LANDAU DAMPING BY NON-LINEAR SPACE-CHARGE FORCES AND OCTUPOLES

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Summary and Conclusions

Increasing the PSB and CPS intensities to 10^{13} ppp will require more powerful octupoles in order to overcome the transverse instabilities that are to be expected. Unfortunately, theory fails to give quantitative predictions at low energies, where direct space-charge forces dominate over image forces. To be able to specify the future needs more precisely, an extension of the theory would be desirable, particularly since experimental evidence is such that it is hard to extrapolate to higher intensities.

The theory presented here solves the dispersion relation, including the influence of external and space-charge induced non-linearities in both transverse dimensions as well as a spread in the LNS coefficient U. Results are preliminary in the sense that modulation of the space-charge forces due to synchrotron motion is not included. They show that the combined effect of external and space-charge non-linearities in the two transverse planes can considerably enhance Landau damping in low-energy machines (up to 10 GeV, say). The sign of the octupole current is important especially for a flat beam, and should be chosen in such a way that ν increases with amplitude in the direction where the beam is wide.

Stability diagrams for several typical conditions are presented and applied to the PSB.

1. Introduction

Octupole lenses have been installed both in the CPS and its Booster years ago, and they have successfully cured transverse instabilities. It is planned to use more powerful lenses in the CPS, and possibly in the Booster too, in order to tame future intensities. It is puzzling, however, to observe that the currents needed to avoid instabilities are considerably lower than predicted by theory[1,2]. In addition, lower thresholds for radial rather than for vertical resistive wall instability were observed frequently in the PSB. These observations suggest that the influence of space-charge non-linearities and the spreads due to motion in the second transverse direction might be important.

An attempt to include these effects[3] already showed the importance of the variations due to the betatron amplitudes in the second transverse plane. This approach, however, still predicted too high octupole strengths and failed to give the correct behaviour in the limit of vanishing external octupole force. In this limit incoherent space-charge forces should have no influence on dipole motion, a fact which can be deduced from the single particle-equation[4] and is fully confirmed by computer simulation[5].

To make more reliable extrapolations, it was felt necessary to extend the theory in order to remove these deficiencies. The main difference of the present work in comparison with Ref. 3 is that we include space-charge non-linearity in both the driving term (the U-term of Ref. 1) and in the single-particle frequency.

A general model would have to include:

i) frequency spreads due to external non-linearities in both transverse (x and y) directions;

ii) spreads due to x- and y-non-linearities of space-charge forces;

iii) spreads due to the energy distribution of the beam;

iv) spreads due to longitudinal variation of transverse space-charge forces;

and probably other effects. In the present paper we content ourselves with considering the effects (i) and (ii). This is a valid approximation at least for the coasting beam experiments which were performed in the Booster and which exhibited the "anomalities" discussed above. We shall find that the combination of external and space-charge non-linearities can considerably enhance the stability conditions, and that the spreads from both transverse directions are important.

2. Dispersion Relation

Let us discuss dipole oscillations in one transverse (x) direction. Neglecting energy spread, the dispersion relation of Laslett, Neil and Sessler (LNS) can be written as

$$(U + V + iV)\int_0^\infty \frac{-\frac{1}{2}\, h'(a)a^2\, da}{\omega - (n - \nu_x)\Omega} = 1\ . \tag{1}$$

Here the tune $\nu_x = \nu_x(a)$ depends on a, the amplitude of the incoherent betatron oscillation; h(a) is the corresponding distribution function, $\int_0^\infty h(a)a\, da = 1$. The coefficient U + V + iV can be expressed in terms of the revolution frequency (Ω) and the incoherent ($\Delta\nu_{ic}$) and coherent ($\Delta\nu_c$) Laslett tune shifts (generalized to include wall resistivity and equipment interaction) :

$$U + V + iV = \Omega(\Delta\nu_c - \Delta\nu_{ic})\ . \tag{2}$$

Finally $\omega = (n - \nu)\Omega$ is the mode frequency (coasting beam mode n) to be obtained by solving Eq. (1); Im(ω) is the growth rate, ν being the collective betatron frequency.

