The Twofold Way
a Short Disquisition of LEP Physics *

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Abstract

A condensed survey of various aspects of LEP 1 physics, $e^+e^- \to \overline{f}f$, is presented, with a critical examination of the ingredients that are available for theoretical predictions. A prototype of comparisons for 2f calculations can be found in hep-ph/9902452. [1]

1 Prolegomena

Consider the LEP 1 measurements of $\sigma_{\mu,\mathrm{had}}$ and A^l_{FB} as an example to discuss the data analysis strategy: each LEP experiment extracts a set of

$$\mathrm{PO} \stackrel{\mathrm{def}}{=} \{M_Z, \Gamma_Z, R_{e,\mu,\tau}, A^{0,e,\mu,\tau}_{\mathrm{FB}}\}, \tag{1}$$

from their measured σ and A_{FB} (*realistic observables*). The four sets of PO are combined, taking correlated errors between the LEP experiments into account, in order to obtain a LEP-average set of PO. The latter is then interpreted, for example within the frame-work of the Minimal Standard Model. The practical attitude of the experiments is to stay with a *Model-Independent* (MI) fit, i.e.

$$\text{from} \quad \mathrm{RO} \to \mathrm{PO} \oplus \text{a SM remnant} \tag{2}$$

for each experiment, and these sets of PO are averaged. The extraction of Lagrangian parameters,

$$M_Z, m_t, M_H, \alpha_S(M_Z^2), \quad \text{and} \quad \alpha(M_Z^2), \tag{3}$$

is based on the LEP-averaged PO. Since PO are determined by fitting RO, one has to clarify what is actually taken from the SM, such as imaginary parts, parts which have been moved to interference terms and photon-exchange terms making the MI results dependent on the SM.

An alternative way to extract SM parameters from the LEP measurements is to average directly the SM parameters which have been obtained by each experiment through a SM fit to its own RO. This alternative should also be pursued by the experiments.

The 2f emergency kit is illustrated in Fig. 1.

2 Exegesis of comparison

All experimental informations are passed through *decoders* like `TOPAZ0` and `ZFITTER`. Therefore, it is of the upmost importance to critically upgrade and to compare the complete `TOPAZ0` and `ZFITTER` predictions [3] with a particular emphasis on demanding the following criteria:

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[1] The numerical results are based on the work of `TOPAZ0` [1] and `ZFITTER` [2] teams. At present, the following physicists are active members of the two teams: `TOPAZ0`: G. Montagna, O. Nicrosini, G. Passarino and F. Piccinini; `ZFITTER`: D. Bardin, P. Christova, M. Jack, L. Kalinovskaya, A. Olshevski, S. Riemann, T. Riemann.

- Comparisons, after the upgrading, should be consistently better than what they where in earlier studies.
- At the peak all relative deviations among total cross-sections and absolute deviations among asymmetries should be below 0.1 per-mill.
- At the wings, typically $\sqrt{s} = M_Z \pm 1.8$ GeV, they should be below 0.3 per-mill.

It is important to observe that today's comparisons are at the level of 10^{-4}, never been attempted before.

3 PO, RO and all that

Within the context of the SM the RO are described in terms of some set of amplitudes

$$A_{\rm SM} = A_\gamma + A_Z + \text{non-factorizable}, \tag{4}$$

where the last term is due to all those contributions that do not factorize into the Born-like amplitude. Once the matrix element $A_{\rm SM}$ is computed, squared and integrated, a convolution with initial- and final-state QED and final-state QCD radiation follows:

$$\sigma(s) = \int dz\; H_{\rm in}(z,s)\, H_{\rm fin}(z,s)\, \hat{\sigma}(zs), \tag{5}$$

After de-convoluting RO of QED and QCD radiation aN set of approximations transform RO into PO.

4 The inner circle

The effective couplings $\in C$, due to the imaginary parts of the diagrams. In the past this fact had some relevance only for realistic distributions while for PO they were *conventionally* taken to be real. The above convention has changed lately with the introduction of next-to-leading (NLO) corrections: imaginary parts, although not NLO in a strict sense, are sizeable two-loop effects.

The explicit formulae for the $Zf\overline{f}$ vertex are always written starting from a Born-like form of a pre-factor $\times$ fermionic current, where the Born parameters are promoted to effective, scale-dependent parameters,

$$\rho^f_Z\, \gamma_\mu \left[\left(I^{(3)}_f + i\, a_L \right) \gamma_+ - 2\, Q_f \kappa^f_Z s^2 + i\, a_Q \right] = \gamma_\mu \left(\mathcal{G}^f_V + \mathcal{G}^f_A\, \gamma_5 \right), \tag{6}$$

where $\gamma_+ = 1 + \gamma_5$ and $a_{Q,L}$ are the SM imaginary parts. By definition, the total and partial widths of the Z boson include also QED and QCD corrections. The partial decay width is therefore described by the following expression:

$$\Gamma_f \equiv \Gamma\left(Z \to f\overline{f}\right) = 4\, c_f\, \Gamma_0 \left(|\mathcal{G}^f_V|^2\, R^f_V + |\mathcal{G}^f_A|^2\, R^f_A \right) + \Delta_{\rm EW/QCD}\,, \tag{7}$$

where $c_f = 1$ or 3 for leptons or quarks $(f = l, q)$, and R^f_V and R^f_A describe the final state QED and QCD corrections and take into account the fermion mass m_f. The last term,

$$\Delta_{\rm EW/QCD} = \Gamma^{(2)}_{\rm EW/QCD} - \frac{\alpha_S}{\pi}\, \Gamma^{(1)}_{\rm EW}\,, \tag{8}$$

accounts for the non-factorizable corrections. The standard partial width, Γ_0, is

$$\Gamma_0 = \frac{G_F M^3_Z}{24\sqrt{2}\,\pi} = 82.945(7)\ \text{MeV}. \tag{9}$$

The peak hadronic and leptonic cross-sections are defined by

$$\sigma^0_h = 12\pi\, \frac{\Gamma_e \Gamma_h}{M^2_Z \Gamma^2_Z} \qquad \sigma^0_\ell = 12\pi\, \frac{\Gamma_e \Gamma_l}{M^2_Z \Gamma^2_Z}\,, \tag{10}$$

where Γ_Z is the total decay width of the Z boson, i.e, the sum of all partial decay widths. The effective electroweak mixing angles (*effective sinuses*) are always defined by

$$4\,|Q_f|\sin^2\theta_{\rm eff}^f = 1 - \frac{{\rm Re}\,\mathcal{G}_V^f}{{\rm Re}\,\mathcal{G}_A^f} = 1 - \frac{g_V^f}{g_A^f}\;, \tag{11}$$

where we define

$$g_V^f = {\rm Re}\,\mathcal{G}_V^f, \qquad g_A^f = {\rm Re}\,\mathcal{G}_A^f. \tag{12}$$

