STOCHASTIC EFFECTS IN LONGITUDINAL PHASE SPACE

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Some estimates of the stochasticity domain of a particular Hamiltonian dynamic system with a continuously increasing non-linearity have been given elsewhere[1,2]. These estimates are not directly applicable to longitudinal motion in a circular accelerator because in the latter case the non-linearity is periodic and bounded. An extension of the method used in Refs. 1 and 2 shows that in accelerators such as the CERN PS and PSB the accelerating voltage non-linearity produces stochastic behaviour only in the close vicinity of the separatrix. The usual description of a microtron is shown to imply the existence of a large stochastic domain.

Introduction and Statement of the Problem

The longitudinal motion in an accelerator can be described in a straightforward manner if the assumption is made that the ring is composed of discrete elements only. One formulation without space charge, taking into account one RF-cavity gap, is[3]

$$\begin{aligned} \Delta p_{n+1} &= \Delta p_n + \frac{eV}{\beta_s c}\left(\sin\phi_n - \sin\phi_s\right) \\ \phi_{n+1} &= \phi_n + \frac{2\pi h}{p_s}\left(\alpha - \gamma_s^{-2}\right)\Delta p_{n+1} \end{aligned} \tag{1}$$

where p is the particle momentum, Δp the momentum deviation, α the momentum compaction factor, ϕ the phase, n the number of cavity gap traversals, V the amplitude of a sinusoidal accelerating voltage, h the harmonic number, e the particle charge, and β, γ, c the usual relativistic quantities. The subscript s refers to synchronous values. When radiation damping is taken into account, the recurrence (or point-mapping) (1) is equivalent to the known alternate forms[4,5]. It is also equivalent to the differential equation[6]

$$\frac{d\phi}{d\tau} = \psi \ , \quad \frac{d\psi}{d\tau} = \Omega^2\left(-\sin\phi + 2\,\mathrm{tg}\,\phi_s \cdot \sin^2\frac{\phi}{2}\right)\left(1 + 2\sum_{m=1}^{\infty}\cos m\tau\right) \tag{2}$$

where τ and Ω are the normalized time and small-amplitude synchrotron frequency, respectively.

The object of this paper is the determination of the solution structure of the recurrence (1) in the discrete phase plane Δp_n, ϕ_n, and the application of the results to the CERN PS and PSB. When the differential equation (2) is solved by a method of series truncation[6] (perturbations with averaging, or a finite number of canonical transformations which reject the explicit time-dependence toward higher-order terms), the solution structure so obtained is valid only in the region where the recurrence (1) admits closed trajectories (invariant curves without "stochastic" intersections). The region where the invariant curves of (1) intersect is called stochastic, because the intersection pattern is so complex that superficially it appears to be random. Since this pattern is unambiguously defined by (1), it is of course completely deterministic.

Determination of the solution structure of the recurrence (1)

The approach used in this paper is based on the previously tested conjecture that the qualitative features of the solution structure of (1) can be deduced from the distribution in the phase plane of zero- and one-dimensional singular solutions (point- and line-singularities) [1,2,7-11]. Consider, in fact, the particular conservative recurrence

$$\begin{aligned} x_{n+1} &= y_n + F(x_n) = g(x_n, y_n) \\ y_{n+1} &= -x_n + F(x_{n+1}) = f(x_n, y_n) \end{aligned} \tag{3}$$

where F(x) is an arbitrary differentiable function. From (3) it is possible to construct unambiguously the iterated recurrence

$$\begin{aligned} &x_{n+k} = g_k(x_n, y_n) \ , \quad y_{n+k} = f_k(x_n, y_n) \\ &g_1 = g \ , \quad f_1 = f \ , \quad k = \text{integer} \end{aligned} \tag{4}$$

The simplest point-singularity of (3) is a cycle of order k (periodic point of order k), defined by a real root x,y of the two algebraic equations

$$x_{n+k} = x_n = x \ , \quad y_{n+k} = y_n = y \tag{5}$$

deduced from (4), provided x,y is not simultaneously a root of (5) when k is replaced by one of its divisors. A cycle of order k=1 is called a fixed point.

