# PROJECTION QUASICRYSTALS II: VERSUS SUBSTITUTION TILINGS

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**Abstract.** This is the second paper in a short series devoted to the study and application of topological invariants for projection (strip) method quasiperiodic tilings and patterns. In the first paper we study in detail a range of commutative and non-commutative spaces that can be associated to such patterns. In this paper we use these constructions to define and discuss topological invariants for projection patterns variously in terms of groupoid cohomology, C* K-theory, Čech cohomology and dynamical or group cohomology. We show that, up to order, all these invariants are essentially the same, hence providing convenient computational methods for the non-commutative invariants. We also show that these invariants give a useful obstruction to a pattern being a substitution system and we analyse the qualitative nature of these invariants with this property in mind.

**Key words.** Quasicrystal, projection method, tiling dynamical system, tiling groupoid, K-theory, dynamical cohomology, substitution tiling.

## §1 Introduction

In [**FHK1**] we study in detail a range of commutative and non-commutative spaces that can be associated to projection (strip) method quasiperiodic tilings and patterns [**KN**], [**dB**], [**KD**]. We discuss the relationships between these various objects and their dependance on the initial projection data. In this paper we use these constructions and descriptions to define, discuss and apply topological invariants for such patterns.

Reserving detail and elaborations for later, recall that a *projection point pattern* $\mathcal{T}$ on $E = \mathbb{R}^d$ is the pattern of points given by the orthogonal projection of points in a strip $(K \times E) \cap \mathbb{Z}^N \subset \mathbb{R}^N$, where $\mathbb{Z}^N$ is the integer lattice in some higher dimensional Euclidean space $\mathbb{R}^N$ containing $E$, and $K \times E$ is the so-called *acceptance strip*, a fattening of $E$ in $\mathbb{R}^N$ defined by some suitably chosen region $K$ in the orthogonal complement $E^\perp$ of $E$ in $\mathbb{R}^N$. The pattern $\mathcal{T}$ thus depends on the dimension $N$, the positioning of $E$ in $\mathbb{R}^N$ and the shape of the *acceptance domain* $K$. When this construction was first made [**dB**], [**KD**] the domain $K$ was taken to be the projected image onto $E^\perp$ of the unit cube $I^N$ in $\mathbb{R}^N$ and this choice gives rise to the so-called *canonical* patterns, but following [**FHK1**] we allow $K$ to be any compact subset of $E^\perp$ which is the closure of its interior (so, with possibly even fractal boundary, a case of current physical interest [**BKS**]).

To such a pattern and data we associate in [**FHK1**] the pattern groupoid $\mathcal{GT}$ and its reduced $C^*$ algebra $C^*(\mathcal{GT})$; we also associate an $\mathbb{R}^d$ dynamical system with space

$M\mathcal{T}$ (the space of all translations on the pattern $\mathcal{T}$ completed with respect to a particular metric $D$) and a $\mathbb{Z}^d$ dynamical system with space a Cantor set $X_{\mathcal{T}}$. Each of these gives rise to a way of attaching an invariant to the pattern $\mathcal{T}$: respectively, we have the continuous groupoid cohomology $H^*(\mathcal{G}\mathcal{T};\mathbb{Z})$, the $C^*$ $K$-theory $K_*(C^*(\mathcal{G}\mathcal{T}))$, the topological $K$-theory of the space $M\mathcal{T}$, $K^*(M\mathcal{T})$, closely related to Čech cohomology of $M\mathcal{T}$, $H^*(M\mathcal{T};\mathbb{Z})$, and the dynamical or group (co)homology of the Cantor system, $H^*(\mathbb{Z}^d; C(X_{\mathcal{T}},\mathbb{Z}))$ or $H_*(\mathbb{Z}^d; C(X_{\mathcal{T}},\mathbb{Z}))$, namely the (co)homology of the group $\mathbb{Z}^d$ with coefficients $C(X_{\mathcal{T}},\mathbb{Z})$ the continuous integer valued functions on $X_{\mathcal{T}}$.

The first main result of this paper is to demonstrate that all these invariants are isomorphic as groups. The non-commutative invariant, $K_*(C^*(\mathcal{G}\mathcal{T}))$, contains the richer structure of an ordered group (and which appears likely to contain information relevant to subsequent investigations), only some of which is recoverable from the other invariants, but on the other hand, the group (co)homology invariants admit greater ease of computation. We give as an example a complete computation of the cohomology invariants of a projection method pattern with $N = d + 1$ and an arbitrary acceptance domain $K$.

The second main result of the paper is to demonstrate that all these common invariants provide an obstruction to the property of *self similarity* of a pattern. We say that a pattern $\mathcal{T}$ is self similar if there is a constant $\lambda > 1$ such that when $\mathcal{T}$ is magnified by $\lambda$ the original pattern can be derived from the magnified one, $\lambda\mathcal{T}$ say, by replacing each element with an arrangement of points determined only by the local structure of $\lambda\mathcal{T}$. Such a pattern can in fact be defined and constructed as a so-called *substitution system* and the Penrose tilings are perhaps the best known examples of patterns that can be constructed as both projection patterns and also as substitution systems. The question naturally arises as to which projection method patterns are self-similar. We show that the $\mathbb{Q}$ rank of the rationalised invariants mentioned above provides a necessary condition on self similarity: if $\mathcal{T}$ is self similar and translationally finite, then the $\mathbb{Q}$ rank of, for example, $H_*(\mathbb{Z}^d; C(X_{\mathcal{T}},\mathbb{Z})) \otimes \mathbb{Q}$ is finite. Recall that a tiling is translationally finite if it has only a finite number of translation classes of tile. Most examples of substitution tilings in the literature are translationally finite, but we note the exceptional example of the "Pin Wheel" tiling **[GS]** **[Rd]**.

Much of the final part of this paper is devoted to giving a qualitative description of the cohomology of canonical projection patterns. The main result is Theorem 8.9 which gives a purely geometric criterion for infinite generation (or infinite rank) of (rationalised) pattern cohomology. As a corollary of this, we show that almost all canonical projection method patterns fail to be substitution systems and in fact for vast swathes of initial data *all* such patterns fail to be self similar.

Nevertheless, there are interesting examples of projection patterns which do exhibit finite rank rational cohomology. The third paper in this series, **[FHK2]**, examines the computation of these invariants in greater detail, using tools and techniques from algebraic topology. We give methods there for the computation of these invariants in finer detail than merely deciding whether they are of finite or infinite rank and we show how to compute the integral invariants for specific, individual patterns. Those methods also shed further light on the qualitative behaviour of the cohomology of general projection patterns. Our series is complemented by an article [**FHK3**] in which some of the arguments have been simplified at the cost of generality and the description of projection tilings is given in terms of the dualization method. This article contains also a computation for the invariants in the case where $N-d$ is smaller than or equal to 2.

The organisation of this paper is as follows. In §2 we review some of the notation and results used from [**FHK1**], and in §3 we introduce the notion of *continuous similarity* of topological groupoids. This is an important equivalence relation for us as continuously similar groupoids have the same groupoid cohomology, and we extend some of the results of [**FHK1**] showing that many of the groupoids constructed there from projection patterns are continuously similar. In §4 we define all our invariants and prove them to be additively equivalent. In §§5 and 6 we illustrate the computability of these invariants by considering projection patterns arising from data with $d = N - 1$. In §7 we establish the role our invariants play in discussing self similarity properties of patterns and prove that a pattern fails to be self similar if the rationaised homology is infinitely generated.

The remainder of the paper is devoted to the canonical case, showing that the invariants we construct are computable and effective discriminators of tiling properties in more general situations. In §8 we describe the topology of the groupoids above in a geometric way, setting up the notation and definitions sufficient to state the main theorem (8.9), giving sufficient conditions under which there are infinitely many independent generators in rationalised homology. From here until the end of §11, our aim is to prove Theorem 8.9. In §§9,10, we construct (in the *indecomposible case* independent generators in rationalised homology, explicitly represented as indicator functions of convex polytopes in Euclidean space. §11 completes the analysis for the decomposible cases. §12 gives some general classes of patterns where these conditions are satisfied, so combining with §7 to show failure of self-similarity in such cases.

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## §2 Review of projection method tilings and patterns

In this section we review the notation and results from **[FHK1]** which we use in this paper. We begin by recalling the basic construction of the objects to be studied.

Suppose that $E$ is a $d$ dimensional subspace of $\mathbb{R}^N$ and $E^\perp$ its orthocomplement. Suppose that $\mathbb{Z}^N$ is placed in canonical position and write $\Delta = E^\perp \cap \mathbb{Z}^N$. Let $\langle \Delta \rangle$ be the real subspace generated by $\Delta$.

We write $\pi$ for the orthonormal projection onto $E$ and $\pi^\perp$ for the projection onto $E^\perp$. We write $Q$ for the Euclidean closure $\overline{E + \mathbb{Z}^N}$.

**Lemma 2.1** (**[FHK1]** §2) *Suppose that $F$ is a subspace of $\mathbb{R}^N$ complimentary to $E$ and containing $\Delta$. Then there is a real subspace $V$ of $F$ with $Q \cap F = V \oplus \widetilde{\Delta}$ as groups, where $\widetilde{\Delta} = Q \cap \langle \Delta \rangle$; moreover, $\widetilde{\Delta}$ contains $\Delta$ with finite index.* •

Now let $K$ be a compact subset of $E^\perp$ which is the closure if its interior in $E^\perp$. Thus the boundary of $K$ in $E^\perp$ is compact and nowhere dense. We write $\Sigma$ for $K + E \subset \mathbb{R}^N$, the *strip with acceptance domain $K$*.

