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IRREVERSIBILITY AND HIGHER-SPIN CONFORMAL FIELD THEORY

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Abstract

I discuss the idea that quantum irreversibility is a general principle of nature and a related "conformal hypothesis", stating that all fundamental quantum field theories should be renormalization-group (RG) interpolations between ultraviolet and infrared conformal fixed points. In particular, the Newton constant should be viewed as a low-energy effect of the RG scale μ. This approach leads naturally to consider higher-spin conformal field theories, which are here classified, as candidate high-energy theories. Bosonic conformal tensors have a positive-definite action, equal to the square of a field strength, and a higher-derivative gauge invariance. The central charges c and a are well defined and positive. I calculate their values and study the operator-product structure. Fermionic theories have no gauge invariance and can be coupled to Abelian and non-Abelian gauge fields in a renormalizable way. At the quantum level, they contribute to the one-loop beta function with the same sign as ordinary matter, admit a conformal window and non-trivial interacting fixed points. The propagation of the ghost degrees of freedom is visible through the existence of composite operators of high spin and low dimension, violating the Ferrara–Gatto–Grillo theorem. These ghosts might disappear above the Planck length, thanks to the irreversibility of the RG flow, the Nachtmann theorem or equivalent mechanisms. There might also be applications to the nuclear physics of hadronic resonances. Other theories, such as conformal antisymmetric tensors and higher-derivative theories, are shown to be less promising, because of more severe internal problems.

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1 Statement of the problem

In the approach to quantum field theory as a radiative interpolation between pairs of fixed points (see [1] for a brief survey) a natural question is the definition of the central charges c and a for higher-spin fields, in particular gravity. In conformal field theory, c and a are defined by the trace anomaly in external gravity. The quantity c multiplies the square of the Weyl tensor $W^2_{\mu\nu\rho\sigma}$ and is the coefficient of the two-point function of the stress tensor. The quantity a multiplies the Euler density $\mathrm{G}_4 = \varepsilon_{\mu\nu\rho\sigma}\varepsilon_{\alpha\beta\gamma\delta}R^{\mu\nu\alpha\beta}R^{\rho\sigma\gamma\delta}$. A third term, $\Box R$, is multiplied by a coefficient a':

$$\Theta = \frac{1}{(4\pi)^2}\left[-cW^2 + \frac{a}{4}\mathrm{G}_4 - \frac{2}{3}a'\Box R\right]. \tag{1.1}$$

Two orders of problems arise when attempting to define c and a for gravity. First, gravity cannot be embedded in "external gravity" in a non-trivial way. More importantly, the theory is not conformal at the classical level. In particular, the massless spin-2 free-field theory is not a conformal field theory and we cannot formulate a radiative interpolation between pairs of conformal fixed points if the infrared (IR) limit contains a spin-2 free field, no matter what the definition of gravity at high energies is. This is a bit uncomfortable, since it forbids a "unified" description of all interactions in the approach of [1]. That approach can be generalized by the following *conformal hypothesis*:

every quantum field theory describing the phenomena of nature should be a renormalization-group (RG) interpolation between a ultraviolet (UV) conformal field theory and an IR conformal field theory.

By RG flow I mean the flow induced by the dynamical scale μ. This means that not only should the theory have well-defined conformal UV and IR limits, but it should also be conformal as a classical theory, which implies it being strictly renormalizable at the quantum level. These are the theories considered in [2, 3], for which a general formula for the irreversibility of a holds. The conformal hypothesis is the assumption that all the fundamental theories of nature are of this type.

In some sense, the conformal hypothesis puts a restriction on the correspondence principle: the fundamental theories of nature should be obtained by the quantization of classically conformal field theories. Here I discuss the plausibility of the conformal hypothesis as a general principle of physics, both in its strong form (no dimensionful parameter is fundamental; all dimensionful parameters should descend from μ in some way) and in some weaker forms (Newton's constant and, eventually, other non-renormalizable parameters, descend from μ, as well as Λ_{QCD}, but there might be fundamental super-renormalizable parameters and masses).

These ideas are suggested to me by our present knowledge of low-energy QCD [4]. If the quarks are massless, the masses of hadrons are proportional to Λ_{QCD} and therefore descend from μ: massless QCD obeys the conformal hypothesis in its strong form. The pion masses and corrections to hadron masses are due to the quark masses. Therefore massive QCD obeys the conformal hypothesis in its weak form. The strong conformal hypothesis might explain the pion masses in a more fundamental theory, but not in QCD. Massless QCD is the prototype of "perfect" theory from the point of view of the conformal hypothesis. It is tempting to think that the ultimate theory of the world is of the same type. Theories not obeying the (strong or weak) conformal hypothesis should be viewed as low-energy effective theories descending from high-energy fundamental theories obeying the conformal hypothesis.

A first question is: How can we describe gravity without violating the conformal principle? It is conceivable that a non-vanishing cosmological constant can make the graviton disappear in the far IR limit, which would be consistent with the conformal hypothesis. More difficult is to say what gravity should look like at high energies if the conformal hypothesis is true.

The conformal hypothesis is naturally suggested by the idea that quantum irreversibility, i.e. the statement that a decreases along the RG flow, in particular $a_{\rm UV} - a_{\rm IR}$, is equal to the (scheme-invariant) area of the graph of the beta function between the fixed points [2], is a fundamental principle of nature. If this idea is correct, gravity should obey the irreversibility principle.

The question is theoretically relevant for the following reason. Because of the irreversible loss of degrees of freedom along the RG flow, it is natural to expect that the formulation of quantum field theory starting from its low-energy limit might find major obstacles, in perturbation theory. Think, for example, of a free massive scalar field, with lagrangian $\mathcal{L} = \frac{1}{2}[(\partial_\mu\varphi)^2 + m^2\varphi^2]$: its far IR limit ($m \to \infty$) is the empty theory. The theory cannot be recovered unambiguously from the empty theory. In the UV limit, instead, we have a free field and the low-energy limit can be reached with a φ^2-perturbation. Similar considerations apply to RG flows, because the irreversibility of an RG flow is qualitatively (but not quantitatively) similar to the irreversibility of a massive flow. In conclusion, a perturbative formulation of quantum field theory is meaningful only from the UV.

Concretely, this idea implies that, in QED, problems of the type of the Landau poles are not removable; actually they are the sign that the theory is formulated from the wrong limit, or that it is the low-energy effective limit of a more fundamental theory. The triviality of $\lambda\phi^4$ might have a similar explanation. Similarly, the relationship between *a*) QCD expressed in its natural high-energy variables (quarks and gluons) and *b*) QCD expressed in its natural low-energy variables (hadrons) – would not be a well-defined "change of variables": it would not be invertible. We understand that it is very important to answer the question of whether quantum irreversibility is a fundamental principle of nature or a coincidence.

Naively, however, it is tempting to think that gravity, and actually every non-renormalizable interaction, violates the irreversibility principle, because a coupling constant with negative dimension in units of mass kills degrees of freedom in the UV and leaves the IR unchanged. This effect is in some sense dual to the effect of masses or, in general, super-renormalizable terms, where degrees of freedom are killed in the IR and the UV is left unchanged. The intermediate case, a classically conformal theory flowing only due to the RG scale μ, is less trivial, but much more interesting; and it does obey the irreversibility property [2].

In [5] it was shown that above two dimensions the behaviour of a is, at the quantitative level, sensitive to the dimensionality of parameters (marginal, relevant, irrelevant deformations). In particular, strictly renormalizable interactions cannot be assimilated to super-renormalizable interactions (in which I include the mass terms), although they are qualitatively similar. Only in two dimensions, or when $c = a$ in higher even dimensions, can we disregard this difference. I stress that this fact does not contradict, or restrict, the irreversibility statement, which is a statement about the irreversibility of the RG flow, i.e. the flow induced by the RG scale μ (see introduction of [2]), not a generic phenomenon of decreasing of a and the degrees of freedom. A dependence on the dimensionality of the parameters is natural: it would be upsetting if μ behaved as an ordinary scale.

In summary, we have the following four situations:

1) Classically conformal theories, where the flow is due only to μ at the quantum level; there is no explicit dimensionful parameter and μ does not become "real" (for example, in the form of the expectation value of a condensate) at low energies. Examples are the theories of the conformal window.

2) Classically conformal theories having an explicit dimensionful parameter at low energies. An example is massless QCD, and the dimensionful constant is $\Lambda_{\rm QCD}$. However, $\Lambda_{\rm QCD}$ is just the scale μ, which has "come to reality", at low energies, owing to the mechanism of chiral-symmetry breaking and dimensional transmutation. We have

$$\Lambda_{\rm QCD} = \mu \exp\left(-\int^{\alpha(\mu)} \frac{{\rm d}\alpha'}{\alpha' \beta_{\rm QCD}(\alpha')} \right).$$

This case can be assimilated to (1) from the point of view of quantum irreversibility. The formula expressing the difference $a_{\rm UV} - a_{\rm IR}$ [2] as the invariant area of the graph of the beta function between the two fixed holds in this case.

3) Massive terms and super-renormalizable interactions. They obey the inequality $a_{\rm UV} \geq a_{\rm IR}$, but the actual value of $a_{\rm UV} - a_{\rm IR}$ is measured differently [5]. These cases can be assimilated to cases (1) and (2) at the quantitative level when $c = a$ [5].

4) Non-renormalizable interactions and in particular the Newton constant κ. This case remains to be clarified from the point of view of quantum irreversibility.

Gravity will obey the irreversibility property if κ itself is generated by the RG scale μ:

$$\kappa = \frac{1}{\mu} \exp\left(\int^{\lambda(\mu)} \frac{{\rm d}\lambda'}{\beta_G(\lambda')} \right),$$

as a low-energy effect of an unknown, classically conformal high-energy theory having beta function β_G. This is one possible solution to our problem: if Newton's constant descends from μ, then it is constrained by the arguments of [2] to obey the irreversibility principle. The graviton might be the analogue of the pion in massless QCD.

Another wayout might be the following. We consider here the "$c = a$ flows", which means flows connecting UV and IR fixed points in such a way that the difference $c - a$ remains constant (not necessarily zero), in particular $a_{\rm UV} - a_{\rm IR} = c_{\rm UV} - c_{\rm IR}$. The fixed points might or not have $c = a$. For example, taking a direct product between a $c = a$ flow connecting two $c = a$ fixed points and a free-field theory, we can obtain a $c = a$ flow connecting $c \neq a$ fixed points.

