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TRICRITICAL POINT IN THE RANDOM $2+p$–SAT PROBLEM

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Abstract

The tricritical point of the random $2+p$-Satisfiability problem is analytically computed using the replica approach and found to correspond to $p_t \simeq 0.41$. The agreement of this result with previous numerical simulations and rigorous results, as well as its relevance for 'typical' computational complexity issues are briefly recalled.

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The satisfiability (SAT) problem [1] is the paradigm of hard (NP–complete) computational problems arising in complexity theory. A pedagogical introduction to the K-SAT problem [1], a version of SAT, and to some of the current open issues in theoretical computer science may be found in [2].

Briefly speaking, one is given N Boolean variables and a set of M clauses to be satisfied simultaneously. A clause refers to a logical constraint on K Boolean variables, randomly chosen among the N ones. For large instances ($M, N \to \infty$), K-SAT exhibits a striking threshold phenomenon as a function of the intensive ratio $\alpha = M/N$. Numerical simulations show that the probability of finding an assignment of the Boolean variables satisfying all clauses, falls abruptly from one down to zero when α crosses a critical value $\alpha_c(K)$ of the number of clauses per variable [3]. This scenario is rigorously established in the (Polynomial) $K = 2$ case, where $\alpha_c(2) = 1$ [4]. For $K \geq 3$, much less is known; K(≥ 3)-SAT belongs to the NP–complete class, roughly meaning that running times of search algorithms are thought to scale exponentially in N when the problem instances are critically constrained. Recent numerical works have provided an estimate for $\alpha_c(3) \simeq 4.2 - 4.3$ [3].

A statistical mechanics approach has been attempted to get insights on the K-SAT problem, by mapping the latter onto a disordered diluted spin-glass model [5–7]. Replica Symmetric (RS) theory gives the correct value of the threshold for $K = 2$ but fails in predicting the critical α_c for $K \geq 3$ [6,7]. This stems from the nature of the transition taking place at α_c, which is continuous for $K = 2$ and appears discontinuous when $K \geq 3$. In the latter case, the precise location of the critical point for the first order transition would require an appropriate replica symmetry breaking scheme. For interacting models with finite connectivity, the latter issue is still an open problem under many aspects [8].

In the context of combinatorial optimization, the nature of the phase transition characterizing the different problems might be strictly connected with the appearance of computationally hard instances, and hence to the onset of exponential regimes in search algorithms [10]. Recent numerical studies on the so-called $2+p$-SAT problem [9], that smoothly interpolates between 2-SAT ($p = 0$) and 3-SAT ($p = 1$) [7], have strongly supported this statement. It follows that the interest in the precise analytical localization of discontinuous transitions in random SAT models goes much beyond the purely technical aspects of the replica formalism.

In this note, we present the analytical calculation of the tricritical point p_t of the $2+p$-SAT model, separating second-order phase transitions ($0 \leq p < p_t$) from first-order ones

($p_t < p \leq 1$). The 2+p-SAT model is a mixed version of 2-SAT and 3-SAT including $(1-p)M$ (resp. pM) clauses constraining two (resp. three) Boolean variables [7]. This model has a threshold behaviour as usual K–SAT instances [9,11] at a critical ratio $M/N = \alpha_c(2+p)$. In addition, a given set of clause cannot be satisfied if the number of 2–clauses (respectively 3–clauses) exceeds N (resp. $\alpha_c(3)N$). As a consequence, we obtain the following simple upper bound

$$\alpha_c(2+p) \leq \min\left(\frac{1}{1-p}, \frac{\alpha_c(3)}{p}\right) \tag{1}$$

Our calculation shows that

$$\alpha_c(2+p) = \frac{1}{1-p} \qquad , \qquad (0 \leq p < p_t), \tag{2}$$

i.e. that the upper bound is reached when p is smaller that the tricritical value

$$p_t \simeq 0.41 \qquad . \tag{3}$$

Most remarkably, since an earlier presentation of our result [9], a rigorous proof of the equality (2) has been derived for $p < 2/5$ based on the analysis of the so-called unit clause algorithm [11].