To include space-charge non-linearities as well as y-variation of ν_x we can use the derivation of Eq. (1) given by Hereward[6]. We start from the single-particle equation

$$\ddot{x}_i + \Omega^2\left[\nu_i^2(x_i,y_i) - 2\nu_0\Delta\nu_{ic}(x_i,y_i)\right]x_i = 2\Omega^2\nu_0\left[\Delta\nu_c - \Delta\nu_{ic}(x_i,y_i)\right]\bar{x}\ , \tag{3}$$

where $\bar{x}$ is the dipole motion of the beam and x_i the motion of the test particle (coherent and incoherent). The corresponding dispersion relation is

$$\int_0^\infty \frac{(U + V + iV)(a,b)\quad(-\frac{1}{2}\, h'(a)a^2)\, b\, g(b)\, da\, db}{\omega - [n - \nu(a,b)]\Omega} = 1\ . \tag{1a}$$

Here b is the incoherent y-amplitude, g(b) the corresponding distribution, $\int_0^\infty g(b)b\,db = 1$. The tune $\nu(a,b) = \nu_0(a,b) - \Delta\nu_{ic}(a,b)$ includes the incoherent tune shift, and so does U + V + iV through Eq. (2). The terms $\nu(a,b)$ and $(U + V + iV)(a,b)$ are obtained by averaging over the incoherent betatron motion.

To evaluate Eq. (1a) we make some further approximations: We assume that only the incoherent tune shift is non-linear, whereas the image term $\Delta\nu_c$ is the same for all particles. This is a valid approximation at low energy, where $\Delta\nu_c \ll \Delta\nu_{ic}$ and/or for thin beams. The case where both $\Delta\nu_c = \Delta\nu_c(a,b)$ and $\Delta\nu_{ic} = \Delta\nu_{ic}(a,b)$ but where external non-linearities are negligible (ν_i = const) was discussed in Ref. 4. We shall further expand $\nu(a,b)$ and only retain terms up to the octupole moment:

$$\Delta\nu_{ic}(a,b) = \Delta\nu_{ic}(0) + \frac{\partial\Delta\nu_{ic}}{\partial a^2} a^2 + \frac{\partial\Delta\nu_{ic}}{\partial b^2} b^2$$
$$= \Delta - \Delta_a a^2 - \Delta_b b^2 \ , \qquad (4)$$

and

$$\nu_0(a,b) = \nu_0(0) + \frac{\partial\nu_0}{\partial a^2} a^2 + \frac{\partial\nu_0}{\partial b^2} b^2$$
$$= \nu_0 + \nu_a a^2 + \nu_b b^2 \ .$$

The symbols Δ, Δ_a, Δ_b (all positive) and ν_a, ν_b are hereby defined and will be used throughout the rest of the paper. Quantities such as $\nu_a\hat{a}^2$, $\Delta_a\hat{a}^2$, etc., will be denoted as "external spread" and "space-charge spread", respectively; $\hat{a}$ and $\hat{b}$ are typical amplitudes to be defined below; "a" corresponds to the plane of the instability.

3. Results

Here we shall present the solution of the dispersion relation (1a) for two different distributions and for several typical conditions. We assume $\nu_a = 0$, $\nu_b = 0$ in Section 3.1; $\hat{a} >> \hat{b}$ in Section 3.2 and Figs. 1 and 2; $2\hat{a} = \hat{b}$ in Section 3.3 and Figs. 3 and 4; and $\hat{a} << \hat{b}$ in Section 3.3 and Fig. 5.

3.1 No external non-linearities (i.e. $\nu_a = 0$, $\nu_b = 0$)

In this case incoherent space-charge has no effect, as was found already in Ref. 4. In fact it is easily verified from the single-particle equation (3) that $x_i = \bar{x}$ is a solution provided that $\nu_i^2 = \nu_0^2 - 2\nu_0\Delta\nu_c$ is the same for all particles. Under this condition $\Delta\nu_{ic}$ simply drops out from Eq. (3).

3.2 ν-spread due to betatron amplitudes in the plane of the instability only (i.e. $\hat{a} >> \hat{b}$)

3.2.1 "Semicircular" distribution

It is instructive to start with the distribution which -- without space charge forces -- leads to a circular range of stability and gives the rule-of-thumb criterion

$$4\,\frac{|U + V + iV|}{\Omega} \lesssim \delta\nu_{FWHH} \qquad (5)$$

for the stabilizing ν-spread. The derivative of this function which enters into Eq. (1a) is a half circle:

$$-\frac{a^2}{2}\,h'(a^2)\,da^2 = \frac{2}{\pi\hat{a}^2}\sqrt{1 - \left(\frac{a^2}{\hat{a}^2} - 1\right)^2}\,da^2, \qquad (6)$$
$$0 \le a^2 \le 2\hat{a}^2$$