In the $c = a$ flows, dimensionless parameters can be assimilated to dimensionful coupling constants, for example masses and, conceivably, also the Newton constant [5]. This means that the effects of divergences and their removal (running of the coupling constants, RG flow, etc.) can be understood as a more common, "geometrical" phenomenon in disguise and that the dynamical scale μ can be interpreted "classically" as a mass, the inverse of a compactification radius, or something similar. Therefore in a $c = a$ flow divergences are not really "divergences", and the RG flow is not really an "RG" flow. Running coupling and divergences might be effects of an unconvenient choice of variables. In the appropriate variables the theory might be truly finite and μ be a mass or the inverse of a compactification radius. When $c \neq a$, on the other hand, it is very unlikely that an RG flow can be interpreted geometrically. It is anyway true that a mass M behaves like μ in the limit $M \to \infty$, a property used for example in the Pauli-Villars regularization technique.

This motivates us to state that the concept of "finiteness" in quantum field theory should be extended to include flows in which the divergences admit a "classical" interpretation, in particular the $c = a$ flows.

The relevance of these observations to our present problem are that in $c = a$ flows it is reasonable to expect that parameters of types (1–4) can all be assimilated and we do not need to generate the dimensionful parameters of the low-energy theories via the RG scale μ. Then quantum irreversibility would still be a general principle and Newton's constant would not need to descend from μ.

We have proposed two possible solutions to our problem, that the Newton constant descends from μ or that c is equal to a; but which proposal is more promising? In particular, is our world $c = a$?

It is very unlikely that the restriction $c = a$ is phenomenologically viable. Neither the Standard Model, nor QCD have $c = a$. We can check it in the free-field limits. We use the conventional normalization $c = \frac{1}{120}(N_s + 6N_f + 12N_v)$, $a = \frac{1}{360}(N_s + 11N_f + 62N_v)$ for free field theories of $N_{s,f,v}$ real scalars, Dirac fermions and vectors, respectively. QED has $N_v = 1$, $N_f = 1$ and therefore $c = \frac{3}{20}$, $a = \frac{73}{360}$, $c - a = -\frac{37}{360}$. Massless QCD has $N_v = 8$, $N_f = 18$ and therefore $c = \frac{17}{10}$, $a = \frac{347}{180}$, $c - a = -\frac{41}{180}$. The electroweak theory has $N_v = 4$, $N_f = \frac{9}{2}(6)$, $N_s = 4$ and therefore $c = \frac{79}{120}\left(\frac{11}{15}\right)$, $a = \frac{67}{80}\left(\frac{53}{60}\right)$, $c - a = -\frac{43}{180}\left(-\frac{3}{20}\right)$ (in parenthesis the values for a model with right-handed neutrinos). The Standard Model has $c = \frac{283}{120}\left(\frac{73}{30}\right)$, $a = \frac{1991}{720}$ $\left(\frac{253}{90}\right)$, $c - a = -\frac{293}{720}\left(-\frac{17}{45}\right)$. We see that $c - a < 0$ always, which means that there are many vector fields.

A necessary, but not sufficient, condition for a $c = a$ flow is obtained by comparing the values of c and a at energies admitting (approximate) free-field descriptions. The differences between the numbers of spin-0, 1/2, 1 fields at two such energies should be related by the formula

$$2\Delta N_s + 7\Delta N_f = 26\Delta N_v. \tag{1.2}$$

Comparing the UV and IR limits of massless QCD, we find $\Delta N_s = -n_f^2 + 1$, $\Delta N_f = N_c n_f$, $\Delta N_v = N_c^2 - 1$, where N_c is the number of colors and n_f is the number of quark flavours. It is easy to check that the condition has no solution. Only for $N_c = 1$ has n_f a real value. This means that our "perfect theory" is very far from $c = a$. Similarly, the spectra of the known low-energy physics do not appear to obey (1.2). For example, in the IR we can neglect the electron, but we have to include it at energies comparable to its mass. Formula (1.2) implies that as soon as the electron, or other fermions, becomes important, vector fields should appear also. There is no evidence of such a behavior in nature.

We are therefore led to concentrating on the first solution to our problem: the Newton constant is not a fundamental parameter of nature, but a low-energy effect of the RG scale μ in an unknown high-energy formulation of quantum gravity. This is the first, non-trivial prediction of the conformal hypothesis.

We have already explained that, because of quantum irreversibility, it might be hopeless to quantize gravity from its low-energy limit, i.e. the Einstein theory. What can gravity look like at very high energies, if the conformal hypothesis is true? I believe that, before investigating by which mechanism to generate Newton's constant out of the dynamical scale μ, it is necessary to classify all conformal field theories. This paper is devoted to the results of this investigation. The theories formulated here have a positive-definite action and well-defined, positive central charges c and a, which will be calculated for various cases.

Higher-spin conformal field theories have rather unusual properties, which is why I thought that it was good to devote a considerable part of this introduction to stating the problem and the ideas that inspired it. The theories propagate ghosts [6], but on the other hand have a number of interesting features (of which conformal invariance is just the most important), which make them interesting either as a laboratory for investigations in the spirit of [1] and the questions raised above, or for the description of physical phenomena in limited energy ranges. In some respects, theories having a similar status are the higher-derivative theories. Both have non-trivial renormalizable interactions and propagate ghosts. Both can be conformal at the classical level and might have conformal windows at the quantum level. In some cases, they have a positive-definite action in the Euclidean framework. Yet, higher-derivative theories appear to be less promising for our purposes.

Some of our theories have a special gauge invariance, which is investigated in detail. It is a higher-derivative gauge symmetry, the unique gauge transformation compatible with the conformal symmetry. Moreover, these theories admit proper definitions of field strengths, dual field strengths, Chern–Simons forms, topological invariants, etc. The ghost propagation is a delicate issue in the presence of this higher-derivative gauge symmetry; I discuss some features that should be kept in mind when attempting to remove the ghosts from the theory. This might happen dynamically, at low energies, thanks to quantum irreversibility itself (the ghosts might disappear above the Planck length, far before the physically observable degrees of freedom) or a generalization of the Nachtmann theorem [7].

The ghost propagation is exhibited by violations of the Ferrara–Gatto–Grillo theorem [8], stating that primary composite operators with spin s should have a total dimension greater than or equal to $2 + s$. Indeed, the higher-derivative gauge invariance allows for "multiply-conserved" currents with dimensions $\Delta = 2 + s,\ 1 + s, \ldots, 3$. Some of these operators will be constructed explicitly.

The study of higher-spin conformal field theory is in some sense complementary to the Fradkin–Vasiliev higher-spin field theory [9], which is not conformal, but does not propagate ghosts.

There have been earlier works on conformal field equations of spin 2 [10, 11, 12, 13, 14] and spin 3/2 [13] fields. These theories are particular cases of the ones presented below. To my knowledge the relationship between conformal invariance and higher-derivative gauge invariance was not known. Recently, related theories have received some interest in the domain of nuclear physics, where the purpose is to account for the hadronic resonances, such as the spin-3/2 $\Delta(1232)$ [17, 18]. I believe that the properties outlined here might be useful in this domain, at least in a definite energy range.

The plan of the paper is as follows. I present the bosonic conformal fields in section 2, the fermionic fields in section 4. Section 3 is devoted to a detailed analysis of the spin-2 field, with computations of c and a and a study of the operator product (OPE) structure. In section 4 the contribution to the gauge beta function from conformal spin-3/2 matter fermions is computed. Section 5 contains observations about higher-derivatives conformal field theories. I work in the Euclidean framework throughout this paper.

Before beginning our investigation, I mention that a different approach to the problem of defining c and a for gravity might start from the results of Christensen and Duff [19], where the trace anomaly in external gravity is calculated for fields with arbitrary spin. The results, however, are difficult to interpret in our context, because the trace anomaly contains the square of the Ricci curvature R :

$$\Theta = \frac{1}{(4\pi)^2}\left[-cW^2+\frac{a}{4}\mathrm{G}_4+\zeta R^2-\frac{2}{3}a'\Box R\right].$$

This is a sign that the theory is not conformal. The definition of c and a from the trace anomaly are unambiguous only if there is no such term. Yet, since the difference $c-a$ multiplies the unique term containing the Riemann tensor, $R^2_{\mu\nu\rho\sigma}$, there might still be an appropriate definition of $c=a$ theories of gravitons and gravitinos: these should be the theories whose trace anomaly contains only $R^2_{\mu\nu}$, R^2 and $\Box R$, but not $R^2_{\mu\nu\rho\sigma}$. I recall, indeed, that in arbitrary even dimensions the trace anomaly of the $c=a$ theories contains the "minimal amount" of Riemann tensors, as shown in ref. [5]. Explicitly, for spin-3/2 and spin-2 we find, from table II of ref. [19] (omitting the $\Box R$-term):

$$\Theta_{3/2} = \frac{1}{360(4\pi)^2}\left[255\,W^2-22\,\frac{\mathrm{G}_4}{4}+\frac{61}{2}R^2\right],$$
$$\Theta_2 = \frac{1}{360(4\pi)^2}\left[-297\,W^2-127\,\frac{\mathrm{G}_4}{4}-\frac{717}{2}R^2\right].$$

We see that the graviton and gravitino contributions to the R^2-term have opposite signs. This means that a suitable combination of gravitons and gravitinos can cancel the R^2-term and might be suitable for the IR limit of a theory satisfying the conformal hypothesis. Although

I do not pursue this strategy further in this paper, I emphasize that it would be extremely interesting to re-examine the analysis of Christensen and Duff in this spirit.

2 Conformal bosonic fields

The simplest example of higher-spin conformal field theory is the free spin-2 conformal field.

Let $\chi_{\mu\nu}$ be symmetric and traceless. The action

$$S = \int \mathcal{L}_1 = \int \frac{1}{2}(\partial_\mu \chi_{\nu\rho})^2 - \frac{2}{3}(\partial_\mu \chi_{\mu\nu})^2 \tag{2.3}$$

is invariant with respect to coordinate inversion $x^\mu \to \frac{x^\mu}{|x|^2}$. Under this transformation the tensor $\chi_{\mu\nu}$ transforms as

$$\chi_{\mu\nu}(x) \to |x|^2 I_{\mu\rho}(x) I_{\nu\sigma}(x) \chi_{\rho\sigma}(x),$$

where $I_{\mu\nu}(x) = \delta_{\mu\nu} - 2x_\mu x_\nu / |x|^2$. This invariance fixes uniquely the action S, and the lagrangian $\mathcal{L}_1$ up to total derivatives. A better choice of the total derivatives leads to the lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \chi_{\nu\rho})^2 - \frac{1}{2}\partial_\mu \chi_{\nu\rho} \partial_\nu \chi_{\mu\rho} - \frac{1}{6}(\partial_\mu \chi_{\mu\nu})^2. \tag{2.4}$$

$\mathcal{L}$ transforms as a scalar under coordinate inversion, $\mathcal{L} \to |x|^8 \mathcal{L}$. The action S is invariant under the higher-derivative gauge transformation

$$\delta \chi_{\mu\nu} = \partial_\mu \partial_\nu \Lambda - \frac{1}{4}\delta_{\mu\nu} \Box \Lambda, \tag{2.5}$$

but not with respect to the diffeomorphism-type transformation $\delta \chi_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$. The lagrangian $\mathcal{L}$ is also invariant, while $\mathcal{L}_1$ is invariant up to a total derivative.

