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ON THE SYMMETRY OF THE IMPEDANCE*

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Abstract

The reciprocity theorem is used to prove the symmetry of the longitudinal impedance of an accelerating structure with respect to exchange of the coordinates of the leading and trailing particles.

1 INTRODUCTION

The longitudinal impedance $Z(\omega, r_t, r_l)$ is a Fourier component of the longitudinal wake field generated by a leading particle with transverse offset r_l and experienced by the trailing particle with offset r_t lagging behind the leading particle at the distance s.

Current of a relativistic particle moving in $\pm z$-direction with transverse offset $r_l = (x_l, y_l)$ has only longitudinal component $j^{\pm}(r, r_t, t) = \pm e\delta(r - r_l)\delta(s \mp ct)$.

We use Fourier harmonics defined as $j(t) = \int j(\omega)e^{-i\omega t}d\omega/2\pi$. Fourier harmonics of the current are

$$j^{\pm}(r, r_l) = \pm\delta(x - x_l)\delta(y - y_l)e^{\pm i\omega s/c}. \tag{1}$$

The leading particle moving in an accelerator structure and having charge e may excite in the structure the longitudinal electric field $E_\omega(r, s, r_l)$. The longitudinal impedance is defined as integral

$$Z^{\pm}(\omega, r_t, r_l) = \pm\frac{1}{e}\int ds E_\omega^{\pm}(r_t, s, r_l)e^{\mp i\omega s/c}, \tag{2}$$

for the beam propagating in $\pm z$ directions, respectively.

We want to proof that impedance is symmetric in respect with exchange of the transverse coordinates of the leading and trailing particles,

$$Z^{+}(\omega, r_t, r_l) = Z^{+}(\omega, r_l, r_t). \tag{3}$$

For a particular case of the resistive wall impedance of a straight pipe, the symmetry was noticed before[1] from the explicit form of the longitudinal impedance. We try to proof the theorem for an arbitrary structure but with mirror symmetry in respect with $z \to -z$. The proof is based on the reciprocity theorem and symmetry of EM fields generated by particles in a symmetric structure.

2 SYMMETRY OF THE LONGITUDINAL IMPEDANCE

The reciprocity theorem relates EM fields ($E(r, r_t)$, $H(r, r_t)$) and ($E(r, r_l)$, $H(r, r_l)$) driven by two arbitrary currents $j(r, r_t)$ and $j(r, r_l)$ correspondingly.

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The reciprocity can be deduced directly from Maxwell equations[2] which give the following identity:

$$div\{E(r, r_t) \times H(r, r_l) - E(r, r_l) \times H(r, r_t)\} = \tag{4}$$

$$Z_0[j(r, r_t)E(r, r_l) - j(r, r_l)E(r, r_t)] \tag{5}$$

Cross here means vector product.

Integral taken over the left-hand-side over the volume of a beam pipe can be transformed in the surface integral. On the metallic walls of the beam pipe, the tangential components of electric fields are related to the tangential components of the magnetic fields $E = \zeta H \times n$, where n is unit vector perpendicular to the wall, $\zeta = (1 - i)(\omega\delta/2c)$ is surface impedance, and $\delta(-\omega) = -i\delta(\omega)$ is skin depth.

The surface integral over beam pipe with ideal conductivity ($\zeta \to 0$) is zero.

For a surface with finite conductivity, the surface integral

$$\int dV div(E_t \times H_l) = \int dSn.(E_t \times H_l) \tag{6}$$

$$= \zeta\int dS(H_t \times n)(H_l \times n) \tag{7}$$

is symmetric with respect to indexes t, l. Hence, the surface integral over the left-hand-side of Eq. (4) is zero for the finite conductivity as well.

Consider now the surface integrals over the surfaces closing the beam pipe volume at $s \to \pm\infty$ in the plane perpendicular to the z-axis. In the case of finite conductivity of the walls, the integrals are zero because radiated fields are absorbed in the walls. For ideal walls the integral

$$\int_\infty dS\{E(r, r_t) \times H(r, r_l) - E(r, r_l) \times H(r, r_t)\} \tag{8}$$

is the difference of two integrals

$$\int_\infty dS[E_x(r, r_l)H_y(r, r_t) - E_x(r, r_t)H_y(r, r_l)] \tag{9}$$

$$-\int_\infty dS[E_y(r, r_l)H_x(r, r_t) - E_y(r, r_t)H_x(r, r_l)]. \tag{10}$$

We can assume that the beam pipes at infinity are straight pipes and express transverse components of the fields in the TM waves in terms of the longitudinal components:

$$E_x = \frac{iq}{\kappa^2}\frac{\partial E_z}{\partial x}, E_y = \frac{iq}{\kappa^2}\frac{\partial E_z}{\partial y}, \tag{11}$$

$$H_x = -\frac{ik}{\kappa^2}\frac{\partial E_z}{\partial y}, H_y = \frac{ik}{\kappa^2}\frac{\partial E_z}{\partial x}. \tag{12}$$

