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STRONG CONVERGENCE OF ITERATIVE METHODS TO SOLUTIONS OF CERTAIN NONLINEAR OPERATOR EQUATIONS

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Abstract

Let E be an arbitrary real Banach space, $T : E \to E$ a Lipschitz ϕ-strongly accretive operator and let f be in the range of T. It is proved that the new iteration methods introduced by Xu (J. Math. Anal. Appl. 224 (1998), 91-101) converge strongly to the solution of the equation $Tx = f$. Related results deal with the iterative approximation of fixed points of Lipschitz ϕ-pseudocontractions with the new iteration methods in arbitrary real Banach spaces.

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1. INTRODUCTION Let E be a real Banach space and J the normalized duality mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^* : \langle x, f \rangle = ||x||^2 = ||f||^2\},$$

where E^* denotes the dual space of E and $\langle ., . \rangle$ denotes the generalized duality pairing. It is well known that if E^* is strictly convex then J is single-valued. In the sequel we shall denote the single-valued normalized duality mapping by j.

An operator T is called *strongly accretive* if for all $x, y \in D(T)$ there exist $j(x-y) \in J(x-y)$ and a constant $k > 0$ such that

$$\langle Tx - Ty, j(x-y) \rangle \geq k||x-y||^2. \tag{1}$$

T is called *ϕ-strongly accretive* if for all $x, y \in D(T)$ there exist $j(x-y) \in J(x-y)$ and a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y) \rangle \geq \phi(||x-y||)||x-y||. \tag{2}$$

It is shown in [19] that the class of strongly accretive operators is a proper subclass of the class of ϕ-strongly accretive operators.

If I denotes the identity operator, then T is called *strongly pseudocontractive* (respectively, *ϕ-strongly pseudocontractive*) if and only if $(I-T)$ is strongly accretive (respectively, ϕ-strongly accretive). Thus the mapping theory for strongly accretive operators (respectively, ϕ-strongly accretive operators) is closely related to the fixed point theory for strongly pseudocontractive operators (respectively, ϕ-strongly pseudocontractive operators). Recent interest in mapping theory for strongly accretive operators and ϕ-strongly accretive operators particularly as it relates to the existence theorems for nonlinear ordinary and partial differential equations, has prompted a corresponding interest in fixed point theory for strong pseudocontractions and ϕ-strong pseudocontractions (see for example [8],[9],[17]).

It is well known (see for example [8]) that if E is a real Banach space and $T : E \to E$ is continuous and strongly pseudocontractive, then T has a unique fixed point. Furthermore, It is proved in ([9], Theorem 13.1 p.125) that the equation

$$Tx = f \tag{3}$$

has a unique solution if E is an arbitrary real Banach space and $T : E \to E$ is continuous and strongly accretive, or E is uniformly smooth and $T : E \to E$ is demicontinuous.

Let E be a real Banach space and K a nonempty closed convex subset of E. Recently several authors have applied the Mann iteration method [16] and the Ishikawa iteration method [13] to approximate fixed points of Lipschitz strong pseudocontractions

$T : K \to K$, and to approximate solutions of equation (3) when $T : E \to E$ is a Lipschitz strongly accretive operator (see for example [1-7],[10-13],[15],[18-26]).

Recently Xu [25] introduced the following *Mann and Ishikawa iteration methods with errors*:

(a) **The Ishikawa Iteration Method with Errors** For K a nonempty convex subset of a Banach space E and $T: K \to K$ a given operator. The sequence is defined from an arbitrary $x_0 \in K$ by

$$y_n = a_n x_n + b_n T x_n + c_n u_n, \ \ n \geq 0$$

$$x_{n+1} = a'_n x_n + b'_n T y_n + c'_n v_n, \ \ n \geq 0$$

where $\{a_n\}, \ \{b_n\}, \ \{c_n\}, \ \{a'_n\}, \ \{b'_n\}, \ \{c'_n\}$ are sequences in $[0, 1]$ with $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$ and $\{u_n\}$ and $\{v_n\}$ are bounded sequences in K.

(b) **the Mann Iteration Method with Errors** With K, T and x_0 as in (a), the Mann iteration method with errors defined by

$$x_{n+1} = a'_n x_n + b'_n T x_n + c'_n v_n, \ \ n \geq 0$$

is a special case of (a) for which $a_n = 1, \ b_n = c_n = 0, \ \ \forall n \geq 0$.