The gauge-transformation (2.5) is compatible with the conformal symmetry. This can be proved by observing that Λ has dimension -1 and thus, under coordinate inversion, $\delta\chi_{\mu\nu}$ transforms in the same way as $\chi_{\mu\nu}$:

$$\Lambda \to |x|^{-2}\Lambda, \qquad \partial_\mu \partial_\nu \Lambda - \frac{1}{4}\delta_{\mu\nu}\Box\Lambda \to |x|^2 I_{\mu\rho}(x) I_{\nu\sigma}(x) \left(\partial_\rho \partial_\sigma \Lambda - \frac{1}{4}\delta_{\rho\sigma}\Box\Lambda \right).$$

To check this, observe that the derivative operator transforms as a vector of dimension 1: $\partial_\mu \to |x|^2 I_{\mu\nu}(x) \partial_\nu$.

The field equations read

$$\Box \chi_{\mu\nu} = \frac{2}{3}\partial_\rho \left(\partial_\mu \chi_{\nu\rho} + \partial_\nu \chi_{\mu\rho} \right) - \frac{1}{3}\delta_{\mu\nu} \partial_\rho \partial_\sigma \chi_{\rho\sigma}.$$

Defining the vector field $\mathcal{A}_\mu = \partial_\nu \chi_{\mu\nu}$ and its field strength $\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu$, the field equations and gauge invariance imply

$$\partial_\mu \mathcal{F}_{\mu\nu} = 0, \qquad \delta \mathcal{A}_\mu = \frac{3}{4}\partial_\mu \Box \Lambda.$$

2.1 Field strength

The gauge symmetry (2.5) leads to the introduction of a natural field strength,

$$F_{\mu\nu\alpha} = \partial_\mu \chi_{\nu\alpha} - \partial_\nu \chi_{\mu\alpha} - \frac{1}{3}\delta_{\mu\alpha}\partial_\rho \chi_{\rho\nu} + \frac{1}{3}\delta_{\nu\alpha}\partial_\rho \chi_{\rho\mu},$$

which is easily proved to be gauge-invariant. This field strength satisfies a number of noticeable properties. First of all, we have the identities

$$F_{\mu\nu\alpha} = -F_{\nu\mu\alpha}, \qquad F_{\mu\nu\mu} = 0, \qquad F_{\mu\nu\alpha} + F_{\alpha\mu\nu} + F_{\nu\alpha\mu} = 0. \tag{2.6}$$

The third identity will be called the *cyclic identity*. Secondly, the lagrangian (2.4) can be written as

$$\mathcal{L} = \frac{1}{4}(F_{\mu\nu\alpha})^2,$$

which implies, in particular, that it is positive-definite, a fact that was not evident from (2.3) and (2.4). Since $\mathcal{L}$ is a conformal field of dimension 4, it is evident that the field strength is itself a conformal field of dimension 2 and transforms as

$$F_{\mu\nu\alpha} \to |x|^4 I_{\mu\rho}(x) I_{\nu\sigma}(x) I_{\alpha\beta}(x) F_{\rho\sigma\beta}$$

under coordinate inversion. The field equations read

$$\partial_\mu F_{\mu\nu\alpha} + \partial_\mu F_{\mu\alpha\nu} = 0. \tag{2.7}$$

It is convenient to introduce the dual of the field strength, as well as self-dual and anti-self-dual field strengths:

$$\tilde{F}_{\mu\nu\alpha} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} F_{\rho\sigma\alpha}, \qquad F^{\pm}_{\mu\nu\alpha} = \frac{1}{2}\left(F_{\mu\nu\alpha} \pm \tilde{F}_{\mu\nu\alpha}\right).$$

Each of these tensors satisfies the same symmetry identities (2.6) as $F_{\mu\nu\alpha}$. We can also derive the "Bianchi identity"

$$\partial_\mu \tilde{F}_{\mu\nu\alpha} + \partial_\mu \tilde{F}_{\mu\alpha\nu} = 0.$$

There is a natural topological invariant and a "Chern-Simons" form:

$$F_{\mu\nu\alpha}\tilde{F}_{\mu\nu\alpha} = \partial_\mu \left(\varepsilon_{\mu\nu\rho\sigma} \chi_{\nu\alpha} F_{\rho\sigma\alpha}\right).$$

Non-trivial interactions for conformal higher-spin theories can be constructed, as power series in the field strength:

$$\mathcal{L} = \frac{1}{4}\left(F_{\mu\nu\alpha}\right)^2 + \frac{1}{\Lambda^4}\left\{a\left[\left(F_{\mu\nu\alpha}\right)^2\right]^2 + bF_{\mu\nu\alpha}F_{\nu\rho\alpha}F_{\rho\sigma\beta}F_{\sigma\mu\beta} + cF_{\mu\nu\alpha}F_{\nu\rho\beta}F_{\rho\sigma\alpha}F_{\sigma\mu\beta}\right\} + \cdots \tag{2.8}$$

Λ being some mass scale and a, b, c being dimensionless parameters. These vertices are non-trivial because they do not vanish on the solutions to the field equations (2.7). Our interest, however, is mostly to look for non-trivial renormalizable interactions, which preserve conformality at the classical level. These are more difficult to construct, but are fundamental for the conformal hypothesis stated in the introduction. Certain renormalizable interactions will be studied in this paper (for fermionic higher-spin conformal theories), but a complete classification will not be given here.

The coupling to gravity is not straightforward and might not exist at all. Simple attempts to impose the compatibility between the gauge symmetry (2.5) and gravity generate terms that cannot be reabsorbed. Nevertheless, this does not forbid a correct definition of c and a (see section 3).

2.2 Quantization

In the presence of a higher-derivative gauge invariance, the problem of ghost-propagation is delicate. In this section I calculate the propagators and discuss a number of important features.

The most natural gauge-fixing is

$$\partial_\mu\partial_\nu\chi_{\mu\nu} = \partial_\mu\mathcal{A}_\mu = 0. \tag{2.9}$$

The gauge-fixed lagrangian becomes

$$\mathcal{L}_1 = \frac{1}{2}(\partial_\mu\chi_{\nu\rho})^2 - \frac{2}{3}(\partial_\mu\chi_{\mu\nu})^2 + b\,\partial_\mu\partial_\nu\chi_{\mu\nu} - \frac{3}{4}\overline{C}\Box^2 C$$

and the BRS transformation reads

$$s\chi_{\mu\nu} = \partial_\mu\partial_\nu C - \frac{1}{4}\delta_{\mu\nu}\Box C, \quad sC = 0, \quad s\overline{C} = b, \quad sb = 0.$$

Defining the projectors

$$\begin{aligned}
P_{1\mu\nu,\rho\sigma} &= \frac{1}{2}\left(\delta_{\mu\rho}\delta_{\nu\sigma} + \delta_{\mu\sigma}\delta_{\nu\rho}\right) - \frac{1}{4}\delta_{\mu\nu}\delta_{\rho\sigma}, \\
P_{2\mu\nu,\rho\sigma} &= \frac{1}{3}\left(\partial_\mu\partial_\rho\delta_{\nu\sigma} + \partial_\mu\partial_\sigma\delta_{\nu\rho} + \partial_\nu\partial_\rho\delta_{\mu\sigma} + \partial_\nu\partial_\sigma\delta_{\mu\rho} - \partial_\mu\partial_\nu\delta_{\rho\sigma} - \partial_\rho\partial_\sigma\delta_{\mu\nu} + \frac{1}{4}\Box\delta_{\mu\nu}\delta_{\rho\sigma}\right)\frac{1}{\Box}, \\
\varpi_{\mu\nu} &= \partial_\mu\partial_\nu\frac{1}{\Box} - \frac{1}{4}\delta_{\mu\nu},
\end{aligned}$$

we have the relationships

$$P_1^2 = P_1, \quad P_1P_2 = P_2, \quad P_2^2 = \frac{2}{3}P_2 + \frac{4}{9}\varpi\varpi, \quad P_1\varpi = \varpi, \quad P_2\varpi = \varpi, \quad \mathrm{tr}\varpi\varpi = \frac{3}{4}.$$

The lagrangian, written as

$$\mathcal{L}_1 + \frac{3}{4}\overline{C}\Box^2 C = -\frac{1}{2}\begin{pmatrix}\chi & b\end{pmatrix} Q \begin{pmatrix}\chi \\ b\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}\chi & b\end{pmatrix}\Box\begin{bmatrix}P_1 - P_2 & -\varpi \\ -\varpi & 0\end{bmatrix}\begin{pmatrix}\chi \\ b\end{pmatrix},$$

can be easily inverted to find the propagators, which are

$$\left\langle \begin{pmatrix}\chi \\ b\end{pmatrix}\begin{pmatrix}\chi & b\end{pmatrix}\right\rangle = -\begin{bmatrix}P_1 + 3P_2 - \frac{16}{3}\varpi\varpi & -\frac{4}{3}\varpi \\ -\frac{4}{3}\varpi & 0\end{bmatrix}\frac{1}{\Box}.$$

The x-space propagators can be written using

$$-\left(\frac{1}{\Box}\right)_{(x,0)}=\frac{1}{4\pi^2}\frac{1}{|x|^2},\qquad \frac{1}{\Box^2}=-\frac{1}{16\pi^2}\ln|x|^2\mu^2,\qquad \frac{1}{\Box^3}=-\frac{|x|^2}{128\pi^2}\left(\ln|x|^2\mu^2-\frac{3}{2}\right).$$

The field b does not propagate, because $\langle b(x)\,b(0)\rangle=\left\langle s\left(\overline{C}(x)\,b(0)\right)\right\rangle=0$. Similarly, $\langle b(x)\,\chi_{\mu\nu}(0)\rangle$ vanishes on-shell, i.e. when saturated by χ-polarizations satisfying the gauge-fixing condition $\partial_\mu\partial_\nu\chi_{\mu\nu}=0$. To study the two-point function $\langle\chi_{\mu\nu}(x)\,\chi_{\rho\sigma}(0)\rangle$ we need to remove the gauge freedom completely. We have already imposed one gauge condition (2.9) and there remains the residual gauge freedom for us to fix.

I now show that the ghosts of the theory are precisely $\mathcal{A}_\mu$. If we impose

$$\partial^\mu\chi_{\mu\nu}=0, \tag{2.10}$$

not only the ghosts are killed, but just two helicities effectively propagate.

