) I^∞ is a monomorphism. Actually, let $h_1 \in \bar{H}_K^q(N_1)$, $\bar{\varphi}_1 \in \bar{C}_K^q(N_1)$ and $\varphi_1 \in C^q(N_1)$ such that $\varphi_1 \in \bar{\varphi}_1$, $\bar{\varphi}_1 \in h_1$, and $[h_1] \in \mathrm{Ker} I^\infty$.

First, one can consider that the neighborhood N_1 of A is a paracompact subset of X. For, if N_1 is not so, then there is a paracompact subset M_1 of X such that $M_1 < N_1$ (e.g., take $M_1 = X$) [10]. The inclusion $\pi_{M_1N_1}$ induces an epimorphism $\bar{\pi}^{\neq}_{M_1N_1}$ [3], let $\bar{\pi}^q_{M_1N_1}\bar{\psi}_1 = \bar{\varphi}_1$. Thus the cohomology class of $\bar{H}_K^q(M_1)$ represented by $\bar{\psi}_1$ is $[h_1]$, which shows that N_1 can be taken paracompact.

Now, $\bar{\varphi}_1 \in \mathrm{Ker}\delta^q$, or equivalently, there is $\alpha = \{u_\alpha = \nu_\alpha \cap N_1 : \nu_\alpha \subseteq X \text{ is open}\} \in \Omega(N_1)$ such that

$$(\delta^q \varphi_1)^\tau | \alpha^{q(\tau)+2} = 0 \ . \tag{2.2}$$

On the other hand, the assumption $\bar{i}_{N_1}^* h_1 = 0$ asserts that there exists $\bar{\varphi} \in \bar{C}_K^{q-1}(A)$ such that $i_{N_1}^q \varphi_1 - \delta^{q-1}\varphi \in C_0^q(A)$, where $\varphi \in \bar{\varphi}$. This means that there is such $\beta = \{u_\beta = \omega_\beta \cap A : \omega_\beta \subseteq X$ is open$\} \in \Omega(A)$ that

$$(i_{N_1}^q \varphi_1)^\tau = (\delta^{q-1}\varphi)^\tau \text{ on } \beta^{q(\tau)+1} \tag{2.3}$$

Assume that $\beta_1 = \{u_{\beta_1} = \omega_\beta \cap N_1\} \cup \{N_1 - A\}$. The paracompactness of N_1 asserts the existence of $\gamma_1, \gamma_2 \in \Omega(N_1)$ for which $\alpha <^* \gamma_1$ and $\beta_1 <^* \gamma_2$. The directedness of $\Omega(N_1)$ implies that there $\gamma \in \Omega(N_1)$ for which $\gamma_1, \gamma_2 < \gamma$; and so for each $u_\gamma \in \gamma$ there are $u_{\gamma_i} \in \gamma_i$, $i = 1, 2$ and $u_\alpha \in \alpha$, $u_{\beta_1} \in \beta_1$ such that

$$u_\gamma \subset u_{\gamma_i} \subseteq \mathrm{st}(u_{\gamma_i}, \gamma_i) \subseteq u_\alpha \cap u_{\beta_1} \ ,$$

Then

$$\mathrm{st}(u_\gamma, \gamma) \subseteq u_\alpha \cap u_{\beta_1} \tag{2.4}$$

i.e. $\alpha, \beta_1 <^* \gamma$. According to Lemma 6.6.1 in [18], there is a neighborhood N_2 of N_1 and $f : N_2 \to A$ (not necessarily continuous) such that $f i_{N_2} = \mathrm{id}_A$, and $u_{\beta_1} \in \beta_1$ such that

$$f(u_\gamma \cap N_2) \subseteq \mathrm{st}(u_\gamma, \gamma) \subseteq u_{\beta_1} \subseteq u_{\beta_1} \cap A = u_\beta \tag{2.5}$$

Then, by (2.3), we get

$$(\delta^{q-1} f^{q-1} \varphi)^\tau = (f^q i_{N_1}^q \varphi_1) \text{ on } (\gamma \cap N_2)^{q(\tau)+1} \tag{2.6}$$