An invariant curve of (3) is described by G(x,y) = const. if the function G(x,y) satisfies the functional equation

$$G\left(g_k(x,y),\, f_k(x,y)\right) = G(x,y) \tag{6}$$

If the curve G(x,y) = const. has at least one single-valued branch $y = \theta(x)$ passing through a point (x_0, y_0), then an equivalent formulation of (6) for this branch is

$$f_k\left(x, \theta(x)\right) = \theta\left[g_k\left(x, \theta(x)\right)\right] \ , \quad y_0 = \theta(x_0) \ . \tag{7}$$

The function $\theta(x)$ and a sufficient number of its derivatives are assumed to possess at least an asymptotically convergent Taylor series at and near x_0. The simplest line-singularity of (3) is an invariant curve passing through a point of a cycle.

Cycles are classified according to the eigenvalues $\lambda_{1,2}$ of their characteristic equation. The special form of (3) implies $\lambda_1 \cdot \lambda_2 = 1$. The indices are so chosen that either $\lambda_1 > +1$ or $\lambda_2 < -1$. Cycles are called saddles (or hyperbolic points) if the λ are real (of type 1 if $\lambda > 0$ and of type 3 if $\lambda < 0$) and centres (or elliptic points) if $\lambda = \exp(\pm i\phi)$. The constant ϕ is called the rotation angle of the centre. An additional property of a cycle is its rotation number r (integer), describing the least number of turns made around an interior point when following the k successive points of the cycle in the phase plane. For conciseness, a cycle of order k and rotation number r is designated by $k_{(r)}$.

If the point (x_0, y_0) is a saddle, a singular invariant curve segment passing through (x_0, y_0) can be sought in the form

$$y = \theta(x) = y_0 + \sum_{i=1}^{\infty} \beta_i \,(x - x_0)^i \ , \tag{8}$$

where the β_i, i = 1,2, ... are real constants. Because of the postulated smoothness of $\theta(x)$, β_1 is equal to one of the eigenslopes p_1, p_2 at (x_0, y_0). A known segment $y = \theta(x)$ can be continued by means of (4). The slope β_1 can be continued by means of the recurrence

$$\frac{dy_{n+k}}{dx_{n+k}} = \left(a + b\frac{dy_n}{dx_n}\right)^{-1} \cdot \left(c + d\frac{dy_n}{dx_n}\right) \ , \quad \frac{dy_0}{dx_0} = \beta_1 \ , \tag{9}$$

where a,b,c,d are elements of the Jacobian matrix of (4), whose determinant is unity for any k. Similar recurrences exist for the continuation of the β_i, i > 1. If b > 0, the eigenslopes of a saddle are given by $p_{1,2} = (\lambda_{1,2} - a)/b$.

It is straightforward to show that the recurrence (1) can be transformed into the form (3). In fact, when $\theta_n = b(\phi_n - \phi_s)$, $\psi_n = b(\phi_{n-1} - \phi_s)$, $b = \pi - 2\phi_s$, (1) becomes

$$\psi_{n+1} = \theta_n \ , \quad k_0 = \frac{heV}{E_0} \cdot \frac{\alpha\gamma_t^2 - 1}{\gamma_t(\gamma_t^2 - 1)}$$
$$\theta_{n+1} = -\psi_n + 2\theta_n + \frac{2\pi k_0}{b}\left[\sin(b\theta_n + \phi_s) - \sin\phi_s\right] \tag{10}$$

where E_0 is the particle rest energy. The parameter k_0 is negative below transition. Comparing (10) to the diagonal form of (3):

$$x_{n+1} = u_n \ , \quad u_{n+1} = -x_n + 2F(u_n) \tag{11}$$

obtained by means of the transformation $u_n = y_n + F(x_n)$, yields the required equivalence relation

$$F(x) = x - \frac{1-\mu}{b\cos\phi_s}\left[\sin(bx + \phi_s) - \sin\phi_s\right]$$
$$\mu = 1 + \pi k_0 \cos\phi_s \ , \quad 0 \le \phi_s \le \frac{\pi}{2} \ , \quad -1 < \mu < +1 \tag{12}$$

The function F in (12) involves two independent parameters μ and ϕ_s, rendering a graphical display of the singular solutions of (3), (12) very inconvenient. A preliminary study has shown, however, that the stochastic features of (3), (12) do not change qualitatively as long as $\phi_s \neq 0$. For the purposes of this paper it is therefore possible to choose ϕ_s arbitrarily. For convenience ϕ_s was fixed as follows:

$$\phi_s = \frac{\pi}{2} - \operatorname{arc\,tg} \frac{1-\mu}{\pi} \tag{13}$$

The recurrence defined by (3), (12), (13) is periodic in x. In order to assure uniqueness of the graphical representation for a fixed μ, a cylindrical phase space may be used, i.e. the phase plane is thought to be wrapped around a cylinder of radius $1/\pi$ centred on $x = 0$.

