Cohomology of canonical projection tilings.

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Abstract

We define the cohomology of a tiling as cocycle cohomology of its associated groupoid and consider this cohomology for the class of tilings which are obtained from a higher dimensional lattice by the canonical projection method in Schlottmann's formulation. We relate it to the cohomology of this lattice and discuss one of its qualitative features: it provides a topological obstruction for a generic tiling to be substitutional. For tilings of codimension smaller or equal to 2 we present explicit formulae.

Introduction

Quasiperiodic tilings have become an active area of research in solid state physics due to their rôle in modeling quasicrystals [1, 2, 3, 4], and the projection method in its various formulations [5, 6, 7, 8] is one of the most common techniques to construct candidates for such tilings. This raises the question how can we charactarize tilings and possibly classify them? For that to be investigated one must first decide which properties of a tiling are essential for the physical properties of the solid. We take the point of view here that it is only the local structure of the tiling that matters, and even more, only its topological content. According to this point of view the tight binding model for particle motion in the tiling is not uniquely determined by the tiling but its form is constrained by the topology of the tiling, i.e. the Hamiltonian reflects the long range order of the tiling (but additional information is required to specify the interaction strengths etc.). Therefore we are looking for topological invariants of tilings, one of them being its cohomology which we define to be the cohomology of the tiling groupoid.

Without additional mathematical structure of the tiling it is not clear how to obtain explicit results for cohomology groups. Substitution tilings provide a class of tilings where such results can be obtained [9, 10], because they possess a symmetry which relates different scales. The present article is part of a programme to compute the tiling cohomology of another class, those which may be obtained by projection from higher dimensional lattices. We present quantitative results, but only for small codimension (i.e. small difference between the rank of the lattice and the dimension of the tiling), and discuss qualitative, namely sufficient conditions under which the cohomology is infinitely generated. As a matter of fact, these conditions are quite often met and since the cohomology of substitution tilings is finitely generated (when tensored with the rationals) we can conclude that projection method tilings are rarely substitutional. Unfortunately, we cannot offer yet an interpretation of the fact that some tilings produce only finitely many generators for their cohomology whereas others do not. But if understood, it could well lead to a criterion to single out a subset of tilings relevant for quasicrystal physics from the vast set of tilings which may be obtained from the canonical projection method. In this context we point out that no projection method tiling is known to us which has infinitely generated cohomology but allows for local matching rules, c.f. [11].

Apart from the classification problem there is another strong motivation to study tiling cohomology. Tilings obtained by the projection method belong to a large class of tilings for which it can be shown that their cohomology is isomorphic to (unordered) K-theory of the associated groupoid-C^*-algebra

[12]. This (non-commutative) aspect of the topology of tilings has a direct interpretation in physics. The above mentionned C^*-algebra is the algebra of observables for particles moving in the tiling and its ordered K_0-group (or its image on a tracial state) may serve to "count" (or label) the possible gaps in the spectrum of the Hamilton operator which describes its motion [13, 14, 15]. In this context it is even more challanging to find an interpretation of the generators of the K_0-group, in the case where there are infinitely many. At first sight, all but finitely many of them appear to be infinitesimal.

With the important exception of Section 6 most of this article parallels the first two articles of a little series [16, 17, 18] of which the last one will contain quantitative results for tilings of higher codimension. But the main difference is that we use here a description of the tilings (by Laguerre complexes, due to Schlottmann [19]) which, at the cost of generality (when it comes to acceptance domains of quite arbitrary shape), is a lot simpler when it comes to some of the technicalities. The article is organized as follows. We first describe the continuous dynamical system which can be assigned to any reasonable tiling (Section 1). Its associated transformation groupoid has orbits homeomorphic to the space in which the tiling is embedded. We derive here the tiling groupoid as a reduction of this groupoid (Section 2). It is an r-discrete groupoid and we define tiling cohomology to be the cohomology of this groupoid. Again, this can be done for arbitrary tilings but one of the main features of projection method tilings which make a computation of the cohomology feasable is that one can find a $\mathbb{Z}^d$ Cantor dynamical system whose associated transformation groupoid is continuously similar to the tiling groupoid (Section 3). This has as a consequence that the tiling cohomology may be formulated as group cohomology of the group $\mathbb{Z}^d$. It parallels work of Bellissard etal. [20] on the K-theoretic level. After two illustrating examples we review the qualitative results on tiling cohomology that were obtained in [17] (Section 5). In Section 6 we present the calculation of the cohomology for tilings of small codimension. Finally we add a section on the non-commutative topological approach.

1 Continuous tiling dynamical systems

A tiling is a covering of $\mathbb{R}^d$ by closed subsets, called its tiles, which overlap at most at their boundaries and usually are subject to various other constraints, as e.g. being connected, bounded in size and closures of their interiors. They may even be decorated. For the purpose of this work, however, in which we focus attention on canonical projection tilings, it is sufficient to consider tiles which are (possibly decorated) polytopes (with non-empty interior) which touch face to face. Moreover, we require that the tilings are of finite (pattern) type, a notion which we explain below.

Given a tiling $\mathcal{T}$, $\mathbb{R}^d$ acts naturally on it by translation, we denote the tiling translated by x as $\mathcal{T} - x$, and the closure of the orbit $\mathcal{T} - \mathbb{R}^d$ of $\mathcal{T}$ with respect to an appropriate metric gives rise to a dynamical system [21]. There are several proposals for such a metric on spaces of tilings which all are based on comparing patches around the origin of $\mathbb{R}^d$. This may be done as follows: represent a tiling $\mathcal{T}$ as a closed subset of $\mathbb{R}^d$ by the boundaries of its tiles and its decorations by small compact sets, let B_r be the open ball of radius r around $0 \in \mathbb{R}^d$ and $B_r(\mathcal{T}) := (B_r \cap \mathcal{T}) \cup \partial B_r$, a closed set. Two tilings, $\mathcal{T}$ and $\mathcal{T}'$ should be close to each other if $B_r(\mathcal{T})$ and $B_r(\mathcal{T}')$ coincide possibly up to a small discrepancy for large r. The different ways to quantify the allowed discrepancy lead to the different spaces which may be found in the literature. One option is to not allow any discrepancy,

$$D_0(\mathcal{T}, \mathcal{T}') := \inf\{\frac{1}{r+1} | B_r(\mathcal{T}) = B_r(\mathcal{T}')\}.$$

The closure of the orbit of $\mathcal{T}$ under $\mathbb{R}^d$ would then always be a non compact space. If one looks instead at

$$D(\mathcal{T}, \mathcal{T}') = \inf\{\frac{1}{r+1} | d_r(B_r(\mathcal{T}), B_r(\mathcal{T}')) < \frac{1}{r}\},$$

where d_r is the Hausdorff metric defined among closed subsets of the closed r-ball, then completion of the orbit with respect to this metric yields a compact space under very general conditions [21, 22]. Note that D is not invariant under the action of $\mathbb{R}^d$ by translation, but the action is continuous and can thus be extended to the completion.

Definition 1 *The continuous dynamical system associated to $\mathcal{T}$ is $(M\mathcal{T}, \mathbb{R}^d)$, the closure $M\mathcal{T}$ of the orbit of $\mathcal{T}$ with respect to the metric D, with the action of $\mathbb{R}^d$ induced by translation. We call $M\mathcal{T}$ the continuous hull of $\mathcal{T}$.*

Let $M_r(\mathcal{T})$ be the subset of tiles of $\mathcal{T}$ which are contained in the closure of B_r. Like for $\mathcal{T}$ we may think of $M_r(\mathcal{T})$ as the closed subset defined by the boundaries and decorations of its tiles. A tiling $\mathcal{T}$ is called of finite type (or of finite pattern type, or of finite local complexity) if for any r the set of translational congruence classes of sets $M_r(\mathcal{T} - x)$, $x \in \mathbb{R}^d$, is finite.

The elements of the space $M\mathcal{T}$ may again be interpreted as tilings. If $\mathcal{T}$ is of finite type these elements are those tilings in which each finite part can be identified with a finite part of a translate of $\mathcal{T}$. In other words: for all $T \in M\mathcal{T}$ and all r exists an $x \in \mathbb{R}^d$ such that $B_r(T) = B_r(\mathcal{T} - x)$. If, given

$T \in M\mathcal{T}$, for all r exists an $x \in \mathbb{R}^d$ such that $B_r(\mathcal{T}) = B_r(T - x)$ then T is called locally isomorphic to $\mathcal{T}$. If any element of $M\mathcal{T}$ is locally isomorphic to $\mathcal{T}$ then $\mathcal{T}$ is called minimal. This is the case for the tilings we are interested in here. It directly implies that each orbit of the dynamical system is dense.

We say that a finite subset P of tiles of a tiling T is a patch (or pattern, or cluster) of it and write $P \subset T$. Then we define

$$\mathcal{U}_P := \{T \in M\mathcal{T} | P \subset T\},$$

subsets of the continuous hull which will play a major role in what follows.

We mention a third option for a metric on the orbit of $\mathcal{T}$. The metric given in [9] defines the same topology as

$$D_t(\mathcal{T}, \mathcal{T}') := \inf\{\frac{1}{r+1} | \exists x, x' \in B_{\frac{1}{2r}} : B_r(\mathcal{T} - x) = B_r(\mathcal{T}' - x')\},$$

i.e. discrepancy is allowed only for small translations. As soon as here two tilings differ by a rotation however small it is they will have a certain minimal non zero distance. Because of this, closure w.r.t. D_t leads, for instance, for the Pinwheel tilings to a non-compact space whereas closure w.r.t. D would still lead to a compact space. Which kind of metric is to be used has, of course, to be adapted to the problem, but for our purposes the distinction between the latter two metrics is inessential:

Theorem 1 *Let $\mathcal{T}$ be a finite type tiling. Then $M\mathcal{T}$ is compact and equal to the completion of $\mathcal{T} - \mathbb{R}^d$ w.r.t. D_t. Furthermore, the collection of sets $\{B_\epsilon + x + \mathcal{U}_P\}$, $\epsilon > 0$, $x \in \mathbb{R}^d$, P a patch of $\mathcal{T}$, is a base for the topology of $M\mathcal{T}$.*

Proof: We start by showing that the two metrics D and D_t yield the same completion for finite type tilings. Clearly $D(T, T') \leq D_t(T, T')$ so we have to show that any D-Cauchy sequence is also a D_t-Cauchy sequence. In fact, if $D(\mathcal{T}, \mathcal{T}') < \frac{1}{R+1}$ then $d_r(B_r(\mathcal{T}), B_r(\mathcal{T}')) < \frac{1}{R}$ for all $r \leq R$, and since there are only finitely many translational congruence classes of the form $M_r(\mathcal{T} - x)$ we find for each r an ϵ such that $d_r(B_r(\mathcal{T}), B_r(\mathcal{T}')) < \epsilon$ implies $\exists x, x' \in B_\epsilon$: $B_r(\mathcal{T} - x) = B_r(\mathcal{T}' - x')$. This implies that D-Cauchy-sequences are also D_t-Cauchy-sequences. In particular $M\mathcal{T}$ is equal to the completion of $\mathcal{T} - \mathbb{R}^d$ w.r.t. D_t. Its compactness for finite type tilings is well known, see e.g. [22].

Let $r(\epsilon) := \frac{1-\epsilon}{\epsilon}$ and $\mathcal{V}_r(T) = \{T' \in M\mathcal{T} | B_r(T) = B_r(T')\}$. Then we can describe the ϵ-neighbourhoods of T w.r.t. D_t as follows

$$\begin{aligned} D_t(T, T') < \epsilon \quad & \text{iff} \quad \exists r > r(\epsilon) \exists x, x' \in B_{\frac{1}{2r}} : B_r(T - x) = B_r(T' - x') \\ & \text{iff} \quad T' \in \bigcup_{r > r(\epsilon)} \bigcup_{x \in B(\frac{1}{2r})} \left(B(\frac{1}{2r}) + \mathcal{V}_r(T - x) \right). \end{aligned} \tag{1}$$

The tiling being of finite type implies that, for every $r > 0$, $T \in M\mathcal{T}$ exists a finite set of pairs (x_i, P_i), $x_i \in \mathbb{R}^d$, P_i a pattern of $\mathcal{T}$, such that $B_r(T') = B_r(T)$ whenever $\exists i : P_i + x_i$ is a pattern of T'. In other words, $\mathcal{V}_r(T) = \bigcup_i \mathcal{U}_{P_i+x_i}$. This shows that (1) is a union of sets of the above collection.

To show that $B_\epsilon + \mathcal{U}_P$ is open in the metric topology (which by continuity of the action implies that also $B_\epsilon + x + \mathcal{U}_P$ is open for $x \in \mathbb{R}^d$) we take a point T in it and show that a whole neighbourhood (w.r.t. D_t) of it lies in $B_\epsilon + \mathcal{U}_P$. Let R be large enough so that $\frac{1}{R} < \epsilon$ and P is a patch of $B_{R-\frac{1}{2R}}(T)$ (we view here P as a closed subset much like a tiling). Then, for all $x \in B_{\frac{1}{2R}}$, $P \subset B_R(T-x) + x$ and hence $\mathcal{V}_R(T-x) \subset \mathcal{U}_P - x$. This implies that the $\frac{1}{R+1}$-neighbourhood of T lies in $B_\epsilon + \mathcal{U}_P$. q.e.d.

Lemma 1 *Let $\mathcal{T}$ be a finite type tiling. Then $\mathcal{U}_P$ is compact.*

Proof: If $D(T, T')$ is small enough, and $T, T' \in \mathcal{U}_P$ then it is equal to $D_0(T, T')$. That $\mathcal{U}_P$ is complete and precompact w.r.t. the D_0-metric is proven in [15]. q.e.d.

2 The groupoid approach to tilings

To a given tiling one may associate an r-discrete groupoid, the tiling groupoid. This groupoid is special among other groupoids which may be assigned to the tiling in that its C^*-algebra plays the role of the algebra of observables for particles moving in the tiling [15, 10]. It determines the tiling up to topological equivalence [23]. Before we describe it we briefly recall some facts about groupoids.

2.1 Generalities

For a traditional definition of a topological groupoid and as a general reference for most of the concepts introduced below like that of reduction, continuous similarity and continuous cocycle cohomology we refer the reader to [24].

