# Partial Regularity for nonlinear elliptic systems

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**Abstract**

We consider nonlinear elliptic systems of divergence type. We provide a new method for proving partial regularity for weak solutions, based on a generalization of the technique of harmonic approximation. This method is applied in two situations: that of quasilinear elliptic systems with inhomogeneity obeying the natural growth condition, and that of fully nonlinear homogeneous systems. In the latter case our methods extend previous partial regularity results, directly establishing the optimal Hölder exponent for the derivative of a weak solution on its regular set.

## 1 Introduction

In this paper we are concerned with partial regularity for the solutions of certain systems of nonlinear elliptic equations. Specifically, we consider systems of the form

$$-\operatorname{div}(A(x,u,Du)) = f(x,u,Du) \qquad \text{in } \Omega \tag{1.1}$$

for $\Omega$ a bounded domain in $\mathbb{R}^n$, $u$ and $f$ taking values in $\mathbb{R}^N$, where each $A(\cdot,\cdot,\cdot)$ is in $\operatorname{Hom}(\mathbb{R}^n,\mathbb{R}^N)$. A weak solution to (1.1) is then an $\mathbb{R}^N$-valued function $u$ such that, for all test-functions $\varphi \in C_c^\infty(\Omega,\mathbb{R}^N)$, we have

$$\int_\Omega A(x,u,Du)\cdot D\varphi\,dx = \int_\Omega f\cdot\varphi\,dx. \tag{1.2}$$

Of course in order for these notions to make sense, one needs to impose certain structural and regularity conditions on $A$ and the inhomogenity $f$, as well as to restrict $u$ to a particular class of functions. We make these notions precise for the specific structures considered in Section 3, where we study quasilinear elliptic systems which are permitted to have an inhomogeneous term, and Section 4, where we consider fully nonlinear, homogeneous equations of divergence type.

Even under reasonable assumptions on $A$ and $f$, in the case of systems of equations (i.e. $N>1$) one cannot, in general, expect that weak solutions of (1.1) will be classical, i.e. $C^2$-solutions. This was first shown by De Giorgi [DeG]; we refer the reader to [G1, Chapter 2.3] for further discussion, as well as additional examples and references. The goal, then, is to establish a partial regularity theory. The *regular set* of a solution $u$ is defined by

$$\operatorname{Reg} u = \{x\in\Omega \mid u \text{ is continuous on a neighbourhood of } x\},$$

and the *singular set* by

$$\operatorname{Sing} u = \Omega \setminus \operatorname{Reg} u.$$

Partial regularity theory involves obtaining estimates on the size of $\operatorname{Sing} u$ (i.e. showing that $\operatorname{Sing} u$ has zero $n$-dimensional Lebesgue measure or better, controlling the Hausdorff dimension of $\operatorname{Sing} u$), and showing higher regularity on $\operatorname{Reg} u$. We refer the reader to the monographs of Giaquinta, [G1] and [G2], for an extensive treatment of partial regularity theory for systems of the form (1.1), as well as more general elliptic systems.

Under the structure and regularity conditions introduced in Sections 3 and 4, the partial regularity results as expressed in Theorems 3.1 and 4.2 is not new. The point of the current paper is to provide a proof of partial regularity which utilizes a technique which is new to this field, the technique of *$A$-harmonic approximation*, which we will explain after a brief discussion of the standard methods of proof; we refer the reader to [EG, Section 1] and again to [G1, G2], for more extensive discussions. Although this method does not yield a new partial regularity result in the case of quasilinear systems (see Theorems 3.1), we are able to improve existing regularity results in the case of fully nonlinear systems, in fact obtaining the optimal Hölderi constant for the derivative on the regular set: see Theorem 4.2.

There are four essential elements in the proof of partial regularity. The first element is an inequality of Caccioppoli, or reverse-Poincaré, type. This enables one to control the $L^2$-norm of a bounded solution on a ball in terms of the structure constants, the $L^\infty$–norm of the solution and the averaged mean-square deviation on a ball of larger radius. The second element of the proof can then be roughly described as a way of improving the Caccioppoli inequality sufficiently in order to be able to proceed to the third step. The third step is then to show that smallness of a particular functional often termed the *excess*, consisting of the sum of the averaged mean square deviation and a term involving the radius (the latter only appearing in the case of inhomogeneous equations) on a particular ball is sufficient for a weak solution of (1.1) to be Hölder continuous on smaller balls. This is generally straightforward for equations with constant coefficients, and the idea is usually to find an appropriate way of applying the technique of "freezing the coefficients".

The existing proofs can broadly be classified into two groups, the "direct" and the "indirect", the distinction essentially referring to the method of proof employed in the second step described above. In the former case the goal is to prove reverse Hölder–type inequalities. Such inequalities go back to Gehring [Ge]; in the current setting this method was used by Giaquinta–Giusti [GG], and simplified by Giaquinta–Modica [GM1]. We refer the reader to [G1, Chapter 5], [G2, Chapter 6] for applications to other systems and for discussions. The direct proofs tend to be very technical, although of course they have the advantage of generating explicit information on the sensitivity of the various estimates to changes in the structure paramaters. Note that there are more elementary, direct proofs for partial regularity for some elliptic systems fulfilling structure conditions which are stricter than those considered here; see e.g. [EG], [U].

In the second type of proof, one proves the desired estimate by contradiction: if the desired inequality were false, one could construct a particular sequence of solutions to (1.1), each of which fails to satisfy the inequality but which, when appropriately rescaled (or "blown–up"), form a sequence which converges to a solution of a simpler – often linear – problem, for which the inequality holds. Compactness arguments then allow one to reach the desired conclusion. These methods were first applied to quasilinear such as (1.1) by Giusti–Miranda [GiM], see also [G1, Chapter 4], [H]; however the blow–up technique goes back to earlier works of De Giorgi, Almgren and others.

The technique of harmonic approximation is a related idea. The point is to show that a function which is "approximately–harmonic", i.e. a function $g$ for which $\int_\Omega Dg \cdot D\varphi \, dx$ is sufficiently small for all test functions $\varphi$, lies $L^2$–close to some harmonic function. This technique has its origins in Simon's proof of the regularity theorem of Allard ([A]), see [S1, Section 23], and cf. [B]. An application lying closer to the current one can be found in [S2, Section 1.6]. Here the author is concerned with finding a so–called epsilon-regularity theorem for energy minimizing harmonic maps; such theorems show that control on the averaged mean square deviation of a given energy minimizer on a small ball leads to Hölder continuity on smaller balls. The technique of harmonic approximation allows the author to simplify the original epsilon-regularity theorem due to Schoen–Uhlenbeck (see [SU, Section 3]).

In Section 2 of the current paper we generalize this technique to elliptic bilinear forms. For such $A \in \mathrm{Bil}(\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N))$ we call $u \in H^{1,2}(\Omega, \mathbb{R}^N)$ $A$–**harmonic** if it satisfies

$$\int_\Omega A(Du, D\varphi)\, dx = 0 \quad \text{for all } \varphi \in C_c^1(\Omega, \mathbb{R}^N)\,;$$

$A$–harmonic approximation then refers to the direct analog of the above situation. A more general form of this technique has been applied in the setting of geometric measure theory by the first author and Steffen; see [DS, Section 3]. There the authors prove a boundary regularity result for almost minimizing rectifiable currents of general elliptic integrands.

The current approach has some useful properties, which we wish to describe briefly. As an indirect proof it avoids the technical difficulties associated with applying Gehring's Lemma; however we obtain a better control of the sensitivity to the structure constants than other indirect proofs, as the $A$–harmonic approximation argument is the only time where we argue indirectly. For example, it is easy to determine the sensitivity of the excess to the inhomogeneous term. In the indirect part of the argument, we only require standard compactness results (Rellich's Theorem): the usual indirect arguments require one to prove compactness results by hand, on a case-by-case basis. In addition, the application of the $A$–harmonic approximation result is accompanied by straightforward, relatively elementary arguments. All of these factors combine to make the method very flexible.

As outlined above, we exhibit this by deriving the partial regularity results in two cases; in Section 3 we consider quasilinear elliptic systems which are permitted to have an inhomogeneous term, and in Section 4, we consider fully nonlinear, homogeneous equations of divergence type. In each case the result is derived completely in the section at hand: apart from the $A$-harmonic approximation Lemma, we only need the standard results of linear theory presented in Section 2, and elementary inequalities.

The partial regularity theory for nonlinear systems in the full generality given by (1.1) requires no major new techniques beyond those introduced in Sections 3 and 4 of the current paper, but for ease of readability we will present that case in a separate work [DG].

The flexibility of the technique also allows us to apply it to parabolic systems; we will take this up in future work.

We close this section by briefly summarizing the notation we will use in this paper. As noted above, we consider a bounded domain $\Omega \subset \mathbb{R}^n$, and maps from $\Omega$ to $\mathbb{R}^N$, where we take $n \geq 2$, $N \geq 1$. For a given set $X$ we denote by $\mathcal{L}^n(X)$ and $\mathcal{H}^k(X)$ its $n$–dimensional Lebesgue measure and $k$–dimensional Hausdorff measure, respectively. We write $B_\rho(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < \rho\}$, and further $B_\rho = B_\rho(0)$, $B = B_1$. For bounded $X \subset \mathbb{R}^n$ we denote the average of a given $g \in L^1(X)$ by $⨍_X g\, dx$, i.e. $⨍_X g\, dx = \frac{1}{\mathcal{L}^n(X)} \int_X g\, dx$. In particular, we write $g_{x_0,\rho} = ⨍_{B_\rho(x_0)} g\, dx$. We let $\alpha_n$ denote the volume of the unit–ball in $\mathbb{R}^n$, i.e. $\alpha_n = \mathcal{L}^n(B)$. We write $\mathrm{Bil}(\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N))$ for the space of bilinear forms on the space $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^N$.

## 2 The $A$–harmonic Approximation Technique

In this section we present the $A$-harmonic approximation lemma, and for completeness also include two standard estimates from linear theory, the Poincaré Lemma, and a result due to Campanato. We refer the reader to Section 1 for comments on the $A$-harmonic approximation lemma. For convenience of later application, we present the lemma in two different scalings (cf. [DS, Lemma 3.3]).