In [25] Xu showed that his Mann and Ishikawa iteration methods with errors are better than the earlier Mann and Ishikawa iteration methods with errors introduced by Liu [15]. He then used these iteration methods to prove convergence theorems for the iterative approximation of fixed points of strong pseudocontractions in uniformly smooth Banach spaces. Xu [25] also proved convergence theorems for the iterative approximation of the equation (3) when E is a uniformly smooth Banach space and $T: E \to E$ is a strongly accretive operator.

Let E be a real Banach space, $T: E \to E$ a Lipschitz ϕ-strongly accretive operator and let f be in the range of T.

It is our purpose in this paper to prove that the new Mann and Ishikawa iteration methods with errors introduced by Xu [25] converge strongly to the solution of the equation $Tx = f$. Furthermore, if K is a closed convex subset of E and $T: K \to K$ is a Lipschitz ϕ-strong pseudocontraction with a fixed point, we prove that these iteration methods converge to the fixed point of T. Our results generalized, extend and unify several important recent results.

2. MAIN RESULTS

For the rest of this paper, L denotes the Lipschitz constant of T, $L_* = (1+L)$ and $R(T)$ denotes the range of T.

Theorem 1 Let E be an arbitrary real Banach space and $T: E \to E$ a Lipschitz ϕ-strongly accretive operator. Let $f \in R(T)$ and generate $\{x_n\}$ from an arbitrary $x_0 \in E$ by

$$y_n = a_n x_n + b_n(f + (I-T)x_n) + c_n u_n, \ \ n \geq 0 \tag{4}$$

$$x_{n+1} = a'_n x_n + b'_n(f + (I-T)y_n) + c'_n v_n, \ \ n \geq 0, \tag{5}$$

where $\{u_n\}$ and $\{v_n\}$ are bounded sequences in E and $\{a_n\}, \ \{b_n\}, \ \{c_n\}, \ \{a'_n\}, \ \{b'_n\}$ and $\{c'_n\}$ are real sequences in $[0, 1]$ satisfying the conditions:

(i) $a_n + b_n + c_n = a'_n + b'_n + c'_n = 1$

(ii) $\lim b_n = \lim b'_n = \lim c_n = 0$
(iii) $\sum_{n=0}^{\infty} b'_n = \infty$ and
(iv) $\sum c'_n < \infty, \ \sum {b'_n}^2 < \infty, \ \sum b'_n b_n < \infty, \ \sum b'_n c_n < \infty.$
Then the sequence $\{x_n\}$ converges strongly to the solution of the equation $Tx = f$.

Proof Define $S : E \to E$ by $Sx = f + (I - T)x$, and let x^* denote the solution of the equation $Tx = f$. Then x^* is a fixed point of S and for all $x, y \in E$ we have

$$\begin{aligned} \langle (I-S)x - (I-S)y, j(x-y)\rangle &\geq \phi(||x-y||)||x-y|| \\ &\geq \frac{\phi(||x-y||)}{(1+\phi(||x-y||)+||x-y||)}||x-y||^2 \\ &= r(x,y)||x-y||^2, \qquad (6) \end{aligned}$$

where $r(x,y) = \frac{\phi(||x-y||)}{(1+\phi(||x-y||)+||x-y||)} \in [0,1) \ \ \forall x, y \in E$. It follows from Lemma 1.1 of Kato [14] and inequality (6) that

$$||x-y|| \leq ||x-y+\lambda[(I-S)x - r(x,y)x - ((I-S)y - r(x,y)y)]||, \qquad (7)$$

for all $x, y \in E$ and for all $\lambda > 0$. Set $\beta_n = b_n + c_n$, and $\alpha_n = b'_n + c'_n$, then (4) and (5) become

$$y_n = (1-\beta_n)x_n + \beta_n Sx_n + c_n(u_n - Sx_n), \ \ n \geq 0 \qquad (8)$$

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Sy_n + c'_n(v_n - Sy_n), \ \ n \geq 0. \qquad (9)$$

From (9) we obtain

$$\begin{aligned} x_n &= (1+\alpha_n)x_{n+1} + \alpha_n[(I-S)x_{n+1} - r(x_{n+1},x^*)x_{n+1}] \\ &\quad -(1-r(x_{n+1},x^*))\alpha_n x_n + (2-r(x_{n+1},x^*))\alpha_n^2(x_n - Sy_n) \\ &\quad +\alpha_n(Sx_{n+1} - Sy_n) - [1+(2-r(x_{n+1},x^*))\alpha_n]c'_n(v_n - Sy_n) \end{aligned}$$