A point $v \in \mathbb{R}^N$ is said to be *non-singular* if the boundary, $\partial\Sigma$, of $\Sigma$ does not intersect $\mathbb{Z}^N + v$. We write $NS$ for the set of non-singular points in $\mathbb{R}^N$. These points are also called *regular* in the literature.

For each non-singular point $v$ this data defines for us two associated patterns; the *strip point pattern* is the set of points $\widetilde{P}_v = \Sigma \cap (\mathbb{Z}^N + v)$ in $\Sigma$ and the *projection point pattern* is the set of points $P_v = \pi(\widetilde{P}_v)$, a subset of $E$.

In the original construction, **[dB]**, **[KD]**, $K = \pi^\perp([0,1]^N)$ and we call this the *canonical* acceptance domain. The *canonical tiling*, defined by **[OKD]** with this choice of acceptance domain, is formed by picking $u \in NS$ and projecting onto $E$ those $d$-dimensional faces lying entirely in $\Sigma$ of the cubical decomposition of $\mathbb{R}^N$ whose vertices lie at the points of the lattice $\mathbb{Z}^N + u$.

Rather than fix attention on just one pattern, we consider instead all its translated images about $E$. If $E \cap \mathbb{Z}^N = 0$ there are no translational symmetries of the pattern and all these images are distinct. However, completion of this set of translations with respect to the following metric encodes topologically properties of their long-range order and their quasiperiodic "symmetries".

Given a locally compact subset, $A$, of $\mathbb{R}^k$, write $A[r] = (A \cap B(r)) \cup \partial B(r)$, where $B(r)$ is the closed ball of radius $r$ centre 0, and $\partial B(r)$ its boundary. Let $d_r$ be the Hausdorff metric defined among closed subsets of $B(r)$ and define a metric on subsets of $\mathbb{R}^k$ by

$$D(A, A') = \inf\{1/(r+1) \mid d_r(A[r], A'[r]) < 1/r\}.$$

We note that this is a metric on the space of all locally compact subsets of $\mathbb{R}^k$, and that the resulting topology is compact.

The general construction is now to take a locally compact set $A$ of $\mathbb{R}^k$ with a closed subgroup, $H \subset \mathbb{R}^k$, acting by translations on $\mathbb{R}^k$. Define $M(A, H)$ to be the closure of the set $\{A + v : v \in H\}$ with respect to the $D$ metric. The space $M(A, H)$ supports an action of $H$ by homeomorphisms and we consider $(M(A, H), H)$ as a dynamical system.

In **[FHK1]** we consider the two dynamical systems $MP_u = M(P_u, E)$ and $M\widetilde{P}_u = M(\widetilde{P}_u, E)$ with the natural $E$ action by translation and prove that the second system is a finite isometric extension of the first $M\widetilde{P}_u \xrightarrow{\pi_*} MP_u$. We regard the two patterns $\widetilde{P}_u$ and $P_u$ to be respectively the *most* elaborate pattern that can be produced from the projection data (short of imposing further decorations not directly connected to the geometry of the construction) and the *least* decorated pattern defined by the projection data.

**Definition 2.2** Given projection data $E$, $\mathbb{R}^N$, $K$ and a point $u \in NS$ , a *projection method pattern* $\mathcal{T}$ in $E$ is a locally compact subset $\mathcal{T} \subset E$ whose associated space $M\mathcal{T} = M(\mathcal{T}, E)$ fits into an $E$-equivariant factorisation $M\widetilde{P}_u \longrightarrow M\mathcal{T} \longrightarrow MP_u$ of $\pi_*$.

We also define the space $\widetilde{\Pi}_u$ as the completion of the set $NS \cap (Q + u)$ with respect to the metric, $\overline{D}(v, w) = \|v - w\| + D(\widetilde{P}_v, \widetilde{P}_w)$. There is a canonical contraction $\widetilde{\mu}: \widetilde{\Pi}_u \longrightarrow Q + u$ which is 1-1 when the image is a point in $NS$.

The variant metric with formula $\|v - w\| + D(P_v, P_w)$ defines a possibly different completion of $NS \cap (Q + u)$, which we write $\Pi_u$ in **[FHK1]**. However, in section 5 of that paper, we explain weak conditions under which the two spaces are identical, and give good reasons to assume this equivalence in general.

**Definition 2.3** We shall say that the pattern $\mathcal{T}$ is *standard* if $E \cap \mathbb{Z}^N = 0$ and the two spaces $\widetilde{\Pi}_u$ and $\Pi_u$ are identical. Recall that, for standard patterns, all the results of **[FHK1]** hold without complication.

The importance of the space $\widetilde{\Pi}_u$ lies in the way it can provide a useful model for $M\mathcal{T}$. To a standard projection method pattern, $\mathcal{T}$, there corresponds a discrete subgroup, $H_{\mathcal{T}}$, of $Q$ which contains the original lattice $\mathbb{Z}^N \subset \mathbb{R}^N$ with finite index (and hence is itself free abelian of rank $N$). This group acts isometrically on $\widetilde{\Pi}_u$ and the action factors by $\widetilde{\mu}$ to the canonical translation action of $H_{\mathcal{T}}$ on $Q+u$. One of the key results of [**FHK1**] is to establish (7.4) the $E$-equivariant equivalence $M\mathcal{T} \equiv \widetilde{\Pi}_u / H_{\mathcal{T}}$ for standard patterns.

It is convenient to split the group $H_{\mathcal{T}}$ in a way which respects the geometry. We can write $H_{\mathcal{T}}$ as the direct sum $G_{\mathcal{T}} \oplus G_u$ where $G_u \cong \mathbb{Z}^d$, $G_{\mathcal{T}} \cong \mathbb{Z}^{N-d}$ and $G_{\mathcal{T}}$ contains $\Delta$; thus the real vector space spanned by $G_{\mathcal{T}}$ has dimension $N-d$ and is complimentary to $E$ in $\mathbb{R}^N$.

Given such a splitting we write $F$ for this complimentary space spanned by $G_{\mathcal{T}}$ and $\pi'$ for the skew projection (idempotent map) onto $F$ parallel to $E$. We write $K'$ for $\pi'(K)$.

Set $F_u^o = NS \cap (Q+u) \cap F$ and let $F_u$ be the $\overline{D}$-closure of $F_u^o$ in $\widetilde{\Pi}_u$. Let $\overline{K}$ be the closure of $NS \cap (Q+u) \cap K'$ in $F_u$. Then $H_{\mathcal{T}}$ acts freely on $F$ by translation by $\pi'(r)$, $r \in H_{\mathcal{T}}$, and this action restricts to $F_u^o$ and completes to $F_u$ naturally. We consider $(F_u, H_{\mathcal{T}})$ as a non-compact dynamical system. The topology on $F_u$ is easily described.

**Lemma 2.4, [FHK1] (9.4)** *The sets $h\overline{K}$, $h \in H_{\mathcal{T}}$, are compact open in $F_u$ and generate a basis for the topology.* •

**Lemma 2.5, [FHK1] (9.2)** *For a standard pattern $\mathcal{T}$ the space $\widetilde{\Pi}_u$ is homeomorphic to $F_u \times E$. Under this homeomorphism, the action of $E$ on $\widetilde{\Pi}_u$ is trivial on the $F_u$ component and is the natural translation on the factor $E$. The action of $H_{\mathcal{T}}$ on $\widetilde{\Pi}_u$ is the diagonal action of the projections of $H_{\mathcal{T}}$ in the directions $F_u$ and $E$,* i.e., *if $h \in H_{\mathcal{T}}$ and $(x,y) \in F_u \times E$ then $h(x,y) = (\pi'(h)(x), \pi(h)(y))$.* •

**Lemma 2.6, [FHK1] (10.10)** *The action by $G_{\mathcal{T}}$ on $F_u$ has an open compact fundamental domain $Y_{\mathcal{T}}$ homeomorphic to the Cantor set $X_{\mathcal{T}} = F_u/G_{\mathcal{T}}$. This defines the Cantor dynamical system, $(X_{\mathcal{T}}, G_u)$ associated to the pattern $\mathcal{T}$ whose mapping torus, $MT(X_{\mathcal{T}}, G_u)$, is homeomorphic to $M\mathcal{T}$.* •

If $W$ is a topological space we write $C(W;\mathbb{Z})$ for the group of all integer-valued continuous functions on $W$ and $CC(W;\mathbb{Z})$ for the subgroup of compactly supported functions. If $W$ has the action of a group $G$ of homeomorphisms, then $C(W;\mathbb{Z})$ and $CC(W;\mathbb{Z})$ are both naturally $\mathbb{Z}[G]$ modules. For a standard pattern $\mathcal{T}$ we define $CF_u$ as $CC(F_u;\mathbb{Z})$ and $CX_{\mathcal{T}}$ as $C(X_{\mathcal{T}};\mathbb{Z}) = CC(X_{\mathcal{T}};\mathbb{Z})$. Then $CF_u$ is a countable $\mathbb{Z}[H_{\mathcal{T}}]$ module and $CX_{\mathcal{T}}$ is a countable $\mathbb{Z}[G_u]$ module.

Now suppose $G$ is a topological abelian group acting by homeomorphisms on $W$. The *transformation groupoid*, $\mathcal{G}(W,G)$, is the topological space $W \times G$ with multiplication $(x,g)(y,h) = (x, g+h)$ defined whenever $y = gx$ and undefined otherwise. The unit space $\mathcal{G}^o(W,G)$ is the subspace $X \times \{0\}$. The *range* $r(x,g)$ of $(x,g) \in \mathcal{G}$ is defined as $gx$ and its *source*, $s(x,g)$, is defined as $x$.