Following the analysis of ref. [7], the critical threshold $\alpha_c(2+p)$ can be identified by studying the ground–state properties of a cost-energy function which measures the fraction of violated clauses in the original combinatorial problem. The free energy of our model can be obtained by a linear combination of the free energies corresponding to the $K=2$ and $K=3$ cases, with respective weights $1-p$ and p, see eq.(13) in ref. [7]. Within the iterative scheme for the RS solution discussed in ref. [7], the order parameter is the distribution of the effective cavity fields [12]

$$P(h) = \sum_{\ell=-\infty}^{\infty} r_\ell \, \delta\left(h - \frac{\ell}{2q}\right) \qquad . \tag{4}$$

In the above equation, $1/q$ is the resolution of the field which eventually goes to zero. The self-consistent equations for the coefficients r_ℓ's read

$$r_\ell = \int_0^{2\pi} \frac{d\theta}{2\pi} \cos(\ell\theta) \exp\left(\sum_{j=1}^{q} \gamma_j (\cos(j\theta) - 1)\right) \tag{5}$$

for all $\ell = 0, \ldots, q-1$ where

$$\gamma_j/\alpha = 2\,(1-p)\,r_j + 3\,p\,r_j\left(1 - r_0 - 2\sum_{\ell=1}^{j-1} r_\ell - r_j\right)\ ,\ \forall j = 1,\dots,q-1$$
$$\gamma_q/\alpha = (1-p)\left(1 - r_0 - 2\sum_{\ell=1}^{j-1} r_\ell\right) + \frac{3}{4}\,p\,\left(1 - r_0 - 2\sum_{\ell=1}^{j-1} r_\ell\right)^2 \qquad . \tag{6}$$

To find the point where the discontinuous transition first takes place, we look, within the RS scheme for the point p_t at which the derivative of the order parameter at α_c diverges. For $p < p_t$, the transition is of second order and we expect the RS solution to be valid. Indeed, some recent rigorous results [11] have reproduced our RS solution for $p \in [0, 0.4]$, whereas the validity of the RS solution in the interval $[0.4, p_t]$, still has to be proven (we cannot exclude the existence of a Replica Symmetry Breaking (RSB) solution which could shift the value of p_t to some lower value). Above p_t, the RS value for $\alpha_c(2+p)$ is presumably erroneous and RSB effects have to be taken into account. Therefore, at the tricritical p_t point, the weights of the functional order parameter in $h \neq 0$, though discontinuous, appear with a vanishingly small value. We may expand the saddle point equations (5,6) to the second order in parameters r_ℓ and $s \equiv 1 - r_0$. We find

$$r_\ell = \alpha(1-p)r_\ell + \frac{3}{2}\,\alpha\,p\,r_\ell\,\left(s - 2\sum_{k=1}^{\ell-1} r_k - r_\ell\right) - \alpha^2\,(1-p)^2\,r_\ell\,s + \frac{1}{2}(1-p)^2\alpha^2\sum_{j=1}^{\ell-1} r_j r_{\ell-j} + \tag{7}$$
$$+\,(1-p)^2\alpha^2\sum_{j=1}^{q-\ell-1} r_j r_{\ell+j} + \frac{1}{2}(1-p)^2\alpha^2 r_{q-\ell}(s - 2\sum_{k=1}^{\ell-1} r_k)\quad ,\ (\ell = 1, ..., q-1)$$

and, for $\ell = 0$,

$$s = \alpha(1-p)s + 3\alpha p\left[\sum_{j=1}^{q-1} r_j(s - 2\sum_{\ell=1}^{j-1} r_\ell - r_j) + \frac{1}{4}(s - 2\sum_{\ell=1}^{q-1} r_\ell)^2\right] - \tag{8}$$
$$-\,\alpha^2(1-p)^2\left[\sum_{j=1}^{q-1} r_j^2 + (\frac{s}{2} - \sum_{\ell=1}^{q-1} r_\ell)^2\right] - \frac{1}{2}\alpha^2(1-p)^2 s^2\ \ .$$