In a slightly different but equivalent way one may say that a groupoid $\mathcal{G}$ is a set with partial, associative, cancellative multiplication and unique inverses. Multiplication being partial refers to the fact that it is not for all elements defined, but only for a subset of $\mathcal{G} \times \mathcal{G}$ (the composable elements). An inverse of x is a solution y of the equations $xyx = x$ and $yxy = y$, and for a groupoid this solution is required to be unique. Hence we may denote the inverse of x by x^{-1}. The inverse map $x \mapsto x^{-1}$ turns out to be an involution. Multiplication is cancellative if, provided it is defined, $xy = xz$ implies $y = z$, and this is the case whenever the composable elements are the pairs (x, y) for which $x^{-1}x = yy^{-1}$. The set $\mathcal{G}^0 = \{xx^{-1} | x \in \mathcal{G}\}$ is called the set of units, it is the image of the map $r : \mathcal{G} \to \mathcal{G}^0$, $r(x) = xx^{-1}$, which is called the range map. The map $s : \mathcal{G} \to \mathcal{G}^0$, $s(x) = x^{-1}x = r(x^{-1})$ is called the source map. On the set of units, $u \sim v$ whenever $r^{-1}(u) \cap s^{-1}(v) \neq \emptyset$ defines an equivalence relation. Its equivalence classes are called the orbits of $\mathcal{G}$.

A topological groupoid is a groupoid with a topology with respect to which multiplication and inversion are continuous maps. Such a groupoid is called r-discrete if $\mathcal{G}^0$ is an open subset, this implies that $r^{-1}(u)$ is a discrete set for any unit u.

A groupoid is called principal, if its elements are uniquely determined by their range and source, i.e. if the map $\mathcal{G} \to \mathcal{G}^0 \times \mathcal{G}^0 : x \mapsto (r(x), s(x))$ is injective.

2.1.1 Transformation groupoids

Let M be a topological space with a right action of a topological group G by homeomorphisms, denoted here $(x, g) \mapsto x \cdot g$. The transformation groupoid[1] $\mathcal{G}(M, G)$ is the topological space $M \times G$ with product topology, two elements (x, g) and (x', g') are composable provided that $x' = x \cdot g$, and their product is then $(x, g)(x', g') = (x, gg')$. Inversion is then given by $(x, g)^{-1} = (x \cdot g, g^{-1})$. Hence, $r(x, g) = (x, 0)$ and we see that $\mathcal{G}(M, G)$ is r-discrete if G is discrete. Furthermore, $\mathcal{G}(M, G)$ is principal whenever G acts fixpoint freely. One of the examples we have in mind here is $\mathcal{G}(M\mathcal{T}, \mathbb{R}^d)$ which, however, is not r-discrete.

2.1.2 Reductions

Let $\mathcal{G}$ be a groupoid, $\mathcal{G}^0$ its unit space and L be a closed subset of $\mathcal{G}^0$. Then ${}_L\mathcal{G}_L := s^{-1}(L) \cap r^{-1}(L)$ is a closed subgroupoid of $\mathcal{G}$ called the reduction of $\mathcal{G}$ to L. Two further conditions on L will play a major role here. First, that every orbit of $\mathcal{G}$ has a non-emtpy intersection with L – such a reduction is called regular – and second, a topological condition, that L is range-open [16]. L is range-open if for all open $U \subset \mathcal{G}$ the set $r(s^{-1}(L) \cap U)$ is open.

A regular reduction of a groupoid $\mathcal{G}$ to a range-open subset L is for many purposes as good as the groupoid itself. Muhly etal. have established a notion of equivalence between groupoids which captures this phenomenon in greater generality [25]. We will not discuss this notion of equivalence here

[1] or transformation group as in [24]

but point out its consequences, namely that the main topological invariants of the corresponding groupoid-C^*-algebras are isomorphic.

2.1.3 Continuous similarity

As we have mentioned above, the concept of reduction is particularly well adapted to yield an equivalence relation on groupoids which carries over to an equivalence relation on the C^*-algebras they define. It turns out that for projection method tilings the K-groups of the C^*-algebras are related to the cohomology of the groupoids, see Sect. 7, but this relation is not clear on the level of arbitrary tiling-groupoids. On the other hand there is a natural equivalence relation on groupoids (coming from viewing these as functorial objects) which immediately gives rise to an equality on cohomology groups as well as implies equivalence in the sense of Muhly etal. [26]: that of continuously similar groupoids.

Definition 2 *Two homomorphisms $\phi, \psi : \mathcal{G} \to \mathcal{R}$ between (topological) groupoids are (continuously) similar if there exists a function $\Theta : \mathcal{G}^0 \to \mathcal{R}$ such that*

$$\Theta(r(x))\phi(x) = \psi(x)\Theta(s(x)). \tag{2}$$

Two (topological) groupoids, $\mathcal{G}$ and $\mathcal{R}$, are called (continuously) similar if there exist homomorphisms $\phi : \mathcal{G} \to \mathcal{R}$, $\phi' : \mathcal{R} \to \mathcal{G}$ such that $\Phi_{\mathcal{G}} = \phi' \circ \phi$ is (continuously) similar to $\mathrm{id}_{\mathcal{G}}$ and $\Phi_{\mathcal{R}} = \phi \circ \phi'$ is (continuously) similar to $\mathrm{id}_{\mathcal{R}}$.

We are mainly interested in establishing continuous similarity of certain principal transformation groupoids. A useful lemma to test this is proved in [17](3.3,3.4):

Proposition 1 *Let $\mathcal{G} = \mathcal{G}(X, G)$ be a principal transformation groupoid (i.e. G acts freely on X). Suppose L is a closed subset of $\mathcal{G}^0 \cong X$ and $\gamma : X \to G$ a continuous function such that $x \cdot \gamma(x) \in L$ for all $x \in X$. Then the reduction of $\mathcal{G}$ to L is continuously similar to $\mathcal{G}$.*

2.1.4 Continuous cocycle cohomology

Given a dynamical system (M, G) one standard topological invariant associated with it is the cohomology of G with coefficients in the G-module $C(M, \mathbb{Z})$ of integer-valued continuous functions on which G acts as $(g \cdot f)(m) = f(m \cdot g)$. This cohomology may be interpreted as a groupoid cohomology, namely of the groupoid $\mathcal{G}(M, G)$. It is continuous cocycle cohomology of r-discrete groupoids and we will recall its definition here for constant coefficients following [24].

Let A be an abelian group and $\mathcal{G}$ be a groupoid. $\mathcal{G}$ acts on the trivial A-bundle $\mathcal{G}^0 \times A \xrightarrow{\rho} \mathcal{G}^0$ (with product topology) partially, namely $x \in \mathcal{G}$ can act only on elements of the form $(s(x), a)$ mapping them to $(r(x), a)$. We denote this action by Φ, i.e. the partial map given by $x \in \mathcal{G}$ is $\Phi(x)$. The action is continuous in the sense that when $f \in C(\mathcal{G}^0, A)$ is a continuous section of the bundle then the function $x \mapsto (r(x), f(s(x)))$ is continuous too.

Let $\mathcal{G}^{(0)} = \mathcal{G}^0$, and, for $n > 0$, $\mathcal{G}^{(n)}$ be the subset of the n-fold Cartesian product of $\mathcal{G}$ (with relative topology) consisting of composable elements $(x_1, \ldots, x_n)$, i.e. $r(x_i) = s(x_{i-1})$. n-cochains are continuous functions $f : \mathcal{G}^{(n)} \to \mathcal{G}^0 \times A$ such that $\rho(f(x_1, \ldots, x_n)) = r(x_1)$ and, for $n > 0$, $f(x_1, \ldots, x_n) = (r(x_1), 0)$ provided one of the x_i is a unit (0 is the neutral element of A which we denote additively). The n-cochains form an abelian group under pointwise addition. The coboundary operator δ^n is defined as

$$\delta^0(f)(x) = \Phi(x) f(s(x)) - f(r(x)),$$

and, for $n > 0$,

$$\begin{aligned}
\delta^n(f)(x_0, \ldots, x_n) &= \Phi(x_0)(f(x_1, \ldots, x_n)) \\
&\quad + \sum_{i=1}^{n} (-1)^i f(x_0, \ldots, x_{i-1}x_i, \cdots, x_n) \\
&\quad + (-1)^{n+1} f(x_0, \ldots, x_{n-1}).
\end{aligned}$$

$H^n(\mathcal{G}, A) := \ker \delta^n / \mathrm{im} \delta^{n-1}$ is the n's degree continuous cocycle cohomology group with (constant) coefficients in A.

Theorem 2 *Continuously similar groupoids have isomorphic cohomology with constant coefficients.*

The proof is given in [24], the maps ϕ and ϕ' which establish the similarity inducing the cochain-homotopies.

Let us consider a transformation groupoid $\mathcal{G}(M, G)$ as an example. In that case, n-cochains are maps $f : M \times G^n \to M \times A$ which are of the form

$$f(m, g_1, \ldots, g_n) = (m, \tilde{f}(g_1, \ldots, g_n)(m))$$

where $\tilde{f} : G^n \to C(M, A)$ is a continuous map which, for $n > 0$, is the zero map when applied to $(g_1, \ldots, g_n)$ with one $g_i = e$. These are precisely n-cochains of the group G with coefficients in $C(M, A)$, a module of G w.r.t.

the action $(g \cdot f)(m) = f(m \cdot g)$ [27]. Hence every n-cochain of the groupoid with coefficients in A determines an n-cochain of the group G with coefficients in $C(M, A)$ and vice versa. Moreover, under this identification δ^n becomes the usual coboundary operator of group cohomology, because the groupoid action is nothing else than the shift of base point given by the action of G. Thus

$$H^n(\mathcal{G}(M,G), A) \cong H^n(G, C(M,A)),$$

the cohomology of the group G with coefficients in $C(M, A)$. In the following we shall be interested in the cases $A = \mathbb{Z}$ and $A = \mathbb{Q}$.

2.2 The tiling groupoid

The tiling groupoid may be defined without refering to continuous tiling dynamical systems, as e.g. in [15, 10], but for the purpose of the present work it is important to draw the connection which has first been realized by [9]. Starting with the groupoid of the continuous tiling dynamical system $\mathcal{G}(M\mathcal{T}, \mathbb{R}^d)$ we construct the tiling groupoid as a reduction of it.

Construct a closed range-open subset $\Omega_\mathcal{T}$ of $M\mathcal{T}$ as follows: Choose a point in the interior of each tile of $\mathcal{T}$ – called its puncture – in such a way that translationally congruent tiles have their puncture at the same position. Let $\Omega_\mathcal{T}$ be the subset of tilings of $M\mathcal{T}$ for which a puncture of one of its tiles coincides with the origin $0 \in \mathbb{R}^d$.

Definition 3 *The tiling groupoid of $\mathcal{T}$, denoted by $\mathcal{G}_\mathcal{T}$ is the reduction of $\mathcal{G}(M\mathcal{T}, \mathbb{R}^d)$ to $\Omega_\mathcal{T}$.*

Note that $\Omega_\mathcal{T}$ intersects each orbit of $\mathbb{R}^d$. Let us sketch why $\Omega_\mathcal{T}$ is closed and range-open and why $\mathcal{G}_\mathcal{T}$ coincides with the groupoid $\mathcal{R}$ defined in [15] provided $\Omega_\mathcal{T}$ contains only non-periodic finite type tilings[2]. Under the latter condition $\mathbb{R}^d$ acts fixed point freely on $M\mathcal{T}$ and hence $\mathcal{G}_\mathcal{T}$ is principal. Therefore the map between $\mathcal{G}_\mathcal{T}$ and $\mathcal{R}$ is given by $(T, x) \mapsto (T, T - x)$, which certainly preserves multiplication and inversion, is an isomorphism provided it preserves the topology. The tiling being of finite type implies that punctures of two different tiles have a minimal distance, let's say δ. Thus there exists an ϵ (which is roughly as large as δ) such that if $D(\mathcal{T} - x, \mathcal{T} - x') < \epsilon$ and $\mathcal{T} - x, \mathcal{T} - x' \in \Omega_\mathcal{T}$ then $D(\mathcal{T} - x, \mathcal{T} - x') = D_0(\mathcal{T} - x, \mathcal{T} - x')$. It follows that $\Omega_\mathcal{T}$ is the metric completion w.r.t. D_0 of the set of all $\mathcal{T}' \in \Omega_\mathcal{T}$ which are translates of $\mathcal{T}$. In particular, it is closed and the existence of

[2] $\mathcal{G}_\mathcal{T}$ coincides for mainly the same reason with $\mathcal{R}(\mathcal{M}_{II})$ of [10], even for arbitrary finite-type tilings.

a minimal distance δ between punctures directly implies range-openess, c.f. [16]. Furthermore, the metric D_0 and the metric used in [15] to define the hull lead obviously to the same completions. This shows that the above map $(T,x) \mapsto (T, T-x)$ restricts to a homeomorphism of the spaces of units of $\mathcal{G}_\mathcal{T}$ and of $\mathcal{R}$. By construction $\mathcal{G}_\mathcal{T}$ is r-discrete and its topology is generated by the sets $U \times \{x\}$, U open in $\Omega_\mathcal{T}$. Images of those sets under the above map generate the topology of $\mathcal{R}$.

Definition 4 *The cohomology of $\mathcal{T}$, denoted by $H(\mathcal{T})$, is the continuous cocycle cohomology $H(\mathcal{G}_\mathcal{T}, \mathbb{Z})$ of $\mathcal{G}_\mathcal{T}$.*

We will see later on that for canonical projection tilings, $H(\mathcal{G}_\mathcal{T}, \mathbb{Z})$ is isomorphic to the Czech cohomology of $M\mathcal{T}$. It seems to be an interesting question whether this is true in general.

3 Quasiperiodic tilings obtained by cut and projection

The (cut and) projection method is a well known method to produce quasiperiodic point sets or tilings by projection of a certain subset of a periodic set in a higher dimensional space.

In earlier versions, e.g. [5], the favorite set was the integer lattice $\mathbb{Z}^N$ but a price for the simplicity of this choice has to be paid later if the kernel of the projection contains non-zero lattice points. An elegant way around this difficulty, which is applicable to almost all interesting examples, is to use root lattices instead of $\mathbb{Z}^N$ [28] and the construction we use here is related to that.

Rather than looking at arbitrary point sets obtained by the projection method (e.g. with fractal acceptance domain) we want to focus in this article on tilings where the acceptance domain is canonical – after all these include the main candidates for the description of quasicrystals – and for these tilings there is another apporach which is a bit more elaborated to start with but easier to handle when it comes to the later steps in the construction of the cohomology groups. Still we keep strong contact with the old fashioned projection method which we used in [16, 17]. The approach we are about to describe is based on polyhedral complexes and their dualization, it is therefore called dualization method. But in the present context where we start with a higher-dimensional periodic set it can be simply considered as a variant of the projection method. We follow in its description the article by Schlottmann [19] and refer the reader also to the examples discussed in [29].