The forward-backward asymmetry $A_{\rm FB}$ is defined via

$$A_{\rm FB} = \frac{\sigma_{\rm F} - \sigma_{\rm B}}{\sigma_{\rm F} + \sigma_{\rm B}}\;, \qquad \sigma_{\rm T} = \sigma_{\rm F} + \sigma_{\rm B}\;, \tag{13}$$

where $\sigma_{\rm F}$ and $\sigma_{\rm B}$ are the cross sections for forward and backward scattering, respectively. Before analyzing the forward-backward asymmetries we have to describe the inclusion of imaginary parts. $A_{\rm FB}$ is calculated as

$$A_{\rm FB} = \frac{3}{4}\frac{\sigma_{\rm VA}}{\sigma_{\rm T}}\;, \tag{14}$$

where

$$\begin{aligned}\sigma_{\rm VA} \;=\;& \frac{G_F M_Z^2}{\sqrt{2}}\sqrt{\rho_e\rho_f}\,Q_eQ_f{\rm Re}\Big[\alpha^*(M_Z^2)\mathcal{G}_V^e\mathcal{G}_A^f\chi(s)\Big]\\ &+\frac{G_F^2M_Z^4}{8\,\pi}\,\rho_e\rho_f{\rm Re}\Big[\mathcal{G}_V^e\,\big(\mathcal{G}_A^e\big)^*\Big]{\rm Re}\Big[\mathcal{G}_V^f\,\big(\mathcal{G}_A^f\big)^*\Big]s\,|\chi(s)|^2.\end{aligned} \tag{15}$$

This result is valid in the realization where ρ_f is a real quantity, i.e., the imaginary parts are not resummed in ρ_f. In this case

$$\begin{aligned}\mathcal{G}_V^f &= {\rm Re}\big(\mathcal{G}_V^f\big) + i\,{\rm Im}\big(\mathcal{G}_V^f\big) = g_V^f + i\,{\rm Im}\big(\mathcal{G}_V^f\big)\,,\\ \mathcal{G}_A^f &= I_f^{(3)} + i\,{\rm Im}\big(\mathcal{G}_A^f\big)\,.\end{aligned} \tag{16}$$

Otherwise $\mathcal{G}_A^f = I_f^{(3)}$ is a real quantity but ρ_f is complex valued and Eq.(15) has to be changed accordingly, i.e., we introduce

$$g_V^f = \sqrt{\rho_f}\,v_f\;, \qquad g_A^f = \sqrt{\rho_f}\,I_f^{(3)}\;, \tag{17}$$

with

$$v_f = I_f^{(3)} - 2Q_f\sin^2\theta_{\rm eff}^f\;. \tag{18}$$

For the peak asymmetry, the presence of ρ's is irrelevant since they will cancel in the ratio. We have

$$\hat{A}_{\rm FB}^{\rm 0f} = \frac{3}{4}\,\hat{A}_e\,\hat{A}_f, \qquad \hat{A}_f = \frac{2\,{\rm Re}\big[\mathcal{G}_V^f\,\big(\mathcal{G}_A^f\big)^*\big]}{\big(\big|\mathcal{G}_V^f\big|^2 + \big|\mathcal{G}_A^f\big|^2\big)}\;. \tag{19}$$

The question is what to do with imaginary parts in Eq.(19). For partial widths, as they absorb all corrections, the convention is to use

$$\big|\mathcal{G}_{V,A}^f\big|^2 = \big({\rm Re}\mathcal{G}_{V,A}^f\big)^2 + \big({\rm Im}\mathcal{G}_{V,A}^f\big)^2. \tag{20}$$

On the contrary, the PO peak asymmetry $A_{\rm FB}^{\rm 0f}$ will be defined by an analogy of equation Eq.(19) where *conventionally* imaginary parts are not included

$$A_{\rm FB}^{\rm 0f} = \frac{3}{4}\,\mathcal{A}_e\,\mathcal{A}_f, \qquad \mathcal{A}_f = \frac{2\big(g_V^f\,g_A^f\big)}{\big(g_V^f\big)^2+\big(g_A^f\big)^2}\;. \tag{21}$$

A definition of the PO heavy quark forward-backward asymmetry parameter which would include mass effects is

$$\mathcal{A}_b = \frac{2\, g_{_V}^b g_{_A}^b}{\frac{1}{2}\,(3-\beta^2)\,(g_{_V}^b)^2 + \beta^2\,(g_{_A}^b)^2}\,\beta, \tag{22}$$

where β is the b-quark velocity. The difference is very small, due to an accidental cancellation of the mass corrections between the numerator and denominator of Eq.(22). This occurs for down quarks where $(g_{_V}^b)^2 \approx (g_{_A}^b)^2/2$ and where

$$A_{_{\rm FB}}^{0b} \approx \frac{3}{4}\,\frac{2\,g_{_V}^e g_{_A}^e}{(g_{_V}^e)^2+(g_{_A}^e)^2}\,\frac{2\,g_{_V}^b g_{_A}^b}{(g_{_V}^b)^2+(g_{_A}^b)^2}\,(1+\delta_{\rm mass}),$$

$$\delta_{\rm mass} \approx 4\,\frac{m_q^2}{s}\,\frac{(g_{_A}^b)^2/2-(g_{_V}^b)^2}{(g_{_V}^b)^2+(g_{_A}^b)^2}. \tag{23}$$

Therefore, our definition of the PO forward–backward asymmetry and coupling parameter will be as in Eq.(21).

5 Status of art

The most important upgradings in the PO calculations consist of the inclusion of higher-order QCD corrections, mixed electroweak-QCD corrections [4], NLO two-loop corrections of $\mathcal{O}\left(\alpha^2 m_t^2\right)$ [5]. The only case that is not covered is the one of final b-quarks, because it involves non-universal $\mathcal{O}\left(\alpha^2 m_t^2\right)$ vertex corrections.

Another development in the computation of radiative corrections to the hadronic decay of the Z is contained in two papers [6] , which together provide complete corrections of $\mathcal{O}\left(\alpha\alpha_{_S}\right)$ to $\Gamma(Z \to q\overline{q})$ with $q = u, d, s, c$ and b.