Some of the singularities of (3), (12), (13) are easily found, such as the two fixed points at (0,0) and (1,0). The former is a centre of rotation angle $\phi = \arccos \mu$ and the latter a saddle of type 1. The locations of other singularities depend on the value of μ and must be sought numerically. A partial list of cycles is given in Tables 1-3. The invariant curves passing through the "main" saddle (1,0) are shown in Figs. 1-3.

From an inspection of Figs. 1-3 it is clear that the recurrence (2), (12), (13) admits homoclinic points. Hence by a theorem of Birkhoff[12] it admits an infinity of cycles. An inspection of Tables 1-3 suggests that the cycles found so far form well-ordered sets. Some properties of the ordering can be found by rearranging the tables so that the ratio k/r is roughly constant, or by tracing the locations of the points in the phase plane, together with the "main" invariant curves traversing the saddle (1,0), as illustrated in Figs. 1-3. The first deduction is that the points of the cycles are located on curves which approach a discrete set of homoclinic points[8]. In addition to the x-axis, one such curve is $y = x - F(x)$. A more fundamental way consists, however, in ordering the cycles according to conditions of their appearance, disappearance, or other change of properties, i.e. in ordering them from the point of view of Poincaré's theory of bifurcation.

By present knowledge, bifurcations take place only when the eigenvalue of a cycle satisfies

$$\lambda^q = +1 \quad \text{or} \quad \lambda^q = -1 \ , \quad q = \text{integer} \geq 1 \tag{14}$$

The case q = 1 is called a critical case[13] and the case q > 1 an exceptional one[14,15]. The latter arises in (3) whenever $\phi = 2\pi p/q$, p = integer. In the case (3), (12), (13) an exceptional case occurs at (0,0) whenever

$$\mu \to \bar{\mu} = \cos(2\pi r/k) \ , \quad k > r \text{ and relatively prime} \tag{15}$$

It has been found[9] that the traversal of $\bar{\mu} \neq -\frac{1}{2}$ in the direction of decreasing μ (increasing strength of nonlinearity) releases from (0,0) a cycle saddle $k_{(r)}$ of type 1 and a cycle centre $k_{(r)}$. The "bifurcated" cycles $k_{(r)}$ exist only for $\mu < \bar{\mu}$ and they merge with (0,0) when $\mu \to \bar{\mu}$ from below. Furthermore

$$k/r < \text{finite constant}, \quad \lambda_1(k_{(r)}) \to +1$$
$$p_1 - p_2 \to 0 \ , \quad \phi(k_{(r)}) \to 0 \quad \text{as} \quad \mu \to \bar{\mu} \ , \tag{16}$$

where $\lambda(k_{(r)})$, $\phi(k_{(r)})$ designate the eigenvalue and rotation angle of the cycles $k_{(r)}$, respectively. The first enumerable set of cycles of (3), (12), (13) consists therefore of cycles bifurcated from (0,0) at the exceptional values (14), provided $r/k \neq 1/3$, according to the schematic rule:

$$\text{centre } (0,0) \to \text{centre } (0,0) + \text{centre } k_{(r)} + \text{saddle } k_{(r)} \text{ of type } 1 \tag{17}$$

The geometric distance s(x,y) between points of a cycle $k_{(r)}$ so bifurcated and (0,0) is found to increase simultaneously with the parametric distance $s(\mu) = \mu - \bar{\mu}$. The parameters $\lambda_1(k_{(r)})$ and the angle between p_1, p_2 also increase with $s(\mu)$.