Furthermore,

$$x^* = (1+\alpha_n)x^* + \alpha_n[(I-S)x^* - r(x_{n+1},x^*)x^*] - (1-r(x_{n+1},x^*))\alpha_n x^*,$$

so that

$$\begin{aligned} x_n - x^* &= (1+\alpha_n)(x_{n+1} - x^*) \\ &\quad +\alpha_n\Big[(I-S)x_{n+1} - r(x_{n+1},x^*)x_{n+1} - ((I-S)x^* - r(x_{n+1},x^*)x^*)\Big] \\ &\quad -(1-r(x_{n+1},x^*))\alpha_n(x_n - x^*) + (2-r(x_{n+1},x^*))\alpha_n^2(x_n - Sy_n) \\ &\quad +\alpha_n(Sx_{n+1} - Sy_n) - [1+(2-r(x_{n+1},x^*))\alpha_n]c'_n(v_n - Sy_n). \end{aligned}$$

Hence

$$
\begin{aligned}
||x_n - x^*|| &\geq (1+\alpha_n)\Big|\Big|x_{n+1} - x^* \\
&+\frac{\alpha_n}{(1+\alpha_n)}\Big[(I-S)x_{n+1} - r(x_{n+1},x^*)x_{n+1} - ((I-S)x^* - r(x_{n+1},x^*)x^*)\Big]\Big|\Big| \\
&-(1-r(x_{n+1},x^*))\alpha_n||x_n - x^*|| - (2-r(x_{n+1},x^*))\alpha_n^2||x_n - Sy_n|| \\
&-\alpha_n||Sx_{n+1} - Sy_n|| - [1+(2-r(x_{n+1},x^*))\alpha_n]c_n'||v_n - Sy_n|| \\
&\geq (1+\alpha_n)||x_{n+1} - x^*|| - (1-r(x_{n+1},x^*))\alpha_n||x_n - x^*|| \\
&-(2-r(x_{n+1},x^*))\alpha_n^2||x_n - Sy_n|| - \alpha_n||Sx_{n+1} - Sy_n|| \\
&-[1+(2-r(x_{n+1},x^*))\alpha_n]c_n'||v_n - Sy_n||, \quad \text{(using (7))}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
||x_{n+1} - x^*|| &\leq \frac{[1+(1-r(x_{n+1},x^*))\alpha_n]}{(1+\alpha_n)}||x_n - x^*|| + 2\alpha_n^2||x_n - Sy_n|| \\
&+\alpha_n||Sx_{n+1} - Sy_n|| + [1+(2-r(x_{n+1},x^*))\alpha_n]c_n'||v_n - Sy_n|| \\
&\leq [1+(1-r(x_{n+1},x^*))\alpha_n][1-\alpha_n+\alpha_n^2]||x_n - x^*|| \\
&+2\alpha_n^2||x_n - Sy_n|| + \alpha_n||Sx_{n+1} - Sy_n|| + 3c_n'||v_n - Sy_n|| \\
&\leq [1-r(x_{n+1},x^*)\alpha_n + \alpha_n^2]||x_n - x^*|| + 2\alpha_n^2||x_n - Sy_n|| \\
&+\alpha_n||Sx_{n+1} - Sy_n|| + 3c_n'||v_n - Sy_n|| \qquad (10)
\end{aligned}
$$

Furthermore, we have the following estimates:

$$
\begin{aligned}
||y_n - x^*|| &\leq (1-\beta_n)||x_n - x^*|| + \beta_n||Sx_n - x^*|| + c_n||u_n - Sx_n|| \\
&\leq [(1-\beta_n) + \beta_n L_*]||x_n - x^*|| + c_n||u_n - x^*|| + c_n L_*||x_n - x^*|| \\
&\leq [1+2L_*]||x_n - x^*|| + c_n||u_n - x^*||
\end{aligned}
$$

$$
||x_n - Sy_n|| \leq ||x_n - x^*|| + L_*||y_n - x^*|| \leq [1+L_*(1+2L_*)]||x_n - x^*|| + L_* c_n||u_n - x^*||
$$