With this condition the field equations reduce to an ordinary wave equation for $\chi_{\mu\nu}$, $\Box\chi_{\mu\nu}=0$. Moreover, the propagator, saturated with χ-polarizations, becomes just $|\chi_{\mu\nu}(k)|^2/k^2$ and we have to show that the numerator $|\chi_{\mu\nu}(k)|^2$ is positive in the Lorentzian framework. Now, the gauge-fixing $\partial_\mu\partial_\nu\chi_{\mu\nu}=0$ gives, in momentum space,

$$\chi_{00}+\hat{n}_i\hat{n}_j\chi_{ij}=2\hat{n}_i\chi_{0i}, \tag{2.11}$$

where $\hat{n}_i=k_i/k_0$, $i=1,2,3$, and $k_0^2=k_i^2$. The additional conditions $\partial_\mu\chi_{\mu i}=0$ give

$$\hat{n}_j\chi_{ij}=\chi_{0i},$$

which, reinserted into (2.11), also give

$$\chi_{00}=\hat{n}_i\chi_{0i},$$

i.e. $\partial_\mu\chi_{\mu 0}=0$, justifying (2.10). The condition of vanishing trace for $\chi_{\mu\nu}$ gives $\chi_{ii}=-\chi_{00}=\hat{n}_i\hat{n}_j\chi_{ij}$. We have therefore

$$|\chi_{\mu\nu}|^2=|\chi_{ij}|^2+|\chi_{ii}|^2-2|\hat{n}_j\chi_{ij}|^2.$$

Let us choose $\hat{n}_i=(0,0,1)$. The condition $\chi_{ii}=\hat{n}_i\hat{n}_j\chi_{ij}$ gives $\chi_{22}=-\chi_{11}$ and finally

$$|\chi_{\mu\nu}|^2=2(|\chi_{11}|^2+|\chi_{12}|^2)\geq 0.$$

Not only do we get positivity, confirming that no ghost propagates when (2.10) in enforced, but we discover that just two helicities propagate. The four conditions (2.10) leave us with five independent χ-components, χ_{11},χ_{12} and χ_{i3}, but there is no physical propagation of the χ_{i3}. This means that there is a sort of hidden symmetry. We have therefore identified the ghosts of the theories in $\mathcal{A}_\mu$. The theory becomes unitary as soon as the vector $\mathcal{A}_\mu$ disappears by some mechanism not known at present.

2.3 Arbitrary integer spin

Let $\chi_{\mu_1\cdots\mu_s}$ be a completely symmetric and completely traceless tensor. Invariance of the action under the transformation

$$\chi_{\mu_1\cdots\mu_s} \rightarrow |x|^2 I_{\mu_1}^{\nu_1}(x)\cdots I_{\mu_s}^{\nu_s}(x)\chi_{\nu_1\cdots\nu_s} \tag{2.12}$$

fixes uniquely the lagrangian

$$\mathcal{L}_1 = \frac{1}{2}(\partial_\alpha\chi_{\mu_1\cdots\mu_s})^2 - \frac{s}{s+1}(\partial_\alpha\chi_{\alpha\mu_2\cdots\mu_s})^2,$$

up to the overall factor and total derivatives. $\mathcal{L}_1$ reduces to the usual vector lagrangian for $s = 1$ and to the free real-scalar theory for $s = 0$. The action is invariant under the gauge-transformation

$$\delta\chi_{\mu_1\cdots\mu_s} = \partial_{\mu_1}\cdots\partial_{\mu_s}\Lambda - \text{traces},$$

which, as before, is compatible with (2.12), when taking into account that Λ has dimension $1-s$ and transforms as $\Lambda \rightarrow |x|^{2(1-s)}\Lambda$ under coordinate inversion.

The field strength reads

$$F_{\mu\nu\mu_2\cdots\mu_s} = \partial_\mu\chi_{\nu\mu_2\cdots\mu_s} - \partial_\nu\chi_{\mu\mu_2\cdots\mu_s} - \frac{1}{s+1}\sum_{i=2}^{s}\left(\delta_{\mu\mu_i}\partial_\alpha\chi_{\alpha\nu\mu_2\cdots\widehat{\mu_i}\cdots\mu_s} - \delta_{\nu\mu_i}\partial_\alpha\chi_{\alpha\mu\mu_2\cdots\widehat{\mu_i}\cdots\mu_s}\right).$$

A hat denotes indices that have to be omitted. As before, the field strength is gauge-invariant and conformal. It is completely symmetric in $\mu_2\cdots\mu_s$ and antisymmetric in $\mu\nu$. Furthermore, it is completely traceless, not only in the indices $\mu_2\cdots\mu_s$, but also with respect to the remaining contraction:

$$F_{\mu\nu\nu\mu_3\cdots\mu_s} = 0. \tag{2.13}$$

Finally, it satisfies the cyclic condition

$$F_{\mu\nu\alpha\mu_3\cdots\mu_s} + F_{\alpha\mu\nu\mu_3\cdots\mu_s} + F_{\nu\alpha\mu\mu_3\cdots\mu_s} = 0. \tag{2.14}$$

The conformal, positive-definite lagrangian can be written as

$$\mathcal{L} = \frac{1}{4}(F_{\mu\nu\mu_2\cdots\mu_s})^2.$$

The field equations and Bianchi identities are

$$\partial_\mu F_{\mu\alpha_1\cdots\alpha_s} + \text{perms}(\alpha_1\cdots\alpha_s) = 0, \qquad \partial_\mu\tilde{F}_{\mu\alpha_1\cdots\alpha_s} + \text{perms}(\alpha_1\cdots\alpha_s) = 0.$$

Dual and self-dual field strengths are defined as

$$\tilde{F}_{\mu\nu\alpha_2\cdots\alpha_s} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma\alpha_2\cdots\alpha_s}, \qquad F^{\pm}_{\mu\nu\alpha_2\cdots\alpha_s} = \frac{1}{2}\left(F_{\mu\nu\alpha_2\cdots\alpha_s} \pm \tilde{F}_{\mu\nu\alpha_2\cdots\alpha_s}\right)$$

and satisfy the traceless and cyclic conditions (2.13) and (2.14).

There is a topological invariant, proportional to the integral of

$$F_{\mu\nu\mu_2\cdots\mu_s}\tilde{F}_{\mu\nu\mu_2\cdots\mu_s}=\partial_\mu\left(\varepsilon_{\mu\nu\rho\sigma}\chi_{\nu\mu_2\cdots\mu_s}F_{\rho\sigma\mu_2\cdots\mu_s}\right).$$

The equality can be proved by using the Bianchi identity and (2.13).

The stress tensor is

$$T_{\mu\nu}=\text{const.}\,F^+_{\mu\alpha_1\cdots\alpha_s}F^-_{\nu\alpha_1\cdots\alpha_s}.$$

Tracelessness is straightforward, while the proof of conservation follows the same line as in the spin-2 case (see section 3). The procedure to fix the overall factor and the relation with the Noether tensor are discussed in detail for $s=2$. Higher-spin tensor currents can be constructed using the recipes of [20, 21].

2.4 Implications of the higher-derivative gauge invariance on correlators

The general form of the two-point function of a conformal composite operator $\mathcal{O}_{\mu_1\cdots\mu_s}$ with spin s is, in the notation of [20, 21]:

$$\langle\mathcal{O}_{\mu_1\cdots\mu_s}(x)\ \mathcal{O}_{\nu_1\cdots\nu_s}(0)\rangle=c_s\frac{1}{(|x|\mu)^{2h_s}}\prod{}^{(s)}_{\mu_1\cdots\mu_s,\nu_1\cdots\nu_s}\left(\frac{1}{|x|^4}\right),\qquad(2.15)$$

where $\prod^{(s)}_{\mu_1\cdots\mu_s,\nu_1\cdots\nu_s}$ is the unique differential operator of degree $2s$ that is completely symmetric and traceless in $\mu_1\cdots\mu_s$ and $\nu_1\cdots\nu_s$, symmetric in the exchange $\mu\leftrightarrow\nu$, conserved with respect to any index. For example, $\pi_{\mu\nu}=\partial_\mu\partial_\nu-\delta_{\mu\nu}\Box$ for $s=1$, while $\prod^{(2)}_{\mu\nu,\rho\sigma}=\frac{1}{2}(\pi_{\mu\rho}\pi_{\nu\sigma}+\pi_{\mu\sigma}\pi_{\nu\rho})-\frac{1}{3}\pi_{\mu\nu}\pi_{\rho\sigma}$. The factor c_s is a constant (higher-spin central charge) and h_s is equal to δ_s-s-2, where δ_s is the total dimension of the operator $\mathcal{O}_{\mu_1\cdots\mu_s}$.

If the operator $\mathcal{O}_{\mu_1\cdots\mu_s}$ couples to a conformal higher-spin field $\chi_{\mu_1\cdots\mu_s}$, via a vertex $\mathcal{O}_{\mu_1\cdots\mu_s}\chi_{\mu_1\cdots\mu_s}$, then the following "multiple-conservation" condition holds:

$$\partial_{\mu_1}\cdots\partial_{\mu_s}\mathcal{O}_{\mu_1\cdots\mu_s}=0.\qquad(2.16)$$

An ordinary conservation condition $\partial_{\mu_s}\mathcal{O}_{\mu_1\cdots\mu_s}=0$ implies $h_s=0$. Instead, applying the multiple-conservation condition (2.16) to the correlator (2.15), we find that h_s can take an arbitrary integer value between 0 and $1-s$. Consequently, we have the following spectrum of allowed dimensions:

$$\delta_s=2+s,1+s,\cdots,3.\qquad(2.17)$$

Observe that only the operators of dimension 3 need s divergences to be annihilated. Operators of higher dimension are allowed to satisfy more restrictive conditions. In particular, operators of dimension $2+s$ can be conserved in the usual sense ($\partial_{\mu_1}\mathcal{O}_{\mu_1\cdots\mu_s}=0$), operators of dimension $1+s$ can be annihilated by two divergences ($\partial_{\mu_1}\partial_{\mu_2}\mathcal{O}_{\mu_1\cdots\mu_s}=0$), etc.

The Ferrara–Gatto–Grillo theorem [8] says that primary conformal operators with spin-s have dimensions $\delta_s\geq 2+s$. This property, a direct consequence of unitarity, is here violated. This is how the non-unitarity of the theory shows up in this context. We see from (2.17) that the minimal allowed dimension is 3. This feature is relevant to the conformal hypothesis stated in the introduction: the interaction vertex

$$\mathcal{O}_{\mu_1\cdots\mu_s}\chi_{\mu_1\cdots\mu_s}$$

is renormalizable if $\mathcal{O}_{\mu_1\cdots\mu_s}$ is such an operator of dimension 3; therefore, in our theories, renormalizable higher-spin interactions are not ruled out in a trivial way.

3 The spin-2 conformal boson in detail

In this section I study the stress-tensor of the spin-2 conformal boson, compute its two-point function and OPE, and extract the central charges c and a. The result is that both c and a have positive values. Since, in particular, a can be regarded as a counter of the (massless) degrees of freedom, positivity means that the physical degrees of freedom prevail over the ghost ones. In higher-derivative theories, on the contrary, we will see that both c and a are typically negative.

