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BUNCH DIFFUSION DUE TO *RF* NOISE*

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We present a treatment of the influence of *rf* noise on a proton bunch valid for an arbitrary *rf* potential. Our approach is based upon the Hamilton-Jacobi transformation, followed by an application of canonical perturbation theory. A general proof is presented for the relation

$$A_1 = \frac{1}{2}\frac{\partial A_2}{\partial J},$$

recently considered by Boussard, Dome and Graziani. We consider some properties of the solutions of the Fokker-Planck equation describing the diffusion of small bunches.

I. INTRODUCTION

The influence of *rf* noise on the lifetime of bunched proton beams was recently studied by Boussard, Dome, and Graziani.[1] They noted that on a long time scale, the average motion is described by a Fokker-Planck equation,[2]

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial J}(A_1\rho) + \frac{1}{2}\frac{\partial^2}{\partial J^2}(A_2\rho), \tag{1}$$

where J is the action variable and $\rho(J, t)\, dJ$ is the probability of finding a particle between J and $J + dJ$ at time t. The coefficients A_1 and A_2 can be expressed as

$$\begin{aligned} A_1 &= \langle\langle \Delta J/\Delta t\rangle\rangle_Q \\ A_2 &= \langle\langle (\Delta J)^2/\Delta t\rangle\rangle_Q. \end{aligned} \tag{2}$$

$\Delta J = J\text{-}J_0$ is the change in the action variable during the time interval Δt. The double bracket represents an average over the *rf* noise and, in addition, an average over the initial value of the angle variable Q.

Boussard, Dome, and Graziani[1] have found that for a harmonic *rf* potential, and for a sinusoidal *rf* potential, the first moment A_1 is related to the second moment A_2 by

$$A_1 = \frac{1}{2}\frac{\partial A_2}{\partial J}, \tag{3}$$

which results in the reduction of the Fokker-Planck equation to a diffusion equation

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial J}\left(\frac{A_2}{2}\frac{\partial \rho}{\partial J}\right). \tag{4}$$

In our paper, we generalize the discussion of Ref. 1. We show that Eq. (3) is valid for an arbitrary *rf* potential, when terms up to second-order in the *rf* noise are retained. When the correlation time τ of the *rf* noise is short compared to Δt, we find an explicit expression for A_2, from which it is easy to obtain the specific results of Ref. 1 (see Appendix).

Our paper is organized in the following manner: In Section II, we introduce the mathematical formalism used to describe the motion of a particle under the combined influence of the *rf* potential and the *rf* noise. The approach we take is based upon the Hamilton-Jacobi transformation, followed by an application of canonical perturbation theory.

The relation $A_1 = \frac{1}{2}\partial A_2/\partial J$ is proved in Section III for white noise, and in Section IV for general stationary noise. Actually the proof does not require stationary noise, but only symmetry of the correlation

$$\langle \xi(t_1)\xi(t_2)\rangle = \langle \xi(t_2)\xi(t_1)\rangle.$$

In Section V, we derive a useful expression for A_2, valid when the correlation time τ of the *rf* noise is short compared to Δt. Here, we do use

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the stationary property

$$\langle \xi(t_1)\xi(t_2)\rangle = \lambda(t_1 - t_2).$$

In Section VI, we show that Eq. (4) can be derived from the Fokker-Planck equation in the two variables J, Q, where Q is the angle variable. The derivation requires that a random phase type of approximation [see Eq. (67)] be made. Next in Section VII, we summarize some properties of the solutions of Eq. (4), which describe the diffusion of small bunches. We emphasize the importance of the fundamental solution, and consider the behavior of the moments of the distribution. Finally in Section VIII we make some concluding remarks.

II. MATHEMATICAL FORMALISM

Let $q(t)$ represent the deviation at time t of a proton's *rf* phase from the synchronous value, and let $p(t) = \dot{q}(t)$ be the conjugate momentum. The system we consider is specified by the Hamiltonian

$$\mathcal{H} = \tfrac{1}{2}p^2 + f(q) + h(q)\xi(t), \tag{6}$$

where $f(q)$ is an arbitrary *rf* potential, and the *rf* noise is described by the stochastic function[2] $\xi(t)$ with

$$\langle \xi(t)\rangle = 0 \tag{7a}$$

and

$$\langle \xi(t)\xi(t')\rangle = \lambda(t - t'). \tag{7b}$$

The equation of motion corresponding to the Hamiltonian of Eq. (6) is

$$\ddot{q} + f'(q) + h'(q)\xi(t) = 0. \tag{8}$$

Phase noise[3] corresponds to

$$h(q) = -q, \tag{9}$$

and amplitude noise to

$$h(q) = f(q). \tag{10}$$

We now introduce the Hamilton-Jacobi transformation whose generating function $W(q, P)$ is determined by

$$\frac{1}{2}\left(\frac{\partial W}{\partial q}\right)^2 + f(q) = P. \tag{11}$$

The new canonical variables Q, P are related to the original q, p via

$$Q = \frac{\partial W}{\partial P} = \int_0^q \frac{dq'}{\sqrt{2[P - f(q')]}}, \tag{12a}$$

$$p = \frac{\partial W}{\partial q} = \sqrt{2[P - f(q)]}. \tag{12b}$$

The solution of Eq. (12a) is denoted $q(P, Q)$, and the transformed Hamiltonian is

$$\tilde{\mathcal{H}} = P + H(P, Q)\xi(t), \tag{13}$$

where $H(P, Q) = h[q(P, Q)]$.