6 Improved I/O Parameters

Since all Renormalization Schemes (RS) use $G_{_F}$, we refer to the recent work of [7] giving an improved value of $G_{_F} = 1.16637(1) \times 10^{-5}\ \mathrm{GeV}^{-2}$. An important issue concerns the evaluation of $\alpha_{\rm QED}$ at the mass of the Z. Define

$$\alpha(M_{_Z}) = \frac{\alpha(0)}{1-\Delta\alpha^{(5)}(M_{_Z})-\Delta_{\rm top}(M_{_Z})-\Delta_{\rm top}^{\alpha\alpha_S}(M_{_Z})}\,, \tag{24}$$

where one has

$$\Delta\alpha^{(5)}(M_{_Z}) = \Delta\alpha_{\rm lept} + \Delta\alpha^{(5)}_{\rm had}. \tag{25}$$

The input parameter is now $\Delta\alpha^{(5)}_{\rm had}$, as it is the contribution with the largest uncertainty, while the calculation of the top contributions to $\Delta\alpha$ is left for the code. Next, include for $\Delta\alpha_{\rm lept}$ the $\mathcal{O}\left(\alpha^3\right)$ terms of [8] and use as default $\Delta\alpha^{(5)}_{\rm had} = 0.0280398$, taken from [9]. Therefore $1/\alpha^{(5)}(M_{_Z}) = 128.877$, to which one must add the $t\bar{t}$ contribution and the $\mathcal{O}\left(\alpha\alpha_{_S}\right)$ correction induced by the $t\bar{t}$ loop with gluon exchange [10].

7 MI Calculus

To summarize the MI ansatz, one starts with the SM, which introduces complex-valued couplings, calculated to some order in perturbation theory; next we define $g_{_V}^f, g_{_A}^f$ as the real parts of the effective couplings and Γ_f as the physical partial width absorbing all radiative corrections including the imaginary parts of the couplings and fermion mass effects. Furthermore,

$$R_q = \frac{\Gamma_q}{\Gamma_{_h}}\,, \qquad R_l = \frac{\Gamma_{_h}}{\Gamma_l}\,, \tag{26}$$

for quarks and leptons, respectively.

The experimental collaborations report PO for the following sets:

$$\left(R_f, A^{0,f}_{\rm FB}\right), \qquad \left(g^f_{_V}, g^f_{_A}\right), \qquad \left(\sin^2\theta^f_{\rm eff}, \rho_f\right). \tag{27}$$

In order to extract $g^f_{_V}, g^f_{_A}$ from Γ_f one has to get the SM-remnant from Eq.(7), all else is trivial. However, the parameter transformation cannot be completely MI, due to the residual SM dependence appearing inside Eq.(7). In conclusion, the flow of the calculation requested by the experimental Collaborations is:

1. pick the Lagrangian parameters $m_t, M_{_H}$ etc. for the explicit calculation of the residual SM-dependent part;
2. perform the SM initialization of everything, such as imaginary parts etc. giving, among other things, the complement $\overline{\rm SM}$;
3. select $g^f_{_V}, g^f_{_A}$;
4. perform a SM-like calculation of Γ_f, but using arbitrary values for $g^f_{_V}, g^f_{_A}$, and only the rest, namely

$$R^f_{_V}, \quad R^f_{_A}, \qquad \Delta_{\rm EW/QCD}, \qquad {\rm Im}\,\mathcal{G}^f_{_V}, \quad {\rm Im}\,\mathcal{G}^f_{_A}, \tag{28}$$

from the SM.

An example of the parameter transformations is the following: starting from $M_{_Z}, \Gamma_{_Z}, R_{e,\mu,\tau}$ and $A^{0,e,\mu,\tau}_{\rm FB}$ we first obtain

$$\Gamma_e = M_{_Z}\Gamma_{_Z}\left[\frac{\sigma^0_h}{12\,\pi R_e}\right]^{1/2}, \qquad \Gamma_{_h} = M_{_Z}\Gamma_{_Z}\left[\frac{\sigma^0_h R_e}{12\,\pi}\right]^{1/2}. \tag{29}$$

With

$$\mathcal{A}_e = \frac{2}{\sqrt{3}}\sqrt{A^{0,e}_{\rm FB}}, \qquad \text{and} \quad \gamma = \frac{G_{_F} M^3_{_Z}}{6\,\sqrt{2}\,\pi}, \tag{30}$$

we subtract QED radiation,

$$\Gamma^0_e = \frac{\Gamma_e}{1 + \frac{3}{4}\,\frac{\alpha(M^2_{_Z})}{\pi}}, \tag{31}$$

and get

$$\begin{aligned}
\sin^2\theta^e_{\rm eff} &= \frac{1}{4}\left(1 + \frac{\sqrt{1-\mathcal{A}^2_e}-1}{\mathcal{A}_e}\right), \\
\rho_e &= \frac{\Gamma^0_e}{\gamma}\left[\left(\frac{1}{2} - 2\,\sin^2\theta^e_{\rm eff}\right)^2 + \frac{1}{4} + \left({\rm Im}\,\mathcal{G}^e_{_V}\right)^2 + \left({\rm Im}\,\mathcal{G}^e_{_A}\right)^2\right]^{-1}.
\end{aligned} \tag{32}$$

With

$$\mathcal{A}_f = \frac{4}{3}\,\frac{A^{0,f}_{\rm FB}}{\mathcal{A}_e}, \tag{33}$$

we further obtain

$$\begin{aligned}
\sin^2\theta^f_{\rm eff} &= \frac{1}{4\,|Q_f|}\left(1 + \frac{\sqrt{1-\mathcal{A}^2_f}-1}{\mathcal{A}_f}\right), \\
\rho_f &= \frac{\Gamma^0_f}{\gamma}\left[\left(\frac{1}{2} - 2|Q_f|\sin^2\theta^f_{\rm eff}\right)^2 + \frac{1}{4} + \left({\rm Im}\,\mathcal{G}^f_{_V}\right)^2 + \left({\rm Im}\,\mathcal{G}^f_{_A}\right)^2\right]^{-1},
\end{aligned} \tag{34}$$

where $f = \mu, \tau$. For quarks one should remember to subtract first non-factorizable terms and then to distinguish between R^f_V and R^f_A.