Define $D^q : C^{q+1}(N_1) \to C^q(N_2)$ by:

$$\text{if } t = (x_0, \dots, x_{q(\tau)}) \in N_2^{q(\tau)+1} \text{ and } \psi_1 \in C^{q+1}(N_1)$$

then

$$(D^q\psi_1)^\tau(t) = \sum_{r=0}^{q(\tau)} (-1)^\gamma \psi_1^\tau(y_0, \dots, y_\tau, z_\tau, \dots, z_{q(\tau)}) \ ,$$

where

$$y_j = \pi_{N_1N_2}(x_j), \ \ z_j = (i_{N_1}f)(x_j) = f(x_j) \ ,$$

and $\check{\tau}(D^q\psi_1) = \check{\tau}(\psi_1)$. A similar calculation as given in [4], we get

$$(\delta^{q-1}D^{q-1}\varphi_1)^\tau = (f^q i^q_{N_1}\varphi_1)^\tau - (\pi^q_{N_1N_2}\varphi_1)^\tau - (D^q\delta^q\varphi_1)^\tau \tag{2.7}$$

By (2.4), (2.5) for each $u_\gamma \in \gamma$, there is $u_\alpha \in \alpha$ such that

$$(u_\gamma \cap N_2) \cup f(u_\gamma \cap N_2) \subseteq u_\alpha$$

Then, by (2.7), (2.2), (2.6) consequently, we have

$$(\delta^{q-1}D^{q-1}\varphi_1)^\tau = (f^q i^q_{N_1}\varphi_1)^\tau - (\pi^q_{N_1N_2}\varphi_1)^\tau \text{ on } (\gamma \cap N_2)^{q(\tau)+1} \ ,$$

and so

$$(\pi^q_{N_1N_2}\varphi_1)^\tau = (\delta^{q-1}(f^{q-1}\varphi - D^{q-1}\varphi_1))^\tau \text{ on } (\gamma \cap N_2)^{q(\tau)+1} \ .$$

Therefore

$$\psi_2 = f^{q-1}\varphi - D^{q-1}\varphi_1 \in C^{q-1}(N_2) \text{ such that}$$

$$(\pi^q_{N_1N_2}\varphi_1)^\tau = (\delta^{q-1}\psi_2)^\tau \text{ on } (\gamma \cap N_2)^{q(\tau)+1} \ ,$$

i.e. $\bar{\pi}_{N_1N_2}h_1 = 0$ which completes the proof.

Corollary 2.2. Any one-point subset of a paracompact is a taut subspace relative to $\bar{H}^*_K$.

The next part is devoted to study the tautness property for $\tilde{H}^*_K$, which is also valid for $\bar{H}^*_K$. The idea and results of $\alpha - \beta$-contiguous maps, introduced in [4] plays an essential role in this study.

The inclusions $\pi_{N_1N_2} : N_2 \hookrightarrow N_1$ corresponding to the relations $N_1 < N_2$ in $\mathcal{N}(A)$, define the direct system $\{\tilde{H}^q_K(N), \tilde{\pi}^*_{N_1N_2}\}$. Also the inclusion $i_N : A \hookrightarrow N$, where $N \in \mathcal{N}(A)$, define a map of direct systems [9]:

$$I_N : \{H^q(M^{\neq}_\alpha), \tilde{\pi}^*_{\alpha\beta}\}_{\Omega(N)} \longrightarrow \{H^q(M^{\neq}_{\tilde{\alpha}}, \tilde{\pi}^*_{\tilde{\alpha}\tilde{\beta}}\}_{\Omega(A)}$$

where $\alpha \in \Omega(N)$, $\tilde{\alpha} = i_N^{-1}(\alpha) = \alpha \cap A$. On the other hand, $\{\tilde{i}^*_N\}$ define a homomorphism

$$\tilde{I}^\infty : \varinjlim \{\tilde{H}^q_K(N), \tilde{\pi}^*_{N_1N_2}\}_{\mathcal{N}(A)} \to \tilde{H}^q_K(A)$$

Theorem 2.3 (Tautness). If A is a closed subset in a Hausdorff space X such that A is a neighborhood retract, then A is a taut subspace relative to the cohomology $\tilde{H}^*_K$.