For any given $\mu \neq -\frac{1}{2}$ the centre (0,0) is thus surrounded by an infinite but enumerable set of concentric cycle pairs $k_{(r)}$, bifurcated according to the scheme (17), with k/r verifying the inequality $\mu < \cos(2\pi r/k)$. When the difference $\lambda_1(k_r)-1$ is sufficiently small, the computed invariant curves passing through the saddles $k_{(r)}$ are for all practical purposes indistinguishable from regular invariant curves, i.e. from invariant curves not traversing any singular points. The assumption that these invariant curves are not "analytically" closed for arbitrarily small values of s(x,y) implies that homoclinic points can exist arbitrarily close to (0,0). It is, however, known[8] that homoclinic points $(\bar{x},\bar{y})$ are accumulation points of a set of cycles $k_{(r)}$ of coordinates $(x_m,y_m)_k$, and that the "effective" order k/r, as well as the eigenvalues $\lambda_1(k_{(r)})$ of the saddles of type 1 of this set increase indefinitely as the geometric distance between $(\bar{x},\bar{y})$ and $(x_m,y_m)_k$ decreases, i.e.:

$$k/r \to \infty \ , \ \lambda_1(k_{(r)}) \to \infty \text{ and } p_1 - p_2 \to \text{finite constant as } (x_m,y_m)_k \to (\bar{x},\bar{y}) \ . \tag{18}$$

The assumption that sufficiently near (0,0) the invariant curves are not (truly) closed leads to a contradiction between the properties (16) and (18). A small neighbourhood of (0,0) is therefore filled with an infinite but enumerable set of island structures [formed by the invariant curves which traverse the saddles $k_{(r)}$ of type 1, surround the centres $k_{(r)}$ and join without other intersections]. Since two close island structures have been bifurcated from (0,0) at two distinct values of $\phi = 2\pi r/k$, and thus of μ (because k and r are relatively prime) the area between them contains either regular closed invariant curves or closed invariant curves passing through an infinity of singular points [points of cycles $k_{(r)}$ for which $k \to \infty$, $r \to \infty$, $\lim k/r <$ finite constant, $\lambda_1(k_{(r)}) \to 1$, $p_1 - p_2 \to 0$, $\phi(k_{(r)}) \to 0$]. The distribution of closed invariant curves near (0,0) is thus extremely irregular in theory but quite smooth in practice.

Because the construction of an iterated recurrence like (4) does not produce any intrinsically new data, the solution structure near a centre $k_{(r)}$, k > 1, is the same as near the centre (0,0), i.e. one has a "box within a box" behaviour. The growth of the rotation angle as a function of $s(\mu)$ of the centre $\bar{k}(\bar{r})$, bifurcated from the centre $k_{(r)}$ and characterized by the property that $\bar{k}/\bar{r}$ contains the factor k/r, is much faster than the growth of $\phi(k_{(r)})$. In other words, the bifurcation speed is higher inside the inner boxes.

Since the rotation angle of a centre increases with the parametric distance from its generating bifurcation, it may eventually reach the value $\phi = \pi$. If the traversal (in the direction of decreasing μ) of the corresponding critical case $\lambda = -1$ is examined from the bifurcation point of view, it is found[11] that the cycle centre $k_{(r)}$ turns into a cycle saddle $k_{(r)}$ of type 3 with a simultaneous release of a cycle centre $(2k)_{(2r)}$, or schematically

$$\text{centre } k_{(r)} \rightarrow \text{saddle } k_{(r)} \text{ of type } 3 + \text{centre } (2k)_{(2r)} \tag{19}$$

Since the companion saddles $k_{(r)}$ of type 1 remain unchanged, the bifurcation (19) converts the centre-saddle of type 1-pair into a saddle of type 3-saddle of type 1-pair. Several examples of such saddle-saddle pairs appear in Tables 2 and 3. The presence of cycles generated by means of the bifurcation (19) appears to be a sufficient condition of the presence of local stochasticity, i.e. of the presence of interesting invariant curves in the neighbourhood of the saddle-saddle pair. Small islands of closed trajectories may persist near the centres $(2k)_{(2r)}$.