Here q is propagating constant, $(\omega/c)^2 = q^2 + \kappa^2$.

It is easy to see that both integrals are zero. Cancellation of the surface integrals for the TE waves can be seen in the same way.

Interference between TM and TE modes depends on position of the flanges because, for a given frequency, these modes have different propagating constants. Therefore, the interference term can be put to zero by proper choice of the flanges location.

Hence, integration of Eq. (4) gives

$$\int dV[j(r,s,r_t)E(r,s,r_l) - j(r,s,r_l)E(r,s,r_t)] = 0. \tag{13}$$

This is correct for arbitrary currents. For the currents $j(r,r_l)$ propagating in z-direction, and $j(r,r_t,s)$ in $-z$ direction $j(r,s,r_t) = -\delta(r-r_t)e^{-i\omega s/c}$, Eq. (9) gives

$$-\int dsE^+_\omega(r_t,s,r_l)e^{-i\omega s/c} = \int dsE^-_\omega(r_l,s,r_t)e^{i\omega s/c}. \tag{14}$$

Expressions in this equation are the same as those in definition of impedance. Hence,

$$Z^+(r_t,r_l,\omega) = Z^-(r_l,r_t,\omega). \tag{15}$$

Now we have to relate impedances Z^- and Z^+.

To do this, let us compare solutions $E^\pm(r,s,r_0)$ of the wave equation driven by currents

$$j^\pm(r,r_0) = \pm\delta(r-r_0)e^{\pm i\omega s/c}, \tag{16}$$

with the same transverse offsets but moving in opposite direction along the z-axis. The boundary conditions on the beam pipe are the same in both cases but two solutions $E^\pm(r,s,r_0)$ have different asymptotics. For z-components, for example,

$$E^+(s) \to D(\omega)e^{iq_\omega s}, \quad s\to\infty, \tag{17}$$

$$E^+(s) \to A(\omega)e^{iq_\omega s} + B(\omega)e^{-iq_\omega s}, \quad s\to-\infty \tag{18}$$

and

$$E^-(s) \to -[A(\omega)e^{-iq_\omega s} + B(\omega)e^{iq_\omega s}], \quad s\to\infty \tag{19}$$

$$E^-(s) \to -D(\omega)e^{-iq_\omega s}, \quad s\to-\infty. \tag{20}$$

Hence, wave equations, boundary conditions and asymptotics are the same for the fields $E^+(s)$ and $E^-(-s)$. Therefore, z-components are related:

$$E^-(r,s,r_0) = -E^+(r,-s,r_0). \tag{21}$$

Using this result and the definition of Z^- Eq. (2), we obtain

$$Z^-(r_l,r_t,\omega) = -\int dsE^-_\omega(r_l,s,r_t)e^{i\omega s/c}, \tag{22}$$

or

$$Z^-(r_l,r_t,\omega) = \int dsE^+_\omega(r_l,-s,r_t)e^{i\omega s/c}. \tag{23}$$

Expression in the right-hand-side after change of the sign of integrand s coincide with Z^+, See Eq. (2). Hence, for a mirror symmetric structure, we get

$$Z^-(r_l,r_t,\omega) = Z^+(r_l,r_t,\omega). \tag{24}$$

and Eq. (11) gives finally

$$Z^+(r_t,r_l,\omega) = Z^+(r_l,r_t,\omega). \tag{25}$$

The symmetry means that beam-pipe radii at ∞ are equal. The self-field of the beam has only transverse components and does not contribute to the right-hand side.

This conclude the proof.

2.1 References

3 REFERENCES

[1] A. Piwinsky, “Impedances in lossy elliptical vacuum chambers“, DESY 94-068, April 1994

[2] L.D. Landau, E.M. Lifshitz, “Electrodynamics of Continuous Media,“, $69, Nauka, 1957