Thus we may define, $\mathcal{G}\widetilde{\Pi}_{\mathcal{T}}$, $\mathcal{G}F_{\mathcal{T}}$, $\mathcal{G}X_{\mathcal{T}}$ and $\mathcal{G}M\mathcal{T}$, the transformation groupoids for the dynamical systems, $(\widetilde{\Pi}_u, E + H_{\mathcal{T}})$, $(F_u, H_{\mathcal{T}})$, $(X_{\mathcal{T}}, G_u)$ and $(M\mathcal{T}, E)$ respectively.

Given a groupoid $\mathcal{G}$ with unit space $\mathcal{G}^o$ and a subset $L$ of $\mathcal{G}^o$, define the *reduction* of $\mathcal{G}$ to $L$ as the subgroupoid ${}_L\mathcal{G}_L = \{g \in \mathcal{G} \mid r(g), s(g) \in L\}$ of $\mathcal{G}$, with unit space, $L$. If $L$ is closed then ${}_L\mathcal{G}_L$ is a closed subgroupoid of $\mathcal{G}$.

The *pattern groupoid*, $\mathcal{G}\mathcal{T}$ may be defined as the reduction of $\mathcal{G}M\mathcal{T}$ to the set $\Omega_{\mathcal{T}} = \{S \in M\mathcal{T} : 0 \in \pi_*(S)\}$. A variation of this can be defined for punctured tilings as in [**K1**]. Suppose that $\mathcal{T}$ is a projection method pattern which happens also to be a punctured tiling, then we define, $\Omega^*_{\mathcal{T}} = \{S \in M\mathcal{T} : 0 \in \tau(S)\}$ where $\tau(S)$ is the set of punctures for the tiling $S$. We write $\mathcal{G}\mathcal{T}^*$ for the reduction of $\mathcal{G}M\mathcal{T}$ to $\Omega^*_{\mathcal{T}}$. This is the punctured tiling groupoid, and we have noted in [**FHK1**] instances where it is isomorphic to the corresponding pattern groupoid.

From a groupoid $\mathcal{G}$ we can define the $C^*$ algebra $C^*(\mathcal{G})$ [**Ren**], and [**FHK1**] §11 showed that many of the algebras produced from the groupoids above are equivalent at the level of ordered K-theory. In the following we denote by $C(W)$ the $C^*$ algebra of complex valued functions on a compact space $W$, and for a non-compact space $U$, $C_o(U)$ denotes the $C^*$ algebra of complex valued functions tending to zero at infinity.

**Theorem 2.7, [FHK1] §11** *Suppose that $\mathcal{T}$ is a standard projection method pattern. Then the following algebras are strong Morita equivalent* [**MRW**] *and their ordered K-theory agrees (without attention to scale)*
$C(X_{\mathcal{T}}) \rtimes G_u$, $\quad C^*(\mathcal{G}\mathcal{T})$, $\quad C(M\mathcal{T}) \rtimes E$, $\quad C_o(\Pi_u) \rtimes (H_{\mathcal{T}} + E)$, $\quad C_o(F_u) \rtimes H_{\mathcal{T}}$.
*If $\mathcal{T}$ is a translationally finite tiling, then $C^*(\mathcal{G}\mathcal{T}^*)$ is also strong Morita equivalent to these algebras.* •

## §3 Continuous similarity of transformation groupoids

The aim of this section is to show that many of the groupoids we associate with a projection pattern are related by the important concept of *continuous similarity.* Further background facts about groupoids and their cohomology and the idea of similarity may be found in [**Ren**].

**Definition 3.1** Two homomorphisms, $\phi, \psi\colon \mathcal{G} \longrightarrow \mathcal{H}$ between topological groupoids are *continuously similar* if there is a continuous function, $\Theta\colon \mathcal{G}^o \longrightarrow \mathcal{H}$ such that

$$\Theta(r(x))\phi(x) = \psi(x)\Theta(s(x)).$$

Two topological groupoids are *continuously similar* if there exist homomorphisms $\phi\colon \mathcal{G} \longrightarrow \mathcal{H}$, $\psi\colon \mathcal{H} \longrightarrow \mathcal{G}$ such that $\Phi_{\mathcal{G}} = \psi\phi$ is continuously similar to $id_{\mathcal{G}}$ and $\Phi_{\mathcal{H}} = \phi\psi$ is continuously similar to $id_{\mathcal{H}}$.

Our interest in this relation lies in the following fact which we exploit in §4; see **[Ren]** for the definition of continuous cohomology $H^*(\mathcal{G};\mathbb{Z})$ of a topological groupoid $\mathcal{G}$.

**Proposition 3.2** ([**Ren**], with necessary alterations for the continuous category) *If $\mathcal{G}$ and $\mathcal{H}$ are continuously similar then $H^*(\mathcal{G};\mathbb{Z}) = H^*(\mathcal{H};\mathbb{Z})$.* •

We are unaware of any necessary relations between reductions and continuous similarity in the most general case, but it turns out that the construction of continuous similarities follows closely the reduction arguments in the examples that interest us.

**Lemma 3.3** *Suppose that $(X,H)$ is a free topological dynamical system (i.e., $hx = x$ implies that $h$ is the identity), with transformation groupoid $\mathcal{G} = \mathcal{G}(X,H)$, and that $L, L'$ are two closed subsets of $\mathcal{G}^o$. Suppose there are continuous functions, $\gamma\colon L \longrightarrow H$, $\delta\colon L' \longrightarrow H$ which define continuous maps $\alpha\colon L \longrightarrow L'$ and $\beta\colon L' \longrightarrow L$ by $\alpha x = \gamma(x)x$ and $\beta x = \delta(x)x$. Then ${}_L\mathcal{G}_L$ and ${}_{L'}\mathcal{G}_{L'}$ are continuously similar.*

**Proof** We construct the two homomorphisms, $\phi\colon {}_L\mathcal{G}_L \longrightarrow {}_{L'}\mathcal{G}_{L'}$ and $\psi\colon {}_{L'}\mathcal{G}_{L'} \longrightarrow {}_L\mathcal{G}_L$ by putting $\phi(x,g) = (\alpha x, \gamma(gx) + g - \gamma(x))$ and $\psi(y,h) = (\beta y, \delta(hy) + g - \delta(y))$.

A quick check confirms that these are homomorphisms, and they are both clearly continuous. Moreover, $\phi\psi(y,h) = (\alpha\beta y, \gamma((\delta(hy)+h-\delta(y))\beta(y)) + \delta(hy) + h - \delta(y) - \gamma(\beta y)$, a rather complicated expression which can be simplified if we note that $\gamma((\delta(hy) + h - \delta(y))\beta(y)) = \gamma((\delta(hy) + h)y) = \gamma(\beta hy)$, and define $\sigma(y)$ to be the element of $H$ such that $\sigma(y)y = \alpha\beta y$. Then $\sigma(y) = \delta(y) + \gamma(\beta y)$, by definition, and so $\sigma(hy) = \gamma(h\beta(y)) + \delta(hy) = \gamma((\delta(hy) + h - \delta(y))\beta(y)) + \delta(hy)$. This gives $\phi\psi(y,h) = (\alpha\beta y, \sigma(hy) + h - \sigma(y))$.

It is now easy to see that $\phi\psi$ is continuously similar to the identity on ${}_{L'}\mathcal{G}_{L'}$ using the transfer function, $\Theta : L' \longrightarrow {}_{L'}\mathcal{G}_{L'}$ given by $\Theta(y) = (\alpha\beta y, -\sigma(y))$, also clearly continuous.

Reciprocal expressions give the similarity between $\psi\phi$ and the identity on ${}_L\mathcal{G}_L$. •

**Remark 3.4** Note that if $L' = \mathcal{G}^o$, then Lemma 3.4 can be reexpressed in the following form. If $L$ is a closed subset of $\mathcal{G}^o$ for which there is a continuous map $\gamma\colon \mathcal{G}^o \longrightarrow H$ such that $\gamma(x)x \in L$ for all $x \in \mathcal{G}^o$, then ${}_L\mathcal{G}_L$ is continuously similar to $\mathcal{G}$. (The condition on $L$ implies that $L$ intersects every $H$-orbit of $(\mathcal{G}^o, H)$, but the converse is not true.)

We apply this lemma and remark in two ways as we examine continuous similarities between the various groupoids of [**FHK1**].

**Lemma 3.5** *Suppose that $\mathcal{T}$ is a standard projection method pattern and write $\mathcal{G}$ for $\mathcal{G}\widetilde{\Pi}_{\mathcal{T}}$. If $L$ is a clopen subset of $F_u$ then ${}_L\mathcal{G}_L$ is continuously similar to $\mathcal{G}$.*

**Proof** It suffices to find the function $\gamma$ in the remark.

Pick an order $\succ$ on $H_{\mathcal{T}}$ in which every non-empty set has a minimal element. The set $EL = \{vy : v \in E, y \in L\}$ is naturally homeomorphic to $E \times L$ by (2.5), and hence is clopen in $\widetilde{\Pi}_u$. By the minimality of the $E + H_{\mathcal{T}}$ action on $\widetilde{\Pi}_u$, **[FHK1]** (3.9), we have $\cup_{h \in H_{\mathcal{T}}} hEL = \widetilde{\Pi}_u$, so that for each $x \in \widetilde{\Pi}_u$, there is a $\succ$-minimal $h \in H_{\mathcal{T}}$ such that $hx \in EL$. Let $\gamma_0(x)$ be this $g$, and note that by the freedom and isometric action of $G$ and the clopenness of $EL$, this function $x \mapsto \gamma_0(x)$ is continuous and maps $\widetilde{\Pi}_u$ to $EL$.