**2.1 Lemma.** *Consider fixed positive $\lambda$ and $L$, and $n$, $N \in \mathbb{N}$ with $n \geq 2$. Then for any given $\varepsilon > 0$ there exists $\delta = \delta(n, N, \lambda, L, \varepsilon) \in (0, 1]$ with the following property: for any $A \in$ $\mathrm{Bil}(\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N))$ satisfying*

$$A(\xi, \xi) \geq \lambda |\xi|^2 \quad \textit{for all} \quad \xi \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N) \ \textit{ and} \tag{2.1}$$

$$|A(\xi, \eta)| \leq L |\xi|\, |\eta| \quad \text{for all} \quad \xi, \eta \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)\,, \tag{2.2}$$

*for any $g \in H^{1,2}(B_\rho(x_0), \mathbb{R}^N)$ (for some $\rho > 0$, $x_0 \in \mathbb{R}^n$) satisfying*

$$\rho^{2-n} \int\limits_{B_\rho(x_0)} |Dg|^2 dx \leq 1 \ \textit{ and} \tag{2.3}$$

$$\Bigg| \rho^{2-n} \int\limits_{B_\rho(x_0)} A(Dg\,, D\varphi)\, dx \Bigg| \leq \delta \rho \sup_{B_\rho(x_0)} |D\varphi| \quad \textit{for all} \quad \varphi \in C^1_c(B_\rho(x_0), \mathbb{R}^N) \tag{2.4}$$

*there exists an $A$-harmonic function $v \in \mathcal{H} = \{w \in H^{1,2}(B_\rho(x_0), \mathbb{R}^N) \,|\, \rho^{2-n} \int_{B_\rho(x_0)} |Dw|^2\, dx \leq 1\}$ with*

$$\rho^{-n} \int\limits_{B_\rho(x_0)} |v - g|^2\, dx \leq \varepsilon\,. \tag{2.5}$$

**Proof.** We assume first that $x_0 = 0$, $\rho = 1$ (at the end of the proof we will show how a rescaling of this result yields the general result). Were the conclusion false, we could find $\varepsilon > 0$, $\{A_k\}$ each satisfying (2.1), (2.2) and $\{g_k\}$ with $g_k \in H^{1,2}(B, \mathbb{R}^N)$ fulfilling:

$$\int\limits_B |v_k - g_k|^2 dx \geq \varepsilon \text{ for all } A_k - \text{harmonic } v_k \in \mathcal{H} \tag{2.6}$$

(note that there are always $A_k$-harmonic functions in $\mathcal{H}$, for example any constant function)

$$\int\limits_B |Dg_k|^2 dx \leq 1; \quad \text{and} \tag{2.7}$$

$$\Big| \int\limits_B A_k(Dg_k, D\varphi)\, dx \Big| \leq \tfrac{1}{k} \sup_B |D\varphi| \quad \text{for all } \varphi \in C^1_c(B, \mathbb{R}^N). \tag{2.8}$$

Without loss of generality we can assume $⨍_B g_k dx = 0$ (by simply considering the sequence $\{g_k - ⨍_B g_k dx\}$ in place of $\{g_k\}$). Poincaré's inequality and Rellich's lemma then allow us to find a subsequence, also denoted by $\{g_k\}$, $g \in H^{1,2}(B,\mathbb{R}^N)$ and $A \in \mathrm{Bil}(\mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^N))$ (note (2.2)) such that:

$$g_k \to g \text{ weakly in } H^{1,2}(B,\mathbb{R}^N),\ g_k \to g \text{ in } L^2(B,\mathbb{R}^N),\ A_k \to A, \text{ and} \int_B |Dg|^2 dx \le 1\,.$$

We then consider, for $\varphi \in C^1_c(B,\mathbb{R}^N)$,

$$\int_B A(Dg,D\varphi)dx = \int_B A(Dg-Dg_k,D\varphi)dx + \int_B (A-A_k)(Dg_k,D\varphi)dx + \int_B A_k(Dg_k,D\varphi)\,dx.$$

The first term on the right-hand side tends to 0 as $k\to\infty$ due to the weak-$H^{1,2}$ convergence of $g_k$ to $g$; similarly the second term via (2.7) and the convergence of the $A_k$'s, and the third term via (2.8). Thus $g$ is $A$-harmonic on $B$.

We now consider the Dirichlet problem given by

$$\int_B A_k(Dv_k,D\varphi)dx = 0 \quad \text{for all } \varphi \in C^1_c(B,\mathbb{R}^N), \quad v_k - g \in H^{1,2}_0(B,\mathbb{R}^N)\,.$$

This problem has a unique solution (see e.g. [G2, Chapter 1]), which we denote by $v_k$. We then have, using (3.3), the $A_k$-harmonicity of $v_k$, the $A$-harmonicity of $g$, and Hölder's inequality,

$$\begin{aligned}
&\lambda \int_B |Dv_k - Dg|^2 dx \\
&\le \int_B A_k(Dv_k-Dg,Dv_k-Dg)\,dx = -\int_B A_k(Dg,Dv_k-Dg)dx \\
&= \int_B (A-A_k)(Dg,Dv_k-Dg)dx \le |A-A_k| \int_B |Dg|\,|Dv_k-Dg|dx \\
&\le |A-A_k| \Big(\int_B |Dg|^2 dx\Big)^{1/2} \Big(\int_B |Dv_k-Dg|^2 dx\Big)^{1/2}.
\end{aligned}$$

Given the convergence of $A_k$ to $A$ and the fact that $\int_B |Dg|^2 dx \le 1$, we can conclude that $v_k$ converges strongly to $g$ in $H^{1,2}(B,\mathbb{R}^N)$, and in particular we have that $\|v_k - g\|_{L^2(B,\mathbb{R}^N)} \to 0$ as $k\to\infty$. This would provide the desired contradiction if we had that $v_k \in \mathcal{H}$. There is, however, no way of guaranteeing this. We therefore set $m_k = \max\{||Dv_k||_{L^2(B,\mathbb{R}^N)},1\}$, and then define $V_k = \frac{v_k}{m_k}$. Then $V_k$ is also $A_k$-harmonic in $B$, with $V_k \in \mathcal{H}$.

We thus have (where the norms refer to the norm in $L^2(B,\mathbb{R}^N)$)

$$\|V_k - g_k\| \le \|V_k - v_k\| + \|v_k - g\| + \|g - g_k\|\,.$$

We have already established that the second and third terms on the right-hand side approach zero as $k\to\infty$. We note that the strong $H^{1,2}$-convergence of $v_k$ to $g$ shows that $\lim_{k\to\infty}\int_B |Dv_k|^2\,dx$ exists and is bounded above by 1, meaning that $\lim_{k\to\infty} m_k = 1$. Hence the first term on the right-hand side is dominated by $2(1-\frac{1}{m_k})^{\frac{1}{2}}(\|g\|+1)$ for $k$ large (using the $L^2$-convergence of $v_k$ to $g$), which also convergences to zero as $k\to\infty$. This provides the desired contradiction to (2.6).

In order to show the result on a general $B_\rho(x_0)$, we define $G$ on $B$ via $G(y) = g(x_0 + \rho y)$ and see that (2.1) and (2.2) allow us to apply the lemma to conclude the existence of an A-harmonic $V \in H^{1,2}(B, \mathbb{R}^N)$ satisfying (2.3) and (2.4) on $B$ (with $v$ replaced by $V$, $g$ replaced by $G$). Setting $v(x) = V(\frac{x-x_0}{\rho})$ then yields the desired conclusion. □

The second scaling of this result is then

**2.2 Lemma.** *Consider fixed positive $\lambda$ and $L$, and $n$, $N \in \mathbb{N}$ with $n \geq 2$. Then for any given $\varepsilon > 0$ there exists $\delta = \delta(n, N, \lambda, L, \varepsilon) \in (0,1]$ with the following property: for any $A \in \mathrm{Bil}(\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N))$ satisfying (2.1) and (2.2), for any $g \in H^{1,2}(B_\rho(x_0), \mathbb{R}^N)$ (for some $\rho > 0$, $x_0 \in \mathbb{R}^n$) satisfying*

$$\rho^{-n} \int_{B_\rho(x_0)} |Dg|^2 dx \leq 1 \quad \textit{and} \tag{2.9}$$

$$\Big|\rho^{-n} \int_{B_\rho(x_0)} A(Dg\,, D\varphi)\, dx\Big| \leq \delta \sup_{B_\rho(x_0)} |D\varphi| \quad \textit{for all} \quad \varphi \in C_c^1(B_\rho(x_0), \mathbb{R}^N) \tag{2.10}$$

*there exists an A-harmonic function $\widetilde{v} \in \widetilde{\mathcal{H}} = \{w \in H^{1,2}(B_\rho(x_0), \mathbb{R}^N) \mid \rho^{-n} \int_{B_\rho(x_0)} |Dw|^2 dx \leq 1\}$ satisfying*

$$\rho^{-n-2} \int_{B_\rho(x_0)} |\widetilde{v} - g|^2\, dx \leq \varepsilon\,. \tag{2.11}$$

**Proof.** For $x_0 = 0$, $\rho = 1$ this is simply Lemma 2.1. For a general ball $B_\rho(x_0)$ we can apply Lemma 2.1 to the rescaled function $G(x) = \frac{1}{\rho} g(x_0 + \rho x)$ to obtain the existence of an A-harmonic $\widetilde{V} \in H^{1,2}(B, \mathbb{R}^N)$ satisfying (2.11) on $B$ (with $\widetilde{v}$ replaced by $\widetilde{V}$, $g$ replaced by $G$). Rescaling via $\widetilde{v}(x) = \rho \widetilde{V}(\frac{x-x_0}{\rho})$ yields the desired result. □

We next state the Poincaré inequality in the form in which we shall need it.

**2.3 Theorem.** *There exists $c_p = c_p(n)$, without loss of generality $c_p \geq 1$, such that every $u \in H^{1,2}(B_\rho(x_0))$ satisfies*

$$\int_{B_\rho(x_0)} |u - u_{x_0,\rho}|^2 \leq c_p \rho^2 \int_{B_\rho(x_0)} |Du|^2\, dx.$$

For a proof we refer the reader to e.g. [GT, Section 7.8]: note from (7.45) in that book the above result follows with $c_p = 2^{2n}$.

Our final tool is a standard estimate for the solutions to homogeneous second order elliptic systems with constant coefficients, due originally to Campanato, [C2, Teorema 9.2]. The result follows from Caccioppoli's inequality for $h$ and its derivatives for any order, Sobolev's inequality, and Poincaré's inequality. Note that the original result is given for scalar-valued equations, but extends immediately to systems. For convenience we give the estimate in a slightly more general form than that given in [C2] (but one which follows directly, after applying Sobolev's and Poincaré's inequalities).