The new equations of motion are

$$\dot{Q} = 1 + \frac{\partial H(P, Q)}{\partial P}\xi(t), \tag{14a}$$

$$\dot{P} = -\frac{\partial H(P, Q)}{\partial Q}\xi(t), \tag{14b}$$

which can be integrated to yield

$$Q(t) = Q_0 + t + \int_0^t dt'\xi(t') \times \frac{\partial H[P(t'), Q(t')]}{\partial P}, \tag{15a}$$

$$P(t) = P_0 - \int_0^t dt'\xi(t') \times \frac{\partial H[P(t'), Q(t')]}{\partial Q}. \tag{15b}$$

A solution of these equations can be found in the form of a perturbation expansion

$$Q(t) = Q_0 + t + Q_1(t) + Q_2(t) + \dots, \tag{16a}$$

and

$$P(t) = P_0 + P_1(t) + P_2(t) + \dots, \tag{16b}$$

where $P_k(t)$ and $Q_k(t)$ are $O(\xi^k)$. The first-order terms are given by

$$Q_1(t) = \int_0^t dt'\xi(t')\frac{\partial H(P_0, Q_0 + t')}{\partial P_0}, \tag{17a}$$

$$P_1(t) = -\int_0^t dt'\xi(t')\frac{\partial H(P_0, Q_0 + t')}{\partial Q_0}. \tag{17b}$$

It is convenient to define

$$H_0(t) \equiv H(P_0, Q_0 + t), \tag{18a}$$

$$\dot{H}_0(t) \equiv \partial H_0(t)/\partial Q_0 = \partial H_0(t)/\partial t. \tag{18b}$$

Then inserting Eqs. (16) and (17) into Eq. (15b) we find the second-order term

$$P_2(t) = -\int_0^t dt' \int_0^{t'} dt''\xi(t')\xi(t'') \times \{\dot{H}_0(t'), H_0(t'')\}, \tag{19}$$

where the Poisson-Bracket $\{,\}$ is defined by

$$\{A, B\} = \frac{\partial A}{\partial Q_0}\frac{\partial B}{\partial P_0} - \frac{\partial B}{\partial Q_0}\frac{\partial A}{\partial P_0}. \tag{20}$$

The action variable J is related to P by

$$J = \oint p\, dq = \oint \sqrt{2[P - f(q)]}\, dq, \tag{21}$$

and

$$\frac{\partial J(P)}{\partial P} = \oint \frac{dq}{\sqrt{2[P - f(q)]}} = T(P), \tag{22}$$

where $T(P)$ is the period of the synchrotron oscillation. Using the perturbation expansion of Eq. (16b), we can write

$$J(P) = J(P_0) + (P_1 + P_2)T(P_0) + \tfrac{1}{2}(P_1 + P_2)^2 \frac{\partial T(P_0)}{\partial P_0} + \ldots . \tag{23}$$

Let $J_0 = J(P_0)$ and $T_0 = T(P_0)$, then keeping terms to second-order in the noise, we obtain

$$\langle J - J_0\rangle = \langle P_2\rangle T_0 + \tfrac{1}{2}\langle P_1{}^2\rangle \frac{\partial T_0}{\partial P_0}, \tag{24}$$

$$\langle (J - J_0)^2\rangle = \langle P_1{}^2\rangle T_0{}^2, \tag{25}$$

since $\langle P_1\rangle = 0$.

The averages $\langle P_1{}^2\rangle$ and $\langle P_2\rangle$ are found by using Eq. (7) in Eqs. (17b) and (19)

$$\langle P_1{}^2(t)\rangle = \int_0^t dt' \int_0^t dt''\lambda(t' - t'') \times \dot{H}_0(t')\dot{H}(t''), \tag{26}$$

$$\langle P_2(t)\rangle = -\int_0^t dt' \int_0^{t'} dt''\lambda(t' - t'') \times \{\dot{H}_0(t'), H_0(t'')\}. \tag{27}$$

The coefficients A_1 and A_2 appearing in the Fokker-Planck equation (1) are obtained by averaging Eqs. (24) and (25) over Q_0.

$$A_1 = \frac{1}{T_0}\int_0^{T_0} dQ_0\langle \Delta J/\Delta t\rangle \equiv \langle\langle \Delta J/\Delta t\rangle\rangle_Q \tag{28}$$

$$A_2 = \frac{1}{T_0}\int_0^{T_0} dQ_0\langle (\Delta J)^2/\Delta t\rangle \equiv \langle\langle (\Delta J)^2/\Delta t\rangle\rangle_Q, \tag{29}$$

where $\Delta J = J - J_0$ is the change in the action variable during the time interval Δt. Here Δt is taken to be long as compared to the correlation time of the noise $\xi(t)$, but short as compared to the time interval within which J changes appreciably. We have introduced the notation of a double bracket indicating an average over both the *RF* noise and Q.

In the next two sections we shall present derivations of the relation (3), $A_1 = \frac{1}{2}\partial A_2/\partial J$. From Eqs. (24) and (25) it is seen that this relation is equivalent to

$$\langle\langle P_2(t)\rangle\rangle_Q T_0 = \frac{1}{2}\frac{\partial}{\partial P_0}[T_0\langle\langle P_1{}^2(t)\rangle\rangle_Q], \tag{30}$$

which is the form of the relation that will be explicitly proven.