Consider a point set W of a euclidian space $\mathcal{E}$ together with a weight function $w : W \to \mathbb{R}$ on it. For $q \in W$, the set

$$L_{W,w}(q) := \{x \in \mathcal{E} | \forall q' \in W : |x - q|^2 - w(q) \leq |x - q'|^2 - w(q')\} \tag{3}$$

is compact and convex and called the Laguerre-domain of q. Under rather weak conditions on (W, w) all Laguerre-domains are actually polytopes (of dimension smaller or equal to that of $\mathcal{E}$ or even empty sets) and the set of all Laguerre-domains with non-empty interior provide the tiles of a tiling $T(W, w)$ which is of finite type and face to face. Laguerre-domains are a generalization of Voronoi domains which one obtains if the weight function is constant. The construction of Voronoi domains is a familiar one in solid state physics where they arise (though under the name Brouillon-zone or Wigner-Seitz cell) if one takes as W the dual of the crystal lattice. A non-constant weight function gives the means to enlarge certain Laguerre-domains (larger $w(q)$) at the cost of others or even to surpress some.

The faces of the Laguerre-domains define a cell complex structure: this is the so-called Laguerre complex. We denote it by $\mathcal{L}_{W,w}$ and the (closed) cells of dimension k by $\mathcal{L}^{(}_{W,w}k)$. As a cell complex it has a dual, namely the dual ξ^* of a k-cell ξ is the convex hull of the set of $q \in W$ whose corresponding Laguerre-domains contain ξ as a face (ξ^* has codimension k). It is a nice exercise to see that this dual cell complex is again a Laguerre complex, namely $\mathcal{L}_{W^*,w^*}$ where W^* is the set of vertices (0-cells) of $\mathcal{L}_{W,w}$ and $w^* : W^* \to \mathbb{R}$ is given by $w^*(q^*) = |q^* - q|^2 - w(q)$ for some q such that q^* is a vertex of $L_{W,w}(q)$. In particular, also (W^*, w^*) defines a tiling with the above properties.

To come to the projection method we let $\Gamma \in \mathcal{E}$ be a lattice whose generators form a base for $\mathcal{E}$, W be a finite union of Γ-orbits, and w be a Γ-periodic function. Now let $E \subset \mathcal{E}$ be a linear subspace and $\pi : \mathcal{E} \to E$ be the orthogonal projection. We write d for the dimension of E, $d^\perp$ for that of its orthocomplement $E^\perp$, and $\pi^\perp$ for $1 - \pi$. We also write shorter $x^\perp$ for $\pi^\perp(x)$. An element $u \in \mathcal{E}$ is called singular if there is a $\beta \in \mathcal{L}^{(d^\perp-1)}_{W,w}$ such that $\pi^\perp(u) \in \pi^\perp(\beta)$. Hence the set of singular points is $S = S^\perp + E$ where

$$S^\perp := \bigcup_{\beta \in \mathcal{L}^{(d^\perp-1)}_{W,w}} \pi^\perp(\beta).$$

The set of non-singular points is denoted by NS. We more conveniently now collect $\Theta = (W, w)$ and define $\Theta_u = (W_u, w_u)$ as follows $W_u = W + u$, $w_u(q + u) = w(q)$.

Definition 5 *The projection tiling defined by the data* (W, w, E, u) *(*$u \in NS$*) is the tiling* T_u *whose tiles are the elements of the set*

$$\{\pi(\xi^*) | \xi \in \mathcal{L}^{(d^\perp)}_{\Theta_u}, \xi \cap E \neq \emptyset\}.$$

(We surpressed the dependence on W, w, E because that on u is the important one in what follows.) That this is actually a tiling by Laguerre-domains has been shown by Schlottmann [19]. In fact, T_u is the tiling $T(\tilde{W}_u^*, \tilde{w}_u^*)$ defined by the Laguerre-complex dual to $\mathcal{L}_{(\tilde{W}_u, \tilde{w}_u)}$ where $\tilde{W}_u = \pi(W_u)$ and $\tilde{w}_u(\pi(q+u)) = \max\{w(q') - |\pi^\perp(q'+u)|^2 | \pi(q') = \pi(q)\}$ (assuming it exists). Using this description one can see that one loses no generality in restricting to the cases in which $\pi^\perp(\Gamma)$ lies dense in $E^\perp$, and we will do so here. This will save us a lot of extra work later on. For simplicity we will also require $E \cap \Gamma = 0$, which means that the tilings are (completely) non-periodic, and the following conditions:

H1 Up to translation, any $\xi^* \in \mathcal{L}_{\Theta^*}^{(d)}$ is uniquely determined by its projection $\pi(\xi^*)$.

H2 The maximal periodicity lattice of $\mathcal{L}_\Theta$ is Γ.

We will simply call a projection tiling constructed as in the definition satisfying these conditions (with dense $\pi^\perp(\Gamma)$) a canonical projection tiling (tacitly assuming non-periodicity).

Before we pass on let us quickly look at the example $W = \mathbb{Z}^N$, the integer lattice in $\mathbb{R}^N$, and vanishing weight function w. In this highly symmetric case, the dual complex to $\mathcal{L}_{\mathbb{Z}^N, w}$ differs only by a shift about $\delta = \frac{1}{2}\sum_i e_i$ from the original one. Writing $\gamma = \{\sum_{i=1}^N c_i e_i | 0 \le c_i \le 1\}$ for the unit cube, its translates by $\delta + z$, $z \in Z^N$, are its Laguerre-domains and it is not difficult to see that the above construction yields as a result that the vertices of T_u are the points

$$\{\pi(z) |\, z \in (\mathbb{Z}^N + u + \delta) \cap (E + \gamma)\}.$$

This set we refered to in [16] as the canonical projection pattern defined by the data $(\mathbb{Z}^N, E, u')$ with $u' = u + \delta$.

If $E^\perp \cap \mathbb{Z}^N$ were trivial then we had no reason to consider the apparently more elaborated construction with Laguerre-complexes. But the case of non-trivial $E^\perp \cap \mathbb{Z}^N$ occurs in interesting examples such as the Penrose tilings. Let D be the real span of $E^\perp \cap \mathbb{Z}^N$ assuming it is not trivial and V be the orthocomplement of D in $E^\perp$. Then we may compose the projection $\pi : \mathbb{R}^N \to E$ out of two, $\pi = \pi_2 \circ \pi_1$, where $\pi_1 : \mathcal{E} \to E \oplus V$ is the orthogonal projection with kernel D and $\pi_2 : E \oplus V \to E$ with kernel V. Then we perform the construction of the projection method in two steps. In the first we produce the (periodic) tiling defined by the data $(\mathbb{Z}^N, 0, E \oplus V, u)$ using projection π_1. As already mentioned, this tiling can be understood as a Laguerre complex, namely the one defined by $(\pi_1(\mathbb{Z}^N), w)$ where $w(\pi_1(z)) = \max\{w(z') - |\pi_1^\perp(z'+u)|^2 | \pi_1(z') = \pi_1(z)\}$. In the second step we now use this new Laguerre complex and the projection π_2. More precisely, we use the data $(\pi_1(\mathbb{Z}^N), w, E, \pi_1(u))$. Note that w remains zero after the first step in case $\pi_1^\perp(u) \in \mathbb{Z}^N$. In contrast, if $\pi_1^\perp(u) \notin \mathbb{Z}^N$ then we have to expect that the maximal periodicity lattice of the Laguerre complex defined by $(\pi_1(\mathbb{Z}^N), w)$ is a sublattice of $\pi_1(\mathbb{Z}^N)$ containing the lattice $\mathbb{Z}^N \cap (E \oplus V)$.

The most famous class of tilings which may be constructed by the above method are the Penrose tilings. Here $N = 5$, E a two dimensional invariant subspace of the symmetry $e_i \mapsto e_{i+1}$ (i mod 5) and D is the span of δ. If $\pi_1^\perp(u) = -\delta$ then the new Laguerre complex $\mathcal{L}_{\pi_1(\mathbb{Z}^5),w}$ becomes the dual of the Voronoi complex (i.e. the Delaunay complex) of the root lattice A_4 [28]. The resulting tilings are the usual Penrose tilings. Other choices for $\pi_1^\perp(u)$ lead to the so-called generalized Penrose tilings.

Let us describe some important properties of canonical projection tilings. First, for nonsingular u, v, T_u is locally isomorphic to T_v and to any other element of its hull [19] which implies that $MT_u = MT_v$ and that the dynamical system (MT_u, E) is minimal (i.e. any orbit lies dense). We may therefore drop the index u to write $M\mathcal{T}$ for the continuous hull. Given $u \in \mathcal{E}$ (not necessarily non-singular) we define

$$\tilde{P}_u := \{\xi \in \mathcal{L}_{\Theta_u}^{(d^\perp)} | 0 \in \pi^\perp(\xi)\}$$

and call a subset $\tilde{P}$ of some $\tilde{P}_u$ a lift of a tiling T if $T = \{\pi(\xi^*) | \xi \in \tilde{P}\}$. We call $\tilde{P}$ regular if, for all $\xi \in \tilde{P}$, 0 belongs to the interior $\operatorname{Int}\xi^\perp$ of $\xi^\perp$. A regular lift is always of the form $\tilde{P}_u$ for some regular u and vice versa, a regular u yields a regular lift.

Lemma 2 *Let $E \cap \Gamma = 0$ and $u, v \in NS$. Then T_u has a unique lift and $T_u = T_v$ whenever $u - v \in \Gamma$.*

Proof: Let $\tilde{P}$ be some lift of T_u and define

$$A(\tilde{P}) = \bigcap_{\xi \in \tilde{P}} -\xi^\perp.$$

We claim that $A(\tilde{P}) = \{0\}$. Hypothesis H1 implies that T_u determines its lift up to translation in $E^\perp$, in fact, the relative position between the lifts of two neighbouring tiles is fixed since their intersection must be a face which projects onto the intersection of the tiles. Our claim therefore implies that the lift must be unique. Irrationality of E in Γ implies that it intersects of each Γ-orbit of $d^\perp$-cells at least one representative. Hence $\tilde{P}$ contains such a representative for any Γ-orbit and therefore determines uniquely the Laguerre-complex it lies in. From maximality of Γ (H2) follows therefore that $\tilde{P} = \tilde{P}_v$ with $u - v \in \Gamma$. The converse is clear.

So it remains to proof the claim. Clearly $A(\tilde{P})$ is convex and closed. If it is not just the 0 point then it must therefore contain a closed intervall $[0,s]$. Suppose that this is the case. From the definition of singular points and denseness of $\Gamma^\perp$ follows immediately that, first $u+[0,s]$ must contain another regular point which we may assume to be $u+s$, and second, $u+[0,s]$ must contain a singular point in its interior. But, by convexity of the ξ, $u+[0,s] \in \mathrm{Int}\xi^\perp$ for all $\xi \in \tilde{P}$ which shows that all points in $u+[0,s]$ must be regular. This is a contradiction. q.e.d.

For regular u we can also define a lift of a patch P of T_u, namely we let $l_u(P)$ be the collection of all $\xi \in \tilde{P}_u$ for which $\pi(\xi)$ is a tile of P. For a patch P of T_u, $u \in NS$, we let

$$A_u(P) = \bigcap_{\xi \in l_u(P)} -\xi^\perp.$$

$A_u(P)$ is called the acceptance domain for P, for the following reason:

Lemma 3 *Let P be a patch of T_u, $u \in NS$. Then $P \in T_{u+s}$, for $s \in E+\Gamma$ whenever $s \in A_u(P)+\Gamma$.*

Let $s \in E+\Gamma$ which we split $s = s' + s''$ with $s' \in E$, $s'' \in \Gamma$. Then $P \subset T_{u+s}$ whenever $P - s' \in T_u$ and this is the case whenever exists a $v \in E^\perp$ such that $l_u(P) - s' + v \in \tilde{P}_u$ and $\forall \xi \in l_u(P) : 0 \in \pi^\perp(\xi - s' + v)$. The second condition implies that $v \in A_u(P)$. Now by maximality of Γ we deduce from the first condition that $s - v \in \Gamma$. Hence $P \in T_{u+s}$ implies $s \in \Gamma + A_u(P)$ and the converse is anyway clear. q.e.d.

3.1 The topology of $M\mathcal{T}$

For canonical projection tilings we have a much better description of the topology of the continuous hull which is one of the crucial reasons why we can compute their cohomology.

We use the tiling metric to define a metric on the space NS,

$$\bar{D}(v,w) := D(T_u, T_v) + |v-w|,$$

and let Π be the $\bar{D}$-completion of NS.

Lemma 4 *The action of $E+\Gamma$ on NS (by addition), the map $\eta_0 : NS \to M\mathcal{T} : x \mapsto T_x$, and the inclusion $\mu_0 : NS \hookrightarrow \mathcal{E}$ extend to continuous maps to the completion Π. Furthermore, the extension of η_0, $\eta : \Pi \to M\mathcal{T}$ is open and the extension of μ_0 a surjection $\mu : \Pi \to \mathcal{E}$ which is one to one on non-singular points.*

Proof: $\bar{D}$ is invariant under the Γ action and for small $s \in E$ we have that $\bar{D}(u+s, v+s)$ differs very little from $D(u,v)$; this implies that the action of $E+\Gamma$ extends to one by homeomorphisms on Π. Uniform continuity of η_0 and μ_0 is clear, as one can bound the D-metric and the euclidian metric by the $\bar{D}$-metric. Hence both maps extend continuously.

We claim that for $v \in NS$, we can find for all $\epsilon > 0$ a $\delta > 0$ such that $|v-w| < \delta$ ($w \in NS$) implies $D(T_v, T_w) < \epsilon$. This then shows that the preimages of non-singular points under μ are singletons. To assert the claim let, for $R > 0$, $A_R(T_v) = A(l_v(M_R(T_v)))$, a finite intersection of convex compact polytopes. Since v is regular, 0 is an interior point of these polytopes and hence $A_R(T_v)$ contains an open δ-neighbourhood of $0 \in E^\perp$. By Lemma 3 $|v^\perp - w^\perp| < \delta$ implies that $B_R(T_{v^\perp}) = B_R(T_{w^\perp})$, i.e. their D-distance is smaller than $\frac{1}{R+1}$. Taking $|v-w| < \min\{\delta, \epsilon\}$ then implies $\bar{D}(T_v, T_w) < 2\epsilon$.