**2.4 Theorem.** *Consider $A$, $\lambda$ and $L$ as in Lemma 2.1. Then there exists $c_0 = c_0(n, N, \lambda, L)$ (without loss of generality we take $c_0 \geq 1$) such that any $A$–harmonic function $h$ on $B_\rho(x_0)$ satisfies*

$$\rho^2 \sup_{B_{\rho/2}(x_0)} |Dh|^2 + \rho^4 \sup_{B_{\rho/2}(x_0)} |D^2 h|^2 \leq c_0 \rho^{2-n} \int_{B_\rho(x_0)} |Dh|^2 \, dx \, .$$

## 3 Inhomogeneous quasilinear systems

In the special case of an inhomogeneous quasilinear system, (1.1) takes the form

$$- \operatorname{div}(A(x,u)(Du, \cdot)) = f(x, u, Du) \qquad \text{in } \Omega \tag{3.1}$$

for $\Omega$ a bounded domain in $\mathbb{R}^n$, $u$ and $f$ taking values in $\mathbb{R}^N$, where each $A(\cdot, \cdot)$ is a bilinear form on $\operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^N)$. In components (3.1) reads:

$$-D_\alpha (A^{ij}_{\alpha\beta}(x,u) D_\beta u^j) = f^i(x, u, Du),$$

where we sum over repeated indicies, with Greek indicies ranging from 1 to $n$, Roman indicies from 1 to $N$. A weak solution to (3.1) is then an $\mathbb{R}^N$-valued function $u$ such that, for all test-functions $\varphi \in C^\infty_c(\Omega, \mathbb{R}^N)$, we have

$$\int_\Omega A(x,u)(Du, D\varphi)\, dx = \int_\Omega f \cdot \varphi \, dx. \tag{3.2}$$

We commence this section by stating our assumptions on $A$ and $f$, and our notion of a weak solution.

**H1** We assume $A \in C^0(\Omega \times \mathbb{R}^N, \operatorname{Bil}(\operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^N)))$, and further that $A$ is uniformly continuous on sets of the form $\Omega \times \{u : |u| \leq M\}$, for any fixed $M$, $0 < M < \infty$.

**H2** We require that the bilinear forms $A(x,u)$ be uniformly strongly elliptic, i.e. there exists $\lambda > 0$ such that

$$A(x,u)(\xi, \xi) \geq \lambda |\xi|^2 \qquad \text{for all } \xi \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^N),\ (x,u) \in \Omega \times \mathbb{R}^N.$$

**H3** There exists $L > 0$ such that

$$A(x,u)(\xi, \eta) \leq L |\xi|\, |\eta| \qquad \text{for all } \xi, \eta \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^N),\ (x,u) \in \Omega \times \mathbb{R}^N.$$

**H4** We impose the so–called *natural growth condition* on $f$ (cf. [G1, p. 180]), i.e. there exist constants $a$ and $b$, with $a$ possibly depending on $M > 0$, such that

$$|f(x,u,p)| \leq a(M)\, |p|^2 + b \quad \text{for all } x \in \Omega,\ u \in \mathbb{R}^N \text{ with } |u| \leq M, \text{ and } p \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^N).$$

From hypothesis (H1) we have, writing $\omega(\cdot)$ for $\omega(M, \cdot)$, the existence of a monotone nondecreasing concave function $\omega : [0,\infty) \to [0,\infty)$ with $\omega(0) = 0$, continuous at 0, such that (see e.g. [G1, p. 169])

$$|A(x,u) - A(y,v)| \le \omega\left(|x-y|^2 + |u-v|^2\right) \quad \text{for all } x,\, y \in \Omega, u, v \in \mathbb{R}^N,\ |u|, |v| \le M.$$

In this setting, a *weak solution* to (3.1) is defined to be a function $u \in H^{1,2}(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$ such that (3.2) holds for all test-functions $\varphi \in C_c^\infty(\Omega, \mathbb{R}^N)$ and, by approximation, all $\varphi \in H_0^{1,2}(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N)$.

We next quote the partial regularity result. This result is originally due to Giaquinta–Giusti, see [GG, Theorem 2.1].

**3.1 Theorem.** *Let $u$ be a weak solution of (3.1) under the hypotheses (H1)–(H4), $\|u\|_{L^\infty} \le M$, and assume $2a(M)M < \lambda$. Then*

$$\mathcal{H}^{n-2-\varepsilon}(\operatorname{Sing} u) = 0,$$

*for some $\varepsilon > 0$, and further $u \in C^{0,\alpha}(\operatorname{Reg} u, \mathbb{R}^N)$ for all $\alpha \in (0,1)$.*

We remark that the techniques presented here (specifically, combining Theorem 3.3 with a standard covering argument) yield the weaker result of $\mathcal{H}^{n-2}(\operatorname{Sing} u) = 0$, and some form of reverse $L^p$–$L^q$–inequality is then needed to proceed to Theorem 3.1. Note also that there are various higher regularity results, including $u \in C^{1,\sigma}$ on $\operatorname{Reg} u$ for $A$ being $C^{0,\sigma}$, smoothness on $\operatorname{Reg} u$ for smooth $A$, and reduction of the dimension of the singular set, possibly even full regularity (i.e. $\operatorname{Sing} u = \emptyset$) for $A$ having particular structures: see e.g. [GG, Theorem 2.1], [G1, Chapters 6,7], [G2, Chapter 6].

The first result we require in order to establish Theorem 3.1 is a reverse-Poincaré or Caccioppoli-type inequality for weak solutions of (3.1).

**3.2 Lemma.** *Let $u \in H^{1,2} \cap L^\infty(\Omega, \mathbb{R}^N)$ be a weak solution of (3.1) under (H1)–(H4) satisfying $\|u\|_{L^\infty} \le M < \infty$. Further assume $2a(M)M < \lambda$. Then for any $B_\rho(x_0) \subset\subset \Omega$ we have*

$$\int_{B_{\rho/2}(x_0)} |Du|^2 dx \le c_1 \rho^{-2} \int_{B_\rho(x_0)} |u - u_{x_0,\rho}|^2 \, dx + \alpha_n b^2 \rho^{n+2} \tag{3.3}$$

*for $c_1 = c_1(\lambda, L, M, a(M)) \ge 1$.*

**Proof.** Let $a = a(M)$. Consider a fixed $B_\rho(x_0) \subset\subset \Omega$, and a cut-off function $\eta \in C_c^1(B_\rho(x_0))$ satisfying $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_{\rho/2}(x_0)$ and $|\nabla \eta| \le 4/\rho$. The function $\varphi = \eta^2(u - u_{x_0,\rho})$ is admissable as a test-function in (3.2), and we obtain

$$\begin{aligned}\int_{B_\rho(x_0)} & A(x,u)(Du, Du)\eta^2 dx \\ &= \int_{B_\rho(x_0)} \left[ f(x,u,Du) \cdot \eta^2 (u - u_{x_0,\rho}) - 2A(x,u)(Du, \eta(u - u_{x_0,\rho}) \otimes \nabla \eta) \right] dx. \end{aligned} \tag{3.4}$$

Using (H3), (H4),and $\|u\|_{L^\infty} \le M$ the right-hand side of (3.4) can be estimated from above by

$$\int_{B_\rho(x_0)} \Big[a\,|Du|^2\eta^2|u-u_{x_0,\rho}| + b\eta^2|u-u_{x_0,\rho}| + 2L\eta|Du|\,|u-u_{x_0,\rho}|\,|\nabla\eta|\Big]\,dx$$

which, after applying Young's inequality to the second and third terms, is dominated by

$$(2aM+\varepsilon)\int_{B_\rho(x_0)} |Du|^2\eta^2 dx + \frac{16L^2+1/4}{\varepsilon\rho^2}\int_{B_\rho(x_0)} |u-u_{x_0,\rho}|^2 dx + \alpha_n b^2\rho^{n+2}\varepsilon$$

for arbitrary positive $\varepsilon$ (also noting $||u||_{L^\infty} \le M$). From (H2), we further deduce that the left-hand side of (3.4) is bounded from below by $\lambda\int_{B_\rho(x_0)}|Du|^2\eta^2 dx$. Combining these estimates for the choice $\varepsilon = \frac{1}{2}(\lambda - 2aM)$ (which is positive by the conditions of the lemma) and dividing through by $\varepsilon$ yields (3.3) with $c_1 = \frac{64L^2+1}{(\lambda-2aM)^2}$; note from (H2) and (H3) $L \ge \lambda$, so $c_1 \ge 1$. □

We are now in a position to prove the central result for obtaining partial regularity, which is that sufficiently small $L^2$-mean oscillation on sufficiently small balls leads to Hölder continuity on smaller balls.

**3.3 Theorem.** *Consider fixed $\alpha \in (0,1)$. Under the assumptions of Theorem 3.1 there exist positive $R_0$ and $\varepsilon$ (depending on $n$, $N$, $\lambda$, $L$, $b$, $M$, $a(M)$, $\omega(\cdot)$, and $\alpha$) with the property that*

$$\fint_{B_R(x_0)} |u-u_{x_0,R}|^2 dx + R^2 \le \varepsilon^2 \tag{3.5}$$

*for some $R \in (0, R_0]$ implies $u \in C^{0,\alpha}(\overline{B}_{R/2}(x_0), \mathbb{R}^N)$.*

**Proof.** By translation, we consider $x_0 = 0$. Consider $B_R \subset\subset \Omega$, $B_\rho(z) \subset B_R$. We consider fixed $\varphi \in C_c^\infty(B_{\rho/2}(z), \mathbb{R}^N)$, $\sup_{B_{\rho/2}(z)}|D\varphi| \le 1$, as a test-function in (3.2), to obtain

$$\begin{aligned}
&\int_{B_{\rho/2}(z)} A(z,u_{z,\rho})(Du,D\varphi)\,dx \\
&= \int_{B_{\rho/2}(z)} f(x,u,Du)\cdot\varphi\,dx + \int_{B_{\rho/2}(z)} (A(z,u_{z,\rho}) - A(x,u))(Du,D\varphi)\,dx \\
&\le a\int_{B_{\rho/2}(z)} |Du|^2|\varphi|dx + b\int_{B_{\rho/2}(z)} |\varphi|\,dx + \int_{B_{\rho/2}(z)} |A(x,u)-A(z,u_{z,\rho})|\,|Du|\,|D\varphi|\,dx \\
&\le \sup_{B_{\rho/2}(z)}|\varphi|\Big(a\int_{B_{\rho/2}(z)}|Du|^2dx + b\alpha_n(\rho/2)^n\Big) \\
&\quad + \sup_{B_{\rho/2}(z)}|D\varphi|\Big(\int_{B_{\rho/2}(z)}|A(x,u)-A(z,u_{z,\rho})|^2dx\Big)^{1/2}\Big(\int_{B_{\rho/2}(z)}|Du|^2dx\Big)^{1/2},
\end{aligned} \tag{3.6}$$

using first (H4) and then Hölder's inequality. Recalling the definition of $\omega(\cdot)$ and using Lemma 3.1 and (H3), we continue to estimate, for $bR^2 \le 1$, and $c_2 = (1+a)\max\{c_1, b\} \ge 1$ (depending on $\lambda$, $L$, $b$, $M$, $a(M)$):

$$\int_{B_{\rho/2}(z)} A(z,u_{z,\rho})(Du,D\varphi)dx \leq \frac{\rho}{2}\left[a\left(\frac{c_1}{\rho^2}\int_{B_\rho(z)}|u-u_{z,\rho}|^2dx+\alpha_n b^2\rho^{n+2}\right)+\frac{b\alpha_n}{2^n}\rho^n\right]$$