Proof. 1) $\tilde{I}^\infty$ is an epimorphism. Actually, let $h \in \tilde{H}^q_K(A)$. Without loss of generality, the neighborhood retractness of A in X yields that A has an open neighborhood U (in X) such that $U \subseteq N$ and a retraction $\tau_1 : U \to A$ (If U_1 is an open neighborhood of A of which A is retract but $U_1 \not\subseteq N$, take $U = U_1 \cap \mathrm{Int}N$). Let $i_U : A \hookrightarrow U$ then, $\tilde{I}^\infty[\tilde{\tau}_1^*(h)] = \tilde{i}_U^*(\tilde{\tau}_1^* h) = \widetilde{id}_A^*(h) = h$.

2) $\tilde{I}^\infty$ is a monomorphism. Let $[h] \in \mathrm{Ker}\tilde{I}^\infty$,

It is sufficient to construct $V \in \mathcal{N}(A)$ satisfying $N < V$ and $\tilde{\pi}^*_{NV} h = 0$.

Since the cohomology functor commutes with the direct limit [18]. Theorem 1.3 asserts that one may assume that h belongs to $\varinjlim\{H^q(M^{\neq}_\alpha),\, \tilde{\pi}^*_{\alpha\beta}\}_{\Omega(N)}$ with representative $h_\alpha \in H^q(M^{\neq}_\alpha)$, where

$$\alpha = \{u_\alpha = \omega_\alpha \cap N : \omega_\alpha \subseteq X \text{ is open}\} \in \Omega(N)$$

Let $\alpha_1 = \{\omega_\alpha\} \cup \{X - A\}$, $\tilde{\alpha} = \alpha_1 \cap A$,

$$\beta = \{u_\beta = \tau_1^{-1}(u_{\tilde{\alpha}}) \cap (u_\alpha \cap U) : \phi \neq u_{\tilde{\alpha}} \in \tilde{\alpha}\} \ ,$$

$V = \cup u_\beta$, $\tau = \tau_1|V : V \hookrightarrow A$, and $\alpha' = \alpha_1 \cap V$. Then $\tilde{\alpha} \in \Omega(A)$, $\alpha' = \alpha \cap V \in \Omega(V)$, $u_{\tilde{\alpha}} \subseteq u_\beta$ for each $u_{\tilde{\alpha}} \neq \phi$, β is a family of open subsets in $U$and so open in X, V is an open neighborhood of A such that $V \subseteq U$, and $\beta \in \Omega(V)$. Since $u_\beta = u_\beta \cap u_\alpha \subseteq V \cap u_\alpha = u_{\alpha'}$, it follows that $\alpha' < \beta$. Also $\alpha' \cap A = \alpha \cap A = \tilde{\alpha}$ and $j^{-1}\beta = \tilde{\alpha}$, where $j : A \hookrightarrow V$. If $\ell : V \hookrightarrow N$, and $[\varphi] \in H^q(M^{\neq}_\alpha)$, then

$$\begin{aligned}\tilde{j}^*_\beta \tilde{\pi}^*_{\alpha'\beta} \tilde{\ell}^*_\alpha[\varphi] &= \tilde{j}^*_\beta[\{(\varphi^\tau|\alpha'^{q(\tau)+1})|\beta^{q(\tau)+1}\}] \\ &= [\{\varphi^\tau|\tilde{\alpha}^{q(\tau)+1}\}] \ ,\end{aligned}$$

i.e.

$$\tilde{j}^*_\beta \tilde{\pi}^*_{\alpha'\beta} \tilde{\ell}^*_\alpha = \tilde{i}^*_{N,\alpha} \tag{2.8}$$

where $\tilde{i}^{\neq}_{N,\alpha} : M^{\neq}_\alpha \to M^{\neq}_{\tilde{\alpha}}$ is induced by $i_N : A \hookrightarrow N$.

On the other hand, $(j\tau)u_\beta \subseteq u_\beta$ and so $j\tau$, $id_V : V \to V$ are $\beta - \beta$ contiguous [4].

