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# METHODS AND COMPLEX OF PROGRAMS FOR RADIATING PARTICLE 3DOF NONLINEAR DYNAMICS ANALYSIS

Y.Alexahin#, JINR, Dubna, Russia


## *Abstract*

The Lie-transform perturbation theory for non-autonomous non-Hamiltonian systems and its implementation in a complex of ***Mathematica*** programs are described. On the example of LEP 108/90 lattice the radiation effects are shown to play an important effect on particle stability at high energies.


## 1 INTRODUCTION

To meet the strict requirements on the beam quality and lifetime in high energy $e^+e^-$ rings used as circular colliders, synchrotron radiation sources and damping rings of linear colliders, a tool for nonlinear dynamics analysis is necessary which takes into account the synchrotron radiation and coupling of all three degrees of freedom in the presence of errors.

An efficient tool bringing a Hamiltonian dynamical system to the normal form with the help of the Lie-transform perturbation theory was presented in [1]. It permits to evaluate separately contributions of various sources of nonlinearity (kinetic energy, magnetic multipoles, RF field) to the resonance excitation (with a possibility to keep some multipole strengths in symbolic form); the transformation generating function being known all over the lattice may be important for analysis of the off-resonance "smear" and nonlinear emittance production in the case of weak synchrotron radiation.

In the case of strong synchrotron radiation (such as LEP2) the dependence of the radiation reaction force in quadrupoles on the transverse coordinates introduce strong *radiative beta-synchrotron coupling* [2] which should be included in the normalization process. The corresponding generalization of the Lie-transform perturbation theory is outlined in the present report, as well as its implementation in a complex of ***Mathematica*** notebooks permitting to study high order nonlinear effects in machines as complicate as LEP or HERA-e.

## 2 PERTURBATION THEORY

### *2.1 Equation of motion of radiating particle*

Introducing 6D phase space column vector of coordinates and momenta

$$\underline{z} = (x,\ p_x,\ y,\ p_y,\ \sigma,\ \delta_p)^{\mathrm{T}} \tag{1}$$

and $\theta = s/R$ we have the following equation of motion

$$\dot{\underline{z}} \equiv \frac{d}{d\theta}\underline{z} = \underline{F} = \mathrm{S}\cdot\frac{\partial}{\partial \underline{z}}\mathcal{H} + \underline{F}^{(\mathrm{rad})}, \tag{2}$$

$$\mathrm{S} = \mathrm{S}_2 \oplus \mathrm{S}_2 \oplus \mathrm{S}_2, \quad \mathrm{S}_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3}$$

which is driven by the Hamiltonian and radiation reaction $\underline{F}^{(\mathrm{rad})}$, which can be decomposed into the mean (classical) and fluctuating (quantum) parts [3]:

$$\underline{F}^{(\mathrm{rad})} = \underline{F}^{(\mathrm{c})} + \underline{F}^{(\mathrm{q})}\ . \tag{4}$$

We will proceed as follows: solve for the nonlinear dynamics in the Hamiltonian and classical radiative fields and then add the quantum field, $\underline{F}^{(\mathrm{q})}$, to find the distribution of an ensemble of particles.

### *2.2 Linear normal modes*

The first step is subtraction of the finite closed orbit due to misalignments and (the mean part of) the energy losses (see e.g.[4]). We assume it to have been performed so that the power series expansion of the Hamiltonian starts with quadratic terms in the dynamical variables, whereas the mean part of the radiative force starts with linear terms.

For the next step we ignore nonlinearities and introduce eigenvectors $\underline{w}_n$ and eigenvalues $\lambda_n$ of the 1-turn transfer matrix $\mathrm{M}(2\pi+\theta, \theta)$. They form three complex conjugate pairs which we numerate as follows

$$\lambda_n = \exp[2\pi(iQ_n - \gamma_n)], \quad \lambda_{n+1} = \lambda_n^*, \quad n = 1, 3, 5 \tag{5}$$

Alternatively we will use Greek indices for numbering the normal modes, so that $Q_1 = Q_I$, $Q_3 = Q_{II}$, $Q_5 = Q_{III}$, $Q_{III} < 0$ above the transition energy.