The bifurcations (17) and (19) are insufficient to explain the origin of all cycles of the recurrence (3), (12), (13). There are, for example, two cycles saddle $13_{(2)}$ of type 1 in Table 3, and it is found that the cycle pair $3_{(1)}$ exists also when $\mu > -\frac{1}{2}$, whereas according to (17) it should appear at $\mu = -\frac{1}{2}$. The study of this difficulty has disclosed the existence of two additional bifurcations, related to the critical case $\lambda = +1$ $(\phi = 0)$[11]. The first is of the type:

$$\text{composite cycle } k_{(r)} \rightarrow \text{centre } k_{(r)} + \text{saddle } k_{(r)} \text{ of type 1} \tag{20}$$

The composite cycle $k_{(r)}$ appears as an isolated double root of (5), and is unrelated to any close simple root. After traversal of the bifurcation the double root separates into two simple ones, giving rise to the usual pair centre $k_{(r)}$-saddle $k_{(r)}$ of type 1. The second cycle saddle $13_{(2)}$ of Table 3 and the cycle pair $3_{(1)}$ originate in this way (at $\mu \approx 0.23$ and $\mu \approx 0.41$, respectively).

When $\mu \rightarrow \bar{\mu} = \cos(2\pi/3) = -\frac{1}{2}$, only the $3_{(1)}$ saddles merge with the centre (0,0), the $3_{(1)}$ centres remaining some distance away [one point at (-0.33,0)] Furthermore, $\lambda_1(3_{(1)}) \rightarrow 1$ as $\mu \rightarrow \bar{\mu}$. The bifurcation is of the type

$$\left.\begin{array}{l}\text{saddles } k_{(r)} \text{ of type 1}\\ \text{centres } \bar{k}_{(r)}\end{array}\right\} \rightarrow \left\{\begin{array}{c}\text{composite cycle}\\ \lambda(k_{(r)})=1,\ \varphi(\bar{k}_{(r)})=\frac{2\pi p}{q}\end{array}\right\} \rightarrow \cdots$$

$$\cdots \rightarrow \left\{\begin{array}{l}\text{saddles } k_{(r)} \text{ of type 1}\\ \text{centres } \bar{k}_{(r)},\ k \neq \bar{k}\end{array}\right. \tag{21}$$

with $k = 3$ and $\bar{k} = 1$. Six invariant curve segments traverse the composite singular point (0,0) when $\mu = \bar{\mu}$, resulting from the coalescence of the three saddles $3_{(1)}$ of type 1. The bifurcation (21) produces an unstable point singularity.

Consider now the structure of invariant curves of (3), (12), (13) when $\mu \neq -\frac{1}{2}$. According to the theory of Birkhoff[16] the island structures around (0,0) should turn into instability rings (degenerate island structures formed when the invariant curves traversing the saddles of type 1 have other intersections but remain nevertheless inside a domain bounded by a finite closed curve) as their geometric distance from (0,0) increases. Stochasticity thus exists inside an instability ring, but it does not lead to an "orbital" instability. Since instability rings require the existence of a centre-saddle pair, they must give way to a different configuration when the bifurcation (19) is reached.

The corresponding geometric distance from (0,0) can be estimated from Tables 1-3. It has been found, however, that instability rings cease to exist sooner[1,10]. From Tables 1-3 it can be seen that $\lambda_1(k_{(r)})$ and the angle between p_1 and p_2 increase simultaneously with the effective order k/r. Beyond a critical value the geometric distance between (0,0) and a point on a singular invariant curve through a saddle $k_{(r)}$ is found to increase first at an algebraic rate and then at an exponential one. The slow increase has been called diffusion and the fast one stochastic instability[1,10].

Application to longitudinal motion

When the solution structure of the recurrence (3) (12), (13) is expressed in the terminology of accelerator theory [recurrence (1)], RF acceleration is found to produce bunches with a dense structured core (region of closed trajectories, including island structures), surrounded first by a dense but unstructured shell (instability rings) and then by a halo (diffusion region). Since the separatrix of the RF bucket (main invariant curve) is not closed, random particle losses occur from the halo (stochastic instability). The size of each region depends on the specific accelerator parameters. The RF acceleration process is unstable at the 1/3 parametric resonance between the synchrotron and beam revolution frequencies [exceptional case $\phi = 2\pi/3$ at (0,0)].

As far as the PSB is concerned, all longitudinal stochastic effects are negligible ($\mu \geq 0.99$). Analogous circumstances prevail in the CPS ($\mu \geq 0.90$). Stochastic effects may, however, occur inside self-buckets produced by beam-induced high-frequency voltages.