3.1 Computation of c

The field-strength propagator $\langle F_{\mu\nu\alpha}(x)\ F_{\rho\sigma\beta}(0)\rangle$ is, by conformal invariance, $1/|x|^{2d}$ times a linear combination of the following three conformal structures:

$$\begin{aligned}
C^{(1)}_{\mu\nu\alpha,\rho\sigma\beta}(x) &= (I_{\mu\rho}(x)I_{\nu\sigma}(x)-I_{\mu\sigma}(x)I_{\nu\rho}(x))I_{\alpha\beta}(x),\\
C^{(2)}_{\mu\nu\alpha,\rho\sigma\beta} &= (I_{\mu\beta}I_{\nu\rho}-I_{\nu\beta}I_{\mu\rho})I_{\sigma\alpha}-(\rho\leftrightarrow\sigma),\\
C^{(3)}_{\mu\nu\alpha,\rho\sigma\beta} &= (\delta_{\mu\alpha}I_{\nu\rho}-\delta_{\nu\alpha}I_{\mu\rho})\delta_{\sigma\beta}-(\rho\leftrightarrow\sigma),
\end{aligned}$$

where d is the dimension of F (2 in the free-field limit). The trace and cyclic conditions (2.6) fix the combination of $C^{(1)}$, $C^{(2)}$ and $C^{(3)}$ uniquely up to the overall factor, which can be found by direct inspection, using the χ-propagator worked out in the previous section. The final result reads

$$\langle F_{\mu\nu\alpha}(x)\ F_{\rho\sigma\beta}(0)\rangle = \frac{1}{2\pi^2}\frac{1}{|x|^4}\left(2\ C^{(1)}_{\mu\nu\alpha,\rho\sigma\beta}-C^{(2)}_{\mu\nu\alpha,\rho\sigma\beta}+C^{(3)}_{\mu\nu\alpha,\rho\sigma\beta}\right). \tag{3.18}$$

A good check is that this correlator satisfies the field equations (2.7).

For a gauge-invariant stress tensor the natural candidate is

$$T_{\mu\nu}=\frac{8}{3}F^+_{\mu\alpha\beta}F^-_{\nu\alpha\beta}=\frac{4}{3}F_{\mu\alpha\beta}F_{\nu\alpha\beta}-\frac{1}{3}\delta_{\mu\nu}F^2_{\alpha\beta\gamma}$$

but, because the coupling to gravity is problematic and actually might not exist, this form of the stress tensor, and most importantly its overall coefficient, should be checked directly from the algebra of the Poincaré group. The unusual factor will be fixed this way in the next section. The more natural expression $T_{\mu\nu}=2F^+_{\mu\alpha\beta}F^-_{\nu\alpha\beta}$, differring from the above one by a factor 4/3, does not close the algebra correctly.

The Noether method produces a non-gauge-invariant, non-symmetric, traceful stress tensor

$$T^{\rm N}_{\mu\nu}=\partial_\mu\chi_{\alpha\beta}F_{\nu\alpha\beta}-\frac{1}{4}\delta_{\mu\nu}F^2_{\alpha\beta\gamma}.$$

This operator is conserved in ν ($\partial_\nu T_{\mu\nu} = 0$), gauge-invariant and traceless up to total derivatives, and it does not transforms correctly under coordinate inversion. For this reason, it is not simple to use the Noether tensor to extract c and a. Moreover, there exists no improvement term $\psi_{\mu\nu\lambda} = -\psi_{\mu\lambda\nu}$ such that $T_{\mu\nu} = T^{\rm N}_{\mu\nu} + \partial_\lambda \psi_{\mu\nu\lambda}$, because $T_{\mu\nu}$ and $T^{\rm N}_{\mu\nu}$ do not differ by total derivatives.

It is straightforward to show that $T_{\mu\nu}$, instead, is traceless, gauge invariant and conserved on the solutions to the field equations (2.7), and transforms correctly under coordinate inversion. For the proof of conservation we observe that the cyclic identity implies also

$$T_{\mu\nu} = \frac{8}{3} F^{+}_{\mu\alpha\beta} F^{-}_{\nu\beta\alpha}.$$

The difference Δ between the two forms for $T_{\mu\nu}$ is proportional to $F^{+}_{\mu\alpha\beta} F^{-}_{\nu[\beta\alpha]}$, the brackets denoting antisymmetrization. The cyclic identity in (2.6) can be expressed as $F_{\mu\nu\alpha} - F_{\mu\alpha\nu} = F_{\alpha\nu\mu}$. Similar expressions hold for $\tilde{F}$ and $F^{\pm}$. We have therefore $\Delta \sim F^{+}_{\mu\alpha\beta} F^{-}_{\alpha\beta\nu}$. Using the cyclic identity once more on F^{+} we arrive at $\Delta \propto F^{+}_{\alpha\beta\mu} F^{-}_{\alpha\beta\nu} = 0$.

With (3.18), it is relatively simple to calculate the stress-tensor two-point function:

$$\langle T_{\mu\nu}(x)\ T_{\rho\sigma}(0)\rangle = \frac{4}{45\pi^4} \prod\nolimits^{(2)}_{\mu\nu,\rho\sigma} \left(\frac{1}{|x|^4}\right), \qquad \text{i.e.} \quad c = \frac{32}{45}.$$

3.2 OPE structure and computation of a

The OPE structure exhibits novel features with respect to the ordinary theories. In particular, the presence of ghosts is exhibited by higher-spin composite operators of low dimensionality. The OPE of two stress tensors contains: the central charge c, with singularity $1/|x|^8$; the stress-tensor itself, with singularity $1/|x|^4$; higher-spin currents of dimension $2+s, 1+s, \cdots 3$, where s is the spin; descendants and regular terms. The first higher-spin current is a spin-4, dimension-4 operator appearing at the same level as the stress-tensor (singularity $1/|x|^4$). This operator reads

$$\mathcal{O}^{(4)}_{\mu\nu\rho\sigma} = \sum_{\text{perms}(\mu\nu\rho\sigma)}{}^{\prime} \quad F^{+}_{\alpha\mu\nu} F^{-}_{\alpha\rho\sigma} - \text{traces}.$$

The primed sum is understood to be divided by the number of permutations. The operator $\mathcal{O}^{(4)}_{\mu\nu\rho\sigma}$ satisfies the multiple-conservation condition $\partial_\mu\partial_\nu\partial_\rho\partial_\sigma \mathcal{O}^{(4)}_{\mu\nu\rho\sigma} = 0$. The proof of this fact is lengthy and involves repeated use of the cyclic identity and the field equations. Observe in particular that $\partial_\mu \partial_\nu F_{\alpha\mu\nu} = 0$ on the solutions to the field equations. I illustrate the strategy of the proof on the most involved term, which is

$$\partial_\rho \partial_\sigma F^{+}_{\alpha\mu\nu}\, \partial_\mu \partial_\nu F^{-}_{\alpha\rho\sigma}.$$

First, we exchange μ and ρ by using the property of self-duality in $\alpha\mu$ and anti-self-duality in $\alpha\rho$. We then use the cyclic identity on $F^{-}_{\alpha\mu\sigma}$ and arrive at

$$-\partial_\rho \partial_\sigma F^{+}_{\alpha\rho\nu}\, \partial_\mu \partial_\nu (F^{-}_{\mu\sigma\alpha} + F^{-}_{\sigma\alpha\mu}).$$

We use the field equations to replace $\partial_\mu F^-_{\mu\sigma\alpha}$ with $\partial_\mu F^-_{\alpha\mu\sigma}$ and observe that we obtain a term identical to the starting one. We move it on the left-hand side and write

$$\partial_\rho \partial_\sigma F^+_{\alpha\mu\nu}\, \partial_\mu \partial_\nu F^-_{\alpha\rho\sigma} = \frac{1}{2} \partial_\rho \partial_\sigma F^+_{\alpha\rho\nu}\, \partial_\mu \partial_\nu F^-_{\alpha\sigma\mu}.$$

Now we use the cyclic identity on $F^+_{\alpha\rho\nu}$ and get

$$-\frac{1}{2} \partial_\rho \partial_\sigma (F^+_{\rho\nu\alpha} + F^+_{\nu\alpha\rho})\, \partial_\mu \partial_\nu F^-_{\alpha\sigma\mu}.$$

Using the field equation $\partial_\rho F^+_{\rho\nu\alpha} = \partial_\rho F^+_{\alpha\rho\nu}$ we finally arrive at

$$-\frac{1}{4} \partial_\rho \partial_\sigma F^+_{\nu\alpha\rho}\, \partial_\mu \partial_\nu F^-_{\alpha\nu\mu} = -\frac{1}{4} \partial_\rho \partial_\sigma F^+_{\sigma\alpha\rho}\, \partial_\mu \partial_\nu F^-_{\alpha\sigma\mu} = 0.$$

The other non-trace terms in $\mathcal{O}^{(4)}_{\mu\nu\rho\sigma}$ can be shown to vanish in a similar way. Finally, the trace terms always contain the stress tensor and obey the multiple-conservation condition because the stress tensor is conserved.

In the basis of [20] we find the OPE expansion

$$\begin{aligned} T_{\mu\nu}(x)\; T_{\rho\sigma}(0) = & \frac{4}{45\pi^4} \;\prod^{(2)}_{\mu\nu,\rho\sigma} \left(\frac{1}{|x|^4}\right) \\ & + \frac{1}{4\pi^2}\; T_{\alpha\beta}(0) \;\left[\mathrm{SP}_{\mu\nu,\rho\sigma;\alpha\beta} \left(\frac{1}{|x|^2}\right) + \frac{3}{32} \prod^{(2)}_{\mu\nu,\rho\sigma} \partial_\alpha \partial_\beta \left(|x|^2 \ln |x|^2 \mu^2\right) \right. \\ & \qquad \left. - \frac{5}{32} \prod^{(3)}_{\mu\nu\alpha,\beta\rho\sigma} \left(|x|^2 \ln |x|^2 \mu^2\right) \right] \\ & + \frac{1}{4\pi^2}\; \mathcal{O}^{(4)}_{\alpha\beta\gamma\delta}(0) \;\left[-\frac{1}{45} \prod^{(2)}_{\mu\nu,\rho\sigma} \partial_\alpha \partial_\beta \partial_\gamma \partial_\delta \left(|x|^4 \ln |x|^2 \mu^2\right) \right. \\ & \qquad \left. + \frac{5}{126} \prod^{(3)}_{\mu\nu\alpha,\rho\sigma\beta} \partial_\gamma \partial_\delta \left(|x|^4 \ln |x|^2 \mu^2\right) - \frac{1}{216} \prod^{(4)}_{\mu\nu\rho\sigma,\alpha\beta\gamma\delta} \left(|x|^4 \ln |x|^2 \mu^2\right) \right] \\ & + \text{less singular terms} + \text{descendants} + \text{regular terms}, \end{aligned} \tag{3.19}$$

the structure $\mathrm{SP}_{\mu\nu,\rho\sigma;\alpha\beta}\left(\frac{1}{|x|^2}\right)$ being the generator of the Poincaré algebra. The overall coefficient of $T_{\mu\nu}$ has been fixed by matching the coefficient of $T_{\alpha\beta}\mathrm{SP}_{\mu\nu,\rho\sigma;\alpha\beta}\left(\frac{1}{|x|^2}\right)$ in the OPE, which is universal and has to be equal to $1/4\pi^2$.