We use the normalization

$$\underline{w}_\mu^+ \cdot \mathrm{S} \cdot \underline{w}_\mu = i \tag{6}$$

at $\theta$=0 and then propagate the eigenvectors so that to make them $2\pi$-periodic:

$$\underline{w}_n(\theta) = e^{-(iQ_n - \gamma_n)\theta}\,\mathrm{M}(\theta, 0)\cdot\underline{w}_n(0)\,, \tag{7}$$

Using the eigenvectors we can build a matrix, W, with the elements $W_{in}=(\underline{w}_n)_i$ and expand the phase space vector as follows

$$\underline{z} = \mathrm{W}(\theta)\cdot\underline{a} \tag{8}$$

The 6-tuple of the coefficients

$$\underline{a} = \{a_I, a_I^*, a_{II}, a_{II}^*, a_{III}, a_{III}^*\}^T \tag{9}$$

may be regarded as a new phase space vector. In the absence of nonlinearities

$$a_\mu(\theta) = \sqrt{I_\mu}\,\exp[(iQ_\mu - \gamma_\mu)\theta] \tag{10}$$

with $I_\mu$ being the initial values of the action variables of the linear normal modes.

### *2.3 Nonlinear normalization*

In the presence of nonlinearities we may use (8) just as a linear change of variables after which the equation of motion becomes

$$\dot{\underline{a}} = \Lambda^{(0)}\cdot\underline{a} + \underline{R}^{(\mathrm{c})}(\underline{a}, \theta; \varepsilon), \tag{11}$$

# Email: alexahin@sunse.jinr.ru

where $\Lambda^{(0)}$ is a diagonal matrix with elements $\Lambda^{(0)}{}_{kk} = iQ_k - \gamma_k$, vector

$$\underline{R}^{(c)} = \varepsilon\, W^{-1} \cdot [S \cdot (W^{-1})^T \cdot \frac{\partial}{\partial \underline{a}} \mathcal{H}_{\rm hot} + \underline{F}^{(c)}_{\rm hot}] \qquad (12)$$

includes terms of order 2 in $\underline{a}$ and higher (subscript "hot" standing for higher order terms), ε is the perturbation parameter (introduced here in the simplest way).

We may try to treat the nonlinear problem (11) with the method of averaging [5] which does not involve canonical transformations, however it is not convenient for practical calculations in higher orders in the perturbation parameter.

In the Hamiltonian case there is an efficient Lie-transform based algorithm, Deprit's algorithm (see e.g. [6]), which in principle permits to perform calculations to an arbitrarily high order in an automated way. Here we present its generalization for the non-Hamiltonian case.

Let us look for a continuos set of transformations

$$\underline{A} = \underline{A}(\underline{a}, \theta; \varepsilon) \equiv \hat{T}(\underline{a}, \theta; \varepsilon)\underline{a}, \quad \hat{T}(\underline{a}, \theta; 0) = I, \qquad (13)$$

which renders equation of motion in the new dynamical variables

$$\dot{\underline{A}} = \underline{G}(\underline{A}, \theta; \varepsilon), \quad \underline{G}(\underline{A}, \theta; 0) = \Lambda^{(0)} \cdot \underline{A}, \qquad (14)$$

as simple as possible. Defining the transformation by the equation

$$\frac{\partial}{\partial \varepsilon} \underline{A}(\underline{a}, \theta; \varepsilon) = \underline{V}\big(\underline{A}(\underline{a}, \theta; \varepsilon), \theta; \varepsilon\big), \qquad (15)$$

$\underline{V}$ being called *a Lie-dragging field*, we obtain the equation for the inverse operator

$$\frac{\partial}{\partial \varepsilon} \hat{T}^{-1} = -\hat{L}_{\underline{V}} \hat{T}^{-1}, \quad \hat{L}_{\underline{V}} \equiv \underline{V}^T \cdot \frac{\partial}{\partial \underline{u}} \qquad (16)$$