Including now space-charge, the solution of the dispersion relation gives the stability boundary

$$V^2 + \frac{(U + V)^2}{\left(1 + \frac{2}{q_a}\right)^2} = \frac{\Omega^2}{4}\,(\nu_a\hat{a}^2)^2 \simeq \frac{\Omega^2}{16}\,(\delta\nu_{FWHH})^2 \ . \qquad (7)$$

Here we have introduced a parameter $q_a = \nu_a/\Delta_a$, the ratio of external spread to space-charge spread. U involves an average of $\Delta\nu_{ic}$

$$\frac{U}{\Omega} = \Delta\nu_c - (\Delta - \Delta_a\hat{a}^2) . \qquad (8)$$

For $1/q_a = 0$ we recover the rule of thumb (5). With space-charge spread, the stability circle is distorted into an ellipse with real half-axis:

$$\hat{U} = (\Omega/4)\ \delta\nu_{FWHH}\ (1 + 2/q_a) \ .$$

The stable range is thus increased or decreased depending on the sign of q_a, i.e. depending on the polarity of the octupoles. Typically the space-charge non-linearity is such that $2/q_a = \pm |U|/(\Omega\nu_a\hat{a}^2)$ (see Appendix). Hence for $V << |U|$ -- the case of interest in the PSB and in the CPS below 10 GeV -- the stability condition reduces to

$$4\,\frac{|U|}{\Omega} \le \frac{\delta\nu_{FWHH}}{1 \pm \frac{1}{2}} \simeq \delta\nu_{FWHH}\begin{cases}2\\0.67\end{cases} , \qquad (9)$$

and the stabilizing octupole ν-spread is about 0.67 or twice the value obtained from conventional theory ($1/q_a = 0$).

Normalization: In the following we shall use a suitable normalization to reduce the number of parameters. We normalize frequencies (U,V) and spreads to the space-charge spread $S = \Omega\Delta_a\hat{a}^2$ in the plane of the instability. We denote

$$u = \left.\frac{U + V}{S}\right|_{\text{beam centre}} , \qquad v = \frac{V}{S} \ . \qquad (10)$$

This normalization is convenient because the range over which u can reasonably vary is small (values between -3 and -5 depending on the emittance ratio and the plane considered, are typical for the PSB), see Appendix.

Figure 1 displays stability boundaries[2] for the distribution (6) in this normalization. Note that the normalization (10) shifts the ellipse (7) towards more negative u-values because u refers to the beam centre.

The reduction of the stable area for the "wrong" octupole polarity ($q_a < 0$) can be seen from a comparison between Fig. 1a and Fig. 1b for given $|q_a|$, and is obviously due to cancellation of external and space-charge spread.

3.2.2 Parabolic distribution

Figure 2 gives similar results for a parabolic distribution:

$$h(a) = \frac{1}{2\hat{a}^2}\left[1 - \frac{a^2}{2\hat{a}^2}\right] , \qquad 0 \le a^2 \le 2\hat{a}^2 \ . \qquad (11)$$

Again for negative q_a the stable area is largely reduced. One particularity of Fig. 2 is that all curves intersect the point $v = 0$, $u = -2$. It appears that this effect is due to the sharp cut-off of this distribution.

3.3 ν-spread due to betatron amplitudes in both transverse directions (parabolic distribution in each plane)

Now we have to introduce another two parameters:

$$p = \frac{\nu_b\hat{b}^2}{\nu_a\hat{a}^2} , \qquad r = \frac{\Delta_b\hat{b}^2}{\Delta_a\hat{a}^2} ; \qquad (12)$$

p denotes the ratio of the two external spreads and r the ratio of the space-charge spreads; $\hat{b}$ is the equivalent of $\hat{a}$ for the second transverse direction. Note that r is determined by the emittance ratio and varies from 0 for $\hat{a} >> \hat{b}$ to about 2 for $\hat{a} << \hat{b}$; r = 1 corresponds to $\hat{b} = 2\bar{a}$.

Figures 4 and 5 refer to parabolic distributions h(a) g(b) in Eq. (1a) and demonstrate the influence of the second transverse plane. In Figs. 3 and 4, r = 1, i.e. the same spreads $\Delta_a\hat{a}^2$ and $\Delta_b\hat{b}^2$ have been assumed.