Now, given $\gamma_0(x)x \in EL$, there is a unique $\gamma_1(x) \in E$ such that $\gamma_1(x)\gamma_0(x)x \in L$, and it is clear that $x \mapsto \gamma_1(x)$ is continuous as a map $\widetilde{\Pi}_u \longrightarrow E$. The desired map $\gamma$ can now be taken as this composite. •

**Lemma 3.6** *Suppose that $\mathcal{T}$ is a translationally finite tiling for which we have chosen a puncturing and which is also a projection method pattern. Then $\mathcal{GT}$ and $\mathcal{GT}^*$ are continuously similar.*

**Proof** we fit the present hypotheses into the case of Lemma 3.3. Let $X = M\mathcal{T}$ supporting the (free) action of $E$. Let $L = \Omega_{\mathcal{T}}$ and $L' = \Omega^*_{\mathcal{T}}$. We construct the maps $\gamma$ and $\delta$ as follows.

Recall the metric $D_o(A, B) = \inf\{1/(r+1) : A[r] = B[r]\}$ and the argument of Lemma 11.14 **[FHK1]** which shows that $D$ and $D_o$ are equivalent on each of the sets $\Omega_{\mathcal{T}}$ and $\Omega^*_{\mathcal{T}}$. (Actually the argument refers only to the second space, but the fact that $\pi_*\mathcal{T}$ is a Meyer pattern (see **[La]** and remark 11.13 **[FHK1]**) allows it to be applied directly to the first space as well.)

We may assume without loss of generality that, for each $S \in M\mathcal{T}$, each point of $\pi_*(S)$ is in the interior of a tile of $S$ (if not we shift all the tiles in $\mathcal{T}$ by a uniform short generic displacement and start again equivalently).

Suppose that $S \in \Omega_{\mathcal{T}}$. We know that $0 \in \pi_*(S)$ and that by assumption there is a unique tile in $S$ which contains the origin in its interior. This tile has a puncture at a point $v$ say, and so $S - v \in \Omega^*_{\mathcal{T}}$. So we have defined a map from $\Omega_{\mathcal{T}}$ to $E$, $\gamma: S \mapsto -v$ which is clearly continuous with respect to the $D_o$ metric. Moreover the map, $\alpha: S \mapsto S - v$ has range $\Omega^*_{\mathcal{T}}$.

Conversely, let $r$ be chosen so that every ball in $E$ of radius $r$ contains at least one point of $\pi_*(\mathcal{T}) = P_u$. Consider the sets $\pi_*(S) \cap B(r)$, as $S$ runs over $\Omega^*_{\mathcal{T}}$ and note that there are only finitely many possibilities, i.e. the set $J = \{\pi_*(S) \cap B(r) : S \in \Omega^*_{\mathcal{T}}\}$ is a finite collection of non-empty finite subsets of $B(r)$. Furthermore, by the continuity of $\pi_*$ on $\Omega^*_{\mathcal{T}}$ with respect to the $D_o$ metric, for each $C \in J$, the set $\{S \in \Omega^*_{\mathcal{T}} : \pi_*(S) \cap B(r) = C\}$ is clopen.

We define a function, $v$, from sets to points which chooses, for each $C \in J$, an element $v(C) \in C$. Define $\delta(S) = -v(\pi_*(S) \cap B(r))$; this map from patterns to points is continuous by construction. Also by construction, $S + \delta(S) \in \Omega_{\mathcal{T}}$. •

**Lemma 3.7, [FHK1](11.6), (11.8)** *Suppose that $\mathcal{T}$ is a standard projection method pattern. The groupoids $\mathcal{G}F_{\mathcal{T}}$ and $\mathcal{G}X_{\mathcal{T}}$ are each a reduction of $\mathcal{G}\widetilde{\Pi}_{\mathcal{T}}$ to the sets $F_u$ and $Y_{\mathcal{T}}$ respectively and there is a clopen subset $L$ of $F_u$ such that the reduction of $\mathcal{G}\widetilde{\Pi}_{\mathcal{T}}$ to $L$ is isomorphic to $\mathcal{G}\mathcal{T}$.* •

To summarise,

**Corollary 3.8** *Suppose that $\mathcal{T}$ is a standard projection method pattern. Then $\mathcal{G}\mathcal{T}$, $\mathcal{G}\widetilde{\Pi}_{\mathcal{T}}$, $\mathcal{G}F_{\mathcal{T}}$ and $\mathcal{G}X_{\mathcal{T}}$ are all continuously similar. If $\mathcal{T}$ is also a translationally finite tiling, then these are all continuously similar to the tiling groupoid, $\mathcal{G}\mathcal{T}^*$, of* [**K1**].

**Proof** Lemmas (3.5) and (3.7) show that $\mathcal{G}\mathcal{T}$, $\mathcal{G}F_{\mathcal{T}}$ and $\mathcal{G}X_{\mathcal{T}}$ are all continuously similar to $\mathcal{G}\widetilde{\Pi}_{\mathcal{T}}$. The second part is a restatement of (3.6). •

## §4 Pattern cohomology and K-theory

We are now in a position to define our topological invariants for projection method patterns and prove their additive equivalence.

**Definition 4.1** For a standard projection method pattern $\mathcal{T}$ we define for each $m \in \mathbb{Z}$ the following groups.

**(a)** $H^m(\mathcal{G}\mathcal{T}, \mathbb{Z})$, the continuous groupoid cohomology of the pattern groupoid $\mathcal{G}\mathcal{T}$;

**(b)** $H^m(M\mathcal{T})$, the Čech cohomology of the space $M\mathcal{T}$;

**(c)** $H_{d-m}(G_u, CX_{\mathcal{T}})$ and $H^m(G_u; CX_{\mathcal{T}})$, the group homology and cohomology of $G_u$ with coefficients $CX_{\mathcal{T}}$;

**(d)** $H_{d-m}(H_{\mathcal{T}}; CF_u)$ and $H^m(H_{\mathcal{T}}; CF_u)$, the group homology and cohomology of $H_{\mathcal{T}}$ with coefficients $CF_u$;

**(e)** $K_*(C^*(\mathcal{G}\mathcal{T}))$, the $C^*$ $K$-theory of $C^*(\mathcal{G}\mathcal{T})$;

**(f)** $K_*(C(X_{\mathcal{T}}) \rtimes G_u)$, the $C^*$ $K$-theory of the crossed product $C(X_{\mathcal{T}}) \rtimes G_u$;

and, for translationally finite tilings,

**(g)** the continuous groupoid cohomology $H^m(\mathcal{G}\mathcal{T}^*, \mathbb{Z})$.

**Theorem 4.2** *For a standard projection method pattern $\mathcal{T}$ and for each value of $m$, the invariants defined in (4.1)(a) to (d) are all equivalent as groups. If $\mathcal{T}$ is also translationally finite, then these are also equivalent to that defined in (4.1)(g).*

*The invariants defined in (4.1)(e) and (f) are each equivalent as $\mathbb{Z}/2$ graded ordered groups. Finally, all these invariants are related via isomorphisms of groups such as*

$$K_m(C(X_{\mathcal{T}}) \rtimes G_u) = \bigoplus_{j=-\infty}^{\infty} H^{m+d+2j}(G_u; CX_{\mathcal{T}}).$$

*These invariants are, in all cases, torsion free, and those in parts (a) to (d) and (g) are non-zero only for integers $m$ in the range $0 \leqslant m \leqslant d$.*

**Proof** It is immediate from the definition [**Ren**] that if $W$ is a locally compact space on which a discrete abelian group $G$ acts freely by homeomorphisms then the continuous groupoid cohomology $H^*(\mathcal{G}(W,G);\mathbb{Z})$ is naturally isomorphic to the group cohomology $H^*(G, C(W;\mathbb{Z}))$ with coefficients the continuous compactly supported integer-valued functions on $W$, with $\mathbb{Z}[G]$-module structure dictated naturally by the $G$ action on $W$. This proves the equality of (a) (and (g) where appropriate) with the cohomology versions of (c) and (d) from (3.8) and the fact that $\mathcal{G}F_{\mathcal{T}}$ and $\mathcal{G}X_{\mathcal{T}}$ are transformation groupoids.

By (10.10) [**FHK1**] $M\mathcal{T}$ is homeomorphic to the mapping torus $X_{\mathcal{T}} \times_{G_u} EG_u$ and as noted in [**FH**] the Čech cohomology $H^*(M(X_{\mathcal{T}}, G_u))$ is equivalent to the group cohomology $H^*(G_u, CX_{\mathcal{T}})$ (this equivalence is standard and follows, for example, by induction on the rank of $G_u$ with the induction step passing from $\mathbb{Z}^r$ to $\mathbb{Z}^{r+1}$ coming from the comparison of the Mayer-Vietoris decomposition of $X_{\mathcal{T}} \times_{G_u} EG_u$ along one coordinate with the long exact sequence in group cohomology coming from the extension $\mathbb{Z}^r \to \mathbb{Z}^{r+1} \to \mathbb{Z}$). This proves the equivalence of (4.1)(b) with the cohomological invariant of (4.1)(c).

The equation of $H^m(G_u, CX_{\mathcal{T}})$ with $H_{d-m}(G_u; CX_{\mathcal{T}})$ is simply Poincaré duality for the group $G_u \cong \mathbb{Z}^d$.

By Lemma 2.5, a decomposition of $CF_u$ as a $\mathbb{Z}[H_{\mathcal{T}}] = \mathbb{Z}[G_u] \otimes \mathbb{Z}[G_{\mathcal{T}}]$ module is given by $CX_{\mathcal{T}} \otimes \mathbb{Z}[G_{\mathcal{T}}]$ where $CX_{\mathcal{T}}$ is a trivial $\mathbb{Z}[G_{\mathcal{T}}]$ module. Standard homological algebra now tells us that

$$\begin{aligned} H_p(H_{\mathcal{T}}; CF_u) &= H_p(G_u \oplus G_{\mathcal{T}}; CX_{\mathcal{T}} \otimes \mathbb{Z}[G_{\mathcal{T}}]) \\ &= H_p(G_u; CX_{\mathcal{T}}) \end{aligned}$$

establishing the equality of (4.1)(c) and (d) in homology. (A similar argument also works in cohomology.)

