Vacuum effective action and inflation

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Abstract [4]. *We consider vacuum quantum effects in the Early Universe, which may lead to inflation. The inflation is a direct consequence of the supposition that, at high energies, all the particles can be described by the weakly interacting, massless, conformally invariant fields. We discuss, from the effective field theory point of view, the stability of inflation, transition to the FRW solution, and also possibility to study metric and density perturbations.*

The aim of the present article is to discuss the importance of the quantum effects of vacuum for the evolution of the Early Universe. In particular, we shall present a strong arguments that the effective action of vacuum induced by conformal anomaly produces inflation in a very natural way. This problem was previously discussed in [1, 2] and studied in details in [3, 4] (see also [5]), but we hope to shed some more light on it and also draw some prospects and frameworks for the future studies.

Let us start from the description of possible physical situation in the Early Universe, in which the trace anomaly plays the leading role. The main supposition is that there is a desert in the particle spectrum between the Planck mass $M_P \approx 10^{19} GeV$ and the heaviest massive particle. This assumption has serious backgrounds, because the existing particle theories indicate that the unification point lies as $M_X \approx 10^{14} - 10^{16} GeV$. For greater simplicity one can suppose that the $U(1) \times SU(2)_L \times SU(3)$ Standard Model of the Particle Physics is valid till M_P scale, that makes a desired desert very large. Then, all the matter fields below the M_P scale can be described by the effective local quantum field theory. Moreover, if the desert is long enough, at high energies all the particles can be safely treated as massless. Then we can apply the standard relation $p = 1/3 \, \rho$ between pressure and energy density for the massless particles. If we turn to the field description, this relation indicates that all the fields possess not only global, but also local conformal invariance [5]. On the other hand, for very large kinetic energy, the potential energy of the interactions between quantum fields is disregardable, except the vacuum effects. Thus, the most natural model for the Early Universe is the curved manifold filled by the free, massless, conformal invariant fields. Suppose, the underlying matter theory has N_0 scalars, $N_{1/2}$ Dirac spinors (Weyl spinors may be treated in a similar way), and N_1 vectors. The quantum effects of these fields on curved background will be the object of our present study.

The classical solution for the pure radiation case is usual $a \sim t^{1/2}$. However, in the high energy region below M_P the vacuum quantum effects of matter fields must be taken into account, and the geometry of the background may change. For the situation described above,

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the leading quantum effect is the one related to the renormalization of the vacuum action [6, 7]. This action includes, for the conformal case, three necessary structures (conformal invariant and surface terms):

$$S_{vac} = \int d^4x \sqrt{-g} \left\{ l_1 C^2 + l_2 E + l_3 \Box R \right\},$$ (1)

where $l_{1,2,3}$ are some parameters, C^2 is the square of the Weyl tensor and E is the integrand of the Gauss-Bonnet topological (in $d = 4$) invariant. One has to notice that the introduction of the non-conformal terms like $\int \sqrt{-g} R$ or $\int \sqrt{-g} R^2$ is possible but not necessary for the renormalization of the free conformal invariant theories. In other words, non-conformal terms do not renormalize.

The renormalization of the action (1) leads to conformal anomaly [8] and to the anomaly-induced effective action [9, 10] (see also [7]). This action can be written in the form:

$$\bar{\Gamma} = S_c[\bar{g}_{\mu\nu}] + \int d^4x \sqrt{-\bar{g}} \left\{ w\sigma \bar{C}^2 + b\sigma(\bar{E} - \frac{2}{3}\Box \bar{R}) + 2b\sigma\bar{\Delta}\sigma - \right.$$

$$\left. - \frac{3c + 2b}{36} [\bar{R} - 6(\bar{\nabla}\sigma)^2 - 6\Box\sigma]^2) \right\}.$$ (2)