$$+\sqrt{2L}\left(\int_{B_\rho(z)}\omega\left(|x-z|^2+|u-u_{z,\rho}|^2\right)dx\right)^{1/2}\left(\frac{c_1}{\rho^2}\int_{B_\rho(z)}|u-u_{z,\rho}|^2dx+\alpha_n b^2\rho^{n+2}\right)^{1/2}$$

$$\leq \frac{\rho}{2}\alpha_n\rho^{n-2}(a+1)\left(c_1\fint_{B_\rho(z)}|u-u_{z,\rho}|^2dx+b\rho^2\right)$$

$$+\sqrt{2L}\alpha_n\rho^{n-1}\left(\fint_{B_\rho(z)}\omega(\rho^2+|u-u_{z,\rho}|^2)dx\right)^{1/2}\left(c_1\fint_{B_\rho(z)}|u-u_{z,\rho}|^2dx+b\rho^2\right)^{1/2}$$

$$\leq \frac{\rho}{2}\alpha_n\rho^{n-2}\Bigg[c_2\left(\fint_{B_\rho(z)}|u-u_{z,\rho}|^2dx+\rho^2\right)$$

$$+2\sqrt{c_2}\sqrt{2L}\omega^{1/2}\left(\fint_{B_\rho(z)}|u-u_{z,\rho}|^2dx+\rho^2\right)\left(\fint_{B_\rho(z)}|u-u_{z,\rho}|^2dx+\rho^2\right)^{1/2}\Bigg]$$

$$=\alpha_n\rho^{n-2}\left(\fint_{B_\rho(z)}|u-u_{z,\rho}|^2dx+\rho^2\right)^{1/2}\frac{\rho}{2}\Bigg[c_2\left(\fint_{B_\rho(z)}|u-u_{z,\rho}|^2dx+\rho^2\right)^{1/2}$$

$$+\sqrt{8Lc_2}\omega^{1/2}\left(\fint_{B_\rho(z)}|u-u_{z,\rho}|^2dx+\rho^2\right)\Bigg],$$

where we have used Jensen's inequality in the second last estimate.

For $B_\sigma(y)\subset\Omega$ we introduce the notation

$$I(y,\sigma)=\fint_{B_\sigma(y)}|u-u_{z,\sigma}|^2dx+\sigma^2;$$

The above estimate can then be applied to yield, for arbitrary $\varphi\in C^1_c(B_{\rho/2}(z),\mathbb{R}^N)$:

$$\int_{B_{\rho/2}(z)} A(z,u_{z,\rho})(Du,D\varphi)\,dx$$
$$\leq \alpha_n\rho^{n-2}\sqrt{c_2}\sqrt{I(z,\rho)}\frac{\rho}{2}\sup_{B_{\rho/2}(z)}|D\varphi|\left[\sqrt{c_2}\sqrt{I(z,\rho)}+\sqrt{8L}\omega^{1/2}(I(z,\rho))\right].$$

Multiplying through by $(\frac{\rho}{2})^{2-n}$, we obtain (noting that $\alpha_n\leq 2^n$)

$$\left|\left(\frac{\rho}{2}\right)^{2-n}\int_{B_{\rho/2}(z)} A(z,u_{z,\rho})(Du,D\varphi)\,dx\right|$$

$$\leq \sqrt{I(z,\rho)}\frac{\rho}{2} \sup_{B_{\rho/2}(z)} |D\varphi| 2^{n-2}\alpha_n \Big[c_2\sqrt{I(z,\rho)} + \sqrt{8Lc_2}\,\omega^{1/2}(I(z,\rho))\Big]$$

$$\leq c_3\sqrt{I(z,\rho)}\frac{\rho}{2} \sup_{B_{\rho/2}(z)} |D\varphi| \Big[\sqrt{I(z,\rho)} + \omega^{1/2}(I(z,\rho))\Big] \tag{3.7}$$

for $c_3 = 2^{2n-2}(c_2 + 2L) \geq 1$ (depending on $n$, $\lambda$, $L$, $b$, $M$, $a(M)$).

From Lemma 3.2 we note, recalling $b\rho^2 \leq 1$,

$$\left(\frac{\rho}{2}\right)^{2-n} \int_{B_{\rho/2}(z)} |Du|^2\,dx \leq \left(\frac{\rho}{2}\right)^{2-n} \left(\frac{c_1}{\rho^2} \int_{B_\rho(z)} |u - u_{z,\rho}|^2\,dx + b^2\alpha_n\rho^{n+2}\right) \leq c_3 I(z,\rho). \tag{3.8}$$

We define now $v = \frac{u}{\gamma}$, for $\gamma = c_3\sqrt{I(z,\rho)}$. From (3.7) we see

$$\left|\left(\frac{\rho}{2}\right)^{2-n} \int_{B_{\rho/2}(z)} A(z, u_{z,\rho})(Dv, D\varphi)\,dx\right| \leq \Big[\sqrt{I(z,\rho)} + \omega^{1/2}(I(z,\rho))\Big]\frac{\rho}{2} \sup_{B_{\rho/2}(z)} |D\varphi| \tag{3.9}$$

and, from (3.8) (recalling also the definition of $c_3$),

$$\left(\frac{\rho}{2}\right)^{2-n} \int_{B_{\rho/2}(z)} |Dv|^2 dx \leq c_3^{-1} \leq 1\,. \tag{3.10}$$

Now consider a fixed, arbitrary $\varepsilon > 0$, and let $\delta = \delta(n, N, \lambda, L, \varepsilon) \in (0,1]$ be given from Lemma 2.1. If

$$\sqrt{I(z,\rho)} + \omega^{1/2}(I(z,\rho)) \leq \delta \tag{3.11}$$

then we see from (3.9) and (3.10) that $v$ satisfies the conditions of this lemma, allowing us to conclude the existence of an $A(z, u_{z,\rho})$-harmonic function $h \in H^{1,2}(B_{\rho/2}(z), \mathbb{R}^N)$ satisfying

$$\left(\frac{\rho}{2}\right)^{2-n} \int_{B_{\rho/2}(z)} |Dh|^2\,dx \leq 1, \tag{3.12}$$

$$\left(\frac{\rho}{2}\right)^{-n} \int_{B_{\rho/2}(z)} |h - v|^2 dx \leq \varepsilon\,. \tag{3.13}$$

Using Theorem 2.4 and (3.12) we obtain the following interior estimate for $h$, for $\theta \in (0, 1/4]$:

$$\sup_{B_{\theta\rho}(z)} |h - h(z)|^2 \leq \theta^2\rho^2 \sup_{B_{\rho/4}(z)} |Dh|^2 \leq \theta^2\rho^2 c_0 \left(\frac{\rho}{2}\right)^{-n} \int_{B_{\rho/2}(z)} |Dh|^2 dx \leq 4c_0\theta^2\,. \tag{3.14}$$

We now calculate, using (3.13) and (3.14):

$$\begin{aligned}(\theta\rho)^{-n} \int_{B_{\theta\rho}(z)} |v - h(z)|^2 dx &\leq 2(\theta\rho)^{-n} \int_{B_{\theta\rho}(z)} (|v - h|^2 + |h - h(z)|^2)dx \\ &\leq 2(\theta\rho)^{-n} \left(\left(\frac{\rho}{2}\right)^n \varepsilon + \alpha_n(\theta\rho)^n \sup_{B_{\theta\rho}(z)} |h - h(z)|^2\right) \\ &\leq 2^{1-n}\theta^{-n}\varepsilon + 8\alpha_n c_0\theta^2\,.\end{aligned}$$

Multiplying this through by $\gamma^2$ and recalling the definition of $v$, we see

$$(\theta\rho)^{-n} \int\limits_{B_{\theta\rho}(z)} |u - \gamma h(z)|^2 dx \le \gamma^2 \left(2^{1-n}\theta^{-n}\varepsilon + 8\alpha_n c_0 \theta^2\right) .$$

The left-hand side of this inequality can be estimated from below by

$$(\theta\rho)^{-n} \inf_{\xi\in\mathbb{R}^N} \int\limits_{B_{\theta\rho}(z)} |u-\xi|^2 dx = (\theta\rho)^{-n} \int\limits_{B_{\theta\rho}(z)} |u - u_{z,\theta\rho}|^2 \, dx.$$

Combining these estimates we have, since $\gamma^2 = c_3^2 I(z,\rho)$ and since $(\theta\rho)^2 \le c_3^2 c_0 \theta^2 I(z,\rho)$,

$$I(z,\theta\rho) \le c_3^2 \left(2^{1-n}\alpha_n^{-1}\theta^{-n}\varepsilon + 9c_0\theta^2\right) I(z,\rho). \tag{3.15}$$

We first fix

$$\theta = \min\left\{\tfrac{1}{4}, \left(18c_0c_3^2\right)^{-\frac{1}{2(1-\alpha)}}\right\}$$

(depending on $(n,\ N,\ \lambda,\ L,\ b,\ M,\ a(M),\ \alpha)$), so that, in particular, $9c_0c_3^2\theta^2 \le \frac{1}{2}\theta^{2\alpha}$, and then set

$$\varepsilon = \alpha_n 2^{n-2} c_3^{-2}\theta^{n+2\alpha} ,$$

so that $2^{1-n}\alpha_n^{-1}c_3^2\theta^{-n}\varepsilon = \frac{1}{2}\theta^{2\alpha}$. With this choice of $\varepsilon$, and $\delta$ being the corresponding $\delta(n, N, \lambda, L, \varepsilon)$ from Lemma 2.1, we see from (3.15) that we have

$$I(z,\theta\rho) \le \theta^{2\alpha} I(z,\rho) \tag{3.16}$$

provided that (3.11) holds.