Arbitrary vector fields are transformed with the help of the matrix operator

$$\hat{\mathcal{T}}^{-1} = \hat{T}^{-1} \frac{\partial \underline{A}}{\partial \underline{a}}, \qquad (17)$$

which satisfies the equation

$$\frac{\partial}{\partial \varepsilon} \hat{\mathcal{T}}^{-1} = -\hat{\mathcal{L}}_{\underline{V}} \hat{\mathcal{T}}^{-1}, \quad \hat{\mathcal{L}}_{\underline{V}} \underline{U} \equiv \hat{L}_{\underline{V}} \underline{U} - \hat{L}_{\underline{U}} \underline{V} = -\hat{\mathcal{L}}_{\underline{U}} \underline{V}, \qquad (18)$$

The Lie-dragging field $\underline{V}$ is related to the original and new vector fields by the following basic equation [1]

$$\frac{\partial}{\partial \theta} \underline{V} + \hat{\mathcal{L}}_{\underline{G}^{(c)}} \underline{V} = \frac{\partial}{\partial \varepsilon} \underline{G}^{(c)} - \hat{\mathcal{T}}^{-1} \frac{\partial}{\partial \varepsilon} \underline{R}^{(c)}, \qquad (19)$$

which in principle permits to find $\underline{V}$ for a given $\underline{G}^{(c)}$ or *vice versa*. But it is not clear in advance for which $\underline{G}^{(c)}$ solution of (19) may exist. Perturbation theory allows to specify $\underline{G}^{(c)}$ in the process of normalization so as to assure existence of the (formal) solution.

Expanding everything in power series

$$\underline{V} = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \underline{V}_{n+1}, \quad \underline{G}^{(c)} = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \underline{G}^{(c)}_n, \quad \underline{R}^{(c)} = \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \underline{R}^{(c)}_{0n}, \ldots \qquad (20)$$

we can reduce general equation (19) to the following set of homology equations

$$\frac{\partial}{\partial \theta} \underline{V}_n + \hat{\mathcal{L}}_{\underline{G}_0} \underline{V}_n = \underline{G}^{(c)}_n - \underline{R}^{(c)}_{0n} + \underline{\Sigma}_n \qquad (21)$$

where

$$\underline{\Sigma}_n = \sum_{m=1}^{n-1} \left\{ \binom{n-1}{m-1} \hat{\mathcal{L}}_{\underline{V}_m} \underline{G}^{(c)}_{n-m} - \binom{n-1}{m} \underline{R}^{(c)}_{m,n-m} \right\}, \quad \underline{R}^{(c)}_{m,n} = -\sum_{j=1}^{m} \binom{m-1}{j-1} \hat{\mathcal{L}}_{\underline{V}_j} \underline{R}^{(c)}_{m-j,n}$$

Eqs.(21) for the autonomous system (∂/∂θ=0) were first obtained by Kamel [7]. In the Hamiltonian case they reduce to Deprit's equations [6].

Introducing unit 6-vectors $\underline{e}_i$ with the components $(\underline{e}_i)_k = \delta_{ik}$ we can build the vector basis functions

$$\underline{\Phi}_{k,\underline{m}} = \underline{e}_k \prod_i A_i^{m_i}, \quad \hat{\mathcal{L}}_{\underline{G}_0} \underline{\Phi}_{k,\underline{m}} = (\sum_i m_i \Lambda^{(0)}_{ii} - \Lambda^{(0)}_{kk})\, \underline{\Phi}_{k,\underline{m}}, \qquad (22)$$

to solve eqs.(21) by expansion in them. We may set $\underline{G}_n^{(c)}$ to zero unless $(\underline{R}_{0n}^{(c)} - \underline{\Sigma}_n)$ contains a term with the basis function which eigenvalue is close to an integer times $i$, these being the detuning ($m_{2i} = m_{2i-1}$, $i \neq k$, $m_{2k} = m_{2k-1} - 1$) and the resonance terms. The corresponding Fourier harmonics of such terms should be added to $\underline{G}_n^{(c)}$ to avoid small denominators in $\underline{V}_n$.