An accelerator likely to be subject to considerable stochastic effects is the microtron. Assume that the recurrence[4,17,18]

$$x_{n+1} = x_n + y_n \ , \quad y_{n+1} = y_n + 2\pi\left(\frac{\cos x_{n+1}}{\cos \phi_s} - 1\right) \tag{22}$$

is a valid description of the longitudinal motion (some doubts have been expressed in a paper by Turrin[19]). By a linear change of variables (22) transforms into

$$z_{n+1} = v_n \ , \quad v_{n+1} = -z_n + 2v_n + \frac{2\pi}{\cos\phi_s}\left[\cos(v_n - \phi_s) - \cos\phi_s\right] \tag{23}$$

Since an RF voltage $V \sin\phi$ below transition is equivalent to $-V \sin(\phi - 2\phi_s)$ above transition[20], transforming (22) into an equivalent form "below transition" and replacing $\cos\phi$ by $\sin\phi$ yields the recurrence (3), (12), (13) with $0 < \phi_s < \text{arctg}\,(2/\pi) \approx 32.5^\circ$. The solutions of (22) therefore exhibit considerable stochasticity even for relatively small ϕ_s. Figure 3 corresponds to $\phi_s = 15.6^\circ$; on the x-axis, stochastic instability starts at $x \approx 0.51$.

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Table 1

Cycles of the recurrence (3), (12), (13), $\mu = 0.8$

		Centres or Saddles of type 3					Saddles of type 1				
k	r	x	y	ϕ or λ_2 (> 0) (< 0)	p_1	p_2	x	y	λ_1	p_1	p_2
10	1	0.250529	0	0.0001			0.241025	0.036592	1.0001	-1.900	-1.893
11	1	0.541454	0.049673	0.0037			0.553773	0	1.0037	138.1	
12	1	0.701253	0	0.0103			0.690883	0.042725	1.0103	-2.135	-2.021
13	1	0.784250	0.033848	0.0188			0.792358	0	1.0190	48.78	
14	1	0.852864	0	0.0296			0.846697	0.025965	1.0300	-2.269	-2.023
15	1	0.889912	0.019596	0.0434			0.894539	0	1.0444	25.71	
16	1	0.923864	0	0.0616			0.920418	0.014651	1.0635	-2.425	-1.964
17	1	0.942223	0.010888	0.0859			0.944776	0	1.0897	14.19	
18	1	0.959820	0	0.1187			0.957934	0.008060	1.1258	-2.706	-1.840
19	1	0.969314	0.005949	0.1631			0.970704	0	1.1767	7.818	

Table 2

Cycles of the recurrence (3), (12), (13), $\mu = 0.5$

		Centres or Saddles of type 3					Saddles of type 1				
k	r	x	y	ϕ or λ_2 (> 0) (< 0)	p_1	p_2	x	y	λ_1	p_1	p_2
7	1	-0.453247	0	0.4374			0.611988	0	1.531	2.349	
8	1	-0.519721	0	1.1355			0.763198	0.090499	2.766	1.084	-4.284
9	1	-0.541178	0	2.9517			0.872783	0	5.685	1.277	
10	1	-0.548847	0	-9.366	30.47		0.915006	0.038910	13.10	1.057	-1.375
11	1	-0.551694	0	-29.21	25.90		0.931111	0.054332	32.41	1.040	-1.419
12	1	0.971773	0	-81.08	-1.043		0.968120	0.015436	82.94	1.120	-1.082
13	1	0.961612	0.037471	-216.9	-1.072	0.881	0.973774	0.021412	215.3	1.127	-1.086
13	2	-0.364999	0	0.0319			0.455375	0	1.032	36.39	
15	2	-0.495642	0	0.5326			0.707595	0	1.685	5.355	
17	2	-0.532975	0	-2.357	128.0		0.848055	0	6.938	1.665	
19	2	0.899388	0.024261	-43.33	-0.732	1.318	0.885209	0	53.98	1.345	
19	3	-0.313851	0	0.0013			0.375926	0	1.0013	986.2	