To show that η is open recall that $\eta_0^{-1}(T_u) = u + \Gamma$. In particular, different preimages of one single point have a minimal distance. The strategy is to look at restrictions of η_u to small open balls (w.r.t. $\bar{D}$ in the relative topology), smaller than the above distance, and show that their inverses map Cauchy sequences onto Cauchy sequences. Let T_{u_ν} be a D-Cauchy sequence with $(u_\nu)_\nu$ belonging to such a ball ($u_\nu \in NS$). We claim that $(u_\nu)_\nu$ converges in the euclidian metric and therefore also in the $\bar{D}$-metric. To assert our claim observe that we can choose the ball small enough so that convergence of T_{u_ν} implies that of $|\pi(u_\nu)|$ and hence also $T_{u_\nu^\perp}$ is a Cauchy sequence. But the latter is even a Cauchy sequence with respect to the metric D_0. Now $D_0(T_{u_\nu^\perp}, T_{u_{\nu+\mu}^\perp}) \to 0$ implies that $\sup\{R | B_R(T_{u_\nu}) = B_R(T_{u_{\nu+\mu}})\}$ diverges and hence diameter of $A_{R_\nu}(T_{u_\nu})$ shrinks to zero (according to the proof of Lemma 2) which implies, by Lemma 3, $|u_{\nu+\mu}^\perp - u_\nu^\perp| \to 0$. q.e.d.

Corollary 1 *The map η induces an E-equivariant homeomorphism between $M\mathcal{T}$ and Π/Γ, the orbit space.*

Proof: From continuity and Lemma 2 follows immediately that all points in a single Γ-orbit are mapped onto the same tiling. So let us show that $\eta(x) = \eta(y)$ implies $y \in x + \Gamma$. Let $\eta(x) = \eta(y)$ but $x \neq y$. By the Hausdorff property we may find $\bar{D}$-open U and V, with $x \in U$ and $y \in V$, which do not intersect. We may also assume that $\eta(U) = \eta(V)$ (otherwise take $U' = U \cap \eta^{-1}(\eta(V))$ and $V' = V \cap \eta^{-1}(\eta(U))$). Now let x be the limit of a Cauchy-sequence $(x_\nu)_\nu$ in $U \cap NS$. Since different preimages of one single point under η have a minimal distance we can make U so small in diameter that there is a unique $\gamma \in \Gamma$ such that all $x_\nu + \gamma \in V$. Continuity now implies that $(x_\nu + \gamma)_\nu$ converges to y and yields the desired result. E-equivariance is clear. q.e.d.

We have thus another dynamical system $(\Pi, E+\Gamma)$ which plays the role of a "universal covering" (not in its strict sense) of the continuous tiling dynamical system.

Before we proceed and as an aside let us compare this with the so-called torus parametrisation of projection tilings [30]. At the same time we sketch a discussion which was carried out for tilings related to $\mathbb{Z}^N$ (not necessarily canonical) in [16]. There is a surjection $\mu' : MT \to \mathcal{E}/\Gamma$ which makes the following diagram commutative:

$$\begin{array}{ccc} \Pi & \stackrel{\mu}{\to} & \mathcal{E} \\ \eta\downarrow & & \downarrow \\ MT & \stackrel{\mu'}{\to} & \mathcal{E}/\Gamma \end{array} . \tag{4}$$

All maps are E-eqivariant and μ is $E+\Gamma$ equivariant. μ' is as well one to one on (classes of) non-singular points. The dense set NS/Γ of the torus $\mathcal{E}/\Gamma$ therefore yields a parametrization of a dense set (in fact a G_δ-dense set) of tilings. In fact it can be shown that $\mathcal{E}/\Gamma$ parametrizes the remaining set of tilings up to changes on sets of tiles having zero density in the tiling. This torus parametrization is very useful for analyzing symmetry properties of the tilings [30].

Next we want to describe the topology of Π. For that recall that a base of the topology of MT is generated by sets $B_\epsilon + x + \mathcal{U}_P$, $\epsilon > 0$, $x \in E$, P a patch in T. Now recall Lemma 3 which for $u \in E^\perp \cap NS$ can be reformulated by saying that for $x \in u + E + \Gamma$: $P \subset T_x$ whenever $x \in A_u(P) + u + \Gamma$. For $u \in E^\perp \cap NS$ we let

$$\mathcal{A}_u = \{(A_u(P) \cap \Gamma^\perp) + u + y | P \subset T_u, y \in \Gamma^\perp\}.$$

Then, by the interpretation of $A_u(P)$ we see that $\mathcal{A}_u$ is closed under intersection. In fact, $A_u(P) \cap (A_u(P') + y) = A_u(P \cup (P' + \pi(y)))$ provided $P \cup (P' + \pi(y)) \subset T_u$ and $\emptyset$ otherwise. It is useful, to have another description of $\mathcal{A}_u$ which at the same time shows that the following collection of closed subsets in Π,

$$\mathcal{B} := \{\overline{A} | A \in \mathcal{A}_u\},$$

(closure in Π) does not depend on u. Let, for $X \subset \mathcal{L}_\Theta^{(d^\perp)}$, $A(X) := \bigcap_{\xi \in X} -\xi^\perp$ and

$$\mathcal{A}'_u := \{A(X) \cap (\Gamma^\perp + u) | X \subset \mathcal{L}_\Theta^{(d^\perp)} \text{ finite}\}.$$

Then $A_u(P) + u = A(l_u(P) + u)$ which shows that $\mathcal{A}_u \subset \mathcal{A}'_u$. On the other hand let $v \in A(X) \cap (\Gamma^\perp + u)$. Then $\forall \xi \in X$: $\pi(\xi^*) \in T_v$ and $v - u = \gamma^\perp$ for some $\gamma \in \Gamma$. It follows that $\{\pi(\xi^*) | \xi \in X\} + \pi(\gamma)$ is a patch in T_u. Hence $\mathcal{A}_u = \mathcal{A}'_u$. But from the form of $\mathcal{A}'_u$ it is clear that $\mathcal{B}$ does not depend on u.

Theorem 3 *The collection* $\{B_\epsilon + x + U | U \in \mathcal{B}, \epsilon > 0, x \in E\}$ *is a base of the topology of* Π. *In particular,* $\Pi \cong E_c^\perp \times E$ *(product topology) where* $E_c^\perp = \overline{E^\perp \cap NS}$ *($\bar{D}$-closure in* Π*).*

Proof: Let P be a patch of T_u, $u \in \mathcal{E}^\perp \cap NS$. From Lemma 3 follows that for $x \in u + E + \Gamma$, $P \in T_x$ whenever $x \in A_u(P) + u + \Gamma$. Since $A_u(P)$ is compact in $E^\perp$ it follows from closedness of $\mathcal{U}_P$ that $\eta^{-1}(\mathcal{U}_P) = \overline{A_u(P) + u} + \Gamma$. Furthermore, if $\gamma \in \Gamma$ is not trivial then $\bar{D}(A_u(P)+u, \gamma + A_u(P)+u) > \delta$, for some $\delta > 0$ (here we mean the obvious extension of $\bar{D}$ to subsets). Hence, for all $x \in E + \Gamma$, $B_\epsilon + x + \overline{A_u(P) + u}$ is an open set. We conclude that the above collection consists indeed of open sets and its image under η is the collection of sets of which has been said that they form a base of the topology of MT. Now let $V \in \Pi$ open and of diameter smaller than $\frac{\delta}{2}$. Then $\eta(V)$ is a union of open sets U_i and we may assume that $U_i = B_\epsilon + x + \mathcal{U}_P$, $P \subset T_u$, $x \in E + \Gamma$, $\epsilon > 0$, is such that, for non-trivial $\gamma \in \Gamma$, the component $B_\epsilon + x + \overline{A_u(P) + u}$ has distance at least $\frac{\delta}{2}$ to $B_\epsilon + x + \gamma + \overline{A_u(P) + u}$ (otherwise we take a union over larger patches and decrease ϵ). Then V is a union of these components of which we have already shown that they belong to the collection in question. That Π has the above form of a product space is now clear. q.e.d.

Corollary 2 *The collection* $\mathcal{B}$ *is a base of compact open neighbourhoods for* $E_c^\perp$. *In particular,* $E_c^\perp$ *is a totally disconnected set without isolated points.*

Proof: That $\mathcal{B}$ is a base of the topology follows directly from the last theorem. That its sets are compact follows from compactness of the sets $\mathcal{U}_P$, $P \subset T_u$. q.e.d.

Remark. We saw that the sets of $\mathcal{B}$ have the interpretation of acceptance domains: if a nonsingular point u belongs to such a set then this can be interpreted by saying that a certain patch occurs at T_u. If we artificially introduce additional faces in the projected (on $E^\perp$) Laguerre-complex we started with, making out of one $d^\perp$-cell $\xi^\perp$ finitely many, we can encode this in the tiling by means of decorations. Each component of $\xi^\perp$ arising in that way may serve as acceptance domain for the tile $\pi(\xi^*)$ together with a label for that component (understood as a small compact set like an arrow). This is a decorated tile. To insure minimality of the decorated tiling we may require this additional cutting to be Γ-invariant. If we now take the new faces into account by taking as a base for the topology the sets corresponding to the components then we end up with a similar description of the continuous hull in the decorated case. Such a description is important if one wants to describe tilings like the decorated version of the octagonal and decagonal tiling which only after decoration have matching rules.

3.2 A description of the topology by cut-planes

Under the following hypothesis we get another description of the topology of $E_c^\perp$ which will turn out to be crucial.

H3 For all $\beta \in \mathcal{L}_\Theta^{(d^\perp-1)}$, the (affine) hyperplane H_β which is tangent to $\beta^\perp$ is a subset of $S^\perp$.

We call the hyperplanes H_β cut-planes. We do not have a general criterion under which this is true, but H3 is satisfied in many interesting cases, including those in which $W = \mathbb{Z}^N$, $w = 0$. Note that H3 allows us to write the singular points in $E^\perp$ as $S^\perp = \bigcup_{\beta\in\mathcal{L}_\Theta^{(d^\perp-1)}} H_\beta$ which is clearly invariant under the action of Γ given by $\lambda \mapsto \lambda + \gamma^\perp$. The set $\mathcal{C}$ of all cut-planes is invariant under Γ as well and since $\mathcal{L}_\Theta^{(d^\perp-1)}$ contains only a finitely many Γ-orbits $\mathcal{C}$ consists of a finite number of Γ-orbits, too.

A compact polytope in $E^\perp$ is called a $\mathcal{C}$-tope if it is the closure of its interior and if all its boundary faces are subsets of cut-planes. A subset of $E_c^\perp$ is called a $\mathcal{C}$-tope if it is the $\bar{D}$-closure of the set of non-singular points of a $\mathcal{C}$-tope in $E^\perp$.

Theorem 4 *The characteristic functions on $\mathcal{C}$-topes generate $C_c(E_c^\perp, \mathbb{Z})$.*

Proof: $\mathcal{C}$-topes form the set of finite unions of sets of $\mathcal{B}$. The latter being clopen and forming a base of the topology their corresponding characteristic functions generate $C_c(E_c^\perp, \mathbb{Z})$. Since $1_{U\cup V} + 1_{U\cap V} = 1_U + 1_V$ the statement follows. q.e.d.

For $\Gamma = \mathbb{Z}^{d+d^\perp}$ Le gave a description of the topology of $E_c^\perp$ which we want to relate to the above [11]. For $x \in E^\perp$ let c_x be a connected component of $E^\perp \backslash \bigcup_{x\in H\in\mathcal{C}} H$, an open subset of $E^\perp$ called a corner. Note that $c_x = E^\perp$ if $x \in NS$. Let $E_L^\perp = \{(x, c_x) | x \in E^\perp\}$ with topology generated by the sets

$$\mathcal{U}_{(x,c_x)} = \{(y, c_y) | y \in \overline{c_x}, c_x \cap c_y \neq \emptyset\}.$$

Clearly, the projection onto the first factor is a continuous surjective map $E_L^\perp \to E^\perp$. This is Le's description of the cut up space. Let U be a $\mathcal{C}$-tope in $E^\perp$. Then

$$U_L := \{(x, c_x) | x \in U, c_x \cap \mathrm{Int}U \neq \emptyset\}$$

is a preimage of U in $E_L^\perp$ which is a finite union of $\mathcal{U}_L$'s and hence open. Let $\mathcal{B}_L$ be the collection of all sets obtained in this way. Then the topology of $E_L^\perp$ is generated by $\mathcal{B}_L$ since we can realize the sets $\mathcal{U}_{(x,c_x)}$ as (infinite) unions. We leave it to the reader to verify that the map $\mathcal{B} \to \mathcal{B}_L$: $U \mapsto \mu(U)_L$ is a bijection preserving the operations intersection, union, and symmetric difference. Therefore $C_0(E_c^\perp)$ is isomorphic to $C_0(E_L^\perp)$ and $E_c^\perp$ homeomorphic to $E_L^\perp$.

3.3 A variant of the tiling groupoid for canonical projection tilings

For canonical projection tilings it is convenient to use a slightly different groupoid which is isomorphic to a reduction of the tiling groupoid. It is also continuously similar to it. In [17] it is called the pattern groupoid.

Let ϵ be a small vector in E which is not parallel to any of the faces of tiles. Define the following injection between the vertices of a projection tiling and its tiles: to a vertex v we associate the tile which contains in its interior $v + \epsilon$. We assume that ϵ is small enough so that the associated tile contains this vertex. Let $\Omega\mathcal{T}$ be the subset of $M\mathcal{T}$ given by those tilings which have a vertex on $0 \in E$. As for $\Omega_{\mathcal{T}}$ one shows that $\Omega\mathcal{T}$ is a closed range-open subset which intersects each orbit of $\mathcal{G}(M\mathcal{T}, E))$. Thus we define

$$\mathcal{G}\mathcal{T} := {}_{\Omega\mathcal{T}}\mathcal{G}(M\mathcal{T}, E))_{\Omega\mathcal{T}}$$

a reduction of $\mathcal{G}(M\mathcal{T}, E))$. Now consider a new set of punctures for $\mathcal{T}$, a subset of the old one, namely give only those tiles a puncture which are associated to vertices as described above. This choice can be made locally since we only have to test the vertices of the tile itself to decide whether we select its puncture to become a new one. Call $\Omega'_{\mathcal{T}}$ the subset of tilings of $M\mathcal{T}$ for which a new puncture lies on 0. By letting the new punctures tend to the corresponding vertices one immediately these that the reduction ${}_{\Omega'_{\mathcal{T}}}\mathcal{G}(M\mathcal{T}, E))_{\Omega'_{\mathcal{T}}}$ is isomorphic to $\mathcal{G}\mathcal{T}$. Furthermore, ${}_{\Omega'_{\mathcal{T}}}\mathcal{G}(M\mathcal{T}, E))_{\Omega'_{\mathcal{T}}}$ is the reduction to $\Omega'_{\mathcal{T}}$ of $\mathcal{G}_{\mathcal{T}}$ which by a remark in [10] is continuously similar to it. A similar argument can also be found in [17].