We now choose $s_0 > 0$ (depending on $n,\ N,\ \lambda,\ L,\ b,\ M,\ a(M),\ \alpha,\ \omega(\,\cdot\,)$) such that $0 < \omega(s_0) \le \frac{1}{4}\delta^2$, and assume

$$I_R = I(0,R) \le 2^{-n} \min\left\{\tfrac{1}{4}\delta^2, s_0\right\} \tag{3.17}$$

for some $R \in (0, R_0]$, where $R_0 = \min\{\sqrt{2s_0}, 1/\sqrt{b}\}$ (in the case $b=0$ we take $R_0 = \sqrt{2s_0}$). Then for any $z \in B_{R/2}$ we have, noting $I(z, \frac{1}{2}R) \le 2^n I_R$:

$$\sqrt{I(z,\tfrac{1}{2}R)} + \omega^{1/2}(I(z,\tfrac{1}{2}R)) \le \sqrt{2^n I_R} + \omega^{1/2}(2^n I_R) \le \tfrac{1}{2}\delta + \omega^{1/2}(s_0) \le \delta ,$$

so that under the smallness condition (3.17), (3.11) holds with $\rho = \frac{1}{2}R$ for all $z \in B_{R/2}$. We can thus apply (3.16) in this situation to conclude

$$\sqrt{I(z,\tfrac{1}{2}\theta R)} + \omega^{1/2}(I(z,\tfrac{1}{2}\theta R)) \le \sqrt{I(z,\tfrac{1}{2}R)} + \omega^{1/2}(I(z,\tfrac{1}{2}R)) \le \delta,$$

i.e. we can apply (3.16) to $B_{\theta R/2}(z)$, as well, yielding $I(z, \frac{1}{2}\theta^2 R) \le \theta^{4\alpha} I(z, \frac{1}{2}R/2)$, and inductively

$$I(z,\tfrac{1}{2}\theta^k R) \le \theta^{2\alpha k} I(z,\tfrac{1}{2}R) . \tag{3.18}$$

Given $\rho \in (0, \frac{1}{2}R]$, we can find $k \in \mathbb{N}$ such that $\frac{1}{2}\theta^k R < \rho \leq \frac{1}{2}\theta^{k-1}R$, yielding $2\rho > \theta^k R$, and allowing us to estimate $I(z,\rho) \leq \theta^{-n} I(z, \frac{1}{2}\theta^k R)$. Combining these with (3.18) we have

$$I(z,\rho) \leq \theta^{-n-2\alpha} \left(\frac{2\rho}{R}\right)^{2\alpha} I(z, \tfrac{1}{2}R) \leq \left(\frac{2}{\theta}\right)^{n+2\alpha} \left(\frac{\rho}{R}\right)^{2\alpha} I_R,$$

and more particularly

$$\fint_{B_\rho(z)} |u - u_{z,\rho}|^2 \, dx \leq \left(\frac{2}{\theta}\right)^{n+2\alpha} \left(\frac{\rho}{R}\right)^{2\alpha} I_R$$

for all $z \in B_{R/2}$, $0 < \rho \leq \frac{1}{2}R$. The Campanato Theorem [C1, Teorema 1.3] (see also [G1, Chapter 3.1]) then yields

$$|u(x) - u(y)| \leq C(n,\alpha)\sqrt{(2/\theta)^{n+2\alpha} I_R} \left(\frac{|x-y|}{R/2}\right)^{\alpha} \qquad \text{for all } x, y \in \overline{B}_{R/2},$$

i.e. $u \in C^{0,\alpha}(\overline{B}_{R/2}, \mathbb{R}^N)$. □

The partial regularity result Theorem 3.1 now follows, modulo the comments after the statement of that theorem.

## 4 The fully nonlinear homogeneous case

In this section we consider the case of a general homogeneous system of second-order elliptic equations, i.e. we consider weak solutions of

$$\operatorname{div} A(x, u, Du) = 0 \text{ in } \Omega$$

for $\Omega$ a bounded domain in $\mathbb{R}^n$, and $A : \Omega \times \mathbb{R}^n \times \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^N) \to \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^N)$.

In analogy to Section 3, a *weak solution* here means $u \in H^{1,2}(\Omega, \mathbb{R}^N)$ such that

$$\int_\Omega A(x, u, Du) \cdot D\varphi \, dx = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega, \mathbb{R}^N). \tag{4.1}$$

We assume the following structure-conditions on $A$ (cf. the conditions in Section 3):

**H1** $A(x, \xi, p)$ are differentiable functions in $p$ with bounded and continuous derivatives

$$\left|\frac{\partial A}{\partial p}(x, \xi, p)\right| \leq L \quad \text{for all } (x, \xi, p) \in \Omega \times \mathbb{R}^N \times \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^N), \text{ for some } L > 0;$$

**H2** $A$ is uniformly strongly elliptic, i.e. for some $\lambda > 0$ we have

$$\left(\frac{\partial A}{\partial p}(x, \xi, p)\nu\right) \nu \geq \lambda |\nu|^2 \quad \text{for all } x \in \Omega,\ \xi \in \mathbb{R}^N \text{ and } p, \nu \in \operatorname{Hom}(\mathbb{R}^n, \mathbb{R}^N); \text{ and}$$

**H3** there exists $\beta \in (0,1)$ and $K : [0,\infty) \to [0,\infty)$ monotone nondecreasing such that

$$\left|A(x,\xi,p) - A(\widetilde{x},\widetilde{\xi},p)\right| \le K(|\xi|)\left(|x-\widetilde{x}|^2 + |\xi - \widetilde{\xi}|^2\right)^{\beta/2}(1+|p|)$$

for all $x, \widetilde{x} \in \Omega$, $\xi, \widetilde{\xi} \in \mathbb{R}^N$, and $p \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$; without loss of generality we take $K \ge 1$.

From (H1) and (H2) we immediately deduce the following (cf. Section 3):

$$|A(x,\xi,p) - A(x,\xi,\nu)| \le L|p-\nu|; \tag{4.2}$$

$$(A(x,\xi,p) - A(x,\xi,\nu)) \cdot (p-\nu) \ge \lambda |p-\nu|^2 \tag{4.3}$$

for $x \in \Omega$, $\xi \in \mathbb{R}^N$ and $p, \nu \in \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$.

Further (H1) allows us to deduce the existence of a function $\omega : [0,\infty) \times [0,\infty) \to [0,\infty)$ with $\omega(t,0) = 0$ for all $t$ such that $t \mapsto \omega(t,s)$ is monotone nondecreasing for fixed $s$, $s \mapsto \omega(t,s)$ is concave and monotone nondecreasing for fixed $t$, and such that for all $(x,\xi,p)$, $(\widetilde{x},\widetilde{\xi},\widetilde{p})$ in $\Omega \times \mathbb{R}^N \times \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$ with $|\xi| + |p| \le M$ we have

$$\left|\frac{\partial A}{\partial p}(x,\xi,p) - \frac{\partial A}{\partial p}(\widetilde{x},\widetilde{\xi},\widetilde{p})\right| \le \omega\left(M, |x-\widetilde{x}|^2 + |\xi - \widetilde{\xi}|^2 + |p-\widetilde{p}|^2\right);$$

cf. [GG, p. 124], as well as Section 3.

As in Section 3, our first goal is to prove an inequality of Caccioppoli, or reverse-Poincaré, type. We require the inequality in a more general form than that needed in the case of a quasilinear system.

**4.1 Lemma.** *Consider $\nu$ fixed in $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^N)$, $\xi$ fixed in $\mathbb{R}^N$ and let $u \in H^{1,2}(\Omega, \mathbb{R}^N)$ be a weak solution to (4.1). Then for all $x_0 \in \Omega$ and $\rho \le 1$ such that $B_\rho(x_0) \subset\subset \Omega$, there holds*

$$\int_{B_{\rho/2}(x_0)} |Du - \nu|^2 dx$$

$$\le \frac{c_1}{\rho^2} \int_{B_\rho(x_0)} |u - \xi - \nu(x-x_0)|^2 dx + c_2 \alpha_n \rho^{n+2\beta} (K(|\xi| + |\nu|)\,(1+|\nu|))^{\frac{2}{1-\beta}}$$

*for $c_1 = c_1(\lambda, L)$, $c_2 = c_2(\lambda, \beta)$.*

**Proof.** We denote $u - \xi - \nu(x-x_0)$ by $v$, and consider a standard cut-off function $\eta \in C_c^\infty(B_\rho(x_0))$ satisfying $0 \le \eta \le 1$, $|\nabla \eta| < \frac{4}{\rho}$, $\eta \equiv 1$ on $B_{\rho/2}(x_0)$. Then $\varphi = \eta^2 v$ is, by approximation, admissible as a test-function in (4.1), and we obtain

$$\int_{B_\rho(x_0)} A(x,u,Du) \cdot (Du - \nu)\eta^2 dx = -2 \int_{B_\rho(x_0)} A(x,u,Du) \cdot \eta v \otimes \nabla \eta dx.$$

We further have

$$-\int_{B_\rho(x_0)} A(x,u,\nu) \cdot (Du-\nu)\eta^2\, dx$$

$$= 2\int_{B_\rho(x_0)} A(x,u,\nu) \cdot \eta v \otimes \nabla\eta dx - \int_{B_\rho(x_0)} A(x,u,\nu) \cdot D\varphi\, dx,$$

and

$$0 = \int\limits_{B_\rho(x_0)} A(x_0,\xi,\nu)\cdot D\varphi\,dx.$$

Adding these three equations yields

$$\begin{aligned}
&\int\limits_{B_\rho(x_0)} (A(x,u,Du)-A(x,u,\nu))\cdot(Du-\nu)\eta^2 dx\\
&= -2\int\limits_{B_\rho(x_0)} (A(x,u,Du)-A(x,u,\nu))\eta\cdot v\otimes\nabla\eta\,dx\\
&\quad -\int\limits_{B_\rho(x_0)} (A(x,u,\nu)-A(x,\xi+\nu(x-x_0),\nu))\cdot D\varphi\,dx\\
&\quad -\int\limits_{B_\rho(x_0)} (A(x,\xi+\nu(x-x_0),\nu)-A(x_0,\xi,\nu))\cdot D\varphi\,dx\\
&\le I+II+III+IV,
\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}
I &= 2L\int\limits_{B_\rho(x_0)} |Du-\nu|\,|v|\,|\nabla\eta|\,\eta dx,\\
II &= K(|\xi|+|\nu|)(1+|\nu|)\int\limits_{B_\rho(x_0)} |v|^\beta|Du-\nu|\eta^2\,dx,\\
III &= 2K(|\xi|+|\nu|)(1+|\nu|)\int\limits_{B_\rho(x_0)} |v|^{1+\beta}|\nabla\eta|\eta\,dx,\quad\text{and}\\
IV &= K(|\xi|+|\nu|)(1+|\nu|)\int\limits_{B_\rho(x_0)} \Big(|x-x_0|^2+|\nu(x-x_0)|^2\Big)^{\beta/2}\\
&\qquad\qquad\Big(\eta^2|Du-\nu|+2\eta|v|\,|\nabla\eta|\Big)\,dx,
\end{aligned}$$

after using (4.2),(H3), and $\rho\le 1$.