Observable	TOPAZO	ZFITTER	$10^3 \times \frac{T-Z}{T}$
$1/\alpha^{(5)}(M_z)$	128.877	128.877	
$1/\alpha(M_z)$	128.887	128.887	
M_W [GeV]	80.3731	80.3738	-0.009
σ_h^0 [nb]	41.4761	41.4777	-0.04
σ_l^0 [nb]	1.9995	1.9997	-0.12
Γ_h [GeV]	1.74211	1.74223	-0.07
Γ_z [GeV]	2.49549	2.49573	-0.10
Γ_ν [MeV]	167.207	167.234	-0.16
Γ_e [MeV]	83.983	83.995	-0.14
Γ_μ [MeV]	83.983	83.995	-0.14
Γ_τ [MeV]	83.793	83.805	-0.14
Γ_u [MeV]	300.129	300.154	-0.08
Γ_d [MeV]	382.961	382.996	-0.09
Γ_c [MeV]	300.069	300.092	-0.08
Γ_b [MeV]	375.997	375.993	0.01
$\Gamma_{\rm inv}$ [GeV]	0.50162	0.50170	-0.16
R_l	20.7435	20.7420	0.07
R_b^0	0.215829	0.215811	0.08
R_c^0	0.172245	0.172246	-0.01
$\sin^2\theta_{\rm eff}^{\rm lept}$	0.231596	0.231601	-0.02
$\sin^2\theta_{\rm eff}^{b}$	0.232864	0.232950	-0.37
$\sin^2\theta_{\rm eff}^{c}$	0.231491	0.231495	-0.02
ρ_e	1.00513	1.00528	-0.15
ρ_b	0.99413	0.99424	-0.11
ρ_c	1.00582	1.00598	-0.16
Observable	TOPAZO	ZFITTER	$10^3 \times (T-Z)$
$A_{\rm FB}^{0,l}$	0.016084	0.016074	0.01
$A_{\rm FB}^{0,b}$	0.102594	0.102617	-0.02
$A_{\rm FB}^{0,c}$	0.073324	0.073300	0.02
$\mathcal{A}_e$	0.146440	0.146396	0.04
$\mathcal{A}_b$	0.934654	0.934607	0.05
$\mathcal{A}_c$	0.667609	0.667595	0.01

Table 1: Complete table of PO, from TOPAZO and ZFITTER.

8 Results for PO

Having established a common input parameter set (IPS) we now turn to discussing the results for pseudo-observables (PO). The full list contains more PO and is given in Tab.(1), where we have included the relative and absolute difference between TOPAZO and ZFITTER in units of per-mill.

9 Theretical Uncertainties for PO

Here we discuss the theoretical uncertainties associated with PO. In Tabs. 2–3. We give the central value, the minus error and the plus error as predicted by TOPAZO and compare with the current total experimental error if available. The procedure is straightforward: both codes have a preferred calculational setup and options to be varied, options having to do with the remaining theoretical uncertainties and the corresponding implementation of higher order terms. To give an example, we have now LO and NLO two-loop EW corrections but we are still missing the NNLO ones and this allows for variations in the final recipe for ρ_f, etc.

TOPAZO has been run over all the remaining (after implementation of NLO) options and all the results for PO have been collected. We will use

- *central* for PO evaluated at the preferred setup;
- minus error for $\mathrm{PO}_{central} - \min_{\mathrm{opt}}$ PO;
- plus error for $\max_{\mathrm{opt}}$ PO $- \mathrm{PO}_{central}$.

Observable	central	minus error	plus error	total exp. error
$1/\alpha^{(5)}(M_z)$	128.877	-	-	
$1/\alpha(M_z)$	128.887	-	-	
M_W [GeV]	80.3731	5.8 MeV	0.3 MeV	90 MeV
σ_h^0 [nb]	41.4761	1.0 pb	1.6 pb	58 pb
Γ_ν [MeV]	167.207	0.017	0.001	
Γ_e [MeV]	83.983	0.010	0.0005	0.10^*
Γ_μ [MeV]	83.983	0.010	0.0005	
Γ_τ [MeV]	83.793	0.010	0.0005	
Γ_u [MeV]	300.129	0.047	0.013	
Γ_d [MeV]	382.961	0.054	0.010	
Γ_c [MeV]	300.069	0.047	0.013	
Γ_b [MeV]	375.997	0.208	0.077	
Γ_{had} [GeV]	1.74211	0.26 MeV	0.11 MeV	2.3 MeV*
Γ_{inv} [GeV]	0.50162	0.05 MeV	0.002 MeV	1.8 MeV*
Γ_z [GeV]	2.49549	0.34 MeV	0.11 MeV	2.4 MeV

Table 2: Theoretical uncertainties for PO from TOPAZO. $*$) assumes lepton universality.

10 Realistic Observables

The RO are computed in the context of the SM, however one of the goals will be to pin down the definition of PO; the calculation of RO in terms of the defined PO for the purpose of MI fits, showing that for PO with values as calculated in the SM, the RO are *by construction* identical to the full SM RO calculation. The last point requires expressing ρ's and effective mixing angles in terms of PO, assuming the validity of the SM. One should remember that gauge invariance (GI) at the Z pole (on-shell GI) is entirely another story from GI at any arbitrary scale (off-shell GI).

Some of the re-summations that are allowed at the pole and that heavily influence the definition of effective Z couplings are not trivially extendible to the off-shell case. Therefore, the expression for RO = RO(PO), at arbitrary s, requires a careful examination and should be better understood as

$$\mathrm{RO} = \mathrm{RO}\left(\mathrm{PO}, \overline{\mathrm{SM}}\right), \tag{35}$$

that is, for example:

$$\begin{aligned}\sigma_{\mathrm{MI}} = \sigma_{\mathrm{SM}}\Big(R_l, A^{0,l}_{\mathrm{FB}}, \dots \quad &\rightarrow \quad g^f_{V}, g^f_{A} \\ &\rightarrow \quad \rho_f, \sin^2\theta^f_{\mathrm{eff}}; \text{residual SM}\Big)\,.\end{aligned} \tag{36}$$

Observable	central	minus error	plus error	total exp. error
R_l	20.7435	0.0020	0.0013	0.026
R_b^0	0.215829	0.000100	0.000031	0.00074
R_c^0	0.172245	0.000005	0.000024	0.0044
$\sin^2\theta_{\mathrm{eff}}^{\mathrm{lept}}$	0.231596	0.000035	0.000033	0.00018
$\sin^2\theta_{\mathrm{eff}}^{b}$	0.232864	0.000002	0.000048	
$\sin^2\theta_{\mathrm{eff}}^{c}$	0.231491	0.000029	0.000033	
$A_{\mathrm{FB}}^{0,l}$	0.016084	0.000057	0.000060	0.00096
$A_{\mathrm{FB}}^{0,b}$	0.102594	0.000184	0.000195	0.0021
$A_{\mathrm{FB}}^{0,c}$	0.073324	0.000142	0.000149	0.0044
$\mathcal{A}_e$	0.146440	0.000259	0.000275	0.0051
$\mathcal{A}_b$	0.934654	0.000032	0.000005	0.035
$\mathcal{A}_c$	0.667609	0.000114	0.000103	0.040
ρ_e	1.00513	0.00010	0.000005	0.0012
ρ_b	0.99413	0.00048	0.000001	
ρ_c	1.00582	0.00010	0.000005	

Table 3: Theoretical uncertainties for PO from `TOPAZO`.

As long as the procedure does not violate GI and the PO are given SM values, there is nothing wrong with the calculations. It is clear that in this case the SM RO coincide with the MI RO.