The equality of (4.1)(e) and (f) follows from the Morita equivalence of the underlying $C^*$ algebras in (2.7) and the equality

$$K_m(C(X_{\mathcal{T}}) \rtimes G_u) = \bigoplus_{j=-\infty}^{\infty} H^{m+d+2j}(G_u; CX_{\mathcal{T}})$$

is one of the main results of [**FH**].

The torsion-freedom of these invariants also follows from the results of [**FH**], while the vanishing of the (co)homological invariants outside the range of dimensions stated is immediate from their identification with the (co)homology of the group $G_u = \mathbb{Z}^d$. •

We make one further reduction of the complexity of the computation of these invariants. Recall first the construction of section 2, in particular the equation $F \cap Q = V \oplus \widetilde{\Delta}$ splitting

$F$ into continuous and discrete directions, and in which the projection $\pi'(\mathbb{Z}^N)$ is dense. Recall also the map $\widetilde{\mu}\colon \Pi_u \longrightarrow Q+u$ defined in (2.2 ff.) for each $u \in NS$.

**Definition 4.3** The restriction of $\widetilde{\mu}$ to $F_u$ is written $\nu\colon F_u \longrightarrow F\cap(Q+u) = (F\cap Q)+\pi'(u)$; this map is $\pi'(\mathbb{Z}^N)$-equivariant and $|\nu^{-1}(v)| = 1$ precisely when $v \in NS \cap F \cap (Q+u)$ (see Lemma 9.2 of [**FHK1**]).

Let $\Gamma_{\mathcal{T}} = \{v \in H_{\mathcal{T}} : \pi'(v) \in V\}$ and $CV_u = \{f \in CF_u : \nu(supp(f)) \subset V + \pi'(u)\}$, where *supp* refers to the support of the function. This is consistent with setting $V_u = \{x \in F_u : \nu(x) \in V + \pi'(u)\}$ and taking $CV_u$ as $CC(V_u;\mathbb{Z})$. There is a natural decomposition $CF_u = CV_u \otimes_{\mathbb{Z}} \mathbb{Z}[\widetilde{\Delta}]$.

**Lemma 4.4** *As a subgroup of $H_{\mathcal{T}} = G_{\mathcal{T}} \oplus G_u$, $\Gamma_{\mathcal{T}}$ satisfies $\Gamma_{\mathcal{T}} = (\Gamma_{\mathcal{T}} \cap G_{\mathcal{T}}) \oplus G_u$. Moreover, $\Gamma_{\mathcal{T}}$ is complemented in $H_{\mathcal{T}}$ by a group $\Gamma_\Delta$, naturally isomorphic to $\widetilde{\Delta}$.*

*With this splitting, the action of $\mathbb{Z}[H_{\mathcal{T}}] = \mathbb{Z}[\Gamma_{\mathcal{T}}] \otimes \mathbb{Z}[\Gamma_\Delta]$ on $CF_u = CV_u \otimes_{\mathbb{Z}} \mathbb{Z}[\widetilde{\Delta}]$ is the obvious one, and hence there is an isomorphism of homology groups $H_*(H_{\mathcal{T}};CF_u) \cong H_*(\Gamma_{\mathcal{T}};CV_u)$.*

**Proof** The decompositions and restrictions on $G_{\mathcal{T}}$ and $G_u$ follow from the definition and the original construction of (10.1) in [**FHK1**]. The conclusion in homology it the same homological argument as used in the previous proof. •

We note that since $H_{\mathcal{T}} \supset \mathbb{Z}^N$ and $\pi^\perp(\mathbb{Z}^N)$ is dense in $Q \cap F$, the group $\Gamma_{\mathcal{T}}$ acts minimally on $V$ and hence on $V_u$.

**Corollary 4.5** *With the data above,*

$$K_n(C^*(\mathcal{G}\mathcal{T})) = \bigoplus_{j=-\infty}^{\infty} H_{n+2j}(\Gamma_{\mathcal{T}};CV_u). \qquad \bullet$$

This is, in fact, the most computationally efficient route to these invariants and, with the exception of section 6, the one we shall use in the remainder of this paper and in [**FHK2**].

## §5 Inverse limit acceptance domains

Our immediate goal is to illustrate the computation of the invariants introduced in §4 by examining projection method patterns on $\mathbb{R}^d$ arising as projection from $\mathbb{R}^{d+1}$ for more or less arbitrary acceptance domains $K$. To facilitate those computations we examine in this section a general technique which sometimes simplifies the computations of projection pattern cohomology when the acceptance domain is disconnected. We assume throughout that all the projection patterns are standard.

Suppose that $K$ and $K_i$, $i = 1, 2, \ldots$, are compact subsets of $E^\perp$ each of which is the closure of its interior, and suppose that $IntK = \cup_i IntK_i$ is a disjoint union and that $\partial K = \cup \partial K_i$. This occurs if the $K_i$ are all disjoint, for example, but the setup can be a little more general. Let $K_i^* = \underset{j\leqslant i}{\cup} K_j$, so that $IntK_i^* = \underset{j\leqslant i}{\cup} IntK_j$ is a disjoint union and $\partial K_i^* = \underset{j\leqslant i}{\cup} \partial K_j$.

We define $NS^i = \mathbb{R}^N \setminus (E + \mathbb{Z}^N + \partial K_i^*)$ and $\Sigma^i = K_i^* + E$. So for each $u \in NS^i$ we have $\widetilde{P}_u^i = \mathbb{Z}^N \cap \Sigma^i$ and $P_u^i = \pi(\widetilde{P}_u^i)$. From these we construct $\widetilde{\Pi}_u^i$, $\Pi_u^i$, $MP_u^i$ and so on, as usual. In fact, in the following, we shall be interested only in the strip pattern $\widetilde{P}_u^i$.

Provided $u$ is non-singular for $K$, and hence is non-singular for all $K_i^*$, we can take a space $F$ complimentary to $E$ in $\mathbb{R}^N$ and a corresponding group $G_u$ which will play their usual roles for all sets of projection data $(E, \mathbb{R}^N, K_i^*)$ and $(E, \mathbb{R}^N, K)$. For each domain, $K_i^*$, we construct the corresponding $F_u^i$ *etc.* The following lemma follows easily from the definitions.

**Lemma 5.1** *Suppose $j < i$ throughout the statement of this lemma. Then $NS^i$ is a dense subset of $NS^j$ and $NS = \cap_i NS^i$. Moreover, for $u \in NS^i$, we have a natural continuous $E + \mathbb{Z}^N$ equivariant surjection $\widetilde{\Pi}_u^i \longrightarrow \widetilde{\Pi}_u^j$, and a natural continuous $E$ equivariant surjection $M\widetilde{P}_u^i \longrightarrow M\widetilde{P}_u^j$; this latter is described equivalently by the formula $S \mapsto S \cap \Sigma^j$.*

*We also have an $\mathbb{Z}^N$-equivariant map $F_u^i \longrightarrow F_u^j$, and a $G_u$-equivariant map $X_u^i \longrightarrow X_u^j$. All these maps respect the commutative diagrams of* [**FHK1**] *and they map many-to-one only when the image is in (the appropriate embedding of) $NS^j \setminus NS^i$.* •

**Theorem 5.2** *With the notation and assumptions above, we have the following equivariant homeomorphisms.*

**(a)** $\widetilde{\Pi}_u \cong \lim\limits_{\leftarrow} \widetilde{\Pi}_u^i$, $\quad E + \mathbb{Z}^N$ *equivariantly;*

**(b)** $M\widetilde{P}_u \cong \lim\limits_{\leftarrow} M\widetilde{P}_u^i$, $\quad E$*-equivariantly;*

**(c)** $F_u \cong \lim\limits_{\leftarrow} F_u^i$, $\quad \mathbb{Z}^N$*-equivariantly;*

**(d)** $X_u \cong \lim\limits_{\leftarrow} X_u^i$, $\quad G_u$*-equivariantly.*

**Proof** Once again the results are straightforward from the definitions. The map $M\widetilde{P}_u \longrightarrow M\widetilde{P}_u^i$ is again equivalently written $S \mapsto S \cap \Sigma^i$. •

The following is now a direct consequence of (4.2), (5.2)(b) and the behaviour of Čech cohomology on inverse limits.

**Corollary 5.3** *There is a natural equivalence $H^*(\mathcal{G}\widetilde{P}_u) = \lim\limits_{\rightarrow} H^*(\mathcal{G}\widetilde{P}_u^i)$.*

**§6 Cohomology of the case d=N–1**

In this section we determine the cohomology for standard projection method patterns when $d = N - 1$. It is the only case for which we have such a complete answer.

Here $E$ is a codimension 1 subspace of $\mathbb{R}^N$ and so $E \cap \mathbb{Z}^N = 0$ implies that $\Delta = 0$. So assuming $E \cap \mathbb{Z}^N = 0$, the pattern considered is automatically standard, **[FHK1]** (5.2). Moreover, $MP_v = MP_u = M\widetilde{P}_u$ for all $u, v \in NS$ and so, given $E$ and $K$, there is only one projection pattern torus $MP$ to consider, no need to parametrise by $u$, and an equivalence between $H_P$ and the original lattice $\mathbb{Z}^N$ **[FHK1]** (8.2). With this in mind, we shall avoid further explicit mention of any particular non-singular point $u$. Write $e_1, ..., e_N$ for the usual unit vector basis of $\mathbb{R}^N$, which are also the generators of $\mathbb{Z}^N$. Choose the space $F$ as that spanned by $e_N$, and so $G_{\mathcal{T}} = \langle e_N \rangle$ and $G_u = \langle e_1, e_2, ..., e_{N-1} \rangle$. Recall that we write $K' = \pi'(K) \subset F$, where $\pi'$ is the skew projection onto $F$ parallel to $E$ and that $\pi'$ maps $K$ homeomorphically to $K'$, preserving the boundary, $\partial K' = \pi'(\partial K) \cong \partial K$.