It follows that $(\widetilde{id}_V)^q_{\beta-\beta}$, $(\widetilde{j\tau})^q_{\beta-\beta} : M^q_\beta \to M^q_\beta$ are cochain homotopic [4]. Then $(\widetilde{id}_V)^*_{\beta-\beta} = (\widetilde{j\tau})^*_{\beta-\beta} = \tilde{\tau}^*_{\tilde{\alpha}-\beta}\tilde{j}^*_\beta$, which yields that $\tilde{j}^*_\beta$ is a monomorphism. Because $\tilde{i}^*_{N,\alpha} h_\alpha = 0$, (2.8) yields that $\tilde{\pi}^*_{\alpha'\beta}\tilde{\ell}^*_\alpha h_\alpha = 0$. Since $\tilde{\ell}^*_\alpha h_\alpha$, $\tilde{\pi}^*_{\alpha'\beta}(\tilde{\ell}_\alpha h_\alpha)$ represent the zero element of $\varinjlim\{H^q(M^{\neq}_\alpha),\, \tilde{\pi}^*_{\alpha\beta}\}_{\Omega(N)}$, it follows that $\tilde{\pi}^*_{NV} h = [\tilde{\ell}^*_\alpha h_\alpha] = 0$.

The rest of this article is centered around a special case of the continuity property for $\bar{H}^*_K$. As an application of the continuity property the cohomology groups satisfy a much stronger form of the excision axiom.

The following results can be deduced from those given in [9].

Lemma 2.4. Let X be the intersection of a nested system $\{X_\alpha, \pi_{\beta\alpha}\}_\Lambda$, then (i) X and $\varprojlim\{X_\alpha, \pi_{\beta\alpha}\}_\Lambda$ are homeomorphic

(ii) If the nested system consists of compact Hausdorff spaces then X is a closed subset of each X_α.

(iii) If N is an open neighborhood of X in X_α (for some $\alpha \in \Lambda$), then there is $\beta > \alpha$ in Λ such that $X_\beta \subseteq N$.

The inclusions $i_\alpha : X \hookrightarrow X_\alpha$ define a map

$$I : \{\bar{H}^q_K(X_\alpha), \bar{\pi}^*_{\alpha\beta}\}_\Lambda \to \bar{H}^q_K(X) ,$$

its direct limit is denoted by $\bar{I}^\infty$.

Theorem 2.5 (Weak continuity). If X is the intersection of a nested system $\{X_\alpha, \pi_{\beta\alpha}\}_\Lambda$ of compact Hausdorff spaces, then $\bar{I}^\infty$ is an isomorphism.

Proof. Since each X_α is a paracompact Hausdorff space [10] and X_α is closed in X (Lemma 2.4), it follows, by Theorem 2.1, that X is a taut subspace in X_α relative to $\bar{H}^*_K$.

(1) $\bar{I}^\infty$ is an epimorphism. Let $h \in \bar{H}^q_K(X)$, then, according to Theorem 2.1, there exists an open neighborhood N of X in X_α and $h_N \in \bar{H}^q_K(N)$, such that $\bar{i}^*_N(h_N) = h$. By Lemma 2.4, there is $\beta > \alpha$ in Λ such that $X_\beta \subseteq N$. Let $i_\beta : X \hookrightarrow X_\beta$, $j_\beta : X_\beta \hookrightarrow N$. Because $\bar{i}^*_\beta(\bar{j}^*_\beta h_N) = (\overline{j_\beta i_\beta})^* h_N = \bar{i}^*_N h_N = h$, then $\bar{I}^\infty[\bar{j}^*_\beta h_N] = h$.

(2) $\bar{I}^\infty$ is a monomorphism. Let $[h_\alpha] \in \text{Ker}\bar{I}^\infty$, i.e. $\bar{i}^*_\alpha h_\alpha = 0$. The tautness of X in X_α yields, by Theorem 2.1, an open neighborhood N of X in X_α such that h_N is the unique element for which $\bar{i}'^*_N h_N = 0$, where $i'_N : X \hookrightarrow N$. Because $\bar{i}'^*_N(\bar{i}^*_N h_\alpha) = \bar{i}^*_\alpha h_\alpha = 0$, then $\bar{i}^*_N h_\alpha = 0$. Let $\beta > \alpha$ in Λ such that $X_\beta \subseteq N$, then $\bar{\pi}^*_{\alpha\beta} h_\alpha = (\overline{i_N i_\beta})^* h_\alpha = \bar{j}^*_\beta(\bar{i}^*_N h_\alpha) = 0$, i.e. $[h_\alpha] = 0$.