Here $g_{\mu\nu} = a^2(\eta) \cdot \bar{g}_{\mu\nu}$, $\sigma(\eta) = \ln a(\eta)$, and the fiducial metric $\bar{g}_{\mu\nu}$ has fixed determinant. Δ is conformal invariant self-adjoint operator

$$\Delta = \Box^2 + 2R^{\mu\nu}\nabla_\mu\nabla_\nu - \frac{2}{3} R\Box + \frac{1}{3} (\nabla^\mu R)\nabla_\mu,$$

coefficients w, b, c are

$$w = \frac{N_0 + 6N_{1/2} + 12N_1}{120 \cdot (4\pi)^2}, \quad b = -\frac{N_0 + 11N_{1/2} + 62N_1}{360 \cdot (4\pi)^2}, \quad c = \frac{N_0 + 6N_{1/2} - 18N_1}{180 \cdot (4\pi)^2}, \quad (3)$$

and $S_c[g_{\mu\nu}]$ is some unknown conformal-invariant functional. In general, exact calculation of this functional is impossible. As far as we are interested in the conformally flat cosmological solutions, this functional has no importance. However it becomes relevant for we intend to study some non-isotropy, and in particular explore metric perturbations. The situation with the perturbations is quite similar to the one for the black-hole physics. In the last case one can apply the Reigert solution (2) and achieve "correct" result for the Hawking radiation, but only for the particular choice of $S_c[g_{\mu\nu}]$ [11].

It is important to notice that the solution (2) can be rewritten in the covariant but non-local form [9]. In turn, this can be transformed again into local expression depending on two auxiliary scalar fields φ and ψ [9, 11].

$$\Gamma = S_c[g_{\mu\nu}] - \frac{c - \frac{2}{3}b}{12(4\pi)^2} \int d^4x \sqrt{-g(x)} R^2(x) + \int d^4x \sqrt{-g(x)} \left\{ \frac{1}{2} \varphi\Delta\varphi - \frac{1}{2} \psi\Delta\psi + \right.$$

$$\left. + \varphi \left[\frac{\sqrt{-b}}{8\pi} (E - \frac{2}{3}\Box R) - \frac{w}{4\pi\sqrt{-b}} C^2 \right] + \frac{w}{4\pi\sqrt{-b}} \psi C^2 \right\}.$$ (4)

Indeed this local covariant formulation is more suitable for the study of metric perturbations. However the isotropic solution can be achieved on the basis of the action (2). For the Early Universe the curvature (positive or negative) of the space section of the space-time has no importance, and thus one can restrict consideration by the conformally flat case.

Let us derive the equation of motion for the $\sigma(\eta)$, put $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ and pass to the new variables: physical time $d\eta = a(\eta) \cdot dt$ and $H(t) = \dot{a}(t)/a(t)$. The equation for $H(t)$ has the form [5]

$$\dddot{H} + 7\ddot{H}H + 4\left(3 - \frac{b}{c}\right)\dot{H}H^2 + 4\dot{H}^2 - 4\frac{b}{c}H^4 - \frac{2M_{Pl}^2}{c}\left(H^2 + \dot{H}\right) = 0\,, \tag{5}$$

The special inflationary solution corresponds to $H = const$:

$$H = \pm\frac{M_P}{\sqrt{-b}}\,, \qquad a(t) = a_0 \cdot \exp Ht\,, \tag{6}$$

positive sign corresponds to inflation. The solution (6) was first discovered and studied in [2, 3, 4] [5]. Here we shall discuss the behaviour of the $a(t)$ on the basis of the effective approach to quantum theory. The inflationary solution for the local covariant version of the induced action (4) can be achieved, and the behaviour of the conformal factor is the same as in (6). At the same time the inflationary solution of (4) contains the arbitrariness related to the boundary conditions for the auxiliary fields φ, ψ. We remark that in the black-hole case this arbitrariness allows one to classify the vacuum states [11], and therefore it is natural to expect that it can be successfully used in the study of the metric perturbations around the solution (6).

The detailed analysis shows [3, 5] that the special solution is stable with respect to the variations of $a(t)$, if the parameters of the underlying quantum theory satisfy the condition $\frac{b}{c} < 0$, that leads, according to (3), to the relation[6]

$$N_1 < \frac{1}{3}N_{1/2} + \frac{1}{18}N_0\,. \tag{7}$$

With this constraint satisfied, the theory goes to inflation independent on the initial conditions, and the inflation is eternal – unless the masses of the particles become seen and change the structure of effective action. We remark that the coefficient c in (3) can be modified by adding finite $\int \sqrt{-g}R^2$-term to the vacuum action. This coefficient does not renormalize in the free theory and has very week running in the interaction theory [13], but it can provide the stability of inflation for any particle content. Below we do not consider this term.

Let us remind that the basis for the above solution is an effective action induced by conformal anomaly – that is the quantum effect of the massless conformal invariant fields. In order to achieve better understanding of the approximation, let us suppose that the fields possess masses, and call typical mass m. For simplicity, one can imagine that we have just a massive scalar with $\xi = 1/6$, but the consideration is the same for the spinor case. One

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[5]In [3] two other similar solutions for the FRW metric with $k = \pm 1$ were found.