III. WHITE NOISE

White noise is characterized by a delta function correlation

$$\lambda(t - t') = \lambda\delta(t - t'). \tag{31}$$

Inserting Eq. (31) into Eq. (26) and averaging over Q_0, we obtain

$$\begin{aligned}\langle\langle P_1{}^2(t)\rangle\rangle_Q &= \frac{\lambda}{T_0}\oint dQ_0 \int_0^t dt' \times \left[\frac{\partial H(P_0, Q_0 + t')}{\partial Q_0}\right]^2 \\ &= \frac{\lambda}{T_0}\int_0^t dt' \oint dQ_0 \times \left[\frac{\partial H(P_0, Q_0)}{\partial Q_0}\right]^2 \\ &= \frac{\lambda t}{T_0}\oint dQ_0 \left[h'(q(P_0, Q_0)) \times \frac{\partial q(P_0, Q_0)}{\partial Q_0}\right]^2,\end{aligned}$$

and finally

$$\langle\langle P_1^2(t)\rangle\rangle_Q = \frac{\lambda t}{T_0}\oint dq[h'(q)]^2\sqrt{2[P_0 - f(q)]}. \quad (32)$$

Next we use Eq. (31) in Eq. (27), and find

$$\langle P_2(t)\rangle = -\frac{\lambda}{2}\int_0^t dt'\{\dot{H}_0(t'), H_0(t')\}. \quad (33)$$

Defining

$$q_0(t) = q(P_0, Q_0 + t),$$

$$q_0(t) = \partial q_0(t)/\partial Q_0 = \partial q_0(t)/\partial t,$$

it is straightforward to show that the Poisson Bracket

$$\{\dot{H}_0(t'), H_0(t'')\} = h'[q_0(t')]h'[q_0(t'')] \times \{\dot{q}_0(t'), q_0(t'')\} + h''[q_0(t')]h'[q_0(t'')]\dot{q}_0(t') \times \{q_0(t'), q_0(t'')\}.$$

For $t' = t''$, the Poisson Brackets on the right hand side become $\{\dot{q}_0, q_0\} = \{p, q\} = -1$ and $\{q_0, q_0\} = 0$, so

$$\langle P_2(t)\rangle = \frac{\lambda}{2}\int_0^t dt'[h'(q_0(t'))]^2. \quad (34)$$

Averaging over Q_0, one obtains

$$\langle\langle P_2(t)\rangle\rangle_Q = \frac{\lambda t}{2T_0}\oint dQ_0[h'(q_0(t'))]^2. \quad (35)$$

We are now ready to establish Eq. (30) and hence Eq. (3).

$$\frac{\partial}{\partial P_0}[T_0\langle\langle P_1^2(t)\rangle\rangle_Q] = \lambda t\oint dq\frac{[h'(q)]^2}{\sqrt{2[P_0 - f(q)]}}$$

$$= \lambda t\oint dq\frac{[h'(q)]^2}{\partial q(P_0, Q_0)/\partial Q_0}$$

$$= \lambda t\oint dQ_0[h'(q_0(t'))]^2$$

$$= 2T_0\langle\langle P_2(t)\rangle\rangle_Q$$

This proves the validity of Eq. (3).

The conclusion of this section is that for white noise

$$A_2 = \lambda T_0\oint p\, dq[h'(q)]^2 \quad (36)$$

and

$$A_1 = \frac{1}{2}\frac{\partial A_2}{\partial J_0}. \quad (37)$$

IV. PROOF OF $A_1 = \frac{1}{2}\frac{\partial A_2}{\partial J}$

Now without imposing the constraint of white noise, let us proceed with the proof of the relation [of Eq. (3)] between the first and second moments. Recall from Eqs. (26) and (27) that

$$\langle P_1^2(t)\rangle = \int_0^t dt'\int_0^t dt''\lambda(t' - t'')\dot{H}_0(t')\dot{H}_0(t'')$$

and

$$\langle P_2(t)\rangle = \int_0^t dt'\int_0^{t'} dt''\lambda(t' - t'') \times\left[\frac{\partial\dot{H}_0(t')}{\partial P_0}\dot{H}_0(t'') - \ddot{H}_0(t')\frac{\partial H_0(t'')}{\partial P_0}\right],$$

where $H_0(t) = H(P_0, Q_0 + t)$ and $\dot{H}_0(t) = \partial H/\partial Q_0$.

Using the expression for the derivative of a product,

$$\frac{\partial\dot{H}_0(t')}{\partial P_0}\dot{H}_0(t'') = \frac{\partial}{\partial P_0}[\dot{H}_0(t')\dot{H}_0(t'')] - \dot{H}_0(t')\frac{\partial\dot{H}_0(t'')}{\partial P_0},$$

and the fact that for a symmetric function $f(t', t'') = f(t'', t')$,

$$\int_0^t dt'\int_0^{t'} dt''f(t', t'') = \int_0^t dt'\int_{t'}^t dt''f(t'', t') = \frac{1}{2}\int_0^t dt'\int_0^t dt''f(t', t''),$$

it follows that

$$\langle P_2(t)\rangle = \frac{1}{2}\frac{\partial}{\partial P_0}\langle P_1{}^2(t)\rangle - R, \tag{38}$$

where

$$R = \frac{\partial}{\partial Q_0}\int_0^t dt' \int_0^{t'} dt''\lambda(t' - t'') \times \frac{\partial H(P_0, Q_0 + t')}{\partial Q_0}\frac{\partial H(P_0, Q_0 + t'')}{\partial P_0}. \tag{39}$$

The next step is to average Eq. (38) over one period. Define the average $\langle\ \rangle_Q$ by

$$\langle f\rangle_Q = \frac{1}{T_0}\int_0^{T_0} dQ_0 f(Q_0). \tag{40}$$

Note that since the period T_0 is a function of P_0, the derivative $\partial/\partial P_0$ does not commute with the average $\langle\ \rangle_Q$. In fact, one easily shows that

$$T_0\left\langle\frac{\partial f}{\partial P_0}\right\rangle_Q = \frac{\partial}{\partial P_0}[T_0\langle f\rangle_Q] - \frac{dT_0}{dP_0}f(T_0). \tag{41}$$

