Available at: `http://www.ictp.trieste.it/~pub_off` IC/98/123

United Nations Educational Scientific and Cultural Organization
and
International Atomic Energy Agency

THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

HOMOTHETIES OF CYLINDRICALLY SYMMETRIC STATIC SPACETIMES

Asghar Qadir [1]
Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy,

M. Sharif
Department of Applied Mathematics, IMECC, State University of Campinas,
CEP 13081-970 Campinas, Brazil
and
Department of Mathematics, Punjab University, Lahore, Pakistan

and

M. Ziad
Department of Mathematics, Quaid-i-Azam University, Islamabad, Pakistan
and
The Abdus Salam International Centre for Theoretical Physics, Trieste, Italy.

Abstract

In this note we consider the homotheties of cylindrically symmetric static spacetimes. We find that we can provide a *complete list* of all metrics that admit non-trivial homothetic motions and are cylindrically symmetric static.

MIRAMARE – TRIESTE

August 1998

[1]Senior Associate of the ICTP.

Homotheties [1] have been argued to have special physical significance [2]. There is a large body of literature on this subject [3]. Of particular relevance for our present purpose is a classification of homotheties allowed in spacetimes according to the order of the homothety Lie algebra [4]. In this paper it was shown that for order 6 or more the problem is completely resolved, while for order 5 or less only some general results could be provided. An alternative approach had been adopted for obtaining complete classifications of spacetimes admitting some minimal isometry [5]. By introducing the constraint on the spacetime that it must possess that minimal symmetry, one could obtain *all* metrics (or classes of metrics) possessing higher symmetries. This "bottom-up" approach was used to study homotheties of spherically symmetric spacetimes [6]. Here we use the same method for cylindrically symmetric static spacetimes.

A *homothety*, H is a solution of the homothety equations

$$\mathcal{L}_H g_{\mu\nu} = \psi g_{\mu\nu} \tag{1}$$

In the case $\psi = 0$, H becomes an *isometry* or *Killing vector* (KV). As such $\psi \neq 0$ are called *proper homotheties*. We solve Eq. (1) for $\psi \neq 0$ with the metric written in generalised cylindrical coordinates

$$ds^2 = e^{\nu(\rho)}dt^2 - d\rho^2 - e^{\lambda(\rho)}a^2 d\theta^2 - e^{\mu(\rho)}dz^2 \ , \tag{2}$$

where a^2 is introduced to keep track of dimensions, a being taken to have dimensions of length.

Consider the above metric for the case $\nu = \lambda = \mu = 0$. This is clearly a flat spacetime. In fact, if we were to replace "$a\theta$" by "y" and "ρ" by "x" this would be easily recognised as Minkowski space. However, we obtained it by requiring the minimal isometry group to be $SO(2) \otimes \mathbb{R} \otimes \mathbb{R}$ for cylindrical symmetry and staticity (one $\mathbb{R}$ being spacelike along the axis and one timelike) not $\mathbb{R} \otimes \mathbb{R} \otimes \mathbb{R}$. Consequently, we need to insist that θ is an *angle* and hence runs from O to 2π. We thus have to take a topological construction, keeping the space flat we identify $\theta = 0$ with $\theta = 2\pi$. When this procedure was used in the classification of the isometries of cylindrically symmetric static spacetimes [7], the above space was called "wrapped Minkowski". It is a cylindrical analogue of the spherically symmetric static Bertotti-Robinson metric [8]

$$ds^2 = dt^2 - dr^2 - a^2 d\Omega^2 \ . \tag{3}$$

Topological constructions of the above type acquired special interest with their application [9] to "topological field theories".

There are many other Bertotti-Robinson metrics and correspondingly their cylindrical analogues. It would be useful to be able to identify them in the cylindrical case. Two more forms of the above "wrapped Minkowski" space are directly identifiable for $\nu = \lambda = 0$, $\mu = 2\ell n(\rho/\rho_0)$ and $\nu = 2\ell n(\rho/\rho_0)$, $\lambda = \mu = 0$ (where ρ_0 is a constant with units of length). Since a homothety essentially involves a rescaling, the "wrapped Minkowski" space will admit *local* homotheties which would be *globally* prohibited due to the topological construction. (In effect a deficit, or an excess, angle will be provided.) This feature will hold for all cylindrical Bertotti-Robinson like spacetimes admitting a proper homothety. By deriving the local homotheties and checking which must be globally disallowed, we can identify the special spacetimes referred to above.