For $\varepsilon$ positive to be fixed later we have, using Young's inequality,

$$I\le\varepsilon\int\limits_{B_\rho(x_0)} |Du-\nu|^2\eta^2 dx+\frac{16L^2}{\varepsilon\rho^2}\int\limits_{B_\rho(x_0)} |v|^2 dx.$$

Using Young's inequality twice in II, we have

$$\begin{aligned}
II &\le \varepsilon\int\limits_{B_\rho(x_0)} |Du-\nu|^2\eta^2 dx+\frac{1}{\varepsilon}K^2(|\xi|+|\nu|)(1+|\nu|)^2\int\limits_{B_\rho(x_0)} \rho^{2\beta}\Big(\frac{1}{\rho}|v|\Big)^{2\beta}dx\\
&\le \varepsilon\int\limits_{B_\rho(x_0)} |Du-\nu|^2\eta^2 dx+\frac{1}{\varepsilon}\Bigg(\Big(K(|\xi|+|\nu|)(1+|\nu|)\Big)^{\frac{2}{1-\beta}}\alpha_n\rho^{n+\frac{2\beta}{1-\beta}}+\frac{1}{\rho^2}\int\limits_{B_\rho(x_0)} |v|^2 dx\Bigg),
\end{aligned}$$

and similarly we see

$$III \le \frac{1}{\rho^2} \int\limits_{B_\rho(x_0)} |v|^2 dx + \Big(8K(|\xi| + |\nu|)(1+|\nu|)\Big)^{\frac{2}{1-\beta}} \alpha_n \rho^{n+\frac{2\beta}{1-\beta}}$$

and

$$\begin{aligned} IV &\le \int\limits_{B_\rho(x_0)} K(|\xi| + |\nu|)\rho^\beta (1+|\nu|)^{\beta+1} \left(\eta |Du - \nu| + \frac{8}{\rho}|v|\right) dx \\ &\le \varepsilon \int\limits_{B_\rho(x_0)} |Du - \nu|^2 \eta^2 dx + \frac{1}{\rho^2} \int\limits_{B_\rho(x_0)} |v|^2 dx \\ &\qquad + \left(64 + \frac{1}{\varepsilon}\right) K(|\xi| + |\nu|)^2 (1+|\nu|)^{2(1+\beta)} \alpha_n \rho^{n+2\beta}. \end{aligned}$$

Combining these estimates in (4.4) and using (4.3), we have

$$\begin{aligned} &(\lambda - 3\varepsilon) \int\limits_{B_\rho(x_0)} |Du - \nu|^2 \eta^2 dx \\ &\le \left(\frac{16L^2+1}{\varepsilon} + 2\right) \frac{1}{\rho^2} \int\limits_{B_\rho(x_0)} |v|^2 dx + \left(\frac{1}{\varepsilon} + 8^{\frac{2}{1-\beta}}\right) \Big(K(|\xi| + |\nu|)(1+|\nu|)\Big)^{\frac{2}{1-\beta}} \alpha_n \rho^{n+\frac{2\beta}{1-\beta}} \\ &\qquad + \left(\frac{1}{\varepsilon} + 64\right) K(|\xi| + |\nu|)^2 (1+|\nu|)^{2(1+\beta)} \alpha_n \rho^{n+2\beta}. \end{aligned} \tag{4.5}$$

Noting that $\rho^{\frac{2\beta}{1-\beta}} \le \rho^{2\beta}$ for $\rho \le 1$, $K^2 \le K^{\frac{2}{1-\beta}}$ (since $K \ge 1$), $(1+|\nu|)^{2(1+\beta)} \le (1+|\nu|)^{\frac{2}{1-\beta}}$, and $64 \le 8^{\frac{2}{1-\beta}}$ we can estimate from (4.5)

$$\begin{aligned} &(\lambda - 3\varepsilon) \int\limits_{B_\rho(x_0)} |Du - \nu|^2 \eta^2 \, dx \\ &\le \left(\frac{16L^2+1}{\varepsilon} + 2\right) \frac{1}{\rho^2} \int\limits_{B_\rho(x_0)} |v|^2 dx + 2\left(8^{\frac{2}{1-\beta}} + \frac{1}{\varepsilon}\right) \alpha_n \rho^{n+2\beta} \Big(K(|\xi| + |\nu|)(1+|\nu|)\Big)^{\frac{2}{1-\beta}}. \end{aligned} \tag{4.6}$$

Setting $\varepsilon = \frac{\lambda}{6}$ in (4.6) and multiplying through by $\frac{2}{\lambda}$, we obtain the desired inequality with

$$c_1 = c_1(\lambda, L) = \frac{12(16L^2+1) + 4\lambda}{\lambda^2} \quad \text{and} \quad c_2 = c_2(\lambda, \beta) = \frac{4}{\lambda}\left(8^{\frac{2}{1-\beta}} + \frac{6}{\lambda}\right).$$

□

We now consider $\varphi \in C_c^\infty(B_\rho(x_0), \mathbb{R}^N)$ with $\sup\limits_{B_\rho(x_0)} |D\varphi| \le 1$, and we henceforth restrict to $\rho \le 1$. We further fix $\xi = u_{x_0,\rho} = ⨍_{B_\rho(x_0)} u \, dx$, and (as in the proof of Lemma 4.1) set $v = u - \xi - \nu(x - x_0)$. We have, noting that $\int\limits_{B_\rho(x_0)} A(x_0, \xi, \nu) \cdot D\varphi \, dx = 0$ (since $A(x_0, \xi, \nu)$ is constant) and using (4.1):

$$\begin{aligned} &\int\limits_{B_\rho(x_0)} \left[\int\limits_0^1 \frac{\partial A}{\partial p}(x_0, \xi, \nu + t(Du - \nu)) dt\right] (Du - \nu) D\varphi \, dx \\ &= \int\limits_{B_\rho(x_0)} (A(x_0, \xi, Du) - A(x_0, \xi, \nu)) \cdot D\varphi \, dx = \int\limits_{B_\rho(x_0)} (A(x_0, \xi, Du) - A(x, u, Du)) \cdot D\varphi \, dx. \end{aligned}$$

Rearranging this, we have

$$\int\limits_{B_\rho(x_0)} \frac{\partial A}{\partial p}(x_0,\xi,\nu)(Du-\nu)D\varphi\,dx = \int\limits_{B_\rho(x_0)} \Big[\int\limits_0^1 \frac{\partial A}{\partial p}(x_0,\xi,\nu)dt\Big](Du-\nu)D\varphi\,dx$$

$$= \int\limits_{B_\rho(x_0)} \Big[\int\limits_0^1 \Big(\frac{\partial A}{\partial p}(x_0,\xi,\nu) - \frac{\partial A}{\partial p}(x_0,\xi,\nu+t(Du-\nu))\Big)dt\Big](Du-\nu)D\varphi\,dx +$$

$$+\int\limits_B (A(x_0,\xi,Du)-A(x,u,Du))\cdot D\varphi\,dx. \tag{4.7}$$

Using (H1) and the estimate for the modulus of continuity of $\frac{\partial A}{\partial p}$, we have

$$\left|\frac{\partial A}{\partial p}(x_0,\xi,\nu) - \frac{\partial A}{\partial p}(x_0,\xi,\nu+t(Du-\nu))\right| \le \sqrt{2L}\omega^{1/2}\left(|\xi|+|\nu|,|Du-\nu|^2\right)\,,$$

and hence (4.7) yields, recalling $\|D\varphi\|_{L^\infty}\le 1$,

$$\int\limits_{B_\rho(x_0)} \frac{\partial A}{\partial p}(x_0,\xi,\nu)(Du-\nu)D\varphi\,dx \le I+II+III, \tag{4.8}$$

where

$$I = \sqrt{2L}\int\limits_{B_\rho(x_0)} \omega^{1/2}\left(|\xi|+|\nu|,|Du-\nu|^2\right)|Du-\nu|\,dx,$$

$$II = \int\limits_{B_\rho(x_0)} |A(x_0,\xi,Du)-A(x,\xi+\nu(x-x_0),Du)|\,dx \quad \text{and}$$

$$III = \int\limits_{B_\rho(x_0)} |A(x,\xi+\nu(x-x_0),Du)-A(x,u,Du)|\,dx\,.$$

We have, using first Cauchy-Schwarz's and then Jensen's inequalities:

$$I \le \sqrt{2L}\sqrt{\alpha_n}\rho^{n/2}\omega^{1/2}\Big(|\xi|+|\nu|, \fint\limits_{B_\rho(x_0)} |Du-\nu|^2dx\Big)\Big(\int\limits_{B_\rho(x_0)} |Du-\nu|^2\,dx\Big)^{1/2}. \tag{4.9}$$

We abbreviate $K(|\xi|+|\nu|)$ by $\kappa$ and estimate, using (H3), Young's inequality, $\kappa\ge 1$, and $\rho\le 1$:

$$\begin{aligned} II &\le \kappa\int\limits_{B_\rho(x_0)} (1+|\nu|)^\beta\rho^\beta(1+|\nu|+|Du-\nu|)dx \\ &\le 2\kappa^2(1+|\nu|)^{1+\beta}\alpha_n\rho^{n+\beta} + \int\limits_{B_\rho(x_0)} |Du-\nu|^2dx\,. \end{aligned}$$

We also estimate III by using (H3) and (repeatedly) applying Young's inequality:

$$III \le \kappa\int\limits_{B_\rho(x_0)} |v|^\beta(1+|Du|)dx$$

$$\leq \kappa \int_{B_\rho(x_0)} |v|^\beta (1+|\nu|)dx + \kappa \int_{B_\rho(x_0)} |v|^\beta |Du-\nu| dx$$
$$\leq \kappa \int_{B_\rho(x_0)} (1+|\nu|)\rho^\beta \left(\frac{1}{\rho}|v|\right)^\beta dx + \kappa^2 \int_{B_\rho(x_0)} \rho^{2\beta}\left(\frac{1}{\rho}|v|\right)^{2\beta} dx + \int_{B_\rho(x_0)} |Du-\nu|^2 dx$$
$$\leq \frac{2}{\rho^2}\int_{B_\rho(x_0)} |v|^2 dx + 2\left(\kappa(1+|\nu|)\right)^{\frac{2}{1-\beta}} \alpha_n \rho^{n+\beta} + \int_{B_\rho(x_0)} |Du-\nu|^2 dx, \tag{4.10}$$

again where we have noted $\kappa \geq 1$, and used the fact that $\rho^{\frac{2\beta}{1-\beta}} \leq \rho^{\frac{2\beta}{2-\beta}} \leq \rho^\beta$ for $\rho \leq 1$. By Theorem 2.3 we can further estimate from (4.10):

$$III \leq (1+2c_p) \int_{B_\rho(x_0)} |Du-\nu|^2 dx + 2\left(\kappa(1+|\nu|)\right)^{\frac{2}{1-\beta}} \alpha_n \rho^{n+\beta}.$$

Since $\kappa^2(1+|\nu|)^{1+\beta} \leq \Big(\kappa(1+|\nu|)\Big)^{\frac{2}{1-\beta}}$, we can combine the above estimates to obtain

$$II + III \leq 2(1+c_p) \int_{B_\rho(x_0)} |Du-\nu|^2 dx + 4\left(\kappa(1+|\nu|)\right)^{\frac{2}{1-\beta}} \alpha_n \rho^{n+\beta}. \tag{4.11}$$

We define

$$\Phi(x_0,\rho,\nu) = \fint_{B_\rho(x_0)} |Du-\nu|^2\, dx \qquad \text{and } H(t) = (K(M+t)(1+t))^{\frac{2}{1-\beta}}.$$

Note that $H$ is monotone nondecreasing, and takes values in $[1,\infty)$; in particular we have that $(\kappa(1+|\nu|))^{\frac{2}{1-\beta}} \leq H(|\xi|+|\nu|)$. We can now combine (4.9) with (4.11) in (4.8) to obtain

$$\begin{aligned}
&\int_{B_\rho(x_0)} \frac{\partial A}{\partial p}(x_0,\xi,\nu)(Du-\nu)D\varphi\, dx \\
&\leq \sqrt{2L}\omega^{1/2}(|\xi|+|\nu|,\Phi(x_0,\rho,\nu))\Phi^{1/2}(x_0,\rho,\nu) + 2(1+c_p)\alpha_n\Phi(x_0,\rho,\nu) + 4\alpha_n H(|\xi|+|\nu|)\rho^\beta \\
&\leq c_3\left(\omega^{1/2}(|\xi|+|\nu|,\Phi(x_0,\rho,\nu))\Phi^{1/2}(x_0,\rho,\nu) + \Phi(x_0,\rho,\nu) + H(|\xi|+|\nu|)\rho^\beta\right)
\end{aligned} \tag{4.12}$$

for $c_3 = 2^{n+1}(1+c_p) + \sqrt{2L}$ (depending on $n$ and $L$).