Proposition 2 *Let $u \in NS$ such that $0 \in \mathcal{E}$ is a vertex of T_u. Let $L = A_u(\{0\}) + u^{\perp}$. Then $\mathcal{G}\mathcal{T}$ is isomorphic to the reduction ${}_L\mathcal{G}(\Pi, E+\Gamma)_L$ which is equal to ${}_L\mathcal{G}(E_c^{\perp}, \Gamma)_L$.*

Proof: We may assume that $u \in E^{\perp}$ otherwise replacing $\Omega\mathcal{T}$ by the set of tilings which have a puncture on $-\pi(u)$ which obviously leads to an isomorphic groupoid. Then Lemma 2 implies that $\eta^{-1}(\Omega\mathcal{T}) = L + \Gamma$. It follows that the map ${}_L\mathcal{G}(\Pi, E+\Gamma)_L \to {}_{\Omega\mathcal{T}}\mathcal{G}(M\mathcal{T}, E)_{\Omega\mathcal{T}}$: $(x, s+\gamma) \mapsto (\eta(x), s)$ is an isomorphism. The other statement is clear. q.e.d.

3.4 Discrete tiling dynamical systems for canonical projection tilings

The projection method provides us with various other dynamical systems related to the tiling among them being also some given by a minimal action of $\mathbb{Z}^d$ on a Cantor-set, most useful in computing tiling cohomology.

Let F be a subspace which is complimentary to E, i.e. $F \cap E = 0$ and $F + E = \mathcal{E}$. We denote by π' the projection onto F which has kernel E, hence it is not orthogonal except if $F = E^\perp$. The restriction of π' to $u + \Gamma^\perp$ ($u \in E^\perp \cap NS$) extends to a homeomorphism between $E_c^\perp$ and $F_c = \overline{F \cap NS}$ (closure in Π) and may as well write $\Pi = F_c \times E$ with product topology. Since $E \cap \Gamma = \{0\}$, $\pi'(\Gamma)$ is isomorphic to Γ so that we have a natural minimal action of Γ on F, $x \cdot \gamma = x - \gamma^\perp$, without fixed points. The extension of this action to F_c defines a minimal dynamical system (F_c, Γ) also without fixed points.

Proposition 3 *$\mathcal{G}(F_c, \Gamma)$ is continuously similar to $\mathcal{G}(\Pi, E + \Gamma)$.*

Proof: We apply Proposition 1 taking $L = F_c$ (which is closed) and $\gamma : \Pi \to E + \Gamma$ to be the extension of $\pi : \mathcal{E} \to E$. q.e.d.

Now we decompose $\Gamma \cong \mathbb{Z}^{d+d^\perp}$ into complementary subgroups, $\Gamma = G_0 \oplus G_1$, where we may assume that $G_0 \cong \mathbb{Z}^{d^\perp}$ and $G_0' := \pi'(G_0)$ spans F. Define

$$X := F_c / G_0$$

so that we obtain (X, G_1), a minimal dynamical system without fixed points.

Proposition 4 *$\mathcal{G}(F_c, \Gamma)$ is continuously similar to $\mathcal{G}(X, G_1)$.*

Proof: We claim that F_c has a clopen fundamental domain Y for G_0. The lemma follows then from Proposition 1 upon using $L = Y$ and $\gamma : F_c \to \Gamma$, $\gamma(x)$ being the unique element of G_0 such that $x \cdot \gamma(x) \in Y$. The latter is indeed continuous since the preimage of a lattice point is a translate of the fundamental domain and therefore open.

To assert the claim pick any $\xi \in \mathcal{L}_\Theta^{(d^\perp)}$ such that $A(\xi)$ has interior. Since G_0' spans F it has a compact fundamental domain Y^0. By density of $\Gamma^\perp$ there is a finite subset $0 \in J \subset \Gamma$ such that $Y^1 = \bigcup_{\gamma \in J}(A(\xi) + \gamma^\perp)$ covers Y^0. It follows that

$$Y_c^1 := \bigcup_{\gamma \in J} (\overline{A(\xi)} + \gamma^\perp)$$

is a compact open subset of F_c and $Y_c^1 + G_0' = F_c$. Now let G_0^+ be a positive cone of G_0' which satisfies $G_0' = G_0^+ \cup (-G_0^+)$ thus implying a total order. We claim that $Y := Y_c^1 \backslash (Y_c^1 + G_0^+ \backslash \{0\}) \cap Y_c^1$ is a clopen fundamental domain. Clopeness is easy to see. So let $x \in F_c$. Clearly, the set of all and $g \in G_0'$ such that $x + g \in Y_c^1$ is non-empty and finite. The unique minimal element g_0 of this set is the only one satisfying $x + g_0 \in Y$. q.e.d.

Proposition 5 *$\mathcal{GT}$ is continuously similar to $\mathcal{G}(E_c^\perp, \Gamma)$.*

Proof: From Proposition 2 we know that $\mathcal{GT}$ is isomorphic to the reduction of $\mathcal{G}(E_c^\perp, \Gamma)$ to a closed subset L of $E_c^\perp$. We claim that there exists a choice of decomposition $\Gamma = G_0 + G_1$ with the properties as above such that L contains a clopen fundamental domain Y for G_0. The proposition then follows from Proposition 1 upon using the same map γ as in Proposition 4 which works since Y is a subset of L.

It remains to prove the claim. Since $\Gamma^\perp$ is dense in $E^\perp$ we can choose $d^\perp$ elements of Γ which generate a group H isomorphic to $\mathbb{Z}^{d^\perp}$, such that $H^\perp$ spans $E^\perp$, and has a fundamental domain Y' in $E^\perp$ contained in $\mu(L)$. Let G_0 be the group generated by H and representatives for the torsion elements of Γ/H. It is a free abelian group of rank $d^\perp$ which contains H and $G_0^\perp$ cannot be dense in $E^\perp$. By the same construction as in the proof of the last proposition we obtain from Y' a fundamental domain Y for G_0 in $E_c^\perp$ which is contained in L since $\mu(Y) \subset Y'$. q.e.d.

Corollary 3 $H(\mathcal{T}) \cong H(\Gamma, C(F_c, \mathbb{Z})) \cong H(G_1, C(X, \mathbb{Z}))$.

A direct consequence of the above corollary is that $H^k(\mathcal{T})$ is trivial, if k exceeds the rank of G_1 which is d, the dimension of the tiling. Furthermore, using that $H^0(G_1, C(X, \mathbb{Z})) = \{f \in C(X, \mathbb{Z}) | \forall g \in G_1 : g \cdot f = f\}$ [27], minimality of the G_1 action implies that $H^0(\mathcal{T}) = \mathbb{Z}$. Finally, if M is a G_1-module then $H^d(G_1, M) = Coinv(G_1, M)$ is the group of coinvariants [27]:

$$Coinv(G_1, M) := M/\langle\{m - g \cdot m | m \in M, g \in G_1\}\rangle.$$

By the corollary $H^d(\mathcal{T})$ is thus equal to $C(X, \mathbb{Z})/E(G_1)$ where $E(G_1)$ is subgroup of $C(X, \mathbb{Z})$ generated by the elements $f - g \cdot f$, $g \in G_1$ ($g \cdot f(x) = f(x \cdot g)$).

3.5 The dynamical systems $(X, \mathbb{Z}^d)$

We pause here to comment on the dynamical systems of the form (X, G_1) which have been defined in the last section. A priori they depend on the position of F and on the choice of G_0. We have seen, however, that they are in a certain sense all equivalent, namely their groupoids are all continuously similar and they are all reductions of one big groupoid. They are not all isomorphic, as an investigation of the order unit of the K_0-group of the C^*-algebra they define shows.

The dependence on F is inessential. π' induces a homeomorphism between $E_u^\perp$ and F_c which intertwines the Γ action. Therefore, different F's lead to isomorphic dynamical systems (F_c, Γ) and, if we keep the decomposition $\Gamma = G_0 \oplus G_1$ fixed, (X, G_1). But taking F as the span of G_0 one verifies directly that $M\mathcal{T}$ is the mapping torus of (X, G_1) [16]. We point out one consequence of this (which we will, however, not make use of below).

Corollary 4 *The tiling cohomology of non-periodic canonical projection tilings is isomorphic to the Czech cohomology of their continuous hull.*

We do not know whether this result is true for general tilings.

If, on the other hand, $\Gamma = \mathbb{Z}^{d+d^\perp}$, $F = E^\perp$ and G_0 generated by, let's say the first $d^\perp$ base elements e_i then the dynamical system is the rope dynamical system of [10].

Finally, we summarize the structure of (X, G_1) in a commutative diagram which is the discrete analogue of (4), refering the reader to [16] for the neccessary proofs:

$$\begin{array}{ccc} F_c & \stackrel{\mu}{\to} & F \\ \eta\downarrow & & \downarrow \\ X & \stackrel{\mu'}{\to} & F/G_0 \end{array}$$

The maps are Γ (resp. G_1) equivariant where the G_1-action on the $d^\perp$-torus F/G_0 is by rotations (constant shifts). X is a Cantor set and the surjection $\mu' : X \to F/G_0$ is one to one for nonsingular points of X which form a dense G_δ subset. Thus (X, G_1) is an almost one to one extension of a relatively simple system: that of rotations on a torus. But the crucial topological information is encoded in the set on which μ' is not injective.

4 Examples

Before we proceed to give a qualitative picture of tiling cohomology we discuss the two simplest examples which we believe show typical features. Both are one-dimensional tilings obtained from an integer lattice. So apart from $H^0(\mathcal{T})$ which is $\mathbb{Z}$ we have to compute the coinvariants only.

In our first example we take $W = \mathbb{Z}^2$, $w = 0$ and $d = 1$. Here E is specified by a vector $(1, \nu)$ and ν has to be irrational to meat the requirement $E \cap \mathbb{Z}^2 = 0$. Clearly, $E^\perp$ is generated by $(-\nu, 1)$ and the cut planes are simply points, namely the points of $\pi^\perp(\mathbb{Z}^2)$ (we ignore the shift by δ). Identifying $E^\perp$ with $\mathbb{R}$ we have $\pi^\perp(\mathbb{Z}^2) = \mathbb{Z} + \nu\mathbb{Z}$ (after a suitable rescaling). Hence $C_c(E_c^\perp, \mathbb{Z})$ is generated by indicator functions $1_{[a,b]}$ (on the $\bar{D}$-closure of $[a, b] \cap NS$)

with $a, b \in \mathbb{Z} + \nu\mathbb{Z}$, $a < b$. How many of them are cohomologous? Clearly, $1_{[a,b]} \sim 1_{[0,b-a]}$ and there are unique $n, m \in \mathbb{Z}$ such that $b - a = n + \nu m$. Defining $1_{[a,b]} = -1_{[b,a]}$ in case $a > b$ we get

$$1_{[0,b-a]} = 1_{[0,n]} + 1_{[n,n+\nu m]} \sim n 1_{[0,1]} + m 1_{[0,\nu]}$$

which shows that the coinvariants are $\mathbb{Z}^2$ provided the two generators given by the classes of $1_{[0,1]}$ and of $1_{[0,\nu]}$ are independent. This will be shown in Section 7. Let us mention in this context that the above tilings are very close to being substitutional [31] (they are strictly substitutional only for ν a quadratic irrationality).

The above result shows that whatever ν is, as long as it is irrational $H^1(\mathbb{Z}^2, C_c(E_c^\perp, \mathbb{Z})) = \mathbb{Z}^2$. This shows that cohomology is not a very fine invariant to distinguish tilings. But we will see in Section 7 how to improve this.

In our second example we take $W = \mathbb{Z}^3$, $w = 0$ and $d = 1$. Here we consider only the case where $E^\perp \cap \mathbb{Z}^3 = 0$, because the other leads essentially to the above situation. The cut planes are lines in this case which $\pi^\perp(\mathbb{Z}^3)$-translates of $H_\alpha = \langle e_\alpha^\perp \rangle$, $\alpha = 1, 2, 3$ (again up to the shift by δ). Any two H_α span $E^\perp$.

We claim that the result for the cohomology differs drastically from the above in that the coinvariants are infinitely generated. Fix $g_1, g_2 \in \pi^\perp(\mathbb{Z}^3)$ and let U be the rhombus (we assume it has interior) whose boundary faces lie in $H_1 \cup (H_1 + g_1) \cup H_2 \cup (H_2 + g_2)$. Clearly, 1_U, the indicator function on the $\bar{D}$-closure of $U \cap NS$, belongs to $C_c(E_c^\perp, \mathbb{Z})$. Let, for $\alpha = 1, 2$, π_1 (π_2) be the projection onto H_1 (H_2) which has kernel H_2 (H_1) and let $\Gamma_\alpha = \pi_\alpha(\pi^\perp(\mathbb{Z}^3))$. Then for all $\lambda_\alpha \in \Gamma_\alpha$ also $1_{U+\lambda_1+\lambda_2} \in C_c(E_c^\perp, \mathbb{Z})$. How many of them are cohomologous? Let us try to repeat the construction of the first example. Clearly

$$1_{U+\lambda_1+\lambda_2} \sim 1_{U+\lambda_1'+\lambda_2'} \quad \text{if } \lambda_1 + \lambda_2 - \lambda_1' - \lambda_2' \in \pi^\perp(\mathbb{Z}^3).$$

But since the rank of Γ_α is at least 2 (because it is dense in H_α) we see that the number of $\pi^\perp(\mathbb{Z}^3)$ orbits of points in $\Gamma_1 + \Gamma_2$ (which is the number of elements in $(\Gamma_1 + \Gamma_2)/\pi^\perp(\mathbb{Z}^3)$) is infinite. Therefore the construction used in the first example cannot be used here to reduce the generators to a finite set. This does not prove our claim but it outlines a crucial point, namely that there are infinitely many orbits of points which are intersections of cut planes. From this we will conclude below that the tilings of the second example cannot be substitutional.