11 Final-State Radiation

One should realize that s' (s' being the centre-of-mass energy of the e^+e^- system after initial state radiation) is not equivalent to the invariant mass of the final-state $f\overline{f}$ system, $M^2(f\overline{f})$, due to final state QED and QCD radiation. Furthermore, in the presence of a s'-cut the correction for final state QED radiation is simply

$$R_{\mathrm{QED}}^{\mathrm{FS}} = \frac{3}{4}\,Q_f^2\,\frac{\alpha(s)}{\pi}\,, \tag{37}$$

while for a M^2-cut the correction is more complicated, see [11].

For full angular acceptance one derives the following corrections:

$$\begin{aligned}\sigma\left(s\right) \quad = \quad & \frac{\alpha}{4\pi}Q_f^2\sigma^0\left(s\right)\Bigg\{-2x^2+4\Bigg[\left(z+\frac{z^2}{2}+2\ln x\right)\ln\frac{s}{m_f^2} \\ & +z\left(1+\frac{z}{2}\right)\ln z+2\zeta(2)-2\mathrm{Li}_2\left(x\right)-2\ln x+\frac{5}{4}-3z-\frac{z^2}{4}\Bigg]\Bigg\}.\end{aligned}$$

Here we have introduced

$$x = 1-z, \qquad z = M^2\left(f\overline{f}\right)/s. \tag{38}$$

For an s'-cut both QED and QCD final-state radiation are included through an inclusive correction factor. For M^2-cut the result remains perfectly defined for leptons, however for hadronic final states there is a problem. This has to do with QCD final-state corrections. Indeed we face the following situation:

1. for $e^+e^- \to f\overline{f}\gamma$ the exact correction factor is known at $\mathcal{O}\left(\alpha\right)$ even in the presence of a $M^2(f\overline{f})$ cut;

2. the complete set of final-state QCD corrections are known up to $\mathcal{O}\left(\alpha_s^3\right)$ [12] only for the fully inclusive setup, i.e., no cut on the $f\overline{f}$ invariant mass;
3. the mixed two-loop QED/QCD final-state corrections are also known only for a fully inclusive setup [13]
4. at $\mathcal{O}\left(\alpha_s\right)$ QCD final-state corrections in presence of a M^2-cut follow from the analogous QED calculation of[14].

12 Convoluted Realistic Observables

We have the following options:

CA3 Complete RO, with QED initial state radiation implemented through an additive $\mathcal{O}\left(\alpha^3\right)$ radiator [15]

CF3 Complete RO, with QED initial state radiation implemented through a factorized $\mathcal{O}\left(\alpha^3\right)$ radiator [16]

The following two equations define cross-sections and forward-backward asymmetry *convoluted* with ISR:

$$\sigma_{\mathrm{T}}\left(s\right) = \int_{z_0}^{1} dz H\left(z;s\right)\hat{\sigma}_{\mathrm{T}}\left(zs\right), \tag{39}$$

where $z_0 = s_0/s$ and

$$A_{\mathrm{FB}}\left(s\right) = \frac{\pi\alpha^2 Q_e^2 Q_f^2}{\sigma_{\mathrm{tot}}} \int_{z_0}^{1} dz \frac{1}{\left(1+z\right)^2} H_{\mathrm{FB}}\left(z;s\right)\hat{\sigma}_{\mathrm{FB}}\left(zs\right), \tag{40}$$

Note that the so-called *radiator* (or *flux function*), $H\left(z;s\right)$, is known up to terms of order α^3 while H_{FB} is only known up to terms of order α^2. The kernel cross-sections $\hat{\sigma}_{\mathrm{T,FB}}$ should be understood as the improved Born approximation (IBA), corrected with all electroweak and possibly all FSR (QED $\otimes$ QCD) corrections, where all coupling constants and effective vector and axial weak couplings are *running*, i.e., they depend on $s' = zs$.

13 Uncertainty on QED Convolution

We define the following quantities:

$$\delta^{\mathrm{dec}}\left(O\right) = \frac{O}{O^{\mathrm{SD}}} - 1, \qquad \Delta^{\mathrm{dec}}\left(O\right) = O - O^{\mathrm{SD}}. \tag{41}$$

For convenience of the reader we have reproduced in Tab.(4) the results for the CA3 and CF3 mode. From Tab.(4) we derive the absolute differences, for A^{μ}_{FB}, and the relative ones, for cross-sections, between additive and factorized versions of the QED radiators. They are shown in Tab.(5).

14 Initial-Final QED Interference

If initial-final QED (ISR-FSR) interference (IFI) is included in a calculation we have a conceptual problem with the meaning of the s'-cut. In this case the definition of the variable s' is unnatural since one does no longer know the origin of the radiative photon (ISR or FSR). Only a cut on the invariant mass of the final-state $f\overline{f}$ system would make sense. There is another option: to select events with little initial-state radiation one can use a cut on the acollinearity angle θ_{acol}, between the outgoing fermion and anti-fermion. A cut on θ_{acol} is roughly equivalent to a cut on the invariant mass of the $f\overline{f}$ system, indeed one may write

$$\frac{s'}{s} \approx z_{\mathrm{eff}} = \frac{1-\sin(\theta_{\mathrm{acol}}/2)}{1+\sin(\theta_{\mathrm{acol}}/2)}\ , \tag{42}$$

therefore a cut of $\theta_{\rm acol} < 10^\circ$ is roughly corresponding to the request that $s'/s > 0.84$. The inclusion of initial-final QED interference in `TOPAZO` and in `ZFITTER` is done at $\mathcal{O}(\alpha)$, i.e., the $\mathcal{O}(\alpha)$ interference term is added linearly to the cross-sections without entering the convolution with ISR and without cross-talk to FSR. For loose cuts the induced uncertainty is rather small and the effect of the interference itself is minute. On the contrary, when we select a tight acollinearity cut the resulting limit on the energy of the emitted photon becomes more stringent and the effect of interference grows. The corresponding theoretical uncertainty, due to missing higher-order corrections, is therefore expected to be larger.

I/F INT is currently treated dubiously by the experiments: The MC used to extrapolate for efficiency and possibly acceptance do not contain I/F interference (eg, KORALZ in multi-photon mode). Thus extrapolated and quoted results somehow miss I/F interference. There is a claim of some serious discrepancy between `KK` [17] / `TOPAZO` / `ZFITTER` in $\sigma_{\rm int}, A^{\rm int}_{\rm FB}$ for muons (see http://home.cern.ch/jadach) It has to do with *exponentiation* of `ISR` $\otimes$ `FSR`.