$$
\begin{aligned}
||Sx_{n+1} - Sy_n|| &\leq L_*||x_{n+1} - y_n|| \\
&\leq L_*\Big[(1-\alpha_n)||x_n - y_n|| + \alpha_n||Sy_n - y_n|| + c_n'||v_n - Sy_n||\Big] \\
&\leq L_*\Big[\beta_n||x_n - Sx_n|| + c_n||u_n - Sx_n|| + c_n'||v_n - Sy_n||\Big]
\end{aligned}
$$

$$
\begin{aligned}
&\quad +L_*(1+L_*)\alpha_n||y_n-x^*|| \\
&\leq L_*\Big[\beta_n(1+L_*)||x_n-x^*||+c_n||u_n-x^*||+c_nL_*||x_n-x^*||+c'_n||v_n-x^*||\Big] \\
&\quad +[c'_nL_*^2+\alpha_nL_*(1+L_*)]||y_n-x^*|| \\
&\leq \Big[L_*(1+L_*)\beta_n+c_nL_*^2+(L_*^2c'_n+L_*(1+L_*)\alpha_n)(1+2L_*)\Big]||x_n-x^*|| \\
&\quad +[L_*c_n+(c'_nL_*^2+L_*(1+L_*)\alpha_n)c_n]||u_n-x^*||+L_*c'_n||v_n-x^*||
\end{aligned}
$$

$$
\begin{aligned}
||v_n-Sy_n|| &\leq ||v_n-x^*||+L_*||y_n-x^*|| \\
&\leq ||v_n-x^*||+L_*(1+2L_*)||x_n-x^*||+L_*c_n||u_n-x^*||.
\end{aligned}
$$

Hence

$$
\begin{aligned}
||x_{n+1}-x^*|| &\leq [1-r(x_{n+1},x^*)\alpha_n+\alpha_n^2]||x_n-x^*|| \\
&\quad +2\alpha_n^2\Big[(1+L_*(1+2L_*))||x_n-x^*||+L_*c_n||u_n-x^*||\Big] \\
&\quad +\alpha_n\Big\{[L_*(1+L_*)\beta_n+c_nL_*^2+(L_*^2c'_n+L_*(1+L_*)\alpha_n)(1+2L_*)]||x_n-x^*|| \\
&\quad +[L_*c_n+(c'_nL_*^2+L_*(1+L_*)\alpha_n)c_n]||u_n-x^*||+L_*c'_n||v_n-x^*||\Big\} \\
&\quad +3c'_n[||v_n-x^*||+L_*(1+2L_*)||x_n-x^*||+L_*c_n||u_n-x^*||] \\
&\leq \Big\{1+[2L_*^3+7L_*^2+3(1+L_*)]\alpha_n^2+L_*(1+L_*)\alpha_n\beta_n+L_*^2(1+2L_*)\alpha_nc'_n \\
&\quad +L_*^2\alpha_nc_n+3L_*(1+2L_*)c'_n\Big\}||x_n-x^*||-r(x_{n+1},x^*)\alpha_n||x_n-x^*|| \\
&\quad +[2L_*\alpha_n^2c_n+L_*\alpha_nc_n+L_*^2\alpha_nc_nc'_n+L_*(1+L_*)\alpha_n^2c_n+3L_*c_nc'_n]||u_n-x^*|| \\
&\quad +(3+L_*)c'_n||v_n-x^*||. \qquad (11)
\end{aligned}
$$

Since $\{u_n\}$ and $\{v_n\}$ are bounded, we have $||u_n-x^*||\leq M_1,\ \ ||v_n-x^*||\leq M_2$ for some positive constants M_1 and M_2. Set $M=\max\{M_1,M_2\}$. Then it follows from (11) that

$$
\begin{aligned}
||x_{n+1}-x^*|| &\leq \Big\{1+[2L_*^3+7L_*^2+3(1+L_*)]\alpha_n^2+L_*(1+L_*)\alpha_n\beta_n+L_*^2(1+2L_*)\alpha_nc'_n \\
&\quad +L_*^2\alpha_nc_n+3L_*(1+2L_*)c'_n\Big\}||x_n-x^*||-r(x_{n+1},x^*)\alpha_n||x_n-x^*|| \\
&\quad +[2L_*\alpha_n^2c_n+L_*\alpha_nc_n+L_*^2\alpha_nc_nc'_n+L_*(1+L_*)\alpha_n^2c_n+3L_*c_nc'_n]M \\
&\quad +(3+L_*)c'_nM \\
&= [1+\delta_n]||x_n-x^*||-r(x_{n+1},x^*)\alpha_n||x_n-x^*||+\sigma_n, \qquad (12)
\end{aligned}
$$