The case p = 1 as taken in Fig. 3 assumes that two sets of octupoles (in F- and D-sections say) are used such that ν_a and ν_b have the same sign. In this case we recover the reduction of the stable area for negative q_a as for the one-dimensional case (Figs. 1 and 2).

In Fig. 4, ν_a and ν_b have been taken of opposite sign, which is typical for simple lens arrangements.

In this case, the reduction of stable areas is less noticeable, because the tendency of cancellation for the one octupole polarity is to some extent smoothed out : cancellation in one plane goes together with addition in the other plane.

Finally Fig. 5 corresponds to a beam which is wide in the direction perpendicular to the plane of instability ($\hat{b} >> \hat{a}$), as is usually the case for vertical instability of a multiturn beam. Hence we take r = 2 and assume in addition "simple octupoles" with large negative p.

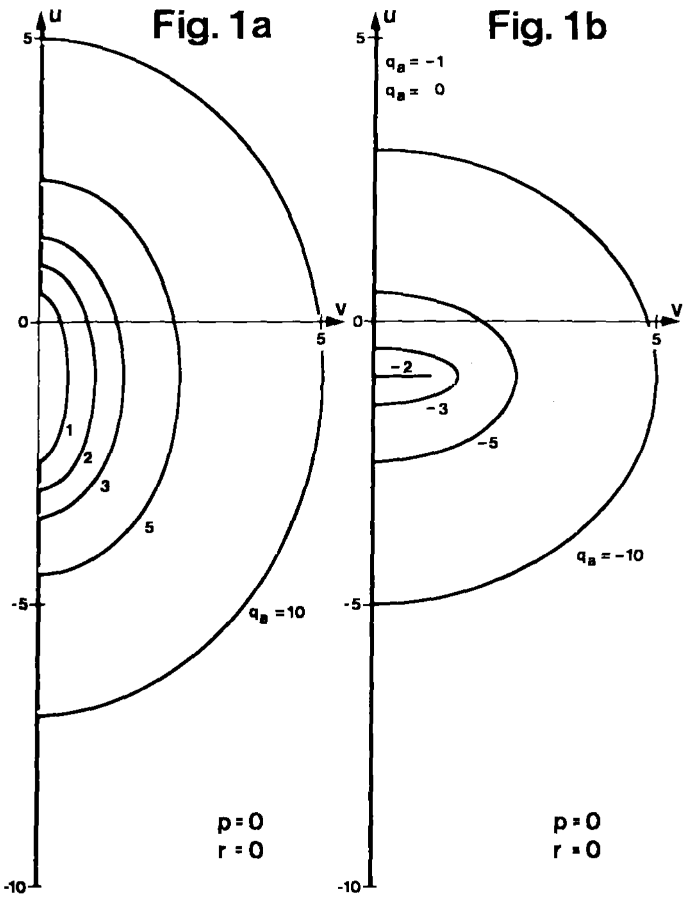
One finds again that one octupole polarity is favourable although the "right" sign is now the opposite to the preferred one in the case of a beam which is wide in the other direction ($p = 0$, $r = 0$). We conclude that the octupole moment in the direction where the beam is wide should be chosen with care.

Application to the PSB

The coasting beam and bunched beam instabilities observed in the PSB[7,8] occur sometimes horizontally, sometimes vertically, in an apparently irregular way. In order to explain this feature, we apply the results of Section 3.3 to the PSB. We compute the thresholds for both planes as a function of the emittance ratio E_H/E_V, assuming the product $E_H E_V$ and hence the area of the beam cross-sections to be a constant. We include image forces and averaging over the strongly varying beam dimensions within a machine period. The arrangement of the octupoles is as described in Ref. 9. Thresholds are expressed by the octupole currents required to stabilize the beam.

Figure 6 shows the result for $N = 2.5\ 10^{12}$ p/p and $E_H E_V = 5200\ (\pi \text{ mrad mm})^2$ at 50 MeV. One observes that positive octupole current ($\partial\nu/\partial r^2 > 0$) is more favourable and that for this polarity vertical stability requires stronger octupole currents for $E_H/E_V \lesssim 1.7$ whereas horizontal stability is more critical for $E_H/E_V \gtrsim 1.7$. The intersection point $E_H/E_V = 1.7$ falls into the region of emittance ratios actually observed in many machine experiments. This might explain why slight differences in beam parameters can favour the one or the other direction.