Choosing $f = \langle P_1{}^2(t)\rangle$, Eq. (41) implies

$$T_0\left\langle\frac{\partial}{\partial P_0}\langle P_1{}^2(t)\rangle\right\rangle_Q = \frac{\partial}{\partial P_0}[T_0\langle\langle P_1{}^2(t)\rangle\rangle_Q] - u, \tag{42}$$

with

$$u = \frac{dT_0}{dP_0}\int_0^t dt' \int_0^t dt''\lambda(t' - t'') \times \frac{\partial H(P_0, t')}{\partial t'}\frac{\partial H(P_0, t'')}{\partial t''}. \tag{43}$$

Averaging Eq. (38) we obtain

$$T_0\langle\langle P_2(t)\rangle\rangle_Q = \frac{1}{2}\frac{\partial}{\partial P_0}[T_0\langle\langle P_1{}^2(t)\rangle\rangle_Q] - \frac{u}{2} - \langle R\rangle_Q. \tag{44}$$

In order to evaluate $\langle R\rangle_Q$, we must note that $\partial H(P_0, Q_0 + t)/\partial P_0$ is not periodic in Q_0. From Eq. (39) we see that

$$\langle R\rangle_Q = \frac{1}{T_0}\int_0^t dt' \int_0^{t'} dt''\lambda(t' - t'') \times \frac{\partial H(P_0, t')}{\partial t'}\Delta(P_0, t''), \tag{45}$$

where

$$\Delta(P_0, t) = \frac{\partial H(P_0, t + T_0)}{\partial P_0} - \frac{\partial H(P_0, t)}{\partial P_0}, \tag{46}$$

is the change in $\partial H/\partial P_0$ over one period.

From the periodicity of $H(P_0, Q_0 + t)$, it is clear that we can write $H(P_0, Q_0 + t) = \eta(P_0, (Q_0 + t)/T_0)$, where $\eta(P_0, x + 1) = \eta(P_0, x)$. Then

$$\frac{\partial H(P_0, Q_0 + t)}{\partial P_0} = \eta_1\left[P_0, \frac{Q_0 + t}{T_0}\right] - \left[\frac{Q_0 + t}{T_0{}^2}\right]\frac{dT_0}{dP_0}\eta_2\left[P_0, \frac{Q_0 + t}{T_0}\right], \tag{47}$$

where η_1 and η_2 are partial derivatives of η with respect to its first and second arguments, respectively. The functions η_1 and η_2 are periodic in their second arguments, so

$$\Delta(P_0, t) = \frac{-1}{T_0}\frac{dT_0}{dP_0}\eta_2\left[P_0, \frac{t}{T_0}\right] \tag{48}$$

$$= -\frac{dT_0}{dP_0}\frac{\partial H(P_0, t)}{\partial t}. \tag{49}$$

Inserting Eq. (49) into Eq. (45), we see that

$$\langle R\rangle_Q = -\frac{u}{2}. \tag{50}$$

Hence, Eq. (44) reduces to

$$T_0\langle\langle P_2(t)\rangle\rangle_Q = \frac{1}{2}\frac{\partial}{\partial P_0}[T_0\langle\langle P_1{}^2(t)\rangle\rangle_Q],$$

which is the relation of Eq. (30) that we set out to prove. The discussion at the end of Section II showed that Eq. (30) is equivalent to

$$A_1 = \frac{1}{2}\frac{\partial A_2}{\partial J},$$

so our proof is complete.

V EVALUATION OF A_2 FOR NOISE WITH SHORT CORRELATION TIME

In order to evaluate $\langle P_1^2(t)\rangle$, we introduce the Fourier expansion,

$$H_0(t) = H(P_0, Q_0 + t) = \sum_{n=-\infty}^{\infty} \gamma_n e^{in\Omega_0 t}, \quad (51)$$

where $\Omega_0 = 2\pi/T_0$ and $\gamma_{-n} = \gamma_n^*$. Also,

$$\lambda(t) = \int_{-\infty}^{\infty} d\omega \Lambda(\omega) e^{-i\omega t}, \quad (52)$$

where $\Lambda(\omega) = \Lambda(-\omega)$ follows from the symmetry $\lambda(t) = \lambda(-t)$. Inserting these expansions into Eq. (26) and averaging over Q_0, we obtain

$$\langle\langle P_1^2(t)\rangle\rangle_Q = 4 \sum_{n=-\infty}^{\infty} n^2\Omega_0^2 |\gamma_n|^2 \times \int_{-\infty}^{\infty} d\omega \Lambda(\omega) \frac{\sin^2\left[\frac{\omega - n\Omega_0}{2} t\right]}{(\omega - n\Omega_0)^2}. \quad (53)$$

To derive Eq. (53) we noted that γ_n is proportional to $e^{in\Omega_0 Q_0}$, so that

$$\frac{1}{T_0}\int_0^{T_0} dQ_0 \gamma_n \gamma_m^* = |\gamma_n|^2 \delta_{nm}. \quad (54)$$

Let us now suppose that the correlation time τ of the *rf* noise is much shorter than t,

$$\tau \ll t. \quad (55)$$

In this case, $\Lambda(\omega)$ is approximately constant over the interval $\Delta\omega \sim 1/t$, so it follows that

$$\langle\langle P_1^2(t)\rangle\rangle_Q \cong 4 \sum_{n=-\infty}^{\infty} n^2\Omega_0^2 |\gamma_n|^2 \times \Lambda(n\Omega_0) \int_{-\infty}^{\infty} d\omega \frac{\sin^2\left[\frac{\omega - n\omega_0}{2} t\right]}{(\omega - n\omega_0)^2}. \quad (56)$$

The integral in this expression is equal to $\pi t/2$, so

$$\langle\langle P_1^2(t)\rangle\rangle_Q \cong 2\pi t \sum_{n=-\infty}^{\infty} n^2\Omega_0^2 |\gamma_n|^2 \Lambda(n\Omega_0). \quad (57)$$