5 Sufficient conditions for infinitely generated cohomology

In this section we review the main results of [17] which provide criteria under which the cohomology of a canonical projection tiling which satisfies H3 is infinitely generated even when rational coefficients are considered. We fix a canonical projection tiling satisfying H3 throughout this section considering all notions relative to that tiling.

Definition 6 *We call a point $x \in S$ an intersection-cut-point if it is the only point in the intersection of $d^\perp$ cut planes.*

Let $\mathcal{P}$ be the set of intersection-cut-points. Clearly, $\mathcal{P}$ is invariant under the action of Γ. We let $\Omega(\mathcal{P}) = \mathcal{P}/\Gamma$ the orbit space. The main result of [17] is the following theorem:

Theorem 5 *[17] If $\Omega(\mathcal{P})$ is an infinite set then $H^d(\mathcal{GT}, \mathbb{Q})$ is infinitely generated.*

We do not repeat its proof here, but let us explain how to obtain criteria under which $\Omega(\mathcal{P})$ is infinite.

Choose $d^\perp$ cut planes H_β, which we index now simply by $\alpha = 1, \dots, d^\perp$, such that their intersection is one single point. Let $S' := \bigcup_\alpha (H_\alpha + \Gamma^\perp)$ and $\mathcal{P}' = \mathcal{P} \cap S'$, a subset which is clearly Γ-invariant. Let $L_\alpha := \bigcap_{\alpha' \neq \alpha} H_{\alpha'}$, a line, and $\pi_\alpha : E^\perp \to L_\alpha$ be the (not necessarily orthogonal) projection with kernel H_α. Then the stabilizer of L_α, $\Gamma^\alpha := \Gamma \cap L_\alpha$ is certainly a subgroup of $\Gamma_\alpha = \pi_\alpha(\Gamma^\perp)$.

Lemma 5 *If* $\mathrm{rank}\Gamma^\alpha < \mathrm{rank}\Gamma_\alpha$ *then $\Omega(\mathcal{P})$ is an infinite set.*

Proof: Let $x \in L_\alpha \cap \mathcal{P}'$. Then, by construction, $x + \Gamma_\alpha \in \mathcal{P}'$, too. The latter set may be decomposed in its Γ^α-orbits and if $\mathrm{rank}\Gamma^\alpha < \mathrm{rank}\Gamma_\alpha$ there are infinitely many. On the other hand, intersection-cut-points of $L_\alpha \cap \mathcal{P}'$ which lie in different Γ^α-orbits lie also in different Γ-orbits. q.e.d.

This gives a criterium which is perhaps most easily checked and at the same time shows that $\Omega(\mathcal{P})$ being an infinite set is a generic feature.

Corollary 5 *If* $\mathrm{rank}\Gamma^\alpha < 2$ *then $\Omega(\mathcal{P})$ is an infinite set.*

Proof: Denseness of $\Gamma^\perp$ implies that of Γ_α. Hence $\mathrm{rank}\Gamma_\alpha \geq 2$. q.e.d.

Lemma 6 *If $d^\perp > d$ then $\Omega(\mathcal{P})$ is an infinite set.*

Proof: We showed above $\mathrm{rank}\Gamma_\alpha \geq 2$. In particular, $\sum_\alpha \mathrm{rank}\Gamma_\alpha \geq 2d^\perp$. The statement of the lemma follows therefore from the observation that $\Omega(\mathcal{P})$ is an infinite set if $(\bigoplus_\alpha \Gamma_\alpha)/\Gamma^\perp$ is infinite and the latter is the case whenever $\sum_\alpha \mathrm{rank}\Gamma_\alpha > d + d^\perp$. q.e.d.

The claim of our second example above follows from this lemma. Finally, with a little more thorough analysis [17] one can show that if $\Omega(\mathcal{P})$ is a finite set then $\frac{d}{d^\perp}$ must be an integer.

5.1 Comparison with substitutional tilings

Apart from those tilings which arrise from the canonical projection method there is another very important class of tilings for which the cohomology can be computed. It is the class of of finite type tilings which allow for a locally invertible (primitive) substitution. We briefly describe these tilings and show that their group of coinvariants with rational coefficients is always finitely generated. The result of the last section then implies that generically canonical projection tilings which satisfy H3 do not allow for a locally invertible substitution.

A substitution of a tiling T (the termini inflation and deflation are also used in this context) is roughly speaking a rule according to which each tile of T gets substituted with a whole collection of tiles (a patch) such that these patches fit together (without overlap or gaps) to form a new tiling which is locally isomorphic to T. Furthermore, the translational congruence class of the patch which substitutes a tile depends only on the translational congruence class of that tile and the relative position between two patches only on the relative position between the two tiles which they substitute. Therefore, the rule is specified if given for any translational congruence class of tiles (of which there are only finitely many) and for all possible relative positions two neighbouring tiles can have (which are also only finitely many). There are other conditions a substitution has to satisfy for it to be useful in computing cohomology, in particular that there is an inverse procedure which is also locally defined which means that it can be formulated as a rule depending on translational congruence classes of patches. But rather then introducing the necessary terminology to formulate these conditions in detail we present one of the major examples, which is by the way also a canonical projection method tiling,

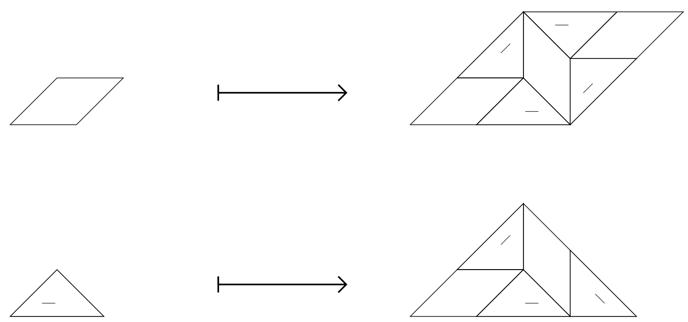

Fig. 1 Substitution of the octagonal tiling (triangle version).

and refer the reader to [9] and [10] where the theories for the computation of cohomology are developed.

Of the two approaches to compute the cohomology of substitution tilings that of [9] is based on the continuous dynamical system $(M\mathcal{T}, \mathbb{R}^d)$ whereas that of [10] is based on the tiling groupoid $\mathcal{G}_\mathcal{T}$. We consider here the latter. The essential observation of this approach is that a primitive invertible substitution gives rise to a homeomorphism Θ (the Robinson map) between $\Omega_\mathcal{T}$ and the space of paths $\mathcal{P}_\Sigma$ on an oriented graph Σ in which for any two vertices exists at least one edge which starts at the first and ends at the second vertex. In the simpler case where the substitution forces its border (see [15]) the connectivity matrix σ of Σ is a power of the substitution matrix. A path on an oriented graph is a sequence of edges which fit together in the sense that the $n+1$th one starts at the vertex where the nth one ends. A natural principal topological groupoid $\mathcal{G}_\Sigma$ comes with path spaces, namely the one given by tail equivalence: two paths are tail equivalent if they agree up to finitely many edges. The tiling groupoid $\mathcal{G}_\mathcal{T}$, which is always principal for such substitution tilings, is via Θ identified with a subset of $\mathcal{P}_\Sigma \times \mathcal{P}_\Sigma$ and hence can be compared with $\mathcal{G}_\Sigma$, in fact, $\mathcal{G}_\Sigma$ is a subset of $G_\mathcal{T}$ (but neither an open nor a closed one). The main result of that construction is that the group of coinvariants with integer coefficients[3] of $\mathcal{G}_\mathcal{T}$ is a quotient of the group of coinvariants of $\mathcal{G}_\Sigma$ and the latter is easy to obtain. It is the direct limit of the system $\mathbb{Z}^N \stackrel{\sigma}{\to} \mathbb{Z}^N \stackrel{\sigma}{\to} \cdots$ where N is the number of vertices of Σ (which in the border forcing case coincides with the translational tile-classes). The

[3]It should be noted that the group of coinvariants was defined in [10] without refering to cocycle cohomology but it coincides with $H^d(\mathcal{T})$ as defined in this article once a relation to group cohomology can be established as e.g. is the case for projection method tilings (in the language of [10] this means that the tiling reduces to a $\mathbb{Z}^d$-decoration). This is also the case when rational coefficients are considered.

direct limit of the above system need not to be finitely generated but when rational coefficients are considered instead of integer ones then the direct limit becomes that of the system $\mathbb{Q}^N \stackrel{\sigma}{\to} \mathbb{Q}^N \stackrel{\sigma}{\to} \cdots$ which equals $\mathbb{Q}^R$ where R is the rank of σ^n for large n (rankσ^n stabilizes). Hence, the group of coinvariants of $\mathcal{G}_\mathcal{T}$ with rational coefficients is finitely generated. To summarize, when rationalized, the degree d cohomology group of a substitution tiling which is a the same time a canonical projection tiling is finitely generated. In particular, a canonical projection tiling which satisfies H3 is generically not substitutional.

It is worth comparing the above result with a similar one due to Pleasants who uses the theory of algebraic number fields [32]. In the context of projection method tilings there is an approach to the construction of substitutions which is based on the torus-parametrization. In fact, it is most powerful not when tilings are considered but when projection point patterns are looked at (the latter are closely related to tilings, see [16]). The projection point pattern given by the data (Γ, E, A), a lattice $\Gamma \subset \mathcal{E}$, a subspace E, and a subset A of $E^\perp$, called the acceptance domain which is subject to rather weak conditions, is the point set $P_A := \pi((E + A) \cap \Gamma)$. The canonical choice for A corresponds to one where $P_A = \{\pi(\xi)|\xi \in \tilde{P}^0\}$ with $\tilde{P}^0$ the set of vertices (0-cells) of the lift of a canonical projection tiling T (constructed from the same data with constant weight function). In that case, A is a polytope. But in [32] A is allowed to be more general. In that case, what comes close to being a substitution after rescaling and is called an inflation is defined to be a linear map [32] (or even affine linear [30]) which has E as one of its eigenspaces (with eigenvalue of modulus greater than 1), preserves Γ, and is contracting in a space F complementary to E. The question under which conditions such a map defines a local inflation in the same sense as above, i.e. an inflation which can be defined as a map on translational congruence classes, leads to a criterion on the acceptance zone A.

The method of Pleasants [32] is designed to construct projection point patterns with given (finite) symmetry group (acting by isometries). It is based on the result that every representation of a finite isometry group acting on $\mathbb{R}^d$ can be written as a matrix representation where the matrices take their entries in a real algebraic number field $\mathcal{K}$ of (finite) degree p. This number field $\mathcal{K}$ is then used to construct a decomposition $\mathbb{R}^{dp} = E \oplus E^\perp$ where $\dim E = d$, and a lattice Γ so that the point pattern with the desired symmetry is the projection point pattern constructed from data (Γ, E) and a (general) acceptance domain in $E^\perp$. Details of the construction can be found in [32]. In that article Pleasants comes to the conclusion that local inflations always exist but, for $p > 2$, never for polytopal acceptance domain (so in particular not for canonical one) whereas this obstruction is absent for $p = 2$. Note that $\dim E^\perp \geq \dim E$ in his construction, equality holding only for $p = 2$. His result, compared with Lemma 6, is therefore in agreement with ours.

6 Explicit formulae for $d^\perp \leq 2$

The purpose of the present section is to present quantitative results for canonical projection tilings of codimension smaller than or equal to 2 where we continue to assume H3. The restriction to small codimension is a matter of simplification. In principle, the calculations can be carried out for any codimension, however they become quite complicated. Algebraic topology provides a tool to organize such calculations, namely spectral sequences, and in a forthcoming article we shall exploit their full power [18]. In this article we can avoid them by restricting to small codimension. For simplicity we rule out the case in which $\Omega(\mathcal{P})$ is infinite, in which case we already saw in the last section that the cohomology is infinitely generated. In fact, the results below show in particular that finite $\Omega(\mathcal{P})$ implies finitely generated cohomology ($d^\perp \leq 2$).

The calculations rely on the description of the topology of $E_c^\perp$ by cut planes and we recall here the general set up. $\mathcal{C}$ is a countable collection of cut planes, in fact, finitely many Γ-orbits and we index the orbits by I. We know that the normals of the cut planes span F and that $\Gamma^\perp$ lies dense in it. We now simplify the notation in writing Γ in place of $\Gamma^\perp$.

The task is to compute the cohomology of the group Γ with values in $C(E_c^\perp, \mathbb{Z})$ and the strategy is a follows. We recognize $C_c(E_c^\perp, \mathbb{Z})$, the compactly supported functions, as an Γ-module in a (finite) exact sequence of Γ-modules and use the functorial properties of the homology functor (we switch from cohomology to homology), in particular that it turns short exact sequences into long exact ones, to perform the calculation. The point is that the other modules of the exact sequence are effectively lower dimensional so that one can proceed recursively.

6.1 Group homology

It turns out to be more convenient to use group homology in place of group cohomology. Using that $E^\perp$ has $d^\perp$ non-compact independent directions and Poincaré duality one proves [17]:

$$H^k(\Gamma, C(E_c^\perp, \mathbb{Z})) \cong H_{d-k}(\Gamma, C_c(E_c^\perp, \mathbb{Z})).$$

As a general reference to group homology we refer to [27]. Group homology is defined using any (projective) resolution of $\mathbb{Z}$ by modules of the group, i.e. $\mathbb{Z}$-modules which carry an action of the group. We choose here the following free resolution. Let $\{e_1, \cdots, e_N\}$ be a base of $\Gamma \cong \mathbb{Z}^N$. The exterior ring over Γ, $\Lambda\Gamma$, is the free graded $\mathbb{Z}$-module $\Lambda\Gamma = \bigoplus_{k=0}^N \Lambda_k\Gamma$, $\Lambda_k\Gamma$ having base $\{e_{i_1} \wedge \cdots \wedge e_{i_k} | 1 \leq i_j < i_{j+1} \leq N\}$ with antisymmetric multiplication (denoted by $\wedge$), i.e. the only relations are $e_i \wedge e_j = -e_j \wedge e_i$. Γ acts on $\Lambda_k\Gamma$ trivially. We may regard the Γ module $\mathbb{Z}\Gamma$ (the free $\mathbb{Z}$-module with base Γ and action of Γ by shift of the base) as integer valued Laurent polynomials in N variables $\{t_1, \cdots, t_N\}$. Addition in $\mathbb{Z}\Gamma$ then corresponds to multiplication of Laurent-polynomials. Now the resolution reads

$$0 \to \Lambda_N\Gamma \otimes \mathbb{Z}\Gamma \stackrel{\partial}{\to} \Lambda_{N-1}\Gamma \otimes \mathbb{Z}\Gamma \stackrel{\partial}{\to} \cdots \Lambda_0\Gamma \otimes \mathbb{Z}\Gamma \stackrel{\Sigma}{\to} \mathbb{Z} \to 0,$$

where ∂ is the unique $\mathbb{Z}\Gamma$-linear derivation of degree 1 determined by $\partial(e_i) = (t_i - 1)$ and $\Sigma(t_i) = 1$. Now, given a Γ-module M, the homology of the group Γ with coefficient module M, $H_*(\Gamma, M)$, is defined as the homology of the complex

$$0 \to \Lambda_N\Gamma \otimes \mathbb{Z}\Gamma \otimes_\Gamma M \stackrel{\partial\otimes 1}{\to} \cdots \Lambda_0\Gamma \otimes \mathbb{Z}\Gamma \otimes_\Gamma M \to 0$$

where, for two Γ-modules M_1, M_2, $M_1 \otimes_\Gamma M_2$ is the quotient of the algebraic tensor product space (over $\mathbb{Z}$) $M_1 \otimes M_2$ by the relations $\gamma \cdot m_1 \otimes m_2 = m_1 \otimes \gamma \cdot m_2$. In particular, $H_k(\Gamma, \mathbb{Z}\Gamma)$ is trivial for all $k > 0$ and equal to $\mathbb{Z}$ for $k = 0$.