For $\varepsilon > 0$ to be determined later, we take $\delta = \delta(n,N,\lambda,L,\varepsilon) \in (0,1]$ to be the corresponding constant from the $A$-harmonic approximation Lemma.

$$w = \frac{u - u_{x_0,\rho} - \nu(x-x_0)}{c_3\left(\Phi(x_0,\rho,\nu) + 4\delta^{-2}\rho^{2\beta}H^2(|u_{x_0,\rho}|+|\nu|)\right)^{1/2}}$$

and obtain from (4.12), now for arbitrary $\varphi \in C_c^\infty(B_\rho(x_0),\mathbb{R}^N)$,

$$\begin{aligned}
&\rho^{-n}\int_{B_\rho(x_0)} \frac{\partial A}{\partial p}(x_0,u_{x_0,\rho},\nu)DwD\varphi\, dx \\
&\qquad \leq \left(\omega^{1/2}\left(|u_{x_0,\rho}|+|\nu|,\Phi(x_0,\rho,\nu)\right) + \Phi^{1/2}(x_0,\rho,\nu) + \tfrac{\delta}{2}\right)\sup_{B_\rho(x_0)}|D\varphi|
\end{aligned} \tag{4.13}$$

(noting that $c_3 > 2^n > \alpha_n$) and

$$\rho^{-n} \int\limits_{B_\rho(x_0)} |Dw|^2 dx \le \frac{\alpha_n}{c_3^2} \le 1. \tag{4.14}$$

For $\varepsilon > 0$ to be determined later, we take $\delta = \delta(n, N, \lambda, L, \varepsilon) \in (0,1]$ to be the corresponding constant from the $A$-harmonic approximation Lemma. If the smallness-condition

$$\omega^{1/2}\left(|u_{x_0,\rho}| + |\nu|, \Phi(x_0,\rho,\nu)\right) + \Phi^{1/2}(x_0,\rho,\nu) \le \delta/2 \tag{4.15}$$

is satisfied, inequalities (4.13) and (4.14) allow us to apply the second scaling of the $A$-harmonic approximation Lemma, Lemma 3.2, to conclude the existence of a $\frac{\partial A}{\partial p}(x_0, u_{x_0,\rho}, \nu)$-harmonic function $h \in H^{1,2}(B_\rho(x_0), \mathbb{R}^N)$ satisfying:

$$\rho^{-n-2} \int\limits_{B_\rho(x_0)} |w-h|^2 dx \le \varepsilon; \tag{4.16}$$

and

$$\rho^{-n} \int\limits_{B_\rho(x_0)} |Dh|^2 dx \le 1. \tag{4.17}$$

From Theorem 2.4 and (4.17) we have

$$\sup_{B_{\rho/2}(x_0)} |D^2 h|^2 \le c_0 \rho^{-n-2} \int\limits_{B_\rho(x_0)} |Dh|^2 dx \le \frac{c_0}{\rho^2}.$$

For $\theta \in (0, 1/4]$ (we will later fix $\theta$), Taylor's theorem applied to $h$ at $x_0$ thus yields

$$\sup_{x \in B_{2\theta\rho}(x_0)} |h(x) - h(x_0) - Dh(x_0)(x-x_0)|^2 \le \frac{c_0}{\rho^2}(2\theta\rho)^4 = 16 c_0 \theta^4 \rho^2. \tag{4.18}$$

We have then

$$\begin{aligned}
&(2\theta\rho)^{-n-2} \int\limits_{B_{2\theta\rho}(x_0)} |w - h(x_0) - Dh(x_0)(x-x_0)|^2 dx \\
&\le 2(2\theta\rho)^{-n-2}\Bigg( \int\limits_{B_{2\theta\rho}(x_0)} |w-h|^2 dx + \int\limits_{B_{2\theta\rho}(x_0)} |h - h(x_0) - Dh(x_0)(x-x_0)|^2 dx \Bigg) \\
&\le 2(2\theta\rho)^{-n-2}\left(\rho^{n+2}\varepsilon + 16 c_0 \alpha_n (2\theta\rho)^n \theta^4 \rho^2\right) \qquad \text{(via (4.16), (4.18))} \\
&= 2^{-n-1}\theta^{-n-2}\varepsilon + 8 c_0 \alpha_n \theta^2.
\end{aligned} \tag{4.19}$$

Setting $\gamma = c_3\left(\Phi(x_0,\rho,\nu) + 4\delta^{-2}\rho^{2\beta} H^2(|u_{x_0,\rho}| + |\nu|)\right)^{1/2}$ and noting that the mean-value of $u - (\nu + \gamma Dh(x_0))(x-x_0)$ on $B_{2\theta\rho}(x_0)$ is $u_{x_0,2\theta\rho}$, we have

$$(2\theta\rho)^{-n-2} \int\limits_{B_{2\theta\rho}(x_0)} |u - u_{x_0,2\theta\rho} - (\nu + \gamma Dh(x_0))(x-x_0)|^2 dx$$

$$\le (2\theta\rho)^{-n-2} \int\limits_{B_{2\theta\rho}(x_0)} |u - u_{x_0,\rho} - \nu(x-x_0) - \gamma(h(x_0) + Dh(x_0)(x-x_0))|^2 dx$$
$$= \gamma^2 (2\theta\rho)^{-n-2} \int\limits_{B_{2\theta\rho}(x_0)} |w - h(x_0) - Dh(x_0)(x-x_0)|^2 dx$$
$$\le c_3^2 \left(2^{-n-1}\theta^{-n-2}\varepsilon + 8c_0\alpha_n\theta^2\right) \left(\Phi(x_0,\rho,\nu) + 4\delta^{-2}\rho^{2\beta} H^2(|u_{x_0,\rho}| + |\nu|)\right) \qquad \text{(from (4.19))}$$
$$\le c_4(\theta^{-n-2}\varepsilon + \theta^2) \left(\Phi(x_0,\rho,\nu) + 4\delta^{-2}\rho^{2\beta} H^2(|u_{x_0,\rho}| + |\nu|)\right) \tag{4.20}$$

where $c_4 = (2^{-n-1} + 8\alpha_n c_0)c_3^2$ (depending on $n$, $N$, $\lambda$ and $L$). Note that $c_4 \ge 1$.

Applying Lemma 4.1 on $B_{2\theta\rho}(x_0)$ with $\xi = u_{x_0,2\theta\rho}$, and $\nu + \gamma Dh(x_0)$ in place of $\nu$ yields

$$\int\limits_{B_{\theta\rho}(x_0)} |Du - (\nu + \gamma Dh(x_0))|^2 dx$$
$$\le \frac{c_1}{(2\theta\rho)^2} \int\limits_{B_{2\theta\rho}(x_0)} |u - u_{x_0,2\theta\rho} - (\nu + \gamma Dh(x_0))(x-x_0)|^2 dx + F\,, \tag{4.21}$$

for

$$F = c_2\alpha_n (2\theta\rho)^{n+2\beta} \Big(K(|u_{x_0,2\theta\rho}| + |\nu + \gamma Dh(x_0)|)(1 + |\nu + \gamma Dh(x_0)|)\Big)^{\frac{2}{1-\beta}}$$
$$\le c_2\alpha_n (2\theta\rho)^{n+2\beta} H\left(|u_{x_0,2\theta\rho}| + |\nu + \gamma Dh(x_0)|\right) \tag{4.22}$$

after taking into account the definition of $H$. We note, using Theorem 2.4 and (4.17)

$$|\gamma Dh(x_0)| = c_3(\Phi(x_0,\rho,\nu) + 4\delta^{-2}\rho^{2\beta} H^2(|u_{x_0,\rho}| + |\nu|))^{1/2} |Dh(x_0)|$$
$$\le c_3\sqrt{c_0}(\Phi(x_0,\rho,\nu) + 4\delta^{-2}\rho^{2\beta} H^2(|u_{x_0,\rho}| + |\nu|))^{1/2}\,. \tag{4.23}$$

Further we have

$$|u_{x_0,2\theta\rho}| \le |u_{x_0,\rho}| + \Big|\fint_{B_{2\theta\rho}(x_0)} (u - u_{x_0,\rho} - \nu(x-x_0))dx\Big|$$
$$\le |u_{x_0,\rho}| + \Big(\fint_{B_{2\theta\rho}(x_0)} |u - u_{x_0,\rho} - \nu(x-x_0)|^2 dx\Big)^{1/2}$$
$$\le |u_{x_0,\rho}| + (2\theta)^{-n/2}\Big(\fint_{B_{\rho}(x_0)} |u - u_{x_0,\rho} - \nu(x-x_0)|^2 dx\Big)^{1/2}$$
$$\le |u_{x_0,\rho}| + \frac{\sqrt{c_p}}{(2\theta)^{n/2}} \rho \Phi^{1/2}(x_0,\rho,\nu)$$
$$\le |u_{x_0,\rho}| + \frac{\sqrt{c_p}}{(2\theta)^{n/2}} \rho \left(\Phi(x_0,\rho,\nu) + 4\delta^{-2}\rho^{2\beta} H^2(|u_{x_0,\rho}| + |\nu|)\right)^{1/2}\,, \tag{4.24}$$

the second-last inequality following from Poincaré's inequality (note that $u - u_{x_0,\rho} - \nu(x-x_0)$ has mean-value 0 on $B_\rho(x_0)$). Thus, noting $\rho \le 1$, we combine (4.23) and (4.24) to obtain

$$|u_{x_0,2\theta\rho}| + |\nu + \gamma Dh(x_0)|$$
$$\le |u_{x_0,\rho}| + |\nu| + \left(c_3\sqrt{c_0} + \frac{\sqrt{c_p}}{(2\theta)^{n/2}}\right) \left(\Phi(x_0,\rho,\nu) + 4\delta^{-2}\rho^{2\beta} H^2(|u_{x_0,\rho}| + |\nu|)\right)^{1/2}\,.$$

Assume that we have

$$\left(c_3\sqrt{c_0}+\frac{\sqrt{c_p}}{(2\theta)^{n/2}}\right)\Phi^{1/2}(x_0,\rho,\nu)\le \tfrac{1}{2} \tag{4.25}$$

and

$$2\left(c_3\sqrt{c_0}+\frac{\sqrt{c_p}}{(2\theta)^{n/2}}\right)\rho^{\beta}H(|u_{x_0}|+|\nu|)\delta^{-1}\le \tfrac{1}{2}\,. \tag{4.26}$$

Then we see from (4.22), recalling also that $H\ge 1$,

$$c_2\alpha_n(2\theta\rho)^{n+2\beta}H\left(1+|u_{x_0,\rho}|+|\nu|\right)\le c_5\alpha_n(\theta\rho)^n\theta^{2\beta}\rho^{2\beta}H^2\left(1+|u_{x_0,\rho}|+|\nu|\right) \tag{4.27}$$

for $c_5=\max\{2^{n+2}c_2,1\}$ depending on $n$, $\lambda$ and $\beta$.