Eq. (57) can be rewritten in a form which makes manifest the agreement with the previously obtained result, Eq. (36), in the white noise limit. We define the functional,

$$J_h = \int_0^{T_0} dt[\dot{H}_0(t)]^2 = \oint p\, dq[h'(q)]^2. \quad (58)$$

Use of the Fourier expansion (51) leads to

$$J_h = T_0\Omega_0^2 \sum_{n=-\infty}^{\infty} n^2 |\gamma_n|^2. \quad (59)$$

Recalling from Eqs. (25) and (29) that

$$A_2 = T_0^2 \frac{1}{t} \langle\langle P_1^2(t)\rangle\rangle_Q, \quad (60)$$

we see that Eqs. (57) and (59) yield

$$A_2 = 2\pi T_0 J_h \frac{\sum_{n=1}^{\infty} n^2 |\gamma_n|^2 \Lambda(n\Omega_0)}{\sum_{n=1}^{\infty} n^2 |\gamma_n|^2}. \quad (61)$$

The result for white noise, Eq. (36), is obtained when the white noise spectral density, $\Lambda(\omega) = \lambda/2\pi$, is used in Eq. (61).

VI. THE FOKKER-PLANCK EQUATION

The Fokker-Planck equation for the distribution function $\psi(P, Q, t)$ is

$$\frac{\partial\psi}{\partial t} = -\frac{\partial\psi}{\partial Q} - \frac{\partial}{\partial Q}[a_1\psi] - \frac{\partial}{\partial P}[a_2\psi] + \frac{1}{2}\frac{\partial^2}{\partial Q^2}[b_{11}\psi] + \frac{\partial^2}{\partial P\partial Q}[b_{12}\psi] + \frac{1}{2}\frac{\partial^2}{\partial P^2}[b_{22}\psi]. \quad (62)$$

The coefficients are defined by

$$1 + a_1 = \langle(\Delta Q)/\Delta t\rangle,$$
$$a_2 = \langle(\Delta P)/\Delta t\rangle,$$
$$b_{11} = \langle(\Delta Q)^2/\Delta t\rangle,$$
$$b_{12} = \langle(\Delta P \Delta Q)/\Delta t\rangle,$$
$$b_{22} = \langle(\Delta P)^2/\Delta t\rangle.$$

Following the discussion of Sections II and IV, it is straightforward to show that

$$a_1 = \frac{1}{2}\frac{\partial b_{11}}{\partial Q} + \frac{\partial c_{21}}{\partial P}, \qquad (63a)$$

$$a_2 = \frac{1}{2}\frac{\partial b_{22}}{\partial P} + \frac{\partial c_{12}}{\partial Q}, \qquad (63b)$$

$$b_{12} = c_{12} + c_{21}. \qquad (63c)$$

Explicit expressions for b_{11}, b_{22}, c_{21} and c_{12} are

$$b_{11}\Delta t = \int_0^{\Delta t} dt' \int_0^{\Delta t} dt'' \lambda(t' - t'') \times \frac{\partial H(P, Q + t')}{\partial P}\frac{\partial H(P, Q + t'')}{\partial P},$$

$$b_{22}\Delta t = \int_0^{\Delta t} dt' \int_0^{\Delta t} dt'' \lambda(t' - t'') \times \frac{\partial H(P, Q + t')}{\partial Q}\frac{\partial H(P, Q + t'')}{\partial Q},$$

$$c_{21}\Delta t = -\int_0^{\Delta t} dt' \int_0^{t'} dt'' \lambda(t' - t'') \times \frac{\partial H(P, Q + t')}{\partial P}\frac{\partial H(P, Q + t'')}{\partial Q},$$

$$c_{12}\Delta t = -\int_0^{\Delta t} dt' \int_0^{t'} dt'' \lambda(t' - t'') \times \frac{\partial H(P, Q + t')}{\partial Q}\frac{\partial H(P, Q + t'')}{\partial P}.$$

Inserting Eqs. (63a, b, c) into the Fokker-Planck equation (62), one finds

$$\frac{\partial \psi}{\partial t} = \frac{1}{2}\frac{\partial}{\partial Q}\left[b_{11}\frac{\partial \psi}{\partial Q}\right] + \frac{1}{2}\frac{\partial}{\partial P}\left[b_{22}\frac{\partial \psi}{\partial P}\right] - \frac{\partial \psi}{\partial Q} + \frac{\partial}{\partial Q}\left[c_{21}\frac{\partial \psi}{\partial P}\right] + \frac{\partial}{\partial P}\left[c_{12}\frac{\partial \psi}{\partial Q}\right]. \qquad (64)$$

Let us write Eq. (64) as

$$\frac{\partial \psi}{\partial t} = L\psi. \qquad (65)$$

Consider the average $\langle\ \rangle_Q$ which we defined in Eq. (40). Taking the average of Eq. (65), we obtain

$$\frac{\partial}{\partial t}\langle\psi\rangle_Q = \langle L\psi\rangle_Q. \qquad (66)$$

In order to obtain an equation involving only $\langle\psi\rangle_Q$, we make the approximation

$$\frac{\partial}{\partial t}\langle\psi\rangle_Q = \langle L\langle\psi\rangle_Q\rangle_Q. \qquad (67)$$

Let us introduce the notation $F(P, t) = \langle\psi\rangle_Q$, and note that applying the approximation described in Eq. (67) to Eq. (64), it follows that

$$\frac{\partial F}{\partial t} = \left\langle\frac{1}{2}\frac{\partial}{\partial P}\left[b_{22}\frac{\partial F}{\partial P}\right]\right\rangle_Q + \left\langle\frac{\partial}{\partial Q}\left[c_{21}\frac{\partial F}{\partial P}\right]\right\rangle_Q. \qquad (68)$$

In order to take the derivative $\partial/\partial P$ in the first term on the right hand side of Eq. (68) outside the average, we use Eq. (41) of Section IV, yielding

$$T\frac{\partial F}{\partial t} = \frac{1}{2}\frac{\partial}{\partial P}\left[T\langle b_{22}\rangle_Q\frac{\partial F}{\partial P}\right] - \frac{1}{2}\frac{dT}{dP}b_{22}(T)\frac{\partial F}{\partial P} + (\Delta c_{21})\frac{\partial F}{\partial P}. \qquad (69)$$

Here, $T = T(P)$ is the oscillation period, and Δc_{21} is the change in c_{21} over one period.