The predicted octupole currents are still higher than measured values. This is probably owing to the neglect of synchrotron motion in our model for the case



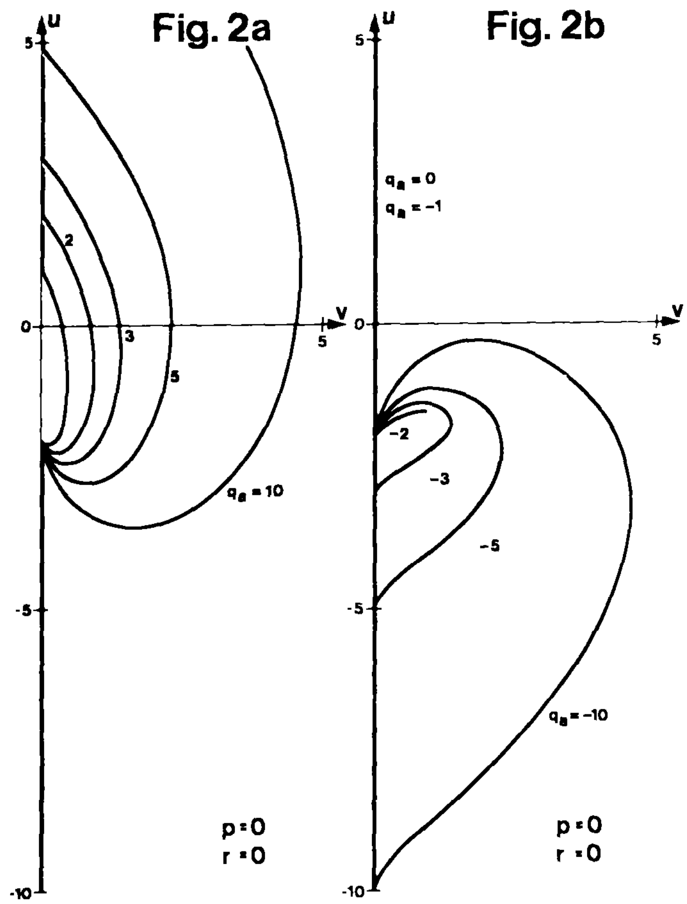


Fig. 1 : Stability diagram for a beam with amplitude distribution (6). The LNS-coefficients u and v are normalized to the space-charge ν-spread (for this distribution and vanishing image forces, u is around -3). The beam is stable if the point (u,v) is on the left-hand side of the curve.

Fig. 2 : Same physical conditions as in Fig. 1, but parabolic amplitude distribution (11). For this distribution u is typically between -3.5 and -5.

q_a is a measure of the octupole ν-spread due to betatron oscillations in the plane of the instability. Parts (a) and (b) of Figs 1-5 refer to the two polarities of the octupoles, respectively; p is the ratio of the octupole ν-spreads due to incoherent motion in the two transverse directions; r is the corresponding ratio for space-charge ν-spreads. In Figs 1 and 2, p and r are 0 because the beam is wide in the direction of the instability ($\hat{a} >> \hat{b}$).