Following our procedure, we found that there are only 3 cylindrically symmetric static spacetimes admitting homothetic motions (globally). They can be generically expressed by the metric with $\nu = 2\alpha\ell n(\rho/\rho_0)$, $\lambda = 2\ell n(\rho/a)$, $\mu = 0$, where α is an arbitrary (real) parameter. With $\alpha = 0$ we get usual Minkowski space with proper homothety

$$\underline{H} = \psi(t\frac{\partial}{\partial t} + \rho\frac{\partial}{\partial\rho} + z\frac{\partial}{\partial z}) \ , \tag{4}$$

in addition to its usual 10 KVs. For $\alpha = 1$ we get a spacetime of Petrov type [8] D, admitting the 5 homotheties

$$\underline{H} = c_0\frac{\partial}{\partial t} + c_1(\theta\frac{\partial}{\partial t} + \frac{t}{\rho_0^2}\frac{\partial}{\partial\theta}) + c_2\frac{\partial}{\partial\theta} + c_3\frac{\partial}{\partial z} + \psi(\beta\frac{\partial}{\partial\rho} + z\frac{\partial}{\partial z}) \ . \tag{5}$$

In the general case $\alpha \neq 0, 1$ we get a spacetime of Petrov type-I and Segré type [8] [1,111], admitting the 4 homotheties

$$\underline{H} = c_0\frac{\partial}{\partial t} + c_2\frac{\partial}{\partial\theta} + c_3\frac{\partial}{\partial z} + \psi[(1-a)t\frac{\partial}{\partial t} + \rho\frac{\partial}{\partial\rho} + z\frac{\partial}{\partial z}] \ . \tag{6}$$

The Bertotti-Robinson like spacetimes admitting local homotheties that are globally ruled out are the following. One admitting 7 local homotheties has $\nu = \lambda = \mu = 2\alpha\ell n(\rho/\rho_0)$ with $\alpha \neq 0, 1$; 5 have $\nu = \mu = 2\ell n(\rho/\rho_0)$, $\lambda = 0$ and $\nu = 0$, $\lambda = \mu = 2\beta\ell n(\rho/\rho_0)$; while 4 have $\nu = 2\alpha\ell n(\rho/\rho_0)$, $\lambda = \mu = 2\beta\ell n(\rho/\rho_0)$ (with $\alpha \neq \beta$ and $\alpha, \beta \neq 0, 1$) and $\nu = 2\alpha\ell n(\rho/\rho_0)$, $\lambda = 0$, $\mu = 2\ell n(\rho/\rho_0)$.

This completes the classification.

Acknowledgments

This work was done within the framework of the Associateship Scheme of the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy. Two of the authors (A.Q. and M.Z.) would like to thank Professor M. Virasoro and the Abdus Salam ICTP for hospitality. One of the authors (M.S.) is grateful to the Third World Academy of Sciences Associateship Scheme for support to travel to Brazil and to the State University of Campinas for local hospitality.

References and Footnotes

1. G.H. Katzin, J. Levine and W.R. Davis, J. Math. Phys. **10**, 617 (1969); **11**, 1518 (1970).

2. M.E. Cahill and A.H. Taub, Commun. Math. Phys. **21**, 1 (1971);
A.H. Taub, in *General Relativity*, ed. L. O'Raifertaigh (Oxford U.P., 1972) 135-150;
J. Winicour, in *Lecture Notes in Physics* (Springer Verlag, 1972) Vol. 14, p. 145;
D.M. Eardley, Comm. Math. Phys. **37**, 287 (1974);
C.B.G. McIntosh, Phys. Lett. **A50**, 429 (1975); Gen. Rel. Grav. **7**, 199, 215 (1976).

3. K. Yano, J. Indian Math. Soc. (N.S.) **15**, 105 (1951);
J. Ehlers and W. Kundt, in *Gravitation: An Introduction to Current Research*, ed. L. Witten (Wiley, 1962) p. 80;
C.D. Collinson and D.C. French, J. Math. Phys. **8**, 701 (1967);
B.B. Godfrey, Gen Rel. Grav. **3**, 3 (1972);
A. Thompson, Phys. Rev. Lett. **34**, 507 (1975);
W.D. Halford and R.P. Kerr, J. Math. Phys. **21**, 120 (1980);
W.D. Halford, J. Math. Phys. **21**, 129 (1980);
G.S. Hall, Gen. Rel. Grav. **20**, 671 (1988);

4. G.S. Hall, Gen. Rel. Grav. **22**, 457 (1990).

5. A.H. Bokhari and A. Qadir, J. Math. Phys. **28**, 1019 (1987); **31**, 1463 (1990);
A. Qadir and M. Ziad, Nuovo Cimento **B110**, 317 (1995); *Proc. Sixth Marcel Grossmann Meeting*, eds. T. Nakamura and H. Sato (World Scientific, 1993) 1115-1118.

6. D. Ahmad and M. Ziad, J. Math. Phys. **38**, 2547 (1997).

7. A. Qadir and M. Ziad, Nuovo Cimento **B110**, 277 (1995).

8. D. Kramers, H. Stephani, M.A.H. MacCallum and E. Herlt, *Exact Solutions of Einstein's Field Equations* (Deutscher Verlag, 1980).

9. The simplest example of a topological field theory constructed on a "wrapped up" Minkowski space was presented by R. Rajaraman at the Second BCSPIN Summer School in Physics held at Kathmandu, Nepal in 1991. However, it does not appear either in his write-up of the lectures in *Current Topics in Condensed Matter and Particle Physics*, eds. J.C. Pati, Q. Shafi and Yu Lu (World Scientific, 1993), or in the standard reference, D. Birmingham, M. Blau, M. Rahowski and G. Thompson, Physics Reports **209**, 129 (1991).