	LEP 1 energy in GeV				
	M_Z-3	$M_Z-1.8$	M_Z	$M_Z+1.8$	M_Z+3
$\delta^{\rm dec}(\sigma_\mu^{\rm CA3})$ T	-23.976	-27.616	-26.257	5.356	30.665
$\delta^{\rm dec}(\sigma_\mu^{\rm CA3})$ Z	-24.007	-27.629	-26.261	5.341	30.631
$\delta^{\rm dec}(\sigma_\mu^{\rm CF3})$ T	-23.973	-27.611	-26.253	5.364	30.671
$\delta^{\rm dec}(\sigma_\mu^{\rm CF3})$ Z	-24.000	-27.624	-26.256	5.348	30.637
$\delta^{\rm dec}(\sigma_{\rm had}^{\rm CA3})$ T	-25.867	-28.501	-26.537	4.952	30.564
$\delta^{\rm dec}(\sigma_{\rm had}^{\rm CA3})$ Z	-25.873	-28.503	-26.538	4.945	30.550
$\delta^{\rm dec}(\sigma_{\rm had}^{\rm CF3})$ T	-25.863	-28.497	-26.533	4.959	30.572
$\delta^{\rm dec}(\sigma_{\rm had}^{\rm CF3})$ Z	-25.869	-28.499	-26.533	4.953	30.558
$\Delta^{\rm dec}(A_{\rm FB}^{\mu{\rm CA3}})$ T	-2.190	-1.967	-1.804	-6.292	-11.486
$\Delta^{\rm dec}(A_{\rm FB}^{\mu{\rm CA3}})$ Z	-2.211	-1.975	-1.803	-6.287	-11.480
$\Delta^{\rm dec}(A_{\rm FB}^{\mu{\rm CF3}})$ T	-2.189	-1.966	-1.804	-6.292	-11.487
$\Delta^{\rm dec}(A_{\rm FB}^{\mu{\rm CF3}})$ Z	-2.215	-1.977	-1.803	-6.286	-11.479

Table 4: RO: the effect in % of initial state QED radiation for CA3 and CF3 modes.

	LEP 1 energy in GeV				
	M_Z-3	M_Z-1	M_Z	M_Z+1	M_Z+3
$10^4\times$ (fact/add-1)					
σ_μ	0.44	0.63	0.61	0.72	0.49
	0.88	0.63	0.68	0.72	0.49
$\sigma_{\rm had}$	0.58	0.58	0.64	0.73	0.59
	0.61	0.62	0.67	0.76	0.62
fact-add [pb]					
σ_μ	0.01	0.03	0.09	0.05	0.02
	0.02	0.03	0.10	0.05	0.02
$\sigma_{\rm had}$	0.26	0.56	1.95	1.04	0.48
	0.27	0.60	2.04	1.08	0.51
$10^5\times$ (fact-add)					
$A^\mu_{\rm FB}$	1.00	1.00	0.00	0.00	-1.00
	-4.00	-2.00	0.00	1.00	1.00

Table 5: Absolute and relative differences in `TOPAZO` and in `ZFITTER` for additive and factorized radiators.

15 Note added: upgrading of ISPP

Here we discuss some recent work connected with Initial State Pair Production (ISPP). We consider two different approaches: the first is based on JMS [18], an ansatz for re-summation of both soft photons and soft lepton pairs. For hadron pairs it is corrected numerically with an uncertainty of 30%.

All versions of `TOPAZ0` < 4.4, use an effective, naive, implementation of ISPP ($\mathbf{ONP} = \mathbf{Y}$) based on KKKS [19] interfaced with soft photon re-summation. The agreement with `JMS` is perfect around the resonance but quickly deteriorates on the high energy side.

In the latest version of `TOPAZ0`, the variable **ONP** has the additional value of $\mathbf{ONP} = \mathbf{I}$, where

- KF approach [20] for virtual pairs and for soft and exponentiated pairs is used; it is also applicable to hadron pairs where we use KKKS results for $\mathcal{O}\left(\alpha^2\right)$ and write them in terms of moments. Then we match it to KF and generalize KF to soft and exponentiated hadrons pairs.

- Next, we use generalized KF for virtual and soft pairs and cut to the same s' cut of IS QED radiation. Once an s' cut is introduced in one place then it should be used everywhere, even for photons + pairs.

- Finally, we use the soft approximation only up to some cut Δ that is compatible with $E \gg \Delta \gg 2\,m$. Above the cut we use KKKS-formalism but not added linearly to the cross-section; instead we use a KKKS correction factor in convolution with IS QED radiation. The radiator used here is a leading-log (LL) radiator, evaluated at the second order or, optionally, at (third order).

Now, some comparison[2], the cross-sections at the MI-point $M_{_Z} = 91.1882\,\mathrm{GeV}, \Gamma_{_Z} = 2.4952\,\mathrm{GeV}$, $\sigma_h^0 = 41.560\,\mathrm{nb}$ and $R_l = 20.728, A^{01}_{\mathrm{FB}} = 0.017319$, for ISPP[3] on and off; ISPP effect was taken as the difference. We report $\sigma_{\mathrm{had}}\,[\mathrm{nb}]$ in Tab.(6): first entry is without pairs, second entry is with pairs and third entry is (with - without).

Next, some SM numbers, i.e. the hadronic cross-section for $M_{_Z} = 91.1867\,\mathrm{GeV}$, $m_t = 173.8\,\mathrm{GeV}$, $M_{_H} = 100\,\mathrm{GeV}$ and $\alpha_{_S}(M_{_Z}) = 0.119$. In Tab.(7) first entry is `ONP = N`, second entry is `ONP = I` and third entry is with/without - 1 in per-mill.

Similar results have been obtained in [21].

For an extension to LEP 2 energies [22] let's start from a simple case, e^+e^- PP-corrections to $e^+e^- \to \overline{b}b$. To avoid confusion we will introduce the following definition: **Multi-Peripheral** or **MP** diagrams; **Initial State Singlet**, or **ISS** diagrams; **Initial State Non-Singlet**, or **ISNS** diagrams; **Final State**, or **FS** diagrams.

Note that we include both γ and Z exchange, so that one could still distinguish between ISNS_γ, ISNS_Z and interference. On top of real pair production one has to include virtual e^+e^- pairs.

So far, `TOPAZ0` (and `ZFITTER` too) only include ISNS_γ plus virtual, with ISS optional. The approximation is well justified around the Z-peak but now we have to move to higher energies, where new thresholds open. Furthermore, one has to add a proper definition of *soft* pairs and of *hard pairs*. We can define Soft(Hard) Invariant Mass pairs, or **SIM(HIM)** pairs, according to some pair mass cut. Of course, one could still use a soft-hard separation based on other variables but, in any case, also very hard SIM pairs are 2f signal.