We evaluate Δc_{21} by following the steps leading from Eq. (45) to Eq. (50) in Section IV. The last two terms in Eq. (69) cancel, leaving

$$\frac{\partial F}{\partial t} = \frac{1}{2}\frac{1}{T}\frac{\partial}{\partial P}\left[T\langle b_{22}\rangle_Q\frac{\partial F}{\partial P}\right]. \qquad (70)$$

Introducing $\rho(J, t) = F(P, t)$, and using $\partial/\partial P = T\partial/\partial J$, we derive

$$\frac{\partial \rho}{\partial t} = \frac{1}{2}\frac{\partial}{\partial J}\left[T^2\langle b_{22}\rangle_Q\frac{\partial \rho}{\partial J}\right]. \qquad (71)$$

Since $T^2\langle b_{22}\rangle_Q = \langle\langle(\Delta J)^2/\Delta t\rangle\rangle_Q = A_2$, as defined in Eq. (2), we see that Eq. (71) can be written as

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial J}\left[\frac{A_2}{2}\frac{\partial \rho}{\partial J}\right], \qquad (72)$$

which is the equation used by Boussard, Dome and Graziani,[1] and presented by us in Eq. (4) of the Introduction.

VII. FUNDAMENTAL SOLUTIONS OF THE FOKKER-PLANCK EQUATION

The fundamental solution[2] $K(J, J_0; t)$ of Eq. (72) is defined to be that solution satisfying the initial condition

$$\lim_{t\to 0+} K(J, J_0; t) = \delta(J - J_0), \tag{73}$$

where $\delta(J - J_0)$ is the Dirac delta function. Any other solution can be written in terms of the fundamental solution, as

$$\rho(J, t) = \int_0^\infty dJ_0 K(J, J_0; t)\rho(J_0, o), \tag{74}$$

where $\rho(J, o)$ is the initial distribution existing at $t = o$. The moments $M_n(t)$ defined by

$$M_n(t) = \int_0^\infty dJ J^n \rho(J, t), \tag{75}$$

are useful in characterizing the distribution $\rho(J, t)$.

For simplicity, we consider the case of white noise as described in Section III, for which

$$A_2(J) = \lambda J T(J), \tag{76}$$

as shown in Eq. (36). For a small bunch, the amplitude dependence of the period can be approximated by

$$T(J) \approx T_0 + T_1 J, \tag{77}$$

so

$$\tfrac{1}{2}A_2(J) \approx aJ + bJ^2, \tag{78}$$

with a and b being the appropriate constants.

In the harmonic limit, $b = 0$, so Eq. (72) reduces to

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial J}\left[aJ\frac{\partial \rho}{\partial J}\right]. \tag{79}$$

The fundamental solution can be determined by Fourier transforming the J-dependence, and solving the resulting first-order partial differential equation of Lagrange type. One finds

$$K(J, J_0; t) = \frac{1}{at} e^{-(J+J_0)/at} I_0\left[\frac{2\sqrt{JJ_0}}{at}\right], \tag{80}$$

where I_0 is the zeroth-order modified Bessel function. It is straightforward to determine the moments corresponding to the fundamental solution, using

$$\int_0^\infty e^{-\alpha x} I_0[2\sqrt{\beta x}]\, dx \doteq \frac{1}{\alpha} e^{\beta/\alpha}, \tag{81}$$

and the integrals obtained by differentiating both sides with respect to α. One finds

$$M_1(t) = J_0 + at, \tag{82a}$$

and

$$M_2(t) = J_0^2 + 4aJ_0 t + 2a^2t^2. \tag{82b}$$

Using Eq. (74), we can also determine the solution whose initial distribution is Gaussian, $\rho(J, o) = 1/\sigma e^{-J/\sigma}$. The result is

$$\rho(J, t) = \frac{1}{\sigma + at} e^{-J/(\sigma + at)}. \tag{83}$$

Hence an initial Gaussian distribution remains Gaussian, and the moments are found to be

$$M_n(t) = n!(\sigma + at)^n. \tag{84}$$

As a second example suppose $a = 0$, which can be realized by using a Landau cavity designed to make $T(J) \approx T_1 J$, for small J. In this case Eq. (72) becomes

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial J}\left[bJ^2\frac{\partial \rho}{\partial J}\right]: \tag{85}$$

This equation can be solved by making the change of variables $J = e^u$. Then the equation can be written as

$$\frac{1}{b}\frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial u} + \frac{\partial^2 \rho}{\partial u^2}, \tag{86}$$

which has constant coefficients, and can be solved by Fourier transform. The fundamental solution is

$$K(J, J_0; t) = \frac{1}{J_0\sqrt{4\pi bt}} \times \exp\left[-\frac{\left[bt + \ln\frac{J}{J_0}\right]^2}{4bt}\right], \tag{87}$$

and the corresponding moments

$$M_n(t) = J_0{}^n e^{n(n+1)bt}. \quad (88)$$

Note that in this case the moments grow exponentially[4] with time.