In the calculation of the above OPE, it is necessary to extract the spin-2 content out of the product $F^+_{\mu\nu\alpha}F^-_{\rho\sigma\beta} + F^-_{\mu\nu\alpha}F^+_{\rho\sigma\beta}$. It can be proved that the stress-tensor content of this expression is fixed uniquely by the symmetry properties in the indices, the cyclic identity, the tracelessness of F, and relations such as $F^+_{\alpha\beta\nu}F^-_{\alpha\beta\sigma} = 0$, $F^+_{\mu\alpha\beta}F^-_{\nu\alpha\beta} = \frac{3}{8}T_{\mu\nu}$, with the result

$$\begin{aligned} F^+_{\mu\nu\alpha}F^-_{\rho\sigma\beta} + F^-_{\mu\nu\alpha}F^+_{\rho\sigma\beta} \rightarrow \quad & \frac{3}{128} (-2\delta_{\mu\sigma}\delta_{\nu\rho}T_{\alpha\beta} + 2\delta_{\mu\rho}\delta_{\nu\sigma}T_{\alpha\beta} + 3\delta_{\beta\sigma}\delta_{\nu\rho}T_{\alpha\mu} - 3\delta_{\beta\rho}\delta_{\nu\sigma}T_{\alpha\mu} \\ & -3\delta_{\beta\sigma}\delta_{\mu\rho}T_{\alpha\nu} + 3\delta_{\beta\rho}\delta_{\mu\sigma}T_{\alpha\nu} - \delta_{\beta\nu}\delta_{\mu\sigma}T_{\alpha\rho} + \delta_{\beta\mu}\delta_{\nu\sigma}T_{\alpha\rho} + \delta_{\beta\nu}\delta_{\mu\rho}T_{\alpha\sigma} - \delta_{\beta\mu}\delta_{\nu\rho}T_{\alpha\sigma} - \delta_{\alpha\sigma}\delta_{\nu\rho}T_{\beta\mu} \\ & +\delta_{\alpha\rho}\delta_{\nu\sigma}T_{\beta\mu} + \delta_{\alpha\sigma}\delta_{\mu\rho}T_{\beta\nu} - \delta_{\alpha\rho}\delta_{\mu\sigma}T_{\beta\nu} + 3\delta_{\alpha\nu}\delta_{\mu\sigma}T_{\beta\rho} - 3\delta_{\alpha\mu}\delta_{\nu\sigma}T_{\beta\rho} - 3\delta_{\alpha\nu}\delta_{\mu\rho}T_{\beta\sigma} + 3\delta_{\alpha\mu}\delta_{\nu\rho}T_{\beta\sigma} \\ & +4\delta_{\alpha\sigma}\delta_{\beta\nu}T_{\mu\rho} - 4\delta_{\alpha\nu}\delta_{\beta\sigma}T_{\mu\rho} + 5\delta_{\alpha\beta}\delta_{\nu\sigma}T_{\mu\rho} - 4\delta_{\alpha\rho}\delta_{\beta\nu}T_{\mu\sigma} + 4\delta_{\alpha\nu}\delta_{\beta\rho}T_{\mu\sigma} - 5\delta_{\alpha\beta}\delta_{\nu\rho}T_{\mu\sigma} \\ & -4\delta_{\alpha\sigma}\delta_{\beta\mu}T_{\nu\rho} + 4\delta_{\alpha\mu}\delta_{\beta\sigma}T_{\nu\rho} - 5\delta_{\alpha\beta}\delta_{\mu\sigma}T_{\nu\rho} + 4\delta_{\alpha\rho}\delta_{\beta\mu}T_{\nu\sigma} - 4\delta_{\alpha\mu}\delta_{\beta\rho}T_{\nu\sigma} + 5\delta_{\alpha\beta}\delta_{\mu\rho}T_{\nu\sigma}\;). \end{aligned}$$

The expression on the left-hand side contains also $\mathcal{O}^{(4)}_{\mu\nu\rho\sigma}$, which is however orthogonal to the stress tensor and so does not contribute to c and a.

We can define our a in the following way. The scalar, spinor and vector OPE terms $(TT)^T$ are a basis for the OPE structure [20]. We use the stress-tensor two-point function and the TT OPE to associate effective numbers $n_{s,f,v}$ of scalars, fermions and vectors to the spin-2 conformal field and then apply the free-fields formulas for c and a.

We write

$$\langle (TT)\, T\rangle = n_s \langle (TT)\, T\rangle_s + n_f \langle (TT)\, T\rangle_f + n_v \langle (TT)\, T\rangle_v .$$

Here (TT) means that we take the limit in which the distance between the first two T-insertions tends to zero, and so we can use the OPE calculated above. On the right-hand side, $\langle (TT)\, T\rangle_{s,f,v}$ denote the corresponding expressions for one free real scalar, one fermion and one vector, which can be read in [20]. Clearly, only the T-content of the OPE is relevant in the limit we are considering: $\langle (TT)\, T\rangle = (TT)^T \langle T\, T\rangle$, where $(TT)^T$ denotes the structure multiplying T in the TT OPE. For example, $(TT)^T$ is the structure contained between the first square brackets in (3.19). We have

$$c\,(TT)^T = \frac{1}{120}\left[n_s (TT)^T_s + 6\, n_f (TT)^T_f + 12\, n_v (TT)^T_v \right]. \tag{3.20}$$

Using the two-point functions and OPEs of free fields [20] we arrive, by comparison, at

$$n_s = 0, \qquad\qquad n_f = \frac{256}{27}, \qquad\qquad n_v = \frac{64}{27}.$$

Observe that $n_s = 0$ can be inferred immediately from the OPE. Scalar fields produce a structure $(TT)^T_s$ with the maximal number of uncontracted x_μ's (six), vector fields give a structure $(TT)^T_v$ with the minimum number (two) and $(TT)^T_f$, for the spinors, contain four uncontracted x_μ's. A quick inspection of the propagator shows that our structure $(TT)^T$ cannot contain more than four uncontracted x_μ's.

The final result is

$$c = \frac{32}{45}, \qquad\qquad a = \frac{848}{1215}, \qquad\qquad \frac{c-a}{c} = \frac{1}{54}.$$

We see that both c and a are positive, as well as n_f and n_v, and that c is “almost” equal to a, but slightly greater.

The procedure used to calculate c and a (3.20) guarantees that these values parametrize the trace anomaly in the appropriate way. However, we cannot write a closed expression for the trace anomaly such as (1.1), which makes use of the coupling to external gravity, and we need to work always at the level of correlators and OPEs. It is meaningful, nevertheless, to truncate the right-hand side of (1.1) to the quadratic terms in an expansion of the gravitational field around flat space.

We have therefore shown that c and a can be appropriately defined in our theories despite the absence of a coupling to external gravity, and that they are positive. Some issues need to be better understood, for example the relationship between the gauge-invariant stress tensor $T_{\mu\nu}$ and the Noether tensor.

The $\mathcal{O}^{(4)}_{\alpha\beta\gamma\delta}$-content of the OPE can be extracted with the replacement

$$F^{+}_{\mu\nu\alpha}F^{-}_{\rho\sigma\beta}+F^{-}_{\mu\nu\alpha}F^{+}_{\rho\sigma\beta}\rightarrow\delta_{\nu\sigma}\mathcal{O}^{(4)}_{\alpha\beta\mu\rho}-\delta_{\nu\rho}\mathcal{O}^{(4)}_{\alpha\beta\mu\sigma}-\delta_{\mu\sigma}\mathcal{O}^{(4)}_{\alpha\beta\nu\rho}+\delta_{\mu\rho}\mathcal{O}^{(4)}_{\alpha\beta\nu\sigma}.$$

The presence of this multiply-conserved, spin-4, dimension-4 operator, absent in ordinary theories, is here emphasized, as a good illustration of the new features of higher-spin conformal field theory and the role of the multiple-conservation condition. The hope is that the ghost degrees of freedom, or spin-s operators with dimension lower than $2+s$, might be controlled in some way. A sufficiently strong interaction might raise the dimensions of all operators. I recall that the Nachtmann theorem [7], in unitary theories, states that the anomalous dimensions of the higher-spin currents generated by the singular terms of the OPE are to some extent correlated [15, 16] (e.g. the anomalous dimensions increase with the spin and the magnitude of the interaction). It is conceivable that a similar result here would assure that below a certain energy threshold, when the interaction is sufficiently strong, the theory is perfectly unitary, i.e. all spin-s operators have dimension greater than or equal to $2+s$.

3.3 Antisymmetric conformal tensors

With antisymmetric tensors, many of the nice features of symmetric tensors disappear. In particular, conformal invariance spoils both the positivity of the action and gauge invariance. A 2-form $A_{\mu\nu}$ has the conformal-invariant action

$$S=\frac{1}{2}\int\left[(\partial_{\alpha}A_{\mu\nu})^{2}-4(\partial_{\alpha}A_{\alpha\nu})^{2}\right].$$

With $A_{\mu\nu}=\partial_{\mu}\zeta_{\nu}-\partial_{\nu}\zeta_{\mu}$ we find $S=-\frac{1}{2}\int(\partial_{\alpha}A_{\mu\nu})^{2}$, so that the action is not positive-definite and gauge invariance is completely lost. The theory can be coupled in a (classically) conformal way to Abelian and non-Abelian gauge fields, as well as gravity. Renormalizable couplings to symmetric higher-spin conformal fields is instead problematic. For example, a coupling of a complex antisymmetric tensor with a spin-3 field of the Pauli type, such as

$$igF_{\mu\nu\alpha\beta}A_{\mu\alpha}\bar{A}_{\nu\beta}=\mathcal{O}^{(3)}_{\mu\nu\rho}\chi_{\mu\nu\rho}+\text{total derivatives}$$

vanishes because of the cyclic identity.

The $A_{\mu\nu}$-field equations and propagator read

$$\Box A_{\mu\nu}-2\partial_{\alpha}(\partial_{\mu}A_{\alpha\nu}+\partial_{\nu}A_{\mu\alpha})=0,\qquad\langle A_{\mu\nu}(x)\,A_{\rho\sigma}(0)\rangle=\frac{-1}{8\pi^{2}|x|^{2}}\left(I_{\mu\rho}(x)I_{\nu\sigma}(x)-I_{\mu\sigma}(x)I_{\nu\rho}(x)\right).$$

Observe that the propagator is reflection-negative. We conclude that antisymmetric conformal tensor fields are much less interesting than the symmetric tensors.