If no close resonance is encountered in the orders of interest, we will obtain in the result of the normalization process

$$\underline{G}^{(c)}(\underline{A}, \theta; \varepsilon) = \Lambda(|A_I|^2, |A_{II}|^2, |A_{III}|^2; \varepsilon) \cdot \underline{A}, \qquad (23)$$

where Λ is again a diagonal matrix independent of θ. Having solved eq.(14) for $\underline{A}$, we can transform back to the original variables as follows:

$$\underline{a} = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \underline{A}_n^{-1}(\underline{A}, \theta), \quad \underline{A}_0^{-1} = \underline{A}, \quad \underline{A}_n^{-1} = -\sum_{m=1}^{n} \binom{n-1}{m-1} \hat{L}_{\underline{V}_m} \underline{A}^{-1}_{n-m} \qquad (24)$$

## *2.4 Distribution of radiating particles*

Let us now consider the fluctuating part of the radiative field $\underline{F}^{(q)}$ as a perturbation of the deterministic motion. Transforming and expanding it on the analogy of the classical radiative field in (12), (20) (note that expansion in ε now starts with zero order) we can project it onto the normal form coordinates $\underline{A}$ by the following recursion scheme:

$$\underline{G}_n^{(q)} = \underline{R}^{(q)}_{0n} + \sum_{m=1}^{n} \binom{n}{m} \underline{R}^{(r)}_{m,n-m}, \quad \underline{R}^{(q)}_{m,n} = -\sum_{j=1}^{m} \binom{m-1}{j-1} \hat{\mathcal{L}}_{\underline{V}_j} \underline{R}^{(q)}_{m-j,n}. \qquad (25)$$

In the result of this projection additional nonlinear transverse components of the fluctuating force can appear from the longitudinal component enhancing diffusion in the transverse planes.

To find particle distribution in the phase space, $\mathcal{F}$, one should resort to the Fokker-Planck equation which has the standard form [3] in the complex variables as well.

With known $\mathcal{F}$ we can obtain the *true emittance* of nonlinear normal modes, $\varepsilon_\mu^{\rm (true)} = \langle |A_\mu|^2 \rangle$, and, with the help of (24), the *apparent emittance* of linear normal modes, $\varepsilon_\mu^{\rm (app)} = \langle |a_\mu|^2 \rangle$, related to the observable beam characteristics via eq.(8).

# 3 COMPLEX OF PROGRAMS

The described above theory was implemented in a complex of ***Mathematica*** notebooks (see Fig.1). One group of the notebooks perform symbolic computations and generate analytical expressions for subsequent numerical calculations.

Another group of the notebooks find linear eigenvectors with radiation, components of the Lie-dragging field, resonance and diffusion coefficients for a particular lattice. As the starting point it uses the closed orbit due to imperfections and classical radiation and the linear eigenvectors of the Hamiltonian motion around it computed by ***MAD*** [8]. Employing ***Mathematica*** for numerical calculations permits to operate with nonlinear element strengths in symbolic form which may be convenient for determination of the multipole correctors strength. At present the treatment is limited to the second order effects in the thin lens approximation.

## 4 LEP2 108/90 LATTICE

The lattice with phase advances in the arc cells $\mu_x/\mu_y$= 108°/90° was once considered as a strong candidate for LEP operation at the highest energy. However some problems were encountered with this lattice tested at the beam energy of 86 GeV: sporadic onsets of particle losses and by almost a factor of three larger vertical emittance than expected from the linear theory (0.5 nm vs. 0.2nm). Though the large vertical emittance (but not particle losses) was obtained also in simulation by quantum tracking with ***MAD*** [9], the physics is not still clear. To get an insight the developed methods were applied for the particular misaligned lattice used in [9].