Fig. 3a

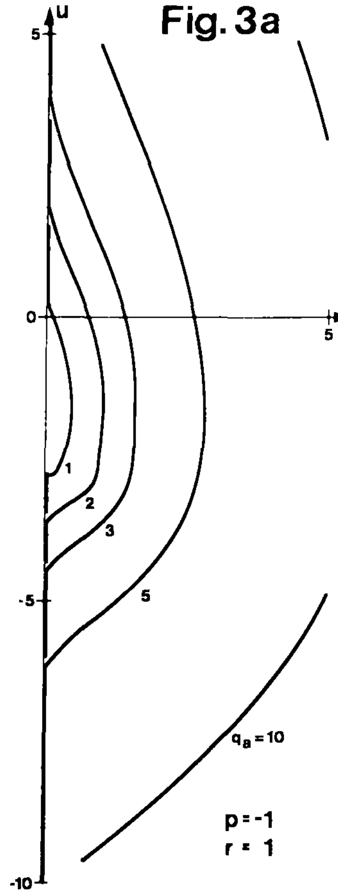


Fig. 3b

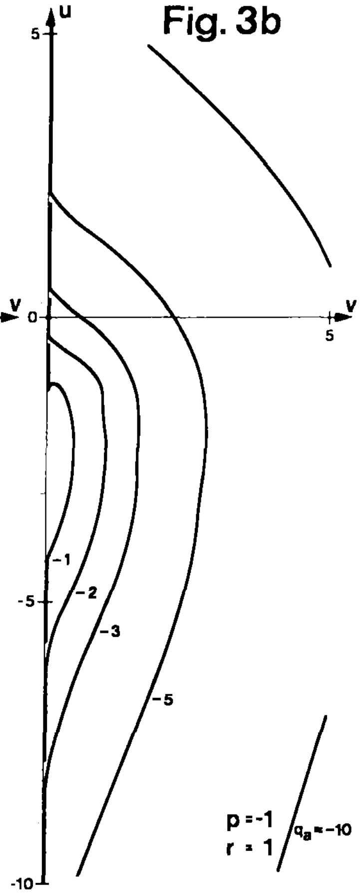


Fig. 4a

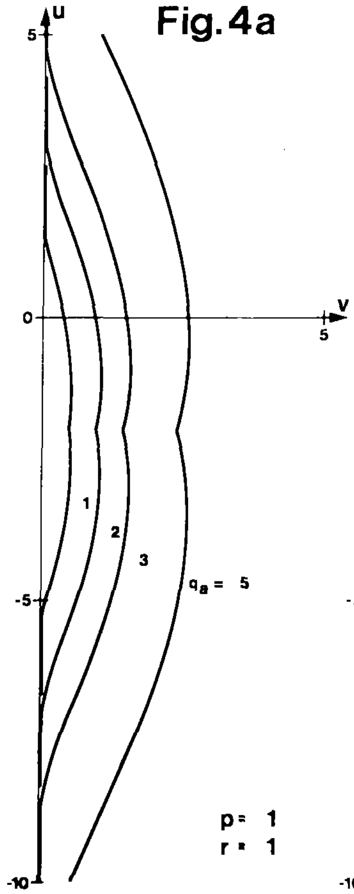


Fig. 4b

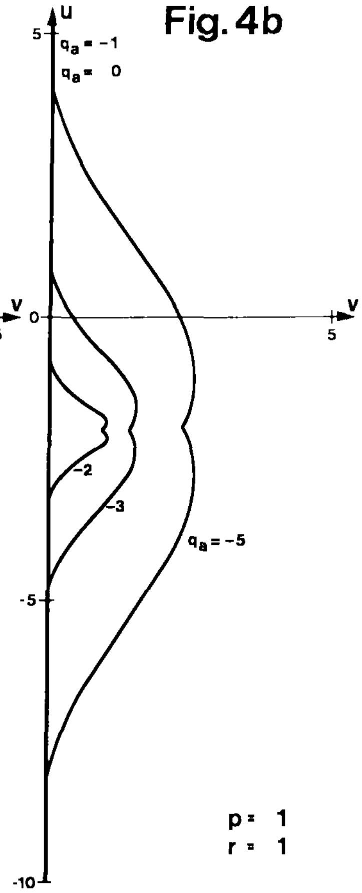


Fig. 5a

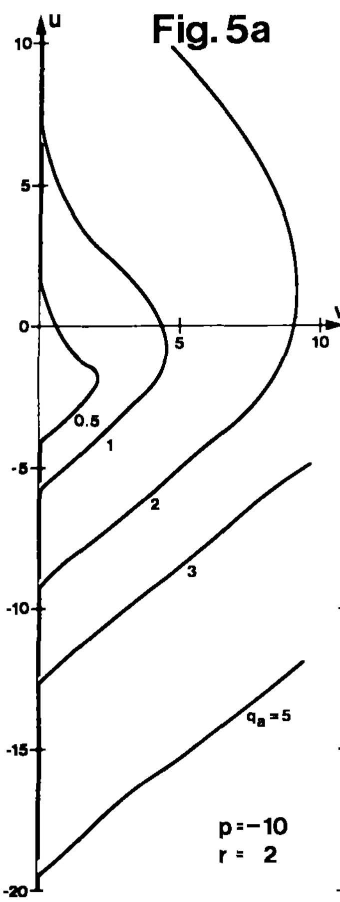


Fig. 5b

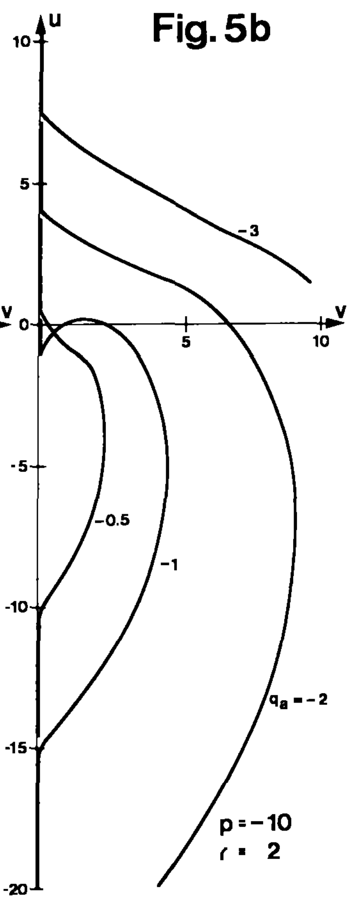


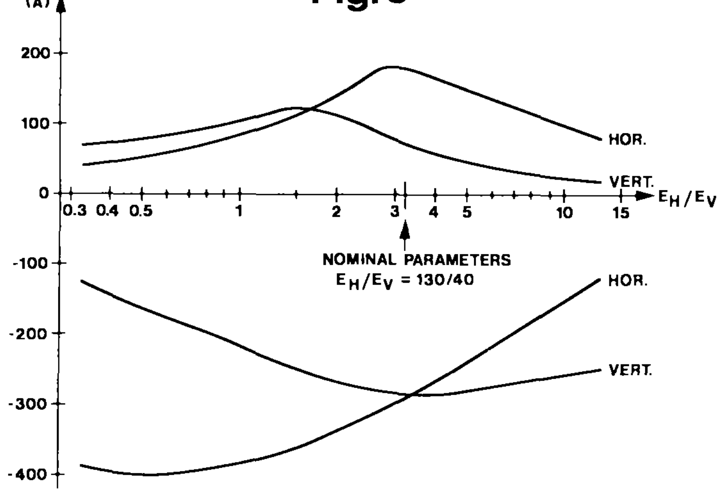
Fig. 3 : Parabolic amplitude distributions (11) in both transverse directions : r=1 means $\hat{b} \approx 2\hat{a}$; p ≈ -1 is typical for one set of octupoles and not too flat a beam

Fig. 4 : As for Fig. 3, but p ≈ 1 requires two sets of octupoles.

Fig. 5 : As for Fig. 3, but for a beam wide in the direction perpendicular to that of the instability ($\hat{b} \gg \hat{a}$).

Fig. 6 : Octupole currents required to stabilize the PSB beam versus emittance ratio, calculated for parabolic amplitude distributions (11) in both transverse planes. Nominal parameters at 50 MeV : $E_H E_V$ = 130×40 (π mrad mm)^2, N = 2.5 10^{12} p/p.

Fig. 6



of bunched beams and because of neutralization in the coasting beam case. To conclude, let us make a comparison with conventional theory (external spread in one transverse direction, no spread in $\Delta\nu_{ic}$) : the latter requires stabilizing octupole currents of 400 A (horizontal) and 520 A (vertical instability), for the nominal $E_H/E_V = 130/40$, all other parameters as for Fig. 6. This is to be compared with the values of 180 A and 70 A taken from Fig. 6.

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Appendix

Calculation of incoherent ν-shift and ν-spreads

In order to assure self-consistency with the parabolic distribution that we mainly use, we should solve the potential problem for the corresponding charge distribution. This seems to be difficult for the kind of factorized amplitude distribution $h(a)g(b)$ assumed above, which represents a beam of rectangular cross-section. Hence we restrict ourselves to the computation of ν-shifts and ν-spreads for a beam of elliptical cross-section and parabolic density

$$\rho(x,y) = \frac{\lambda}{\hat a \hat b \pi}\left[1 - \frac{(x-\bar x)^2}{2\hat a^2} - \frac{y^2}{2\hat b^2}\right], \qquad \text{(A1)}$$

where λ is the linear density and $\hat a\sqrt{2}$, $\hat b\sqrt{2}$ are the beam radii, corresponding to the ellipse $\rho(x,y) = 0$, which contains all particles. This choice provides at least approximate self-consistency: projected densities obviously do agree with those obtained from the factorized amplitude distribution.