3 Applications

One of the good applications of the Alexander-Spanier cohomology of K-types is the study of the 0-dimensional cohomology groups and their relation with the connectedness of the space [4]. In this article two applications are given. In a next work we hope to give more applications. The first application is concentrated to define the partially continuous K-Alexander-Spanier cohomology of an excision map and calculate its value for some dimensions.

Let $\tilde{f}^{\neq} : M^{\neq}_K(Y, B) \to M^{\neq}_K(X, A)$ be the cochain map induced by the map f in Q. Define $M^{\neq}_K(f)$ to be the mapping cone of $\tilde{f}^{\neq}$ by:

$$M^q_K(f) = M^q_K(Y, B) \oplus M^{q-1}_K(X, A) ,$$

$$= M^q_K(Y) \oplus M^{q-1}_K(B) \oplus M^{q-1}_K(X) \oplus M^{q-2}_K(A) ,$$

and the coboundary is

$$\tilde{\Delta}^q(\varphi_2, \psi_2, \varphi_1, \psi_1) =$$

$$= (-\Delta^q(\varphi_2, \psi_2), \Delta^q(\varphi_1, \psi_1) + \tilde{f}^q(\varphi_2, \psi_2))$$

$$= (\delta^q\varphi_2, -\tilde{i}^q\varphi_2 - \delta^{q-1}\psi_2,\ -\delta^{q-1}\varphi_1 + \tilde{f}^q\varphi_2\ ,$$

$$\tilde{i}^{q-1}\varphi_1 + \delta^{q-2}\psi_1 + \widetilde{f|A})^{q-1}\psi_2)$$

Then there is a short exact sequence

$$0 \to \overset{+}{M}{}^{\neq}_K(X, A) \xrightarrow{\lambda^{\neq}} M^{\neq}_K(f) \xrightarrow{x^{\neq}} \bar{M}^{\neq}_K(Y, B) \to \underline{O}_2 \tag{3.1}$$

where $\lambda^{\neq}$, $\chi^{\neq}$ are injection, projection respectively; $\overset{+}{M}{}^{\neq}(X, A)$ is the complex $M^{\neq}_K(X, A)$ with the dimensions all raised by one, and $\bar{M}^{\neq}(Y, B)$ is the complex $M^{\neq}(Y, B)$ with the sign of the coboundary changed [12]. Note that $H^q(\bar{M}^{\neq}_K(Y, B)) = \tilde{H}^q_K(Y, B)$. Let V be an open subset of X such that $\bar{V} \subseteq \mathrm{Int}A, B = X - V$, and $C = A - V$. Put the excision map $e : (B, C) \hookrightarrow (X, A)$ in (3.1) instead of f, and then apply the cohomology functor, we get the long exact sequence:

$$\ldots \to \tilde{H}^q_K(e) \xrightarrow{\tilde{\chi}^*} \tilde{H}^q_K(X, A) \xrightarrow{\tilde{e}^*} \tilde{H}^q_K(B, C)$$

$$\xrightarrow{\tilde{\lambda}^*} \tilde{H}^{q+1}_K(e) \to \ldots \tag{3.2}$$

Thus the groups $\tilde{H}^q_K(e)$, $\tilde{H}^{q+1}_K(e)$ measure how much the cohomological groups deviate from the excision axiom.

Theorem 3.1. If $\dim K = 0$, $e : (B, C) \hookrightarrow (X, A)$ is an excision map, where A is closed and (G, G') any pair of topological abelian groups, then $\tilde{H}^q_K(e) = 0$ when $q = 0$ or $q = 1$.

Proof. (1) Case $q = 0$. We have

$$M^0_K(e) = M^0_K(X, A) = M^0_K(X) = L^0_K(X)$$

Let $\varphi \in M^0_K(e)$ such that $\tilde{\Delta}\varphi = 0$, then $\tilde{i}^0\varphi = 0$, $\tilde{e}\varphi = 0$. Then $\varphi = 0$ [4], which means that Ker $\tilde{\Delta}^0 = 0$.