Suppose that we can split $\Gamma = G \oplus H$ and let us compute $H_*(\Gamma, \mathbb{Z}H)$ where $\mathbb{Z}H$ is the free $\mathbb{Z}$-module generated by H which becomes an Γ-module under the action of Γ given by $(g \oplus h) \cdot h' = h + h'$. Then we can identify

$$\Lambda\Gamma \otimes \mathbb{Z}\Gamma \otimes_\Gamma \mathbb{Z}H \cong \bigoplus_{i+j=k} \Lambda_i G \otimes \Lambda_j H \otimes \mathbb{Z}H \tag{5}$$

and under this identification $\partial \otimes 1$ becomes $(-1)^{deg} \otimes \partial'$ where ∂' is the boundary operator for the homology of H. It follows that

$$H_k(\Gamma, \mathbb{Z}H) \cong \bigoplus_{i+j=k} \Lambda_i G \otimes H_j(H, \mathbb{Z}H) = \Lambda_k G.$$

As a special case, $H_k(\Gamma, \mathbb{Z}) = \Lambda_k\Gamma \cong \mathbb{Z}^{\binom{N}{k}}$. Now let $\Sigma : \mathbb{Z}H \to \mathbb{Z}$ be the sum of the coefficients, i.e. $\Sigma[h] = 1$ for all $h \in H$. We shall later need the following lemma:

Lemma 7 *Under the identifications $H(\Gamma, \mathbb{Z}H) \cong \Lambda G$ and $H(\Gamma, \mathbb{Z}) \cong \Lambda\Gamma$ the induced map $\Sigma_k : H_k(\Gamma, \mathbb{Z}H) \to H_k(\Gamma, \mathbb{Z})$ becomes the embedding $\Lambda_k G \hookrightarrow \Lambda_k\Gamma$.*

Proof: Using the decomposition (5) it is easy to see that the induced map $\Sigma_k : \bigoplus_{i+j=k} \Lambda_i G \otimes H_j(H, \mathbb{Z}H) \to \bigoplus_{i+j=k} \Lambda_i G \otimes H_j(H, \mathbb{Z})$ preserves the bidegree and must be the identity on the first factors of the direct summands. Since $H_k(H, \mathbb{Z}H)$ is trivial whenever $k \neq 0$ and one dimensional for $k = 0$, Σ_k can be determined by evaluating Σ_0 on the generator of $H_0(H, \mathbb{Z}H)$ and one readily checks that this gives a generator of $H_0(H, \mathbb{Z})$ as well. q.e.d.

The basic tool in the calculations below is the following. Whenever we have a short exact sequence of $\Gamma-$modules

$$0 \to A \overset{\varphi}{\to} B \overset{\psi}{\to} C \to 0 \tag{6}$$

we get a long exact sequence of homology groups

$$\cdots \overset{\psi_{k+1}}{\to} H_{k+1}(\Gamma, C) \overset{\gamma_{k+1}}{\to} H_k(\Gamma, A) \overset{\varphi_k}{\to} H_k(\Gamma, B) \overset{\psi_k}{\to} H_k(\Gamma, C) \cdots. \tag{7}$$

The maps φ_k and ψ_k are the induced homomorphisms and the γ_k are the connecting homomorphisms. For details see [27].

6.2 A CW-like complex

Let $\mathcal{C}'$ be an arbitrary countable collection of affine hyperplanes of F', a linear space, and define $\mathcal{C}'$-topes as before: compact polytopes which are the closure of their interior and whose boundary faces belong to hyperplanes from $\mathcal{C}'$. For n at most the dimension of F' let $C^n_{\mathcal{C}'}$ be the $\mathbb{Z}$-module generated by the n-dimensional faces of convex $\mathcal{C}'$-topes satisfying the relations

$$[U_1] + [U_2] = [U_1 \cup U_2]$$

for any two faces U_1, U_2, for which $U_1 \cup U_2$ is as well a convex face and $U_1 \cap U_2$ has no interior (i.e. nonzero codimension in U_1). (The above relations then imply $[U_1] + [U_2] = [U_1 \cup U_2] + [U_1 \cap U_2]$ if $U_1 \cap U_2$ has interior.) If we take $\mathcal{C}' = \mathcal{C}$, our collection of cut planes, then $C^n := C^n_{\mathcal{C}}$ carries an obvious Γ-action, namely $\gamma \cdot [U] = [U + \gamma]$. It is therefore an Γ-module. As Γ-module $C^{d^\perp}$ is isomorphic to $C_c(E_c^\perp, \mathbb{Z})$, the isomorphism being given by assigning to $[U]$ the indicator function on the closure of $U \cap NS$ (which is clopen). Moreover, C^0 is a free $\mathbb{Z}$-module, its above described base is in one to one correspondence to the intersection-cut-points $\mathcal{P}$.

Proposition 6 *There exist Γ-equivariant module maps δ and Σ such that*

$$0 \to C^{d^\perp} \overset{\delta}{\to} C^{d^\perp - 1} \overset{\delta}{\to} \cdots C^0 \overset{\Sigma}{\to} \mathbb{Z} \to 0, \tag{8}$$

is an exact sequence of Γ-modules and $\Sigma[U] = 1$ for all vertices U of $\mathcal{C}$-topes.

Proof: For a subset R of Γ (which we identified with $\Gamma^\perp \subset E^\perp$) let $\mathcal{C}_R := \{H_i + r | r \in R, i \in I\}$ and $S_R = \{x \in H | H \in \mathcal{C}_R\}$. Let $\mathcal{R}$ be the set of subsets $R \subset \Gamma$ such that all connected components of $E^\perp \backslash S_R$ are bounded and have interior. $\mathcal{R}$ is closed under union and hence forms an upper directed system under inclusion. For any $R \in \mathcal{R}$, the $\mathcal{C}_R$-topes define a regular polytopal CW-complex

$$0 \to C^{d^\perp}_{\mathcal{C}_R} \stackrel{\delta_R}{\to} C^{d^\perp-1}_{\mathcal{C}_R} \stackrel{\delta_R}{\to} \cdots C^0_{\mathcal{C}_R} \to 0, \qquad (9)$$

with boundary operators δ_R depending on the choices of orientations for the n-cells ($n > 0$) [33]. Moreover, this complex is acyclic ($E^\perp$ is contractible), i.e. upon replacing $C^0_{\mathcal{C}_R} \to 0$ by $C^0_{\mathcal{C}_R} \stackrel{\Sigma_R}{\to} \mathbb{Z} \to 0$ where $\Sigma_R[U] = 1$, (9) becomes an exact sequence. Let us constrain the orientation of the n-cells in the following way: For each $n < d^\perp$ there are finitely many subsets $J \subset I$ such that $\dim \bigcap_{i\in J} H_i = n$ and J is maximal. Each n-cell belongs to a subspace parallel to one of the $\bigcap_{i\in J} H_i$ and we choose its orientation such that it depends only on the corresponding J (i.e. we choose an orientation for $\bigcap_{i\in J} H_i$ and then the cell inherits it as a subset). By the same principle, all $d^\perp$-cells are supposed to have the same orientation. Then the cochains and boundary operators δ_R share two crucial properties: first, if $R \subset R'$ for $R, R' \in \mathcal{R}$, then we may identify $C^n_{\mathcal{C}_R}$ with a submodule of $C^n_{\mathcal{C}_{R'}}$ and under this identification $\delta_R(x) = \delta_{R'}(x)$ for all $x \in C^n_{\mathcal{C}_R}$, and second, if U and $U + x$ are $\mathcal{C}_R$-topes then $\delta_R[U + x] = \delta_R[U] + x$. The first property implies that the directed system $\mathcal{R}$ gives rise to a directed system of acyclic cochain complexes, and hence its direct limit is an acyclic complex, and the second implies, together with the fact that for all $\gamma \in \Gamma$ and $R \in \mathcal{R}$ also $R + \gamma \in \mathcal{R}$, that this complex becomes a complex of Γ-modules. The statement now follows since $C^n_{\mathcal{C}}$ is the direct limit of $C^n_{\mathcal{C}_R}$ for all n. q.e.d.

6.3 Solutions for $d^\perp = 1, 2$

We now calculate the homology groups $H_k(\Gamma, C^{d^\perp})$ for $d^\perp = 1, 2$.

Lemma 8 *Given the data of a canonical projection tiling*

$$H_k(\Gamma, C^0) = \begin{cases} 0 & \textit{for} \quad k > 0, \\ \mathbb{Z}^L & \textit{for} \quad k = 0, \end{cases} \qquad (10)$$

where L is the number of Γ-orbits of vertices of $\mathcal{C}$-topes, i.e. $L = |\Omega(\mathcal{P})|$.

Proof: Since Γ acts fixpoint-freely we have $\Lambda\Gamma \otimes \mathbb{Z}\Gamma \otimes_\Gamma C^0 \cong \Lambda\Gamma \otimes \mathbb{Z}\Gamma \otimes \mathbb{Z}^L$ which directly implies the result. q.e.d.

Theorem 6 *If $d^\perp = 1$ then*

$$H^{d-k}(\mathcal{T}) = H_k(\Gamma, C^1) = \begin{cases} \mathbb{Z}^{\binom{N}{k+1}} & \textit{for} \quad k > 0, \\ \mathbb{Z}^{N+L-1} & \textit{for} \quad k = 0. \end{cases}$$

Proof: In case $d^\perp = 1$ (8) is a short exact sequence

$$0 \to C^1 \xrightarrow{\delta} C^0 \xrightarrow{\Sigma} \mathbb{Z} \to 0 \tag{11}$$

and we use the resulting long sequence of homology groups for the computation. By the last lemma, apart from the lowest degree every third homology group in that sequence is trivial so that $H_k(\Gamma, C^1) \cong H_{k+1}(\Gamma, \mathbb{Z})$ for $k > 0$. The remaining part of the sequence has the form $0 \to \mathbb{Z}^N \to H_0(\Gamma, C^1) \to \mathbb{Z}^L \to \mathbb{Z} \to 0$ and hence $H_0(\Gamma, C^1)$ has no torsion and equals $\mathbb{Z}^{N+L-1}$. q.e.d.

Note that we did not need to know explicitly the morphisms involved.

Theorem 7 *Let $d^\perp = 2$. Then $H^{d-k}(\mathcal{T}) \cong H_k(\Gamma, C^2)$ and*

$$H_k(\Gamma, C^2) \cong \begin{cases} \mathbb{Z}^{\binom{N}{k+2} - r_k - r_{k+1} + \sum_{\alpha \in I} \binom{\nu_\alpha}{k+1}} & \textit{for} \quad k > 0, \\ \mathbb{Z}^{1 - N + \binom{N}{2} - L - r_1 + \sum_{\alpha \in I} (\nu_\alpha + l_\alpha - 1)} & \textit{for} \quad k = 0, \end{cases} \tag{12}$$

where ν_α is the rank of Γ^α (the stabilizer of H_α), l_α the number of Γ^α-orbits of intersection-cut-points in H_α, and r_k the rank of the module generated by the submodules $\Lambda_{k+1}\Gamma^\alpha \subset \Lambda_{k+1}\Gamma$ for all $\alpha \in I$.