Table 3

Cycles of the recurrence (3), (12), (13), $\mu = 0.125$

		Centres or Saddles of type 3					Saddles of type 1				
k	r	x	y	ϕ or λ_2 (> 0) (< 0)	p_1	p_2	x	y	λ_1	p_1	p_2
5	1	-0.502858	0	2.525			0.551511	0	3.849	1.011	
6	1	-0.577941	0	-25.97	2.731		0.754058	0.163018	20.18	1.297	-1.084
7	1	-0.594668	0	-114.5	2.511		0.892309	0	80.35	1.347	
8	1	0.953152	0	-423.3	-1.458		0.932867	0.054883	291.3	1.423	-1.446
9	1	-0.600442	0	-1494	2.452		0.970033	0	1024	1.514	
9	2	-0.288774	0	0.1467			0.318515	0	1.157	4.245	
11	2	-0.545848	0	-85.46	3.311		0.621978	0	70.28	1.259	
12	2	0.861806	0.029972	-395.1	-1.282	1.141	0.641711	0	251.8	1.377	
13	2						0.647291	0	874.8	1.418	
13	2						0.802214	0.226028	1611	1.202	
14	3	0.455590	0	0.4915			0.440250	0.052857	1.617	-4.440	-1.292
15	3	-0.460141	0	-8.748	1.772		-0.511499	0	5.072	-4.768	
16	3	0.608929	0	-286.8	-1.183		0.597869	0.029811	253.1	1.089	-1.300
17	3						0.613179	0.040647	802.0	1.117	-1.527
19	4	-0.387920	0	1.048			0.505232	0	2.615	1.917	
20	4						-0.525779	0	40.96	-0.413	
25	5	-0.504186	0	0.0378							