Finally, consider $A_2(J)$ to be as given in Eq. (78). Then Eq. (72) becomes

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial J}\left[(aJ + bJ^2)\frac{\partial \rho}{\partial J}\right]. \quad (89)$$

Using the definition (75) of the moments, we write

$$\frac{d}{dt}M_n(t) = \int_0^\infty dJJ^n \frac{\partial}{\partial J}\left[(aJ + bJ^2)\frac{\partial \rho}{\partial J}\right] \quad (90a)$$

$$= an^2 M_{n-1}(t) + bn(n+1)M_n(t). \quad (90b)$$

Note that $M_0(t) = M_0$ is independent of time, take $M_0 = 1$. Then

$$\dot{M}_1(t) = a + 2bM_1(t) \quad (91)$$

and

$$\dot{M}_2(t) = 4aM_1(t) + 6bM_2(t). \quad (92)$$

For the fundamental solution $M_n(o) = J_0{}^n$. In the case $b = 0$, integration of Eqs. (91) and (92) yields the previously obtained results (82a, b). When $b > 0$, we can solve Eqs. (91) and (92) to obtain

$$M_1(t) = \left[J_0 + \frac{a}{2b}\right]e^{2bt} - \frac{a}{2b} \quad (93a)$$

and

$$M_2(t) = \left[J_0{}^2 + \frac{a}{b}J_0 + \frac{a^2}{6b^2}\right]e^{6bt} - \left[\frac{a}{b}J_0 + \frac{a^2}{2b^2}\right]e^{2bt} + \frac{a^2}{3b^2}. \quad (93b)$$

At the expense of some algebra, it is straightforward to determine higher-order moments by successively solving Eq. (90b) for larger n.

VIII. SOME CONCLUDING REMARKS

We have tried to present a simple yet comprehensive treatment of the basic theory needed to describe the diffusion of proton bunches under the influence of *rf* noise. Our discussion generalizes the work of Boussard, Dome, and Graziani,[1] enabling one to consider an arbitrary *rf* potential. In fact, our proof of the relation $A_1 = \frac{1}{2}\partial A_2/\partial J$, is valid for the more general Hamiltonian,

$$\mathcal{H} = \tfrac{1}{2}p^2 + f(q) + h(p, q)\xi(t).$$

One simply defines

$$H(P, Q) = h[q(P, Q), p(P, Q)],$$

and

$$H_0(t) = H(P_0, Q_0 + t).$$

The expression of Eq. (51) for A_2 remains valid in this case upon defining

$$J_h = \int_0^{T_0} dt[\dot{H}_0(t)]^2 = \oint p\, dq \left[\frac{\partial h}{\partial q} - \frac{\partial h}{\partial p}\frac{f'(q)}{p}\right]^2.$$

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2. An excellent review of the theory of stochastic processes is given by M. C. Wang and G. E. Uhlenbeck, Rev. Mod. Phys. *17*, 323 (1945).
3. Our stochastic function $\xi(t)$ is actually the second derivative with respect to time of the function $\phi(t)$ used in Ref. 1.
4. An exponential growth was noted by E. Hartwig, V. K. Neil and R. K. Cooper, 1973 Particle Accelerator Conf. (IEEE Trans. Nucl. Sci. NS-20, 825, 1973), for the case of amplitude noise, see Eq. (5.6).
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APPENDIX: SUMMARY AND DETERMINATION OF $A_2(J)$ IN SOME SPECIAL CASES

Motion Without Noise

$$\ddot{q} + f'(q) = 0$$

$$\tfrac{1}{2}\dot{q}^2 + f(q) = P_0 \text{ constant of motion}$$

$$Q_0 = \int_0^q \frac{dq'}{\sqrt{2[P_0 - f(q')]}}$$

$$q_0(t) = q(P_0, Q_0 + t)$$

$$T_0 = \oint \frac{dq}{\sqrt{2[P_0 - f(q)]}}, \quad \Omega_0 = 2\pi/T_0$$

$$J_0 = \oint dq\sqrt{2[P_0 - f(q)]}$$

Motion With Noise

$$\ddot{q} + f'(q) + h'(q)\xi(t) = 0$$

$$\langle \xi(t) \rangle = 0$$

$$\langle \xi(t)\xi(t') \rangle = \lambda(t - t') = \int d\omega \Lambda(\omega) e^{-i\omega(t-t')}$$

$$A_1 = \langle\langle \Delta J/\Delta t \rangle\rangle_Q$$

(averaged over Q_0)

$$A_2 = \langle\langle (\Delta J)^2/\Delta t \rangle\rangle_Q$$

The moments A_1 and A_2 characterizing the behavior of the system in the presence of *rf* noise are determined in terms of quantities defined by the dynamic behavior in the absence of noise.

$$H_0(t) = h[q(P_0, Q_0 + t)] = \sum_{n=-\infty}^{\infty} \gamma_n e^{in\Omega_0 t}$$

$$J_h = \oint dq\sqrt{2[P_0 - f(q)]}[h'(q)]^2$$

$$A_2 = 2\pi T_0 J_h \frac{\sum_{n=1}^{\infty} n^2 |\gamma_n|^2 \Lambda(n\Omega_0)}{\sum_{n=1}^{\infty} n^2 |\gamma_n|^2}$$

$$A_1 = \frac{1}{2}\frac{\partial A_2}{\partial J}$$

In the following $K(k)$, $E(k)$, $B(k)$, $C(k)$, and $D(k)$ are elliptic integrals,[5] and $sn(u)$, $cn(u)$ and $dn(u)$ are elliptic functions of modulus k.