Proof: Inserting $C^0_0 := \delta(C^1)$ we break the exact sequence (8) into two short ones

$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & C^2 \arrow[r, "\delta"] & C^1 \arrow[rr, "\delta"] \arrow[dr] & & C^0 \arrow[r, "\Sigma"] & \mathbb{Z} \arrow[r] & 0. \\ & & & C^0_0 \arrow[ur] \arrow[dr] & & & \\ & & 0 \arrow[ur] & & 0 & & \end{tikzcd}$$

$0 \to C^0_0 \hookrightarrow C^0 \to \mathbb{Z} \to 0$ can be treated as in the case $d^\perp = 1$. Hence

$$H_k(\Gamma, C^0_0) \cong \begin{cases} \mathbb{Z}^{\binom{N}{k+1}} & \text{for} \quad k > 0, \\ \mathbb{Z}^{N+L-1} & \text{for} \quad k = 0. \end{cases} \tag{13}$$

Furthermore, C^1 is a direct sum of Γ-modules, namely

$$C^1 = \bigoplus_{\alpha \in I} C^1_{\mathcal{C}_\alpha}, \tag{14}$$

$\mathcal{C}_\alpha = \{H_\alpha + \gamma | \gamma \in \Gamma\}$. As before we denote by Γ^α the stabiliser of H_α and we let $\hat{\Gamma}^\alpha$ be a complimentary subgroup, i.e. $\Gamma = \Gamma^\alpha \oplus \hat{\Gamma}^\alpha$ (Γ/Γ^α has no torsion). Now let $\tilde{\mathcal{C}}_\alpha := \{H \cap H_\alpha | H \in \mathcal{C}_\alpha, \operatorname{codim} H \cap H_\alpha = 1\}$, a set of points in H_α which is invariant under Γ^α, and abbreviate $C^n_{\tilde{\mathcal{C}}_\alpha} = C^n_\alpha$. It is naturally a Γ^α-module. Then $C^n_{\mathcal{C}_\alpha} \cong C^n_\alpha \otimes Z\hat{\Gamma}^\alpha$ and the action of $\Gamma^\alpha \oplus \hat{\Gamma}^\alpha$ is such that the first (second) summand acts only on the first (second) factor. In particular, $C^n_{\mathcal{C}_\alpha} \otimes_\Gamma \mathbb{Z}\Gamma \cong C^n_\alpha \otimes_{\Gamma^\alpha} Z\hat{\Gamma} \otimes Z\hat{\Gamma}^\alpha$ which implies

$$H_*(\Gamma, C^1_{\mathcal{C}_\alpha}) \cong H_*(\Gamma^\alpha, C^1_\alpha). \tag{15}$$

Restricting the boundary maps δ and Σ to C^n_α we get a short exact sequence

$$0 \to C^1_\alpha \stackrel{\delta_\alpha}{\to} C^0_\alpha \stackrel{\Sigma_\alpha}{\to} \mathbb{Z} \to 0. \tag{16}$$

As in Theorem 6 and combined with (14,15) we obtain

$$H_k(\Gamma, C^1) \cong \begin{cases} \mathbb{Z}^{\Sigma_\alpha \binom{\nu_\alpha}{k+1}} & \text{for} \quad k > 0, \\ \mathbb{Z}^{\Sigma_\alpha (\nu_\alpha + l_\alpha - 1)} & \text{for} \quad k = 0. \end{cases} \tag{17}$$

with ν_α and l_α as in the statement. Note that the l_α are all finite, since we required L to be finite. Eqns. (13,17) give us part of the information needed to determine $H_*(\Gamma, C^2)$ from the exact sequence

$$0 \to C^2 \stackrel{\delta}{\to} C^1 \stackrel{\delta}{\to} C^0_0 \to 0 \tag{18}$$

but we have to determine explicitly one morphism, because we have no longer enough trivial groups in the resulting long exact sequence of homology groups. We shall therefore determine the induced morphism

$$\beta_* := \delta_* : H_*(\Gamma, C^1) \to H_*(\Gamma, C^0_0). \tag{19}$$

For that look at the following commuting diagram

$$\begin{array}{ccccccc} 0 \to & C^1_\alpha \otimes \mathbb{Z}\hat{\Gamma}^\alpha & \stackrel{\delta_\alpha \otimes 1}{\to} & C^0_\alpha \otimes \mathbb{Z}\hat{\Gamma}^\alpha & \stackrel{\Sigma_\alpha \otimes 1}{\to} & \mathbb{Z}\hat{\Gamma}^\alpha & \to 0 \\ & \downarrow \delta_\alpha \otimes 1 & & \downarrow & & \downarrow \Sigma^\alpha & \\ 0 \to & C^0_0 & \hookrightarrow & C^0 & \stackrel{\Sigma}{\to} & \mathbb{Z} & \to 0 \end{array}$$

where the middle verticle arrow is the inclusion, the right vertical arrow the sum of the coefficients, $\Sigma^\alpha[\gamma] = 1$, and the left vertical arrow the map of interest. In fact, β_k is the direct sum over all α of $(\delta_\alpha \otimes 1)_k : H_k(\Gamma, C^1_\alpha \otimes \mathbb{Z}\hat{\Gamma}^\alpha) \to H_k(\Gamma, C^0_0)$. The above commutative diagram gives rise to two long exact sequences of homology groups together with vertical maps, all commuting,

$(\delta_\alpha \otimes 1)_*$ being one of them. Now we use that for $k > 0$, $H_k(\Gamma, C^0_\alpha \otimes \mathbb{Z}\hat{\Gamma}^\alpha) = H_k(\Gamma, C^0) = 0$ so that we can express $(\delta_\alpha \otimes 1)_*$ through Σ^α_*. In fact, under the identifications $H_k(\Gamma, C^0_\alpha \otimes \mathbb{Z}\hat{\Gamma}^\alpha) \cong H_{k+1}(\Gamma, \mathbb{Z}\hat{\Gamma}^\alpha)$ and $H_k(\Gamma, C^0_0) \cong H_{k+1}(\Gamma, \mathbb{Z})$, which are valid for $k > 0$, we get

$$(\delta_\alpha \otimes 1)_k = \Sigma^\alpha_{k+1}.$$

By Lemma 7 the map Σ^α_k becomes the embedding $\Lambda_k\Gamma^\alpha \hookrightarrow \Lambda_k\Gamma$ under the above identifications. For $k > 0$ therefore, the rank of β_k is equal to the rank of the span of the submodules $\Lambda_{k+1}\Gamma^\alpha$, $\alpha \in I$, in $\Lambda_{k+1}\Gamma$. This is r_k. The long exact sequence corresponding to (18) implies

$$H_k(\Gamma, C^2) \cong H_{k+1}(\Gamma, C^0_0)/\mathrm{im}\,\beta_{k+1} \oplus H_k(\Gamma, C^1) \cap \ker \beta_k.$$

Since, for $k > 0$, $\dim H_k(\Gamma, C^1) \cap \ker \beta_k = \dim H_k(\Gamma, C^1) - r_k$ we get the result of the theorem (the case $k = 0$ needs a little extra care), provided the homology groups are torsion free. That this is the case we know from [12]. q.e.d.

6.4 Example: octagonal tilings

We provide here one example, the octagonal tilings. A whole list of codimension 2 examples will be presented elsewhere [34].

The (undecorated) octagonal tilings are two dimensional tilings which may be constructed from the data $(\mathbb{Z}^4, 0, E)$, the four dimensional integer lattice $\mathbb{Z}^4$ (with standard basis $\{e_i\}_{i=1,\ldots,4}$) and the two dimensional invariant subspace of the eightfold symmetry C_8: $e_i \mapsto e_{i+1}$ for $i = 1, 2, 3$ and $e_4 \mapsto -e_1$ on which C_8 acts by rotation around $\frac{\pi}{4}$ [35, 36]. It consists of squares and 45^0-rhombi all edges having equal length. $E^\perp$ is, of course, also an invariant subspace of the eightfold symmetry and the cut-planes (which are lines) are well known, they are the tangents to the boundary faces of the projection of the unit cube into $E^\perp$ which is a regular octagon. They are translates under $\pi^\perp(\mathbb{Z}^4)$ of the four lines spanned by $e_i^\perp$ which form an orbit under C_8 (we may ignore the shift by δ). From these lines we get all our information, the numbers L, ν_i, l_i, $I = \{1, \ldots, 4\}$, and r_1, r_2, r_3 (higher r_k are unecessary since $d = 2$). Usually it is not so easy to determine L but in our case it is easy to see that apart from the orbit of intersection-cut-point at 0 there only two other ones: the orbit of $\frac{1}{\sqrt{2}}(e_1^\perp + e_3^\perp)$ and that of $\frac{1}{\sqrt{2}}(e_2^\perp + e_4^\perp)$. Hence $L = 3$. Clearly, Γ^1 is spanned by $e_1^\perp$ and $e_2^\perp - e_4^\perp$. Hence $\nu_1 = 2$ and $l_1 = 2$ which carries over to all i by symmetry. Finally, $r_1 = 3$ and $r_k = 0$ for $k \geq 2$ as

$\nu_i = 2$. Inserting the numbers yields

$$\begin{aligned} H^0(\mathcal{T}) &= \mathbb{Z} \\ H^1(\mathcal{T}) &= \mathbb{Z}^5 \\ H^2(\mathcal{T}) &= \mathbb{Z}^9. \end{aligned}$$

This result is in agreement with a calculation we made using Anderson and Putnam's method [9] (the octagonal tiling is substitutional, its substitution is given in Fig. 1).

7 The non-commutative approach

This section is included to connect the cohomology of the tiling with its non-commutative topological invariants. Starting point of the non-commutative approach is the observation that, when translationally congruent tilings are identified, one is forced to consider non-Hausdorff spaces. In fact, for a (completely) non-periodic tiling $\mathcal{T}$, no two points in $M\mathcal{T}/\mathbb{R}^d$ can be separated by open neighbourhoods (in the quotient topology). Connes non commutative geometry was motivated from the desire to analyse such spaces. In the non-commutative topological approach [37] one studies the properties of the (non-commutative) C^*-algebra associated with the dynamical system $(M\mathcal{T}, \mathbb{R}^d)$. This algebra is the crossed product algebra of $C(M\mathcal{T})$, the algebra of continuous functions over $M\mathcal{T}$, with the group $\mathbb{R}^d$. We denote it by $C(M\mathcal{T}) \times \mathbb{R}^d$. Topologically, this algebra may be described by its K-theory [38, 39]. It turns out that the K-groups are closely related to the Czech-cohomology of $M\mathcal{T}$. K-groups, however, contain additional information in form of a natural order structure on the K_0-group and this is the advantage of the non-commutative approach. And we have seen in the first example that cohomology without extra structure is not a very fine invariant.

Equally well from the mathematical point of view, but from a physically motivated point of view less complicated, is to work with the formulation of the quotient $M\mathcal{T}/\mathbb{R}^d$ as the space of orbits of the tiling groupoid $\mathcal{G}_\mathcal{T}$ (or of $\mathcal{GT}$). The C^*-algebra whose K-theory provides the non-commutative topological invariant is then the corresponding groupoid-C^*-algebra [24, 15]. The importance of this groupoid C^*-algebra for physical systems lies in the fact that it provides an abstract definition of the algebra of observables [15, 10] for particles moving in the tiling. A topological invariant of it governs the gap labelling: the scaled ordered K_0-group and its image under a tracial state.

If $\mathcal{T}$ is a projection method tiling $\mathcal{GT}$ (and $\mathcal{G}_\mathcal{T}$) are equivalent in the sense of Muhly et al. to the transformation groupoid $\mathcal{G}(X, G_1)$. This is proven directly in [16] but it also follows from our analysis of Sect. 3.4 where similarity of the two groupoids has been shown. By application of the theory of Muhly etal. [25] we obtain:

Theorem 8 *The K-groups of $C(M\mathcal{T}) \times \mathbb{R}^d$ and of the groupoid-C^*-algebras of $\mathcal{G}_\mathcal{T}$ and of $\mathcal{G}(X, G_1)$ are isomorphic the isomorphism preserving the order on the K_0-group.*

The isomorphism between the first two K-groups was already observed in [9]. Most important, in the present case there is a relation between K-theory and cohomology [12]:

Theorem 9 *Let $(X, \mathbb{Z}^d)$ be a $\mathbb{Z}^d$-dynamical system where X is homeomorphic to the Cantor set. Then*

$$K_i(C(X) \times \mathbb{Z}^d) \cong \bigoplus_j H^{d-i+2j}(\mathbb{Z}^d, C(X, \mathbb{Z}))$$

as unordered groups.

Thus, in view of Corollary 3:

Corollary 6 *For canonical projection method tilings*

$$K_i(C^*(\mathcal{G}\mathcal{T})) \cong \bigoplus_j H^{d-i+2j}(\mathcal{T})$$

as unordered groups.

It is an interesting question whether this result is true for general finite type tilings. As already mentioned, the isomorphism of the Corollary neglects a lot of information contained in the K-groups, namely order on K_0. One can cure for this at least partly by looking at the order on $H^d(\mathcal{T})$, the group of coinvariants, which is induced by the unique invariant probability measure on $\Omega\mathcal{T}$ (the dynamical system $(M\mathcal{T}, \mathbb{R}^d)$ is uniquely ergodic). That measure defines a group homomorphism $C_c(E_c^\perp, \mathbb{Z}) \to \mathbb{R}$ which by invariance induces a homomorphism $\tau : H^d(\mathcal{T}) \to \mathbb{R}$. The subset $\tau^{-1}(\mathbb{R}^{>0})$ is closed under addition and defines a positive cone of $H^d(\mathcal{T})$ which sits inside the positive cone of $K_0(C^*(\mathcal{G}\mathcal{T})$ and contains already a good portion of the information, including that needed for the standard gap-labelling. In fact, for $d = 1$, where $H^1(\mathcal{T}) = K_0(C^*(\mathcal{G}\mathcal{T}))$, this order is precisely the order defined on the K_0-group in the standard way [38].

With this information at hand let us come back to our first example, $W = \mathbb{Z}^2$, $w = 0$, $d = 1$, and E specified by an irrational number ν. To keep track of this dependence we write $\mathcal{T}^{(\nu)}$ for a canonical projection method tiling obtained from these data. The unique invariant probabibity measure on $\Omega\mathcal{T}^{(\nu)}$ is the Lebesgue measure on $E^\perp$ normalized in such the way that $\pi^\perp(\gamma)$ (the projection of the unit cell) has measure 1. From this we see that with $[1_{[a,b]}]$ denoting the coinvariant class of $1_{[a,b]}$,

$$\tau([1_{[a,b]}]) = \frac{b-a}{1+\nu}.$$

In particular, the rank of $\tau(H^1(\mathcal{T}^{(\nu)}))$ is 2 and hence $H^1(\mathcal{T}^{(\nu)}) \cong \mathbb{Z}^2$. Now, $\tau(n[1_{[0,1]}] + m[1_{[0,\nu]}]) > 0$, for $n, m \in \mathbb{Z}$, whenever (n, m) has positive scalar product with $(1, \nu)$ and hence belongs to the upper right half space defined by $E^\perp$ in $\mathbb{R}^2$. It follows that $K_0(\mathcal{G}\mathcal{T}^{(\nu)})$ is order isomorphic to $K_0(\mathcal{G}\mathcal{T}^{(\nu')})$ whenever there exists a matrix $M \in GL(2, \mathbb{Z})$ such that $\nu' = \frac{M_{11}\nu+M_{12}}{M_{21}\nu+M_{22}}$. Note that in the above cases τ is injective. We remark without further explanation that the order unit improves the invariant even more. $K_0(\mathcal{G}\mathcal{T}^{(\nu)})$ and $K_0(\mathcal{G}\mathcal{T}^{(\nu')})$ are order isomorphic with isomorphism preserving the order unit whenever $\nu' = \pm\nu$.

Coming back to our second example, $W = \mathbb{Z}^3$, $w = 0$, $d = 1$, again the unique invariant probabibity measure on $\Omega\mathcal{T}$ is the Lebesgues measure on $E^\perp$ normalized in such the way that $\pi^\perp(\gamma)$ has measure 1. Thus all the elements $[1_{U+\lambda_1+\lambda_2}] - [1_U]$ are mapped to 0 by τ. In fact, one can show that the image of τ is finitely generated so that in this case all but finitely many generators of the K_0-group are neither positive nor negative, i.e. they are infinitesimal.

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