4 Conformal fermionic fields

A spin-$(s+1/2)$ field is described by a spinor $\psi_{\mu_1\cdots\mu_s}$ with s Lorentz indices, completely symmetric and traceless.

The transformation of the spinor under coordinate inversion is

$$\psi_{\mu_1\cdots\mu_s} \to |x|^2 \not{x}\gamma_5 I_{\mu_1}^{\nu_1}(x)\cdots I_{\mu_s}^{\nu_s}(x)\psi_{\nu_1\cdots\nu_s}.$$

The contraction $\gamma_\beta\psi_{\beta\mu_2\cdots\mu_s}$ transforms as a spin-$(s-1/2)$ conformal spinor. Further contractions with gamma matrices are automatically zero, owing to complete tracelessness. Instead $\sum_{i=1}^{s}\gamma_{\mu_i}\gamma_\alpha\psi_{\alpha\mu_1\cdots\hat{\mu}_i\cdots\mu_s}$ transforms as a spin-$(s+1/2)$ spinor. Therefore we can always impose

$$\gamma_\alpha\psi_{\alpha\mu_2\cdots\mu_s} = 0 \tag{4.21}$$

and preserve conformal invariance. Under this condition the most general conformal lagrangian is simply

$$\mathcal{L} = \overline{\psi}_{\mu_1\cdots\mu_s}\not{\partial}\psi_{\mu_1\cdots\mu_s} \tag{4.22}$$

any other possible term vanishing because of (4.21). The proof that (4.22) transforms correctly is rather lengthy, but straightforward. To make (4.21) manifest, we can insert appropriate projectors:

$$\begin{aligned}\mathcal{L} = &\left(\overline{\psi}_{\mu_1\cdots\mu_s} - \frac{1}{2(s+1)}\sum_{i=1}^{s}\overline{\psi}_{\alpha\mu_1\cdots\hat{\mu}_i\cdots\mu_s}\gamma_\alpha\gamma_{\mu_i}\right)\not{\partial}\left(\psi_{\mu_1\cdots\mu_s} - \frac{1}{2(s+1)}\sum_{i=1}^{s}\gamma_{\mu_i}\gamma_\alpha\psi_{\alpha\mu_1\cdots\hat{\mu}_i\cdots\mu_s}\right) = \\ &= \overline{\psi}_{\mu_1\cdots\mu_s}\not{\partial}\psi_{\mu_1\cdots\mu_s} - \frac{s}{s+1}\overline{\psi}_{\alpha\mu_2\cdots\mu_s}\gamma_\alpha\partial_\beta\psi_{\beta\mu_2\cdots\mu_s} - \frac{s}{s+1}\overline{\psi}_{\alpha\mu_2\cdots\mu_s}\gamma_\beta\partial_\alpha\psi_{\beta\mu_2\cdots\mu_s} \\ &+ \frac{s(s+2)}{2(s+1)^2}\overline{\psi}_{\alpha\mu_2\cdots\mu_s}\gamma_\alpha\not{\partial}\gamma_\beta\psi_{\beta\mu_2\cdots\mu_s}.\end{aligned}$$

The field equations are

$$\not{\partial}\psi_{\mu_1\cdots\mu_s} = \frac{1}{s+1}\sum_{i=1}^{s}\gamma_{\mu_i}\partial_\alpha\psi_{\alpha\mu_1\cdots\hat{\mu}_i\cdots\mu_s}.$$

Condition (4.21) is not sufficient to eliminate the ghosts of the theory. We see that no gauge invariance survives and the theory can be straightforwardly coupled to Abelian and non-Abelian gauge fields, as well as gravity. In particular, c and a can be defined in the usual way. In the next section, I discuss the case $s=1$ in detail and compute the contribution of conformal spinors to the gauge beta function.

4.1 Spin 3/2

For $s=1$ the action

$$\begin{aligned}S = \int\mathcal{L} = \int\overline{\psi}_\mu\left[\not{\partial}\psi_\mu - \frac{1}{2}\gamma_\alpha\partial_\mu\psi_\alpha - \frac{1}{2}\gamma_\mu\partial_\alpha\psi_\alpha + \frac{3}{8}\gamma_\mu\not{\partial}\gamma_\alpha\psi_\alpha\right] = \\ = \int\overline{\psi}_\mu P_{\mu\nu}\not{\partial}P_{\nu\rho}\psi_\rho, \qquad P_{\mu\nu} = \delta_{\mu\nu} - \frac{1}{4}\gamma_\mu\gamma_\nu\end{aligned}$$

is invariant under coordinate inversion, the field being transformed as

$$\psi_\mu \to |x|^2 \not{x} \gamma_5 I^\nu_\mu(x) \psi_\nu.$$

The field equations are

$$\not{\partial} \psi_\mu = \frac{1}{2} \gamma_\mu \partial \cdot \psi, \tag{4.23}$$

bearing in mind that $\gamma \cdot \psi = 0$. The field equations imply also

$$\left(\Box \delta_{\mu\nu} - \partial_\mu \partial_\nu\right) \psi_\nu = 0, \qquad \not{\partial} \partial \cdot \psi = 0.$$

The transversal component of ψ_μ obeys an ordinary wave equation, while $\partial \cdot \psi$ obeys the Dirac equation. The transformation $\delta \psi_\mu = \partial_\mu \epsilon$ is not a symmetry, however, since it preserves neither $\gamma \cdot \psi = 0$ nor (4.23).

Our theory coincides with the theory called "singular" by Haberzett in the context of the nuclear theory of hadronic resonances: see formula (40) of ref. [17]. Its conformal invariance, and the unicity of the theory in this respect, is here emphasized.

I investigate in detail the coupling to Abelian and non-Abelian gauge fields, obtained by covariantizing the derivatives:

$$\mathcal{L} = \frac{1}{4} (F^a_{\mu\nu})^2 + \overline{\psi}^i_\mu \left[\not{D}^{ij} \psi^j_\mu - \frac{1}{2} \gamma_\alpha D^{ij}_\mu \psi^j_\alpha - \frac{1}{2} \gamma_\mu D^{ij}_\alpha \psi^j_\alpha + \frac{3}{8} \gamma_\mu \not{D}^{ij} \gamma_\alpha \psi_\alpha \right].$$

Here a is the index of the fundamental representation of the gauge group G and i, j are indices of the matter representation R. The notation for the covariant derivative is $D^{ij}_\mu \psi^j_\nu = \partial_\mu \psi^i_\nu + g(T^a)^{ij} A^a_\mu \psi^j_\nu$, as usual. The spin-3/2 propagator is

$$\begin{aligned} \langle \psi^i_\mu(k) \; \overline{\psi}^j_\nu(-k) \rangle &= -\frac{i\delta^{ij}}{k^2} \left[\not{k} \delta_{\mu\nu} - k_\mu \gamma_\nu - k_\nu \gamma_\mu + \frac{1}{2} \gamma_\mu \not{k} \gamma_\nu + \frac{2}{k^2} k_\mu \not{k} k_\nu \right] \\ &= -\frac{i\delta^{ij}}{k^2} P_{\mu\alpha} \not{k} \left(\delta_{\alpha\beta} + 2 \frac{k_\alpha k_\beta}{k^2} \right) P_{\beta\nu} \end{aligned}$$

and the vertex is

$$\langle \psi^i_\mu \; \overline{\psi}^j_\nu \; A^a_\rho \rangle = -g T^a_{ij} P_{\mu\alpha} \gamma_\rho P_{\alpha\nu}.$$

The theory is conformal at the classical level, and scale invariance is broken, as usual, by the radiative corrections at the quantum level. I have computed the one-loop beta function of this model in two different ways (gluon self-energy and three-gluon vertex), with the result

$$\beta(g) = -\frac{g^3}{48\pi^2} \left[11 C(G) - 20 C(R_{3/2}) - 4 C(R_{1/2}) \right].$$

The correction due to our spin-3/2 field is the term proportional to $C(R_{3/2})$, while the term proportional to $C(R_{1/2})$ is the usual spin-1/2 contribution, here inserted for comparison.

We see that this peculiar type of "matter" contributes to the beta function with the same sign as ordinary matter. For $C(R_{3/2}) \lesssim \frac{11}{20} C(G)$ the one-loop beta function is arbitrarily small with respect to the higher-order corrections, which allow us to conclude that there is a non-trivial IR fixed point, trustable in perturbation theory, and a conformal window, which is the main reason why these theories are an interesting laboratory of models for the ideas of [1]. Similar arguments extend to arbitrary half-integer spin.

4.2 Spin-3/2 couplings

The spin-3/2 theory just studied propagates a spin-1/2 field, namely $\partial \cdot \psi$. Despite this fact, the lagrangian

$$\mathcal{L} = \partial \cdot \overline{\psi}\ \epsilon + \overline{\epsilon}\ \partial \cdot \psi$$

cannot be used to couple our spin-3/2 field to an ordinary spin-1/2 field ϵ. This kinetic term is forbidden by conformal invariance. Indeed, $\partial \cdot \psi$ transforms as

$$\partial \cdot \psi \to |x|^2 \not{x}\gamma_5 \partial \cdot \psi - 4|x|^2 \not{x}\gamma_5 x \cdot \psi,$$

where we have used $\gamma \cdot \psi = 0$. This example shows that conformality is a non-trivial restriction, although certainly less restrictive than the ordinary higher-spin gauge invariance. On the other hand, the vertex

$$\frac{1}{m}\partial_\mu \pi \left(\overline{\psi}_\mu\ \epsilon + \overline{\epsilon}\ \psi_\mu\right),$$

commonly used in nuclear physics to describe the Δ-decay into a nucleon and a pion, has a well-defined conformal weight, but its dimension is 5. It has to be multiplied by a parameter having the dimension of a $(\text{mass})^{-1}$. This kind of coupling does not satisfy the conformal hypothesis, which however is meant only for fundamental theories. It remains to see whether this or similar interactions (see (2.8)) can be generated in a more fundamental way from conformally invariant theories, where conformality is broken only dynamically, by the RG scale μ.