[6]This constraint is not satisfied for the Minimal Standard Model (MSM), but it is may be easily achieved in some generalizations including the supersymmetric MSM. The constraint imposed in [12] also provides stability of inflation.

can derive the anomaly through the Jacobian of the conformal change of variables in the effective action of vacuum.

$$i\Gamma_{vac} = iS_{vac} + \ln \int \mathcal{D}\varphi \exp\{i\,S_{matter}[\varphi, g_{\mu\nu}]\}\,. \tag{8}$$

Then, using the relation $\left[-2g_{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}} + \varphi\frac{\delta}{\delta\varphi}\right]S_{matter} \sim m^2$ we find, for the conformally flat metric, that the effective action of vacuum is given, in the case, by the sum

$$\Gamma_{vac} = \bar{\Gamma} + \frac{i}{2}\,\mathrm{Tr}\,\ln\left\{-\Box + m^2 \cdot e^{2\sigma(t)}\right\}_{g_{\mu\nu}=\eta_{\mu\nu}}\,, \tag{9}$$

where the last contribution is proportional to the massive parameter m^2. Also, in the matter action, only the massive part does not decouple from the $a(t)$, and therefore all "manifestations" of the particle mass is proportional to the m^2. Indeed, the last term in (9) is divergent and leads to the renormalization of the Einstein-Hilbert and cosmological terms, which should be, in the case, included to the S_{vac}. We remark, that the renormalization of the cosmological constant leads to its running and therefore to the appearance of the finite cosmological constant due to the mechanism recently described in [14]. However, this finite cosmological constant is also proportional to m^2. In the equation for $a(t)$ one can pass to the Feynman graf description [15], and then, in the momentum space, the time derivatives will be substituted by the energy μ. Then, in the massive case, the equations which govern the cosmological evolution, have the symbolic form:

$$\mathcal{O}_{ind}(\mu^4) + \mathcal{O}_{GR}(\mu^2 \cdot M_P^2) + \mathcal{O}_{massive}(\mu^2 \cdot m^2) = 0\,. \tag{10}$$

At this point one can apply the effective approach to consider the problem of transition to the power-like FRW Universe. Consider first the very early Universe, with $\mu \approx M_P$. One can, according to our "desert hypothesis", suppose that the heaviest particle has a mass about $100 GeV$. Then the first two terms in (10) are of the same order, while the last one is suppresses by the factor of 10^{-34}. This is the inflation region, and in case of (7) satisfied we meet universal behaviour (6).

Now, let us consider another end of the mass scale, when $\mu \approx 10^{-12} GeV$ (modern cosmic scale). Then the first (anomaly induced) term is (10) is suppressed by the factor of 10^{-28} compared to the last "massive" term, and by the factor of 10^{-62} compared to the Einstein part. Indeed, the Universe does not "see" quantum effects, and evolves by the usual power low. This simple consideration shows that there cannot be any unique equation for the Universe, that could govern its evolution from the start to the end. Instead, one has to divide the evolution for some periods which have essentially distinct dynamics.

Indeed, the most complicated is the intermediate period with μ comparable to the masses of the matter particles. However, in this region the derivation of quantum correction to the effective action meets serious difficulties. First of all, exact calculation of the non-local part of the effective action of the massive fields is unsolved problem, even within the perturbation theory. Besides, the QCD vacuum effects are essentially non-perturbative, and in particular may produce the topological solutions that could also contribute to the dynamics. Indeed, one can consider some simplified models (it is natural to regard [1] as the first attempt of this kind) to investigate the intermediate stage, and thus describe a smooth transition from inflation to the modern Universe.

The model of inflation described above (it is usually referred to as the Starobinsky model), has serious advantages as compared to the usual models based on the inflaton [16]. The main advantage is, of course, the naturalness of inflation, which appears as unavoidable consequence of the desert on the mass scale and the constraint on the particle spectrum (7). On the other hand, in order to compete with the inflaton models, the theory must pass the series of tests, and in particular provide the correct density and metric perturbation spectrum. On the other hand, as it was already mentioned above, when we study the perturbation, the conformal invariant functional $S_c[\bar{g}_{\mu\nu}]$ in (2) becomes relevant, and its proper choice is expected to provide the desirable result [17].

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