(2) Case $q = 1$. We have

$$M^1_K(e) = M'_K(X) \oplus L^0(A) \oplus L^0(B)\ .$$

It is sufficient to show that $\mathrm{Ker}\tilde{\Delta}^1 \subseteq \mathrm{Im}\tilde{\Delta}^0$. Let $(\varphi_2, \psi_2, \varphi_1, 0) \in Ker\tilde{\Delta}^1$, then

$$\delta^1\varphi = 0, \quad \tilde{i}'\varphi_2 = -\delta^0\psi_2$$

$$\tilde{e}^1\varphi_2 = \delta^0\varphi_1 \tag{3.3}$$

$$\tilde{e}^0_1(-\psi_2) = \tilde{j}\varphi_1 \tag{3.4}$$

where $i : A \hookrightarrow X$, $j : C \hookrightarrow B$ and $e_1 = e|C$.

By (3.4), there exists [4], $\varphi \in M^0_K(X) = L^0(X)$ such that

$$\tilde{i}^0\varphi = -\psi_2, \quad \tilde{e}^0\varphi = \varphi_1 \tag{3.5}$$

By (3.3)-(3.5), we get

$$\tilde{i}^1(\delta^0\varphi - \varphi_2) = 0, \quad \tilde{e}^1(\delta^0\varphi - \varphi_2) = 0 \tag{3.6}$$

Then $\delta^0\varphi = \varphi_2$ [4], which with (3.6) yield that $(\varphi, 0, 0, 0) \in M_K^0(e)$ such that $\tilde{\Delta}^0(\varphi, 0, 0, 0) = (\varphi_2, \psi_2, \varphi_1, 0)$.

Combining the sequence (3.2) and the above theorem, we get the following result.

Corollary 3.2 Under the assumptions of Theorem (3.1), the map $\tilde{e}^{*0} : \tilde{H}_K^0(X, A) \to \tilde{H}_K^0(B, C)$ is an isomorphism but $\tilde{e}^{*1}$ is a monomorphism:

The second application is to give attention in our work to use a pair of coefficients groups, an arbitrary locally-finite simplicial complex K, and the condition (k).

Let $\eta : (G, G') \to (F, F')$ be a homeomorphism of pairs of (discrete) abelian groups which is an epimorphism, $(L, L') = \mathrm{Ker}\eta$ and $\lambda : (L, L') \hookrightarrow (G, G')$. Then for each $\bar{\alpha} \in \Omega(X, A)$, the maps η, λ define, naturally a short exact sequence

$$0 \to C^q(\bar{\alpha}, L, L') \to C^q(\bar{\alpha}; G, G') \to C^q(\bar{\alpha}; F, F') \to 0 \ ;$$

its cohomology is a long exact sequence [12] denoted by $S_{\bar{\alpha}}$. One can show that $\{S_{\bar{\alpha}}\}_{\Omega(X,A)}$ is a direct system, its direct limit [3], [4]

$$\ldots \to \bar{H}_K^{q-1}(X, A; F, F') \to \bar{H}_K^q(X, A'; L, L') \to$$

$$\bar{H}_K^q(X, A; G, G') \to \bar{H}_K^q(X, A, F, F') \to \bar{H}_K^{q+1}(X, A; L, L') \to \ldots$$

Now instead of F take the factor group G/G' and so instead of F' will be the null subgroup of G/G'. Then the above sequence yields the following result.

Theorem 3.3 Consider that (X, A) has a trivial $(q-1)$-dimensional K-Alexander-Spanier cohomology group with finite cochains, and a trivial $(q+1)$-dimensional K-Alexander-Spanier cohomology with infinite cochains, taken over the coefficient groups G/G' and G' respectively. Then the group $\bar{H}_K^q(X, A; G, G')$ defined over an arbitrary pair (G, G') of coefficient groups is the extension of the cohomology group $\bar{H}_K^q(X, A; G')$ with infinite cochains over G' by the group $\bar{H}_K^q(X, A, G/G')$ with finite cochains over G/G'.

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