***Table 1***

| resonance | $\lvert v_1\rvert^2$ | $\lvert v_3\rvert^2$ | $\lvert v_5\rvert^2$ |
|---|---|---|---|
| $Q_x - Q_y - \lvert Q_s\rvert$ | 2.76 | 2.41 | $1.50{\cdot}10^6$ |
| $Q_y - 2\lvert Q_s\rvert$ | - | $4.94{\cdot}10^{-3}$ | 6.11 |
| (1-1) $Q_x - Q_s$ | 42.6 | - | $1.55{\cdot}10^7$ |
| (1-1) $Q_y - Q_s$ | - | 205. | $5.02{\cdot}10^5$ |

Table 1 presents for some resonances squared absolute values of the first order Lie-dragging field components averaged over the arcs. One can see that the influence of the transverse motion on the longitudinal one is much stronger than *vice versa*, which is a manifestation of *the radiative beta-synchrotron coupling* [2]. Also, the nearest to the working point $Q_x$ =102.280, $Q_y$ =96.192, $|Q_s|$=0.107 synchro-betatron resonances appear too weak to produce noticeable vertical emittance.

***Table 2***

| | $\partial/\partial I_I$ | $\partial/\partial I_{II}$ | $\partial/\partial I_{III}$ |
|---|---|---|---|
| $\Lambda_I$ | -509+51261 $i$ | -74 - 34702 $i$ | -0.10 - 80 $i$ |
| $\Lambda_{II}$ | -11- 34329 $i$ | 56 +35806 $i$ | 0.01 - 1.2 $i$ |
| $\Lambda_{III}$ | 928 + 4949 $i$ | -127 + 1124 $i$ | -0.02 + 8.5 $i$ |

The second order perturbation theory gives dependence of the tunes and damping rates on the oscillation amplitudes. Table 2 presents derivatives of $\Lambda_\mu = iQ_\mu - \gamma_\mu$ w.r.t. the action variables of the nonlinear normal modes $I_\nu$. Due to large derivative $\partial\mathrm{Re}(\Lambda_{III})/\partial I_I$ the longitudinal damping fails at $I_I$ =2.9 µm. Though this value exceeds the dynamic aperture (~2.5 µm), the weakened longitudinal damping can contribute to particle losses at smaller $I_I$.

## 5 REFERENCES


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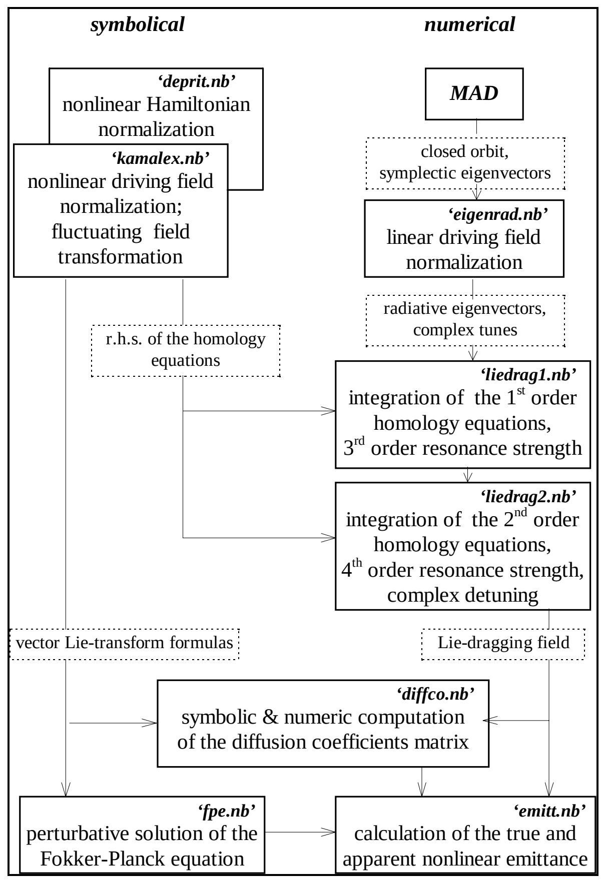


Figure 1. Structure of the complex of programs.