Now any compact subset of $E^\perp$ such as $K$ which is the closure of its interior is a countable union of closed disjoint intervals; thus $\partial K$ and hence $\partial K'$ is countable. Pick $A = \{p_1, p_2, ...\}$, $p_j \in \partial K'$, to be a set of representatives of $\pi'(\mathbb{Z}^N)$ orbits of $\partial K'$, a countable and possibly finite set. Write $k \in \mathbb{Z}_+ \cup \infty$ for the cardinality of $A$.

**Theorem 6.1** *If $\mathcal{T}$ is a projection method pattern with $d = N - 1$ and $E \cap \mathbb{Z}^N = 0$, then*

$$H^m(\mathcal{GT}) = H^m(\mathbb{T}^N \setminus k \text{ points }) = \begin{cases} \mathbb{Z}^{\binom{N}{m}} & \text{for } m \leqslant N-2, \\ \mathbb{Z}^{N+k-1} & \text{for } m = N-1, \\ 0 & \text{otherwise.} \end{cases}$$

*An infinite superscript denotes the countably infinite direct sum of copies of $\mathbb{Z}$.*

**Proof** We know that $IntK'$ is the union of a countable number of open intervals, whose closures, $K_i$, are disjoint. We use the notation and results of §5, setting $K_i^* = \underset{j \leqslant i}{\cup} K_j$ as the finite union of disjoint closed intervals, $\cup_i j[s_j e_N, r_j e_N]$ say. As $M\mathcal{T} = M\widetilde{P}$, by (5.3) it is enough to compute the direct limit $\underset{\rightarrow}{\lim}\, H^*(M\widetilde{P}^i)$.

We consider the process of completion giving rise to the space $M\widetilde{P}^i$ which we consider as $\widetilde{\Pi}^i$ (the completion of the non-singular points $NS^i$) modulo the action of $\mathbb{Z}^N$. The limit points introduced in $\widetilde{\Pi}^i$ arise as the limit of patterns $P^i_{x_n}$ as $x_n$ approaches a singular point, either from a positive $e_N$ direction, or from a negative one. To be more precise, suppose that $x_n = x + t_n e_N \in NS$ is a sequence converging to $x \in \mathbb{R}^N$ in the Euclidean topology. If $(t_n)$ is an increasing sequence, then $\lim_{n\to\infty} \widetilde{P}^i_{x_n}$ exists in the $D$ metric and is the point pattern $(x + \mathbb{Z}^N) \cap (\underset{j \leqslant i}{\cup} (s_j e_N, r_j e_N] + E)$. Likewise, if $(t_n)$ is a decreasing sequence then $\lim_{n\to\infty} \widetilde{P}^i_{x_n}$ is the point pattern $(x + \mathbb{Z}^N) \cap (\underset{j \leqslant i}{\cup} [s_j e_N, r_j e_N) + E)$. These two patterns are the same if and only if $x \in NS^i$. If $x \notin NS^i$ then these two patterns define the two $D$ limit points in $\widetilde{\Pi}^i$ over $x \in \mathbb{R}^N$. Thus the quotient $M\widetilde{P}^i \longrightarrow \mathbb{T}^N$ is 1-1 precisely when mapping to the set $NS^i/\mathbb{Z}^N$, and otherwise it is 2-1; we can picture the map intuitively as a process of "closing the gaps" made by cutting $\mathbb{T}^N$ along the finite set of hyperplanes $(\partial K_i^* + E)/\mathbb{Z}^N$, *c.f.* [**L**].

We examine the space $M\widetilde{P}^i$ more detail. Given $r > 0$, consider the space $M_r^i = \{(S \cap B(r)) \cup \partial B(r) : S \in M\widetilde{P}_u^i\}$ endowed with the Hausdorff metric $d_r$ among the set of all closed subsets of $B(r)$, the closed ball in $\mathbb{R}^N$ with centre 0 and radius $r$. By construction $M_r^i$ is compact and, for $s \geqslant r$ and $i \geqslant j$, there are natural restriction maps $M_s^i \longrightarrow M_r^j$, whose inverse limit for fixed $i = j$ is $M\widetilde{P}^i$, and whose inverse limit over all $i$ and $r$ by (5.2) is $M\widetilde{P}$. The map $M_r^i \longrightarrow \mathbb{T}^N$ given by $\widetilde{P}_v \mapsto v \bmod \mathbb{Z}^N$, $v \in NS^i$, factors the canonical quotient $M\widetilde{P}^i \longrightarrow \mathbb{T}^N$.

Define $C_r^i$ as the set $\{v \in \mathbb{T}^N : (v+\mathbb{Z}^N) \cap (\partial\Sigma^i \cap Int B(r)) \neq \emptyset\}$. As before, $M_r^i \longrightarrow \mathbb{T}^N$ is 2-1 precisely on those points mapped to $C_r^i$ and otherwise is 1-1.

The intersection $\partial\Sigma^i \cap B(r)$ is, for all $r$ large enough compared with the diameter of $K_i^*$, equal to a finite union of codimension 1 discs, parallel to $E$, and of radius at least $r-1$, and at most $r$. Each of these discs has centre $\pi^\perp(a)$ for some $a \in \partial K_i^*$. Consider this collection of discs modulo $\mathbb{Z}^N$ and select two, say with centres $\pi^\perp(a)$ and $\pi^\perp(b)$, where $a, b \in \partial K_i^*$. Then, for $r$ very large and $a - b \in \pi'(\mathbb{Z}^N)$, these discs will overlap modulo $\mathbb{Z}^N$. Since there are a finite number of such pairs in $\partial K_i^*$ to consider, we have a universal $r$ such that if $a, b \in \partial K_i^*$ and $a - b \in \pi'(\mathbb{Z}^N)$, then the disc with centre $\pi^\perp(a)$ overlaps, modulo $\mathbb{Z}^N$, the disc with centre $\pi^\perp(b)$. If $a - b \notin \pi'(\mathbb{Z}^N)$, then these discs will not overlap, modulo $\mathbb{Z}^N$. Hence for $r$ sufficiently large, $\partial\Sigma^i \cap B(r) \bmod \mathbb{Z}^N$ has precisely $|A_i|$ components.

For the same $r$, $C_r^i \bmod \mathbb{Z}^N$ is also a finite union of discs of radius at least $r-1$ and at most $r$; likewise $C_r^i$ has exactly $|A_i|$ components, in direct correspondence with the elements of $A_i$.

The description above of the limiting points in $\widetilde{\Pi}^i$ as we approach $C_r^i$ in a direction transverse to $E$, shows that $M_r^i$ is homeomorphic to $\mathbb{T}^N$ with a small open neighbourhood of $C_r^i$ removed. There is a natural homotopy equivalence with the space $\mathbb{T}^N \setminus C_r^i$.

We can now examine what happens as we let first $r$ and then $i$ tend to infinity in this construction. For the above sufficiently large $r$, the map $M_{r+1}^i \longrightarrow M_r^i$ is, up to homotopy, the injection from $\mathbb{T}^N \setminus C_{r+1}^i$ to $\mathbb{T}^N \setminus C_r^i$, and this is simply, up to homotopy, the identity from $\mathbb{T}^N \setminus |A_i|$ points to itself. Hence $H^*(M_r^i) = H^*(\mathbb{T}^N \setminus |A_i| \text{ points})$ and $H^*(M_r^i) \longrightarrow H^*(M_{r+1})$ is the identity showing that $H^*(M\widetilde{P}^i)$ is the cohomology of the torus with $|A_i|$ punctures.

Finally, for each $i$ and for $r$ sufficiently large (depending on $i$) the map $M_r^{i+1} \longrightarrow M_r^i$ is that induced by the inclusion of $A_i$ in $A_{i+1}$, and this corresponds in the above description to the adding of a new puncture for each element of $A_{i+1} \setminus A_i$. In cohomology, the map

$H^p(M_r^{i-1}) \longrightarrow H^p(M_r^i)$ is thus the identity for $p \neq d$, and in dimension $d = N-1$ gives rise to the direct system of groups and injections $\cdots \longrightarrow \mathbb{Z}^{N-1+|A_i|} \longrightarrow \mathbb{Z}^{N-1+|A_{i+1}|} \longrightarrow \cdots$ which gives the required formula. •

We give an alternative proof of this theorem from a different perspective in [**FHK2**].

We note that the system $(X_u^i, G_u)$ is in fact a Denjoy example [**PSS**] with $\mathbb{Z}^{N-1}$ action and dislocation along $k$ separate orbits.

**Corollary 6.2** *Suppose that* $\Gamma$ *is a dense countable subgroup of* $\mathbb{R}$ *finitely generated by* $r$ *free generators. Suppose we Cantorize* $\mathbb{R}$ *by cutting and splitting along* $k$ $\Gamma$*-orbits (as described e.g. in* [**PSS**]*) to form the locally Cantor space* $R'$ *on which* $\Gamma$ *acts continuously, freely and minimally. Consider the* $\Gamma$*-module* $C$ *of compactly supported integer valued functions defined on* $R'$*. Then* $H^*(\Gamma, C) = H^*(\mathbb{T}^r \setminus k \text{ points}; \mathbb{Z})$*.* •

## §7 Homological conditions for self similarity

This section shows that the (co)homological invariants defined in §4 provide an obstruction to a pattern arising as a substitution system. This result will be used in subsequent sections to show that almost all canonical projection method tilings fail to be self similar.