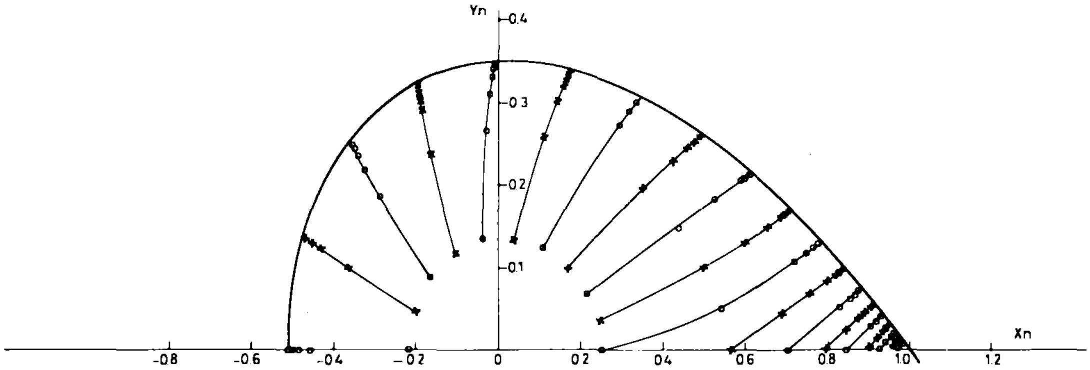

Fig. 1 μ = 0.8. Main invariant curves and positions of some cycles

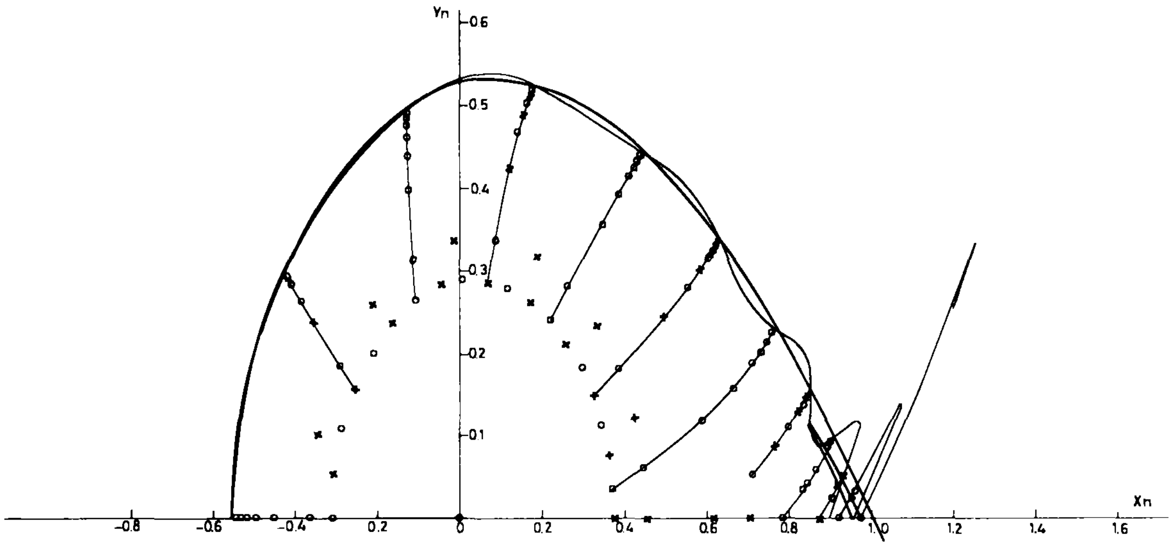

Fig. 2 μ = 0.5. Main invariant curves and positions of some cycles

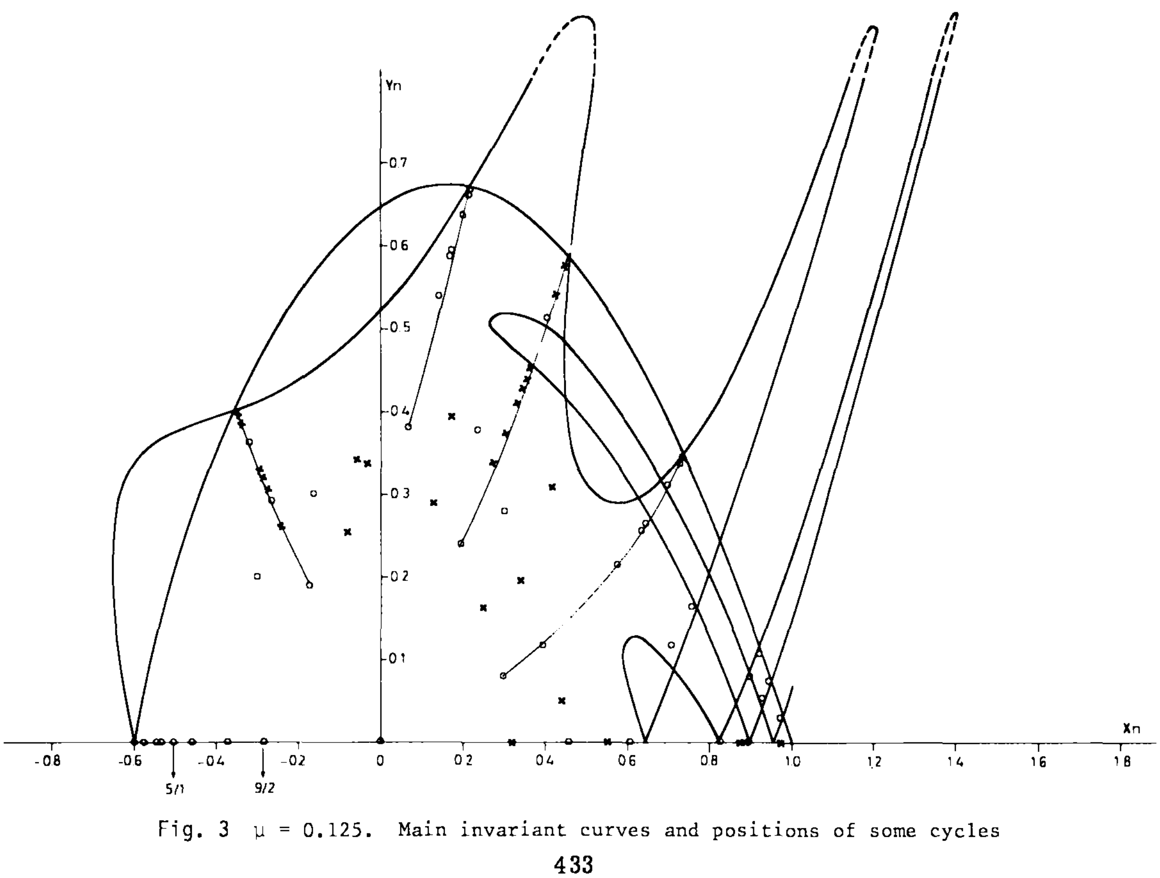

Fig. 3 μ = 0.125. Main invariant curves and positions of some cycles