where

$$\delta_n = [2L_*^3 + 7L_*^2 + 3(1+L_*)]\alpha_n^2 + L_*(1+L_*)\alpha_n\beta_n$$
$$+L_*^2(1+2L_*)\alpha_n c_n + L_*^2\alpha_n c_n + 3L_*(1+2L_*)c_n',$$

$$\sigma_n = [2L_*\alpha_n^2 c_n + L_*\alpha_n c_n + L_*^2\alpha_n c_n c_n' + L_*(1+L_*)\alpha_n^2 c_n + 3L_* c_n c_n' + (3+L_*)c_n']M$$

Observe that condition (iv) implies that $\sum_{n=0}^{\infty}\delta_n < \infty$, and $\sum_{n=0}^{\infty}\sigma_n < \infty$, so that it follows from inequality (12) that $\{||x_n - x^*||\}$ is bounded. Let $||x_n - x^*|| \leq D \;\; \forall\, n \geq 0$. Then it follows from (12) that

$$||x_{n+1} - x^*|| \leq ||x_n - x^*|| + D\delta_n + \sigma_n = ||x_n - x^*|| + \lambda_n,$$

where $\lambda_n = D\delta_n + \sigma_n$. Since $\sum_{n=0}^{\infty}\lambda_n < \infty$, it follows from Lemma 1 of [24] that $\lim_{n\to\infty}||x_n - x^*||$ exists. Let $\lim||x_n - x^*|| = \delta \geq 0$. We prove that $\delta = 0$. Assume that $\delta > 0$. Then there exists a positive integer N_0 such that $||x_n - x^*|| \geq \frac{\delta}{2} \;\; \forall\, n \geq N_0$. Since

$$r(x_{n+1}, x^*)||x_n - x^*|| = \frac{\phi(||x_{n+1} - x^*||)}{1 + \phi(||x_{n+1} - x^*||) + ||x_{n+1} - x^*||}||x_n - x^*|| \geq \frac{\phi(\frac{\delta}{2})\delta}{2(1+\phi(D)+D)}, \;\; \forall\, n \geq N_0,$$

it follows from (12) that

$$||x_{n+1} - x^*|| \leq ||x_n - x^*|| - \frac{\phi(\frac{\delta}{2})\delta}{2(1+\phi(D)+D)}\alpha_n + \lambda_n \;\; \forall\, n \geq N_0.$$

Hence

$$\frac{\phi(\frac{\delta}{2})\delta}{2(1+\phi(D)+D)}\alpha_n \leq ||x_n - x^*|| - ||x_{n+1} - x^*|| + \lambda_n \;\; \forall\, n \geq N_0.$$

This implies that

$$\frac{\phi(\frac{\delta}{2})\delta}{2(1+\phi(D)+D)}\sum_{j=N_0}^{n}\alpha_j \leq ||x_{N_0} - x^*|| + \sum_{j=N_0}^{n}\lambda_j \leq ||x_{N_0} - x^*|| + \sum_{j=0}^{\infty}\lambda_j,$$

so that $\sum_{n=0}^{\infty}\alpha_n < \infty$, contradicting $\sum_{n=0}^{\infty}\alpha_n = \sum_{n=0}^{\infty}(b_n' + c_n') = \infty$.

Hence $\lim||x_n - x^*|| = 0$, completing the proof of Theorem 1. □

Corollary 1 Let E be an arbitrary real Banach space and $T : E \to E$ a Lipschitz ϕ-strongly accretive operator, where ϕ is in addition continuous. Suppose $\liminf_{r\to\infty}\phi(r) > 0$ or $||Tx|| \to \infty$ as $||x|| \to \infty$. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a_n'\}$, $\{b_n'\}$, $\{c_n'\}$, $\{u_n\}$, $\{v_n\}$, $\{y_n\}$ and $\{x_n\}$ be as in Theorem 1. Then for any given $f \in E$, the sequence $\{x_n\}$ converges strongly to the solution of the equation $Tx = f$.