We adopt the construction of substitution tilings in [**AP**] and shall consider only translationally finite tilings. Anderson and Putnam establish the following fact about translationally finite substitution tilings with recognizability.

**Theorem 7.1** [**AP**] *Suppose that* $\mathcal{T}$ *is a translationally finite tiling of* $\mathbb{R}^d$*, stationary under some substitution procedure. Then* $M\mathcal{T}$ *is the inverse limit of a stationary sequence of spaces and maps*

$$Y \xleftarrow{\gamma} Y \xleftarrow{\gamma} \cdots$$

*where* $Y$ *is a finite CW complex and* $\gamma$ *is a cellular map.* •

**Corollary 7.2** *Suppose that* $\mathcal{T}$ *is a translationally finite tiling which is stationary under some substitution procedure. Then for each* $m$*, the rationalise Čech cohomology* $H^m(M\mathcal{T}) \otimes \mathbb{Q}$ *has finite* $\mathbb{Q}$*-dimension.*

**Proof** From (7.1) $H^m(M\mathcal{T}) = \varinjlim H^m(Y)$. So $H^*(M\mathcal{T}) \otimes \mathbb{Q} = \varinjlim (H^*(Y) \otimes \mathbb{Q})$. Thus the $\mathbb{Q}$ dimension of $H^*(M\mathcal{T}) \otimes \mathbb{Q}$ is bounded by that of $H^*(Y) \otimes \mathbb{Q}$ and this is finite since $Y$ is a finite CW complex. •

The conclusion of (7.2) actually applies to much more general pattern constructions. Note that the only principle used is that the space $M\mathcal{T}$ of the tiling dynamical system is the inverse limit of a sequence of maps between uniformly finite CW complexes. We sketch a generalization whose details can be reconstructed by combining the ideas to be found in [**Pr**] and [**Fo**].

Translationally finite substitution tilings are analysed combinatorially by Priebe in her PhD Thesis [**Pr**] where the useful notion of *derivative tiling*, generalised from the 1-dimensional symbolic dynamical concept [**Du**], is developed. We do not pursue the details here except to note that the derivative of an almost periodic translationally finite tiling is almost periodic and translationally finite, and that the process of deriving can be iterated.

Suppose that $\mathcal{T}$ is an almost periodic translationally finite tiling. By means of repeated derivatives, and adapting the analysis of [**Fo**] for periodic lattices, we may build a Bratteli diagram, $\mathcal{B}$, for $\mathcal{T}$. Its set of vertices at level $t$ is formally the set of translation classes of the tiles in the $t^{\text{th}}$ derivative tiling, and the edges relating two consecutive levels, $t$ and $t+1$ say, are determined by the way in which the tiles of the $(t+1)^{\text{th}}$ derivative tiling are built out of the tiles of the $t^{\text{th}}$ derivative tiling. The diagram $\mathcal{B}$ defines a canonical dimension group, $K_0(\mathcal{B})$. Adapting the argument of [**Fo**] we can define a surjection $K_0(\mathcal{B}) \longrightarrow H^d(M\mathcal{T})$ and hence a surjection $K_0(\mathcal{B}) \otimes \mathbb{Q} \longrightarrow H^d(M\mathcal{T}) \otimes \mathbb{Q}$.

In [**Pr**] it is shown that the repeated derivatives of a translationally finite aperiodic substitution tiling have a uniformly bounded number of translation classes of tiles. In particular, the number of vertices at each level of its Bratteli diagram $\mathcal{B}$ is bounded uniformly. Thus $K_0(\mathcal{B}) \otimes \mathbb{Q}$ is finite dimensional over $\mathbb{Q}$, being a direct limit of uniformly finite dimensional $\mathbb{Q}$ vector spaces, and so we reprove (5.2) for the case $p = d$ .

It is worth extracting the full power of this argument since it applies to a wider class than the substitution tilings.

**Theorem 7.3** *Suppose that $\mathcal{T}$ is a translationally finite tiling of $\mathbb{R}^d$ whose repeated derivatives have a uniformly bounded number of translation classes of tiles, then $H^d(\mathcal{G}\mathcal{T}) \otimes \mathbb{Q}$ is finite dimensional over $\mathbb{Q}$.* •

Thus, when we take a pattern, $\mathcal{T}$, computeits rationalized homology $H_0(\Gamma_{\mathcal{T}}; CV_u) \otimes \mathbb{Q}$ and find it is infinite dimensional (see for example section 12), we know we are treating a pattern or tiling which is outwith the class specified in Theorem 7.3, and *a fortiori* outside the class of substitution tilings.

## §8 The canonical projection tiling

For the first time in our studies now, we narrow our attention to the classical projection method tilings of [**OKD**] [**dB**] etc. This section outlines the simplifications to be found in this case, and describes the main result of the remainder of this paper; a sufficient condition for infinitely generated rationalised $H_0$.

In the canonical case, we have data $(K, E, u)$ where $K = \pi^{\perp}([0,1]^N)$ and $u \in NS$. From section 8 in **[FHK1]** we see that, but for a few exceptional cases, we have $H_{\mathcal{T}} = \mathbb{Z}^N$, and if we elect either to exclude these exceptions (as most authors do) or to include them only in their most decorated form ($M\mathcal{T} = M\widetilde{P}_u$), we make $H_{\mathcal{T}} = \mathbb{Z}^N$ a standing assumption.

With Theorem 9.4 **[FHK1]** we have a description of the topology of $CF_u$: it is generated by intersection and differences of the images of a certain compact open set $\overline{K}$. With the canonical choice of $K$ above, we now give a second geometrical description of the elements of $CF_u$ and $CV_u$, following more closely the work of Le **[L]**.

**Definition 8.1** We adopt the notation, $e^J = \langle e_j : j \in J \rangle$, where $J \subset \{1, 2, ..., N\}$ and $\{e_j\}$ is the standard unit basis of $\mathbb{R}^N$ or $\mathbb{Z}^N$. Let $n$ be the dimension of $F$.

Let $\mathcal{I} = \{J \subset \{1, 2, ..., N\} : \dim \pi'(e^J) = n - 1\}$ and define $\mathcal{I}^*$ to be the set of elements of $\mathcal{I}$ minimal with respect to containment.

Define $\mathbb{Z}^N_{n-1} = \cup\{e^J + v : |J| = n-1, v \in \mathbb{Z}^N\}$.

**Lemma 8.2** *i/ $\mathcal{I}^*$ is a sub-collection of the $n-1$-element subsets of $\{1, 2..., N\}$. Also every subspace of $F$ of the form $\pi'(e^J)$, with $|J| = n-1$, is contained in $\pi'(e^{J'})$ for some $J' \in \mathcal{I}^*$.*

*ii/ With the canonical acceptance domain and $F$ chosen transverse to $E$ etc., $\mathbb{R}^N \setminus NS = \pi^{\perp -1}\pi^{\perp}(\mathbb{Z}^N_{n-1})$ and $\pi'(\mathbb{Z}^N_{n-1}) = F \setminus NS$.*

**Proof** Part i/ is straightforward. For part ii/, with the data above, $\pi'(K)$ is a convex polytope in $F$, with interior, each of whose extreme points is of the form $\pi'(v)$ where $v \in \{0,1\}^N$. By i/, each of the faces of $\pi'(K)$ is contained in some $\pi'(e^J + v)$ where $v \in \{0,1\}^N$ and $J \in \mathcal{I}^*$. Also by i/, we have $\pi'(\mathbb{Z}^N_{n-1}) = \pi'(\cup\{e^J + v : v \in \mathbb{Z}^N, J \in \mathcal{I}^*\})$.

However, by definition, $F \setminus NS$ is the union of the faces of those polytopes of the form $\pi'(K+v)$, $v \in \mathbb{Z}^N$. Thus $\pi'(\mathbb{Z}^N_{n-1}) = \pi'(\cup\{e^J + v : v \in \mathbb{Z}^N, J \in \mathcal{I}^*\}) \supset F \setminus NS$.

Conversely, it is easy to show that for each $J \in \mathcal{I}^*$, there is a face of $\pi'(K)$ which is contained in $\pi'(e^J + v)$ for some $v \in \mathbb{Z}^N$. Then for each $J \in \mathcal{I}^*$ and each $v \in \mathbb{Z}^N$, there is some shift of $\pi'(K)$, say $\pi'(K + w)$ for some $w \in \mathbb{Z}^N$, one of whose faces, $\Phi$ say, contains the point $\pi'(v)$ as an extreme point, and $\Phi \subset \pi'(e^J + v)$. However, by letting $v$ run over $\mathbb{Z}^N$ with $w$ accompanying, we cover each $\pi'(e^J + w)$ by shifts of $\Phi$. So we find that $\cup\{\Phi + \pi'(w) : w \in \mathbb{Z}^N\} = \cup\{\pi'(e^J + w) : w \in \mathbb{Z}^N\}$. Thus $\pi'(\mathbb{Z}^N_{n-1}) = \cup\{\pi'(e^J + v) : v \in \mathbb{Z}^N, J \in \mathcal{I}^*\} \subset F \setminus NS$ and we are done. •

**Definition 8.3** Recall the subspace $V$ from (2.1) and write $\dim V = m$. Let $\mathcal{I}^*(V)$ be the set $\{J \in \mathcal{I}^* : \dim(\pi'(e^J) \cap V) = m - 1\}$.

The following is straightforward from Definition 4.3 and Lemma 8.2 above.