Example 1. Sinusoidal *rf* Potential[1]

$$f(q) = 2\omega_0^2 \sin^2 \frac{q}{2}$$

A. Phase Noise: $h(q) = -q$

$$k^2 = P_0/2\omega_0^2$$

$$\sin\frac{q_0(t)}{2} = k\,\text{sn}[\omega_0(t + Q_0); k]$$

$$\dot{q}_0(t) = 2k\omega_0\,\text{cn}[\omega_0(t + Q_0); k]$$

$$T_0 = \frac{4}{\omega_0}K(k), \quad \Omega_0 = 2\pi/T_0$$

$$J_0 = 16\omega_0[E(k) - k'^2K(k)] = 16\omega_0 k^2 B(k)$$

$$B(k) = \int_0^K du\,\text{cn}^2(u)$$

$$\text{cn}(u) = \frac{\pi}{kK}\sum_{l=1,3,5,\ldots}\frac{\cos\left[\frac{l\pi u}{2K}\right]}{\cosh lv}$$

$$v = \frac{\pi}{2}\frac{K'}{K}$$

$$A_2 = 2\pi T_0 J_0 \frac{\sum_{l=1,3,5,\ldots}\frac{\Lambda(l\Omega_0)}{\cosh^2(lv)}}{\sum_{l=1,3,5,\ldots}\frac{1}{\cosh^2(lv)}}$$

B. Amplitude Noise:

$$h(q) = f(q) = 2\omega_0^2 \sin^2\frac{q}{2}$$

$$J_f = \omega_0^5(2k)^4 \oint \Delta \cos^2\theta \sin^2\theta\, d\theta$$

$$\Delta = \sqrt{1 - k^2\sin^2\theta}$$

$$\oint \Delta\cos^2\theta\sin^2\theta\, d\theta = -\frac{4k'^2(1 + k'^2)}{15k^4}K(k) + \frac{8(k^4 - k^2 + 1)}{15k^4}E(k)$$

$$= \frac{4}{15}(D - C + 2E - B)$$

$$\equiv \alpha(k)B(k) \quad (\text{defines}^{(1)}\ \alpha(k))$$

$$J_f = \omega_0{}^5(2k)^4 B(k)\alpha(k)$$

$$H_0(t) = h[q_0(t)] = 2\omega_0{}^2 \sin^2 \frac{q_0(t)}{2}$$

$$= 2\omega_0{}^2 k^2 \,\mathrm{sn}^2[\omega_0(t + Q_0)]$$

$$\mathrm{sn}^2(u) = \frac{K - E}{k^2 K} - \frac{\pi^2}{k^2 K^2} \sum_{n=1}^{\infty} \frac{n \cos\left[\dfrac{n\pi u}{K}\right]}{\sinh(2nv)}$$

$$A_2 = 2\pi T_0 J_f \frac{\displaystyle\sum_{m=2,4,6,\ldots} \frac{m^4 \Lambda(m\Omega_0)}{\sinh^2(mv)}}{\displaystyle\sum_{m=2,4,6\ldots} \frac{m^4}{\sinh^2(mv)}}.$$

C. Small Amplitude Oscillations ($k \ll 1$)

For small amplitude the motion is harmonic.

$$K = \frac{\pi}{2},\ K - E = \frac{\pi}{4}k^2,$$

$$\cosh lv = \sinh lv = \frac{1}{2}\left[\frac{4}{k}\right]^l$$

$$T_0 = \frac{2\pi}{\omega_0}$$

$$J_0 = 4\pi\omega_0 k^2 = \frac{2\pi}{\omega_0} P_0$$

$$J_f = 4\pi\omega_0{}^5 k^4 = \pi\omega_0 P_0{}^2$$

Phase Noise: $A_2 = 2\pi T_0 J_0 \Lambda(\omega_0)$
Amplitude Noise: $A_2 = \pi\omega_0{}^2 J_0{}^2 \Lambda(2\omega_0)$

Example 2. Quartic Potential

$$f(q) = aq^2 + bq^4$$

Define q_m by $P_0 = aq_m{}^2 + bq_m{}^4$.

$$k^2 = \frac{bq_m{}^2}{a + 2bq_m{}^2}$$

$$q_0(t) = q_m \sqrt{\frac{a + bq_m{}^2}{a + 2bq_m{}^2}} \times \mathrm{sd}[\sqrt{2[a + 2bq_m{}^2]}(Q_0 + t); k].$$

$$\mathrm{sd}(u) = \mathrm{sn}(u)/\mathrm{dn}(u)$$

$$\dot{q}_0(t) = q_m \sqrt{2(a + bq_m)} \times \frac{\mathrm{cn}[\sqrt{2(a + 2bq_m{}^2)}(Q_0 + t)]}{\mathrm{dn}^2[\sqrt{2(a + 2bq_m{}^2)}(Q_0 + t)]}$$

$$T_0 = \frac{2\sqrt{2}}{\sqrt{a + 2bq_m{}^2}} K(k)$$

$$J_0 = \frac{4\sqrt{2}}{3b}\sqrt{a + 2bq_m{}^2} \times [(a + bq_m{}^2)K(k) - aE(k)]$$

Phase Noise $h(q) = -q$

$$\mathrm{sd}(u) = -\frac{\pi}{kk'K}\sum_{n=1}^{\infty} \frac{(-)^n \sin[(2n - 1)\pi u/2K]}{\cosh(2n - 1)v}$$

$$A_2 = 2\pi T_0 J_0 \frac{\displaystyle\sum_{l=1,3,5\ldots} \frac{l^2 \Lambda(l\Omega_0)}{\cosh^2(lv)}}{\displaystyle\sum_{l=1,3,5\ldots} \frac{l^2}{\cosh^2(lv)}}$$