Proof The existence of a unique solution to the equation $Tx = f$ follows from Deimling ([8], Corollary 3, p.370) and the result follows from Theorem 1. $\square$

Theorem 2 Let E be a real Banach space and K a nonempty closed convex subset of E. Let $T : K \to K$ be a Lipschitz ϕ-strong pseudocontraction with a nonempty fixed-point set. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{a_n'\}$, $\{b_n'\}$, and $\{c_n'\}$ be as in Theorem 1. Let $\{u_n\}$ and $\{v_n\}$ be bounded sequences in K. Let $\{x_n\}$ be the sequence generated iteratively from an arbitrary $x_0 \in K$ by

$$y_n = a_n x_n + b_n T x_n + c_n u_n, \;\; n \geq 0$$

$$x_{n+1} = a_n' x_n + b_n' T y_n + c_n' v_n, \;\; n \geq 0.$$

Then $\{x_n\}$ converges strongly to the fixed point of T.

Proof As in the proof of Theorem 1, set $\beta_n = b_n + c_n$, and $\alpha_n = b_n' + c_n'$ to obtain

$$y_n = (1 - \beta_n)x_n + \beta_n T x_n + c_n(u_n - T x_n), \;\; n \geq 0$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T y_n + c_n'(v_n - T y_n), \;\; n \geq 0.$$

Since T is a ϕ-strong pseudocontraction, $(I - T)$ is ϕ-strongly accretive so that for all $x, y \in E$, there exist $j(x - y) \in J(x - y)$ and a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle (I - T)x - (I - T)y, j(x - y)\rangle \geq \phi(||x - y||)||x - y|| \geq r(x, y)||x - y||^2.$$

The rest of the argument now follows as in the proof of Theorem 1 and is therefore omitted. $\square$

Remark 1 An operator T with domain $D(T)$ and range $R(T)$ in E is called *ϕ-hemicontractive* (see for example [19]) if $F(T) = \{x \in D(T) : Tx = x\} \neq \emptyset$ and for all $x \in D(T)$ and $x^* \in F(T)$ there exist $j(x - x^*) \in J(x - x^*)$ and a strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - x^*, j(x - x^*)\rangle \leq ||x - x^*||^2 - \phi(||x - x^*||)||x - x^*||.$$

The example in [6] shows that the class of ϕ-strongly pseudocontractive operators with nonempty fixed-point sets is a proper subclass of the class of ϕ-hemicontractive operators. It is easy to see that Theorem 2 easily extends to the class of ϕ-hemicontractive operators.

Remark 2 If we set $a_n = 1,\ b_n = c_n = 0\ \ \forall\ n \geq 0$ in our Theorems and Corollary, we obtain the corresponding results for the Mann iteration method with errors.

Remark 3 Let $\{\alpha_n\}$ and $\{\beta_n\}$ be real sequences satisfying the conditions:

(i) $0 \leq \alpha_n, \beta_n \leq 1,\ \ n \geq 0$ (ii) $\lim \alpha_n = \lim \beta_n = 0$ (iii) $\sum_{n=0}^{\infty} \alpha_n = \infty$ (iv) $\sum_{n=0}^{\infty} \alpha_n \beta_n < \infty$ and (v) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$. If we set $a_n = (1 - \beta_n),\ b_n = \beta_n,\ c_n = 0,\ a'_n = (1 - \alpha_n),$ $b'_n = \alpha_n,\ c'_n = 0,\ \ \forall\ n \geq 0$ in Theorems 1 and 2 respectively, we obtain the corresponding convergence theorems for the original Mann and Ishikawa iteration methods. Thus our results extend, generalize and unify several recent results. In particular, our theorems extend recent results of Chidume and the author [7] (which are themselves generalizations and extensions of several results (see for example [1-3],[5],[10-12],[23],[26])) from the classes of strong pseudocontractions and strongly accretive operators to the more general classes of operators considered here. The results of ([19],[21]) are special cases of our present results.

Remark 4 Suitable choices for the real sequences in our results are:

$$a_n = \frac{n}{n+1},\ b_n = c_n = \frac{1}{2(n+1)},\ a'_n = \frac{n}{n+1},\ b'_n = \frac{n}{(n+1)^2},\ c'_n = \frac{1}{(n+1)^2},\ n \geq 0.$$

Hence $\alpha_n = \beta_n = \frac{1}{n+1},\ n \geq 0$.

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