The analysis of the linear terms in eqs.(7,8) shows that the threshold is given by (2). Next, we expand around the latter by posing $\alpha = \frac{1}{1-p} + x$, $r_\ell = B_\ell x$ and $s = Ax$. At the critical point p_t, the above quantities $\{B_\ell, A\}$ should diverge in order to have a first order jump when $x \to 0^+$. We then assume that $B_\ell = \lambda_\ell A$, with $\lambda_\ell = O(1)$ and $A \to \infty$, discarding irrelevant $O(x^2)$ corrections to the order parameters. We find q equations for p_t and λ_ℓ, $\ell = 1, \dots, q-1$.

$$0 = \frac{3}{4}\,\frac{1-2p_t}{1-p_t} - \sum_{j=1}^{q-1}\lambda_j + \sum_{j=1}^{q-1}\lambda_j^2 + \left(\sum_{j=1}^{q-1}\lambda_j\right)^2 \tag{9}$$

and, for $\ell = 1, \ldots, q-1$,

$$0 = \frac{3}{2} \frac{p_t}{1-p_t} \lambda_\ell \left(1 - 2 \sum_{j=1}^{\ell-1} \lambda_j - \lambda_\ell \right) + \frac{1}{2} \sum_{j=1}^{\ell-1} \lambda_j \lambda_{\ell-j} +$$
$$\sum_{j=1}^{q-\ell-1} \lambda_j \lambda_{\ell+j} + \lambda_{q-\ell} \left(\frac{1}{2} - \sum_{j=1}^{q-1} \lambda_j \right) - \lambda_\ell \qquad (10)$$

Equations (9,10) can easily be solved iteratively. We have computed $p_t(q)$ for $q = 1, \ldots, 37$. The logarithmic plot of $p_t(q) - p_t(q+1)$ versus q is reported in Figure 1. Discarding the small q results, we have found that the last twenty points were well fitted by

$$p_t(q) - p_t(q+1) = \frac{\mathcal{C}}{q^\nu} \qquad (q = 17, \ldots, 36) \qquad , \qquad (11)$$

with $\mathcal{C} = 0.03077$ and $\nu = 1.5427$, see the line in Figure 1. Summing over the whole range $q = 17, ..., \infty$, eq.(11) gives the asymptotic value of p_t

$$p_t = p_t(17) - \mathcal{C} \sum_{q=17}^{\infty} \frac{1}{q^\nu} \simeq 0.412 \qquad . \qquad (12)$$

The estimate (12) of p_t does not strongly depend on the choice of the data to be fitted. We have tried some other reasonable subsets of different sizes and the final results always lie in the range $[0.410; 0.415]$.

From a physical point of view, the nature of the transition manifests itself through the appearance of a finite fraction of completely constrained variables when crossing the threshold [7]. Above p_t, this fraction discontinuously blows up at α_c. The narrow correspondence between this fact and the onset of computational complexity shown by simulations [9] suggests that the underlying mechanisms causing the increase of the typical computational search cost could be related to the fact that search algorithms have to find the precise values of a $O(N)$ number of Boolean variables through an extensive enumeration.

To end with, let us mention that the above results have been derived within an iterative RS scheme allowing for more and more refined field resolutions. The appearance of non integer fields has recently been shown to reflect the existence of RSB solutions [8]. Further work will be necessary to elucidate this point completely.

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[12] the probability distribution $P(h)$ used here is simply related to the order parameter $R(z)$ defined by eq.(36) in ref. [7] through : $P(h) = 2R(2h)$.

FIGURES

FIG. 1. Log–Log plot of $p_t(q) - p_t(q+1)$ versus q. The continuous line is the fit $\frac{\mathcal{C}}{q^\nu}$, with $\mathcal{C} = 0.03077$ and $\nu = 1.5427$.