Neglecting image forces, the force in the x-direction is given by

$$eF_x = \frac{e^2\lambda}{\pi\epsilon_0}\frac{1}{\hat a(\hat a+\hat b)\gamma^2}\left\{(x-\bar x) - \frac{1}{6}\frac{2\hat a+\hat b}{\hat a^2(\hat a+\hat b)}(x-\bar x)^3 - \frac{1}{2\hat b(\hat a+\hat b)}y^2(x-\bar x)\right\}. \qquad \text{(A2)}$$

Performing an averaging process over incoherent betatron motion (e.g. by the method of harmonic balance), $x - \bar x = a\cos(\nu_x\Omega t)$, $y = b\cos(\nu_y\Omega t)$, we take

$$\langle(x-\bar x)^3\rangle = \frac{3}{4}a^3\cos\nu_x\Omega t = \frac{3}{4}a^2(x-\bar x)$$
$$\langle y^2(x-\bar x)\rangle = \frac{1}{2}b^2 a\cos\nu_x\Omega t = \frac{1}{2}b^2(x-\bar x)\,, \qquad \text{(A3)}$$

and obtain for $\Delta\nu_{ic}(a,b)$:

$$\Delta\nu_{ic}(a,b) = \frac{N r_p R}{\epsilon_0\pi\beta^2\gamma^3}\frac{1}{\hat a(\hat a+\hat b)}\left[1 - \frac{2\hat a+\hat b}{8(\hat a+\hat b)}\frac{a^2}{\hat a^2} - \frac{\hat b}{4(\hat a+\hat b)}\frac{b^2}{\hat b^2}\right], \qquad \text{(A4)}$$

where the r.h.s. should be averaged over the circumference. We identify the quantities introduced in Eq.(A4):

$$\Delta = \frac{N r_p R}{\epsilon_0\pi\beta^2\gamma^3}\frac{1}{\hat a(\hat a+\hat b)}$$
$$\Delta_a \simeq \frac{\Delta}{\hat a^2}\,\frac{1+\frac{\hat a}{\hat a+\hat b}}{8} \qquad \text{(A5)}$$
$$\Delta_b \simeq \frac{\Delta}{\hat b^2}\,\frac{1}{4}\,\frac{\hat b}{\hat a+\hat b}\,.$$

This gives a rough estimate of the order of u [of Eq.(10)] to be expected. For our parabolic distribution, u represents the normalized ν-shift in the beam centre:

$$u = \frac{\Delta - \Delta\nu c}{\Delta_a\hat a^2} \simeq -\frac{\Delta}{\Delta_a\hat a^2} = \frac{8}{1+\frac{\hat a}{\hat a+\hat b}}\,.$$

This would imply values of u between −4 and −8, which is valid only for a machine of small wiggle in the β-functions.

Computed values for the PSB, assuming nominal emittances ($E_H = 130\pi$, $E_V = 40\pi$ mrad at 50 MeV) and including image contributions, give values of

$$u = -3.8 \text{ (horizontal)} \quad\text{and}\quad u = -4.8 \text{ (vertical)}.$$

Note that r is approximately given by $r = 2\hat b/(2\hat a + \hat b)$ and can only take values of $0 < r < 2$ (Eqs.(12), (A5)).

For the semicircular distribution (6), a numerical estimate gives $u = -\Delta/\Delta_a\hat a^2 \simeq -3$.

This value of u is used to derive the stability criterion (9). From (8) and (10) we obtain for $V \ll |U|$

$$\frac{1}{q_a} = \frac{\Delta_a\hat a^2}{\nu_a\hat a^2} \simeq \frac{U/\Omega + \Delta}{\nu_a\hat a^2} = \frac{U}{\Omega\nu_a\hat a^2} - \frac{u}{q_a}\,,$$

and

$$\frac{2}{q_a} = \pm\left|\frac{U}{\Omega\nu_a\hat a^2}\right|\frac{2}{|u+1|} = \pm f\left|\frac{U}{\Omega\nu_a\hat a^2}\right|\,, \qquad \text{(A6)}$$

$$f = \frac{2}{|u+1|} \simeq 1\,.$$

Hence

$$|U| < \hat U = \frac{\Omega}{2}\nu_a\hat a^2\left|1 + \frac{2}{q_a}\right|\,,$$

$$2\frac{|U|}{\Omega}\left|1 \pm \frac{f}{2}\right| \le \nu_a\hat a^2 \sim \frac{1}{2}\delta\nu_{FWHH}\,,$$

or

$$4\frac{|U|}{\Omega} \le \delta\nu_{FWHH}\frac{1}{\left|1 \pm \frac{f}{2}\right|} \simeq \delta\nu_{FWHH}\begin{cases}2\\0.67\end{cases}.$$

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