We need an *operative*, universally accepted, separation of various contributions. Consider again the process $e^+e^- \to \overline{b}be^+e^-$; the total cross-section contributes to three different processes: a) genuine 4f events; b) e^+e^- PP-correction to $e^+e^- \to \overline{b}b$; c) $\overline{b}b$ PP-correction to Bhabha scattering.

A *naive* separation, easy-to-implement, would be the following. Let the process be specified by $e^+e^- \to \overline{b}b(Q^2) + e^+e^-(q^2)$. We can integrate and reduce the cross-section to a two-fold integral with the following boundaries:

$$\begin{aligned} 4\,m_b^2 &< Q^2 < (\sqrt{s} - 2\,m_e)^2, \\ 4\,m_e^2 &< q^2 < (\sqrt{s} - \sqrt{Q^2})^2. \end{aligned} \tag{43}$$

Next, we introduce two cuts z_p, z_s, i.e. primary and secondary cuts and define

[2] `JMS` numbers are a courtesy of B. Pietrzyk and of G. Quast

[3] Here ISPP means **NON-SINGLET** Initial State Pair Production. **SINGLET** pairs are, in principle, subtracted together with two-photon contributions

$\sqrt{s}$[GeV]	JMS	TOPAZ0(ONP=Y)	TOPAZ0(ONP=I)
88.464	5.2297	5.2187	-
	5.2170	5.2059	5.2063
	-0.0127	-0.0128	-0.0124
89.455	10.121	10.105	-
	10.094	10.079	10.079
	-0.027	-0.026	-0.026
90.212	18.238	18.227	-
	18.187	18.178	18.179
	-0.051	-0.049	-0.048
91.207	30.529	30.549	-
	30.449	30.471	30.470
	-0.080	-0.078	-0.079
91.238	30.589	30.609	-
	30.510	30.531	30.530
	-0.079	-0.078	-0.079
91.952	25.173	25.166	-
	25.121	25.117	25.112
	-0.052	-0.049	-0.054
92.952	14.503	14.484	-
	14.488	14.476	14.466
	-0.015	-0.008	-0.018
93.701	10.067	10.051	-
	10.064	10.059	10.046
	-0.003	0.008	-0.005

Table 6: Comparison of TOPAZ0 with JMS-approach for the inclusion of ISPP.

$\sqrt{s}$[GeV]	$s'/s = 0.01$	$s'/s = 0.1$	$s'/s = 0.5$	$s'/s = 0.9$
$M_z - 3$ [GeV]	4.45106	4.44444	4.43769	4.38335
	4.44069	4.43391	4.42705	4.37227
	-2.33	-2.37	-2.40	-2.52
$M_z - 1.8$ [GeV]	9.60037	9.59393	9.58742	9.52026
	9.57589	9.56929	9.56267	9.49497
	-2.55	-2.57	-2.58	-2.66
M_z [GeV]	30.43633	30.43005	30.42316	30.31927
	30.35752	30.35109	30.34409	30.23944
	-2.59	-2.59	-2.60	-2.63
$M_z + 1.8$ [GeV]	14.18153	14.17541	14.16813	13.98373
	14.16470	14.15846	14.15117	13.96556
	-1.19	-1.20	-1.20	-1.30
$M_z + 3$ [GeV]	8.19803	8.19206	8.18498	7.87534
	8.19800	8.19193	8.18460	7.87272
	-0.004	-0.02	-0.05	-0.33

Table 7: SM results with the inclusion of ISPP.

- e^+e^- PP-correction to $e^+e^- \to \overline{b}b$:

$$z_p\, s < Q^2 < (\sqrt{s} - 2\, m_e)^2, \qquad 4\, m_e^2 < q^2 < \min\{z_s\, s\,,\, (\sqrt{s} - \sqrt{Q^2})^2\}. \tag{44}$$

- $\overline{b}b$ PP-correction to Bhabha scattering:

$$z_p\, s < q^2 < (\sqrt{s} - 2\, m_b)^2, \qquad 4\, m_b^2 < Q^2 < \min\{z_s\, s\,,\, (\sqrt{s} - \sqrt{q^2})^2\}. \tag{45}$$

- while the remaining portion of the phase space is background.

At the same time, we must address what has to go in the calculation and what is already subtracted from the data. We have to define the exact content of PP-corrections to be inserted in the upgradings of the semi-analytical codes. Then, one can apply a *theory* correction, actually a subtraction obtained from MC, to go to signal definitions as employed by `TOPAZO/ZFITTER`. This is what is done now, the only problem being that the present signal definition in `TOPAZO/ZFITTER` is not good enough for LEP 2. Thus signal definition must be improved and we have to find an agreement on the optimal signal/background separation; it should be realistic enough to be used in the experimental analysis but simple enough to allow its implementation in the codes.

Typical example? Using invariant-mass cuts one finds relatively large corrections (around 5% or more), most of which ares due to the MP process. So, if the situation remains unaltered, one has to apply a rather large subtraction, which is rather unnatural. Of course, we can take the MP into account, but only if some *reasonable* cut is applied to $M(\overline{b}b)$. At 200 GeV the double resonant $e^+e^- \to ZZ \to e^+e^-\overline{b}b$ cross-section, i.e. the NC02 MC, is about 10^{-2} pb, while the total multi-peripheral, that is not accessible with a NC02 (two diagrams) MC, is about 1 pb compared with a $\sigma_{\rm had}$ of $80 - 90$ pb.

However, this huge increase in cross-section is caused by hadronic two-photon collision processes, where the $\overline{b}b$ system is created by the two photons radiated of the e^+e^-. Thus the invariant mass of the $\overline{b}b$ system will be very low, peaking toward $2\, m_b$. The bulk of the cross section will be around 10 GeV invariant $\overline{b}b$ invariant mass.

Finally we come to the most complicate configurations, $e^+e^- \to \overline{q}q\bar{Q}Q$ or even worst, $e^+e^- \to \overline{q}q\overline{q}q$, e.g. *uuuu* etc. configurations. We have to generalize our previous separation of the 2f signal from the 4f background. There are two options: we require that *at least* one pair has an invariant mass greater than $z_p\, s$. But only the $\overline{q}q$ pairs or, should we also include the $\overline{q}Q$ pairs? And what about the $qQ, \overline{q}Q$ pairs? Or, we require that *at least* one pair has an invariant mass greater than $z_p\, s$ while all remaining configurations have invariant mass less than $z_s\, s$.