5 Higher-derivative conformal field theories

I conclude with remarks about higher-derivative conformal field theories, starting from the free higher-derivative scalar field. This case is interesting, because it corresponds to the induced action for the conformal factor ϕ and is described by the lagrangian

$$S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{g}\left[\phi\Delta_4\phi + \frac{Q}{16\pi}\tilde{\mathrm{G}}_4\phi\right],$$

where $\Delta_4 = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3}R\Box + \frac{1}{3}(\nabla^\mu R)\nabla_\mu$ [22, 23] and $\tilde{\mathrm{G}}_4 = \mathrm{G}_4 - \frac{2}{3}\Box R$ is the "pondered" Euler density [3]. Q is the background charge. This theory has been comprehensively studied in rfs. [23] and is the four-dimensional analogue of the free two-dimensional scalar field. Non-unitarity is evident from the fact that c and a are negative:

$$c = -\frac{1}{15}, \qquad a = -\frac{7}{90} - Q^2. \tag{5.24}$$

The values at $Q = 0$ can be read from [23]. We see that no real value of the background charge can give a positive a. Moreover, the background charge has no effect on c. The value of c can be checked by computing the stress-tensor two-point function and does not depend on Q. The stress tensor reads

$$\begin{aligned} T_{\mu\nu} = & -\partial_\mu\Box\phi\partial_\nu\phi - \partial_\nu\Box\phi\partial_\mu\phi - \frac{4}{3}\partial_\mu\partial_\alpha\phi\partial_\nu\partial_\alpha\phi + \frac{2}{3}\partial_\mu\partial_\nu\partial_\alpha\phi\partial_\alpha\phi + 2\Box\phi\partial_\mu\partial_\nu\phi \\ & +\delta_{\mu\nu}\left[\frac{1}{3}\partial_\alpha\Box\phi\partial_\alpha\phi + \frac{1}{3}\left(\partial_\alpha\partial_\beta\phi\right)^2 - \frac{1}{2}\left(\Box\phi\right)^2\right] + \frac{Q}{6\pi}\left(\partial_\mu\partial_\nu - \delta_{\mu\nu}\Box\right)\Box\phi. \end{aligned}$$

The two-point function is

$$\langle T_{\mu\nu}(x)\ T_{\rho\sigma}(0)\rangle = -\frac{1}{120\pi^4}\prod^{(2)}_{\mu\nu,\rho\sigma}\left(\frac{1}{|x|^4}\right),$$

in agreement with the value of c. We see that the non-unitarity of the theory is visible by a severe violation of reflection positivity. Similarly, the non-unitarity of non-conformal higher-derivative theories, such as a scalar field with lagrangian $\mathcal{L} = \frac{1}{2}\Box\phi(\Box + m^2)\phi$, is exhibited by poles with negative residues in the propagator [24].

A trick to change the signs of both central charges it to consider "higher-derivative anti-commuting scalar fields", $\theta,\ \bar{\theta}$. In this case $Q = 0$ and

$$S = \int \mathrm{d}^4x\sqrt{g}\bar{\theta}\Delta_4\theta, \qquad c = \frac{2}{15}, \qquad a = \frac{7}{45}.$$

This theory can be coupled, say, to the electromagnetic field. In a flat gravitational background the most general renormalizable lagrangian has a finite number of parameters due to the statistics of θ :

$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}^2 + |D_\mu D_\mu\theta|^2 + iF_{\mu\nu}\overline{D_\mu\theta}D_\nu\theta + |D_\mu\theta D_\nu\theta|^2 + \cdots,$$

where $D^2 = D_\mu D_\mu$. Each term can be further multiplied by a polynomial $1 + h\bar{\theta}\theta$. Some simplification comes from the invariance of the theory under the renormalizable change of variables

$$A_\mu \to A_\mu + i\alpha\bar{\theta}\overleftrightarrow{\partial_\mu}\theta,$$

α being a parameter of no physical interest.

The change of the statistics of the fields does not eliminate the non-unitarity of the theory. Indeed, the low-dimensionality of $\theta,\ \bar{\theta}$ allows us to construct many operators violating the Ferrara–Gatto–Grillo theorem. There are also operators satisfying reflection positivity before the change of statistics and violating it afterwards. For example

$$\langle(\partial_\mu\bar{\theta}\partial_\mu\theta)(x)\,(\partial_\nu\bar{\theta}\partial_\nu\theta)(0)\rangle = -(\partial_\mu\partial_\nu\langle\bar{\theta}(x)\,\theta(0)\rangle)^2 < 0.$$

Finally, there are also operators having a vanishing two-point function, such as two terms of the electromagnetic current:

$$j_\mu = i\left(\bar{\theta}\Box\partial_\mu\theta - \partial_\mu\Box\bar{\theta}\theta + \frac{1}{3}\partial_\alpha\bar{\theta}\overleftrightarrow{\partial_\mu}\partial_\alpha\theta + \frac{4}{3}\Box\bar{\theta}\partial_\mu\theta - \frac{4}{3}\partial_\mu\bar{\theta}\Box\theta\right), \qquad j'_\mu = -\frac{i}{2}\pi_{\mu\alpha}\left(\bar{\theta}\overleftrightarrow{\partial_\alpha}\theta\right).$$

We find, defining $J_\mu = aj_\mu + bj'_\mu$,

$$\langle j_\mu(x)\,j_\nu(0)\rangle = \langle j'_\mu(x)\,j'_\nu(0)\rangle = 0, \qquad \langle J_\mu(x)\,J_\nu(0)\rangle = -\frac{ab}{4\pi^2}\pi_{\mu\nu}\left(\frac{1}{|x|^4}\right).$$

Despite the unitarity problem, renormalization of this theory is well-behaved and very presumably there is a conformal window, at least when the gauge field is non-Abelian. Theories like these are a useful laboratory for the approach of [1].

For fermionic theories

$$\mathcal{L} = \bar{\psi}\partial\!\!\!/\Box\psi,$$

we have found the stress tensor

$$\begin{aligned} T_{\mu\nu} = h \Big\{ & \bar{\psi}(\gamma_\mu\partial_\nu + \gamma_\nu\partial_\mu)\Box\psi - \Box\bar{\psi}(\gamma_\mu\overleftarrow{\partial_\nu} + \gamma_\nu\overleftarrow{\partial_\mu})\psi + 3\Box\bar{\psi}(\gamma_\mu\partial_\nu + \gamma_\nu\partial_\mu)\psi \\ & -3\bar{\psi}(\gamma_\mu\overleftarrow{\partial_\nu} + \gamma_\nu\overleftarrow{\partial_\mu})\Box\psi - \frac{2}{3}(\partial_\mu\bar{\psi}\overleftrightarrow{\partial\!\!\!/}\partial_\nu\psi + \partial_\nu\bar{\psi}\overleftrightarrow{\partial\!\!\!/}\partial_\mu)\psi + 2\partial_\alpha\bar{\psi}(\gamma_\mu\overleftrightarrow{\partial_\nu} + \gamma_\nu\overleftrightarrow{\partial_\mu})\partial_\alpha\psi \\ & -\frac{10}{3}(\bar{\psi}\overleftarrow{\partial\!\!\!/}\partial_\mu\partial_\nu\psi - \partial_\mu\partial_\nu\bar{\psi}\partial\!\!\!/\psi) - \frac{2}{3}(\bar{\psi}\partial\!\!\!/\partial_\mu\partial_\nu\psi - \partial_\mu\partial_\nu\bar{\psi}\overleftarrow{\partial\!\!\!/}\psi) \\ & +\frac{1}{3}\delta_{\mu\nu}[7(\bar{\psi}\overleftarrow{\partial\!\!\!/}\Box\psi - \Box\bar{\psi}\partial\!\!\!/\psi) - 2\partial_\alpha\bar{\psi}\overleftrightarrow{\partial\!\!\!/}\partial_\alpha\psi] \Big\}, \end{aligned} \tag{5.25}$$

by imposing conservation and tracelessness. It is not straightforward to fix the overall factor h from the coupling to gravity. Indeed, a Weyl-invariant coupling to external gravity might not exist. The factor could be fixed unambiguously with the OPE technique of sect. 3.2 or the Noether method, but here we do not need it, since our primary concern is to show that c is negative, independently of the value of h. We find

$$\langle T_{\mu\nu}(x)\, T_{\rho\sigma}(0)\rangle = -\frac{8h^2}{15\pi^4}\prod\nolimits^{(2)}_{\mu\nu,\rho\sigma}\left(\frac{1}{|x|^4}\right) < 0.$$

We might wonder whether the situation changes in higher dimensions, but it is not so. I have checked that a free scalar field with action $\frac{1}{2}(\Box\phi)^2$ in six dimensions has, again, $c < 0$. The stress tensor reads

$$\begin{aligned} T_{\mu\nu} = h \Big\{ & \frac{3}{4}\partial_\mu\partial_\alpha\phi\partial_\nu\partial_\alpha\phi - \frac{3}{2}\Box\phi\partial_\mu\partial_\nu\phi + \partial_\nu\Box\phi\partial_\mu\phi + \partial_\mu\Box\phi\partial_\nu\phi - \frac{1}{2}\partial_\mu\partial_\nu\partial_\alpha\phi\partial_\alpha\phi \\ & -\frac{1}{4}\phi\Box\partial_\mu\partial_\nu\phi + \delta_{\mu\nu}\left[-\frac{1}{4}\partial_\alpha\Box\phi\partial_\alpha\phi - \frac{1}{8}\left(\partial_\alpha\partial_\beta\phi\right)^2 + \frac{1}{4}\left(\Box\phi\right)^2\right]\Big\} \end{aligned}$$

and the two-point function is

$$\langle T_{\mu\nu}(x)\, T_{\rho\sigma}(0)\rangle = -\frac{25h^2}{86016\pi^6}\prod\nolimits^{(2)}_{\mu\nu,\rho\sigma}\left(\frac{1}{|x|^6}\right) < 0.$$

6 Conclusions

The purpose of the research addressed in the present paper is to show that quantum irreversibility is a fundamental principle of nature and that, on the other hand, the Newton constant is not a fundamental constant, but descends from the RG scale μ, in a similar way as $\Lambda_{\rm QCD}$ does.

The first step is to seek candidate high-energy theories to implement these ideas. These theories might have a very unusual aspect and be more similar to ordinary theories only below a certain energy scale. We have learned that some theories are more promising than others. All of our theories are good toy-models for investigations in the spirit of [1, 5], but, at the level of physical applications, antisymmetric conformal tensors and higher-derivative theories exhibit severe violations of positive definiteness and reflection positivity. The quantities c and a can be turned from negative to positive in higher-derivative theories by changing the statistics of the fields. Nevertheless, even after changing the statistics, the resulting theories appear to be less nice than the higher-spin conformal theories of bosonic and fermionic symmetric fields.

The encouraging results are that both c and a are typically positive in the higher-spin conformal field theories, the action is positive-definite, there is a peculiar gauge symmetry, non-trivial interactions, both renormalizable and non-renormalizable, conformal windows, and so on. The propagation of ghosts is best viewed as a violation of the Ferrara–Gatto–Grillo theorem and is to some extent under control, or, at least, does not seem so severe as to reject these theories right away. Analogues of the Nachtmann theorem, or quantum irreversibility itself, might prove unitarity at sufficiently strong interactions.

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