We have then from (4.20),(4.21), and (4.27), assuming that (4.25) and (4.26) hold (and using Lemma 4.1 with $\xi=u_{x_0,2\theta\rho}$),

$$\begin{aligned}
&\Phi(x_0,\theta\rho,(Du)_{x_0,\theta\rho})\\
&\le \alpha_n^{-1}(\theta\rho)^{-n}\int_{B_{\theta\rho}(x_0)}|Du-(\nu+\gamma Dh(x_0))|^2dx\\
&\le \alpha_n^{-1}(\theta\rho)^{-n}\left(\frac{c_1}{(2\theta\rho)^2}\int_{B_{2\theta\rho}(x_0)}|u-u_{x_0,2\theta\rho}-(\nu+\gamma Dh(x_0))(x-x_0)|^2dx+F\right)\\
&\le 2^nc_1c_4\alpha_n^{-1}\left(\theta^{-n-2}\varepsilon+\theta^2\right)\left(\Phi(x_0,\rho,\nu)+4\delta^{-2}\rho^{2\beta}H^2(|u_{x_0,\rho}|+|\nu|)\right)\\
&\qquad\qquad +c_5\rho^{2\beta}\theta^{2\beta}H^2\left(1+|u_{x_0,\rho}|+|\nu|\right)
\end{aligned} \tag{4.28}$$

We now set $c_6=2^{n+1}c_1c_4\alpha_n^{-1}\ge 1$ (depending on $n$, $N$, $\lambda$, $L$ and $\beta$), and we then fix $\theta\in(0,1/4]$ such that $c_6\theta^2\le\frac{1}{2}\theta^{2\beta}$. We then set $\varepsilon=\theta^{n+4}$, which fixes $\delta\in(0,1]$. Note that $\theta$, $\varepsilon$ and $\delta$ depend on the same parameters as $c_6$.

Taking $\nu=(Du)_{x_0,\rho}$ in (4.28) and writing $\Phi(z,R)$ for $\Phi(z,R,(Du)_{z,R})$ we have

$$\begin{aligned}
\Phi(x_0,\theta\rho)&\le c_6\theta^2\Phi(x_0,\rho)+(2c_6\delta^{-2}+c_5)\rho^{2\beta}\theta^{2\beta}H^2\left(1+|u_{x_0,\rho}|+|\nu|\right)\\
&\le \tfrac{1}{2}\theta^{2\beta}\Phi(x_0,\rho)+c_7\rho^{2\beta}\theta^{2\beta}H^2\left(1+|u_{x_0,\rho}|+|\nu|\right)\ ,
\end{aligned} \tag{4.29}$$

noting that $c_6\theta^2\le\frac{1}{2}\theta^{2\beta}$, and setting $c_7=2c_6\delta^{-2}+c_5$ (with $c_7\ge 1$, depending on $n$, $N$, $\lambda$, $L$ and $\beta$), as long as the smallness conditions (cf. (4.15), (4.25), (4.26))

$$\omega(|u_{x_0,\rho}|+|(Du)_{x_0,\rho}|,\Phi(x_0,\rho))\le\tfrac{\delta^2}{16}\,, \tag{4.30}$$

$$\Phi(x_0,\rho)\le\min\{\tfrac{\delta^2}{16}\,,\,\tfrac{1}{4c_8^2}\} \tag{4.31}$$

and

$$2c_8\rho^{\beta}H(1+|u_{x_0,\rho}|+|(Du)_{x_0,\rho}|)\le\tfrac{\delta}{2} \tag{4.32}$$

are satisfied; here $c_8=c_3\sqrt{c_0}+\frac{\sqrt{c_p}}{(2\theta)^{n/2}}$ (with the same dependancies as $c_7$).

Now for a fixed $M_1 > 0$, we choose $t_0$ positive (depending on $n$, $N$, $\lambda$, $L$, $\beta$, $M_1$ and $\omega(\cdot)$) such that

$$\omega(M_1, t_0) \le \frac{\delta^2}{16} \qquad \text{and} \qquad t_0 \le \min\{\frac{\delta^2}{16}, \frac{1}{4c_8^2}, \frac{M_1^2(1-\theta^\beta)^2}{4(1+\sqrt{c_p})^2\theta^n}\} \tag{4.33}$$

We now set $H_0 = H(1+M_1)$, and choose $\rho_0 > 0$ (depending on the same quantities as $t_0$, and additionally on $K(\cdot)$) such that

$$\rho_0^{2\beta} < \frac{t_0}{(4c_7 + c_8^2)H_0^2};$$

note that this choice ensures $2c_8\rho_0^\beta H_0 < \frac{\delta}{2}$ and $2c_7\rho_0^{2\beta}H_0^2 < \frac{t_0}{2}$.

Assume that we have, for some $\rho \in (0, \rho_0]$,

$$|u_{x_0,\rho}| + |(Du)_{x_0,\rho}| < \tfrac{1}{2}M_1 \quad \text{and} \quad \Phi(x_0,\rho) < \tfrac{1}{2}t_0\,. \tag{4.34}$$

Then (4.30), (4.31)and (4.32) are satisfied, and so we can conclude from (4.29)

$$\Phi(x_0, \theta\rho) \le \theta^{2\beta}\left(\tfrac{1}{2}\Phi(x_0,\rho) + c_7\rho^{2\beta}H_0^2\right).$$

We can iterate this procedure if we can ensure, for every $j \in \mathbb{N}$, that

$$\Phi(x_0, \theta^j\rho) < t_0 \quad \text{and} \quad |u_{x_0,\theta^j\rho}| + |(Du)_{x_0,\theta^j\rho}| < M_1\,. \tag{4.35}$$

Then we would have (4.30), (4.31)and (4.32) with $\rho$ replaced by $\theta^j\rho$, and hence could conclude

$$\Phi(x_0, \theta^{j+1}\rho) \le \theta^{2\beta}\left(\tfrac{1}{2}\Phi(x_0,\theta^j\rho) + c_7(\theta^j\rho)^{2\beta}H_0^2\right). \tag{4.36}$$

We will establish (4.35) by induction. We suppose that (4.35) is valid for $0, \ldots, j-1$. Then

$$\begin{aligned} \Phi(x_0, \theta^j\rho) &\le \frac{\theta^{2j\beta}}{2^j}\Phi(x_0,\rho) + c_7H_0^2\rho^{2\beta}\sum_{\ell=1}^{j}\theta^{2\beta j}2^{1-\ell} \\ &\le \theta^{2j\beta}\left(2^{-j}\Phi(x_0,\rho) + 2c_7H_0^2\rho_0^{2\beta}\right) \\ &\le t_0\theta^{2j\beta}, \end{aligned} \tag{4.37}$$

by the choice of $\rho_0$. We further calculate

$$\begin{aligned} &|u_{x_0,\theta^j\rho}| + |(Du)_{x_0,\theta^j\rho}| \\ &\le |u_{x_0,\rho}| + |(Du)_{x_0,\rho}| + \frac{1+\sqrt{c_p}}{\theta^{n/2}}\sum_{\ell=0}^{j-1}\Phi^{1/2}(x_0,\theta^\ell\rho) \\ &\le \frac{M_1}{2} + \frac{1+\sqrt{c_p}}{\theta^{n/2}}\sum_{\ell=0}^{j-1}\theta^{\ell\beta}\sqrt{t_0} \\ &< \frac{M_1}{2} + \frac{1+\sqrt{c_p}}{\theta^{n/2}(1-\theta^\beta)}\sqrt{t_0} \\ &\le M_1, \end{aligned} \tag{4.38}$$

where we have used (4.37) and (4.34) in obtaining the second inequality, and (4.33) for the final inequality.

From (4.37) and (4.38) we see that we have established (4.35) for all $j \in \mathbb{N}$. As in [GM2, p. 127] (cf. the end of Section 3 of the current paper) this allows us to conclude the desired partial regularity result:

**4.2 Theorem.** *Let* $u \in H^{1,2}(\Omega, \mathbb{R}^N)$ *be a weak solution to (4.1) under the structure-conditions (H1)–(H3). Then* $\operatorname{Reg} u$ *is open in* $\Omega$*, and* $u \in C^{1,\beta}(\operatorname{Reg} u, \mathbb{R}^N)$*. Further* $\operatorname{Sing} u \subset \Sigma_1 \cup \Sigma_2$*, where*

$$\Sigma_1 = \{x \in \Omega \ : \ \liminf_{\rho \to 0^+} \fint_{B_\rho(x_0)} |Du - (Du)_{x_0,\rho}|^2 \, dx > 0\}, \quad \textit{and}$$

$$\Sigma_2 = \{x \in \Omega \ : \ \sup_{\rho > 0} \Big( |u_{x_0,\rho}| + |(Du)_{x_0,\rho}| \Big) = \infty \} \, ,$$

*and in particular,* $\mathcal{L}^n(\operatorname{Sing} u) = 0$*.*

As stated in the introduction, the fact that we obtain the optimal Hölder continuity $C^{1,\beta}$ on the regular set is new, cf. prior proofs such as [GM2, Theorem 1.1]. Note also that our method carries through for the case $\beta = 1$, in this case yielding $u \in C^{1,\alpha}(\operatorname{Reg} u, \mathbb{R}^N)$ for all $\alpha \in (0,1)$.

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