Now we come to the really difficult part of the problem. Since arbitrarily low SIM pairs are allowed, in both cases, a parton-level calculation cannot be accurate enough. Under the simplified assumption that one pair (the Q^2 one) is HIM enough and that the remaining one (the q^2 one) is SIM enough we can write the cross-section as

$$\begin{aligned}
\frac{d\sigma_f}{dq^2 dQ^2} &= \left(\frac{\alpha}{\pi}\right)^2 \sigma_f(Q^2) \frac{R_{\rm had}(q^2)}{3\, sq^2}\, f\left(s, Q^2, q^2\right) \\
f\left(s, Q^2, q^2\right) &= \frac{s + (Q^2 + q^2)}{s - Q^2 - q^2}\, \ln \frac{s - Q^2 - q^2 + \sqrt{\lambda}}{s - Q^2 - q^2 - \sqrt{\lambda}} - 2\sqrt{\lambda} \\
\lambda &= \lambda\left(s, Q^2, q^2\right).
\end{aligned} \tag{46}$$

Clearly we face two problems, one conceptual and another practical. The first problem will be referred to as the *double-counting problem.* At the parton level, we have $e^+e^- \to \overline{q}q\bar{Q}Q$ and, for instance we want to sum over q in order to define the Q-line shape. However, when we also sum over Q in order to have the full hadronic line shape, proper care must be taken in order to avoid double-counting. At the parton level this can be done but one should remember that we actually compute $e^+e^- \to \bar{Q}Q +$ hadrons, through $R_{\rm had}(s)$, and the problem is, by far, more severe. Therefore, we have to agree on what should happen once $R_{\rm had}(s)$ is called for low s.

At least in principle, we can introduce three different cuts, i.e. s'/s for initial state QED radiation and z_p, z_s for pair production. From a theoretical point of view, there is no problem here. We should reach an agreement: one common cut ($s'/s = z_{\min}$), two cuts ($s'/s \neq z_{\min}$ but also $s'/s = z_p$ and z_s) or three cuts ($s'/s \neq z_p \neq z_s$).

Tentative conclusions are as follows: the whole 4f must be included to compute the 2f cross-sections; Or, the whole 4f is to be divided into two components, *signal* and *background*. For our purposes their definition is peculiar, *signal* is what you have implemented into the semi-analytical codes. *Background* is what one subtracts by using a MC. We go from one extreme solution to the other:

Background $= \emptyset$ if everything is included in the semi-analytical calculation. Multi-peripheral is an example of something difficult to implement into the semi-analytical approach if low-invariant mass regions are required.

Signal = ISNS, i.e. everything else (large effects) is subtracted by MC. However, using different MC programs would bring to subtractions that differ by some per-cent, which then would have to be regarded as a *theoretical* systematic uncertainty.

One could decide to count as *background* 4f processes from *true* e^+e^- annihilation (e.g. ZZ, WW) and 4f processes via e-gamma collisions (e.g. $(e)eZ, (e)\nu W$). This distinction is based on selecting sub-sets of diagrams to be used for the 2f signal definition. However, we should not define the *signal* according to the kind diagrams it originates from. We need a definition based on measurable quantities, so we should not exclude a set of diagrams, we should exclude the kinematical region where that particular set dominates.

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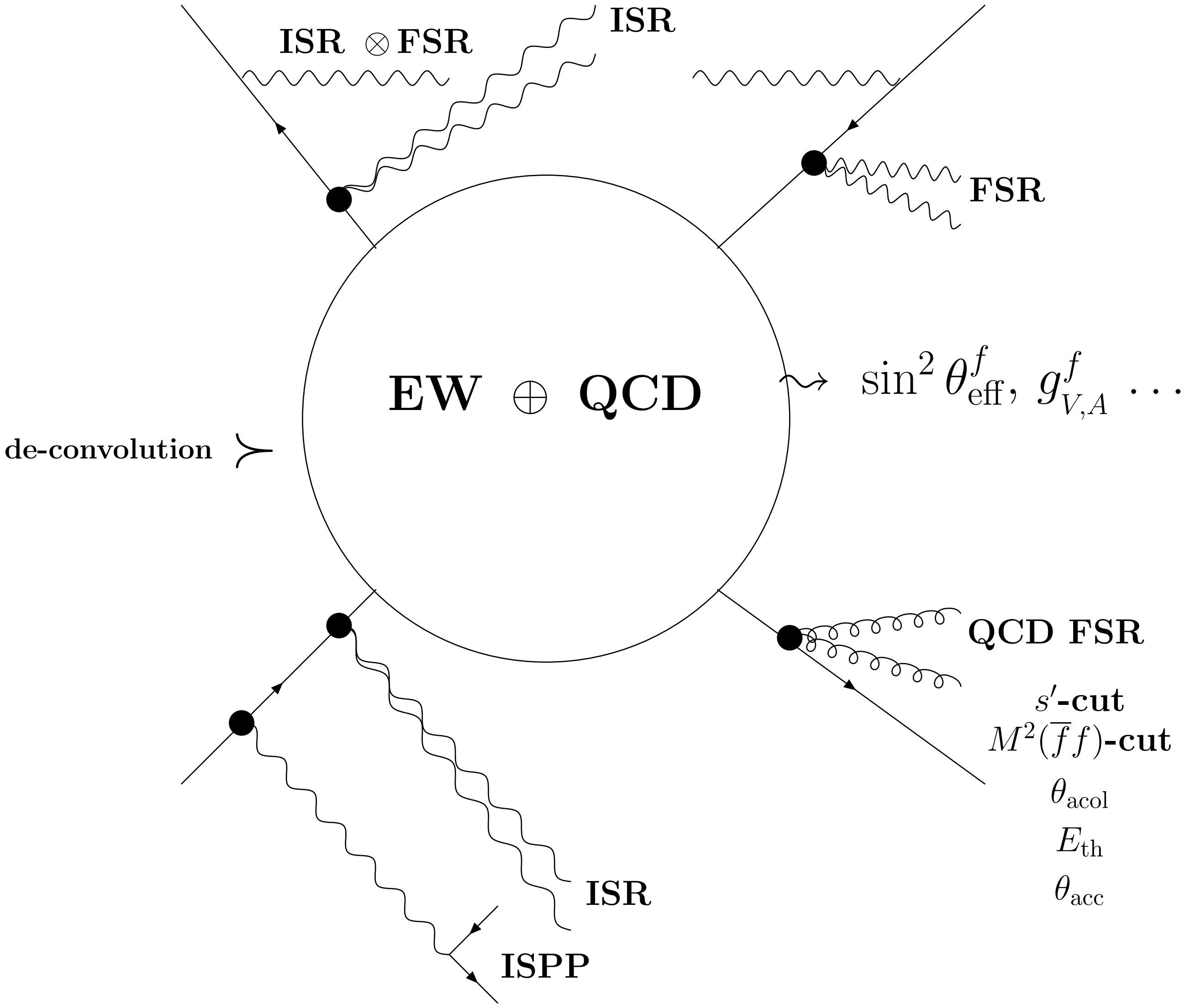

Figure 1: The process $e^+e^- \rightarrow \overline{f}f$.

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