**Lemma 8.4** *i/* $\mathcal{I}^*(V) = \{J \in \mathcal{I}^* : \pi'(e^J) \cap V \neq V\}$.
*ii/ If* $u \notin NS$*, then*

$$\begin{aligned}(V + \pi'(u)) \setminus NS &= (V + \pi'(u)) \cap \pi'(\mathbb{Z}^N_{n-1}) \\ &= (V + \pi'(u)) \cap (\cup\{\pi'(e^J + v) : v \in \mathbb{Z}^N, J \in \mathcal{I}^*(V)\}).\end{aligned}$$ •

With these notations in mind, we are well equipped to describe the topology of $V_u$.

**Definition 8.5** Given $u \in NS$, define a set, $\Lambda_u$, of $m-1$-dimensional affine subspaces of $V$ whose elements are of the form $(\pi'(e^J + v) \cap (V + \pi'(u))) - \pi'(u)$ where $J \in \mathcal{I}^*(V)$ and $v \in \mathbb{Z}^N$. Such a space may be written more conveniently, $\pi'(e^J + v - u) \cap V$.

We say that a subset of $V$ is a $\Lambda_u$*-tope*, if it is a compact polytope which is the closure of its interior and each of whose faces is a subset of some element of $\Lambda_u$.

We shall also say that a subset, $B$, of $V_u$ is a $\Lambda_u$*-tope* if $B$ is clopen and $\nu(B) - \pi'(u)$ is a $\Lambda_u$-tope subset of $V$ in the sense above (recall $\nu$ from (4.3) above).

**Theorem 8.6** *The set of* $\Lambda_u$*-topes is* $\Gamma$ *invariant and the indicator functions of* $\Lambda_u$*-topes generate* $CV_u$ *as a* $\mathbb{Z}$*-module,* $\Gamma$*-equivariantly. The set of* $\Lambda_u$*-topes in* $V_u$ *is thus precisely the collection of compact open subsets of* $V_u$*.*

**Proof** By Theorem 9.3 of [**FHK1**] we know that $CF_u$ is generated by indicator functions of the sets formed by finite intersection, union and difference of sets of the form $\overline{K} + \pi'(v)$; this collection is written $\mathcal{B}_u$. By definition 4.3, $CV_u$ is generated by the indicator functions of sets, $\mathcal{B}'_u = \{B \in \mathcal{B}_u : \nu(B) \subset V + \pi'(u)\}$. So if $B \in \mathcal{B}'_u$, then $\nu(B)$ is the closure of the corresponding intersection, union and difference of sets of the form $(\pi'([0,1]^N) + \pi'(v)) \cap (V + \pi'(u))$.

However, $\nu$ is a map which sends open sets to sets with interior (cf. Lemma 9.3 [**FHK1**]) and so $\nu(B)$, being clearly a polytope, is the closure of its interior. Moreover, each of the faces of $\nu(B)$ is, by the proof of Lemma 8.2, contained in some $\pi'(e^J) + \pi'(v)$. Thus we confirm the conditions needed for a $\Lambda_u$-tope.

The conclusion about the topology and the $\Gamma_{\mathcal{T}}$ equivariance of the construction are immediate. •

**Remark 8.7** We compare this with the topology described in [**L**]. There each open half-space defined by a hyperplane element of $\Lambda_u$ is completed to define a clopen subset of a totally disconnected space. Using Theorem 8.6 above, it is straightforward to see that the two topologies agree since each open half-space in $V$ is a union of the interiors of $\Lambda_u$-topes (in $V$), and each $\Lambda_u$-tope in $V_u$ is an intersection of clopen half-spaces.

**Definition 8.8** Write $\mathcal{P}$ for the set of points in $V$ which can found as the 0-dimensional intersection of $m$ elements of $\Lambda_u$. Note that, under the assumptions on $\Lambda_u$, $\mathcal{P}$ is a countable set, invariant under shifts by $\Gamma_{\mathcal{T}}$.

Say that $\mathcal{P}$ *is finitely generated* if $\mathcal{P}$ is the disjoint union of a finite number of $\Gamma_{\mathcal{T}}$ orbits, and is *infinitely generated* otherwise. Write $\Omega(\mathcal{P})$ for the collection of $\Gamma$ orbits in $\mathcal{P}$.

The main theorem concerning canonical projection method tilings is the following.

**Theorem 8.9** *Given a canonical projection method tiling $\mathcal{T}$ and the constructions above, if $\mathcal{P}$ is infinitely generated, then $H^d(\mathcal{GT})$ is infinitely generated and $H^d(\mathcal{GT})\otimes\mathbb{Q}$ is infinite dimensional.*

We complete the proof of this theorem in Section 11, but the following algebraic observation provides a significant simplifying step.

**Lemma 8.10** *Suppose that $G$ is a torsion-free $\mathbb{Z}$ module. For a $\mathbb{Z}$-module $H$ write $H/2$ for $H \otimes \mathbb{Z}/2$, its reduction modulo 2. Given a $\mathbb{Z}$-module homomorphism $\phi\colon G \longrightarrow H/2$, if $Im\phi$ is infinite dimensional as a $\mathbb{Z}/2$ subspace of $H/2$ then $G$ is infinitely generated as a $\mathbb{Z}$-module and $G \otimes \mathbb{Q}$ is infinitely generated as a $\mathbb{Q}$ vector space.*

**Proof** It is sufficient to prove the statement concerning $G \otimes \mathbb{Q}$. Suppose that $\phi(s_n)$ is a sequence of independent elements in $Im\phi$ and suppose that there is some relation

$$\sum_{n=1}^{m} q_n s_n = 0$$

for $q_n \in \mathbb{Q}$. Since $G$ is torsion-free, we can assume the $q_n$ are integers and have no common factor; in particular, they are not all even. Applying the map $\phi$ then gives a non-trivial relation among the $\phi(s_n)$, a direct contradiction, as required. •

## §9 Constructing $\Lambda$-topes

In section 8, we described the basic compact open subsets of $V_u$ as $\Lambda_u$-topes. In this section we abstract this idea conveniently. Here we develop some constructions based on a general collection of affine hyperplanes, $\Lambda$, of a vector space, $W$, with group action, $\Gamma$. Always, the example in mind is $\Lambda = \Lambda_u$, $W = V$ and $\Gamma = \Gamma_{\mathcal{T}}$. Indeed the first few definitions and constructions are only the slightest generalisation of those of section 8.

**Definition 9.1** Suppose that $W$ is a vector space of dimension $m$ and that $\Gamma$ is a finitely generated free abelian group acting minimally by translation on $W$. We write $w \mapsto w + \gamma$ for the group action by $\gamma \in \Gamma$, and we may think of $\Gamma$ as a dense subgroup of $W$ without confusion.

Suppose that $\Lambda$ is a countable collection of affine subspaces of $W$ such that each $H \in \Lambda$ has dimension $m - 1$, and such that, if $H \in \Lambda$ and $\gamma \in \Gamma$, then $H + \gamma \in \Lambda$. We suppose that the number of $\Gamma$ orbits in $\Lambda$ is finite, and we write $\Omega(\Lambda)$ for the set of orbits.

If $H \in \Lambda$, then we define a unit normal vector, $\lambda(H)$. We suppose that the set $\mathcal{N}(\Lambda) = \{\lambda(H) : H \in \Lambda\}$ is finite and that we have chosen the $\lambda(H)$ consistently so that $-\lambda(H) \notin \mathcal{N}(\Lambda)$. Finally, we suppose that $\mathcal{N}(\Lambda)$ generates $W$ as a vector space.

Now we consider points formed by the intersection of elements of $\Lambda$.

**Definition 9.2** Write $\mathcal{P}$ for the set of points in $W$ which can found as the 0-dimensional intersection of $m$ elements of $\Lambda$. Note that, under the assumptions on $\Lambda$, $\mathcal{P}$ is a countable set, invariant under shifts by $\Gamma$.

Say that $\mathcal{P}$ is *finitely generated* if $\mathcal{P}$ is the disjoint union of a finite number of $\Gamma$ orbits and is *infinitely generated* otherwise. Write $\Omega(\mathcal{P})$ for the collection of $\Gamma$ orbits in $\mathcal{P}$.

We say that a subset, $D$, of $W$ is a $\Lambda$*-tope*, if $D$ is a compact polytope which is the closure of its interior and each of whose faces is a subset of some element of $\Lambda$.

In the space $W' = W \setminus \cup\{H : H \in \Lambda\}$, the collection of sets, $\mathcal{A}$, of the form $A \cap W'$, where $A$ is a $\Lambda$-tope, is an algebra.

We write $CW$ for the ring of integer-valued functions generated by indicator functions of elements of $\mathcal{A}$. This defines a canonical $\Gamma$-equivariant topological extension of $W$ which we shall write $\overline{W}$.

It is at this level of generality that we shall prove Theorem 8.9.

**Definition 9.3** Suppose that $D \subset W$ is a $\Lambda$-tope. A point $p \in W$ is a *vertex of* $D$ if $p \in \partial D$ and we have $H_1, ..., H_m \in \Lambda$ such that $\{p\} = \cap H_i$ and for each $i$ the component of $H_i \cap \partial D$ which contains $p$ is a union of faces of $D$.

If we wish to specify the $H_i$ in the definition above, then we say that $v$ is a $(H_1, ..., H_m)$-*vertex of* $D$. Note that, given the orbit class of each $H_i$, this information defines $H_i$ uniquely for each $1 \leqslant i \leqslant m$.

It is clear that a vertex is an element of $\mathcal{P}$. Conversely, each element of $\mathcal{P}$ is a vertex in some $\Lambda$-tope – this uses the fact that $\mathcal{N}(\Lambda)$ generate $W$ and that each $\Gamma$-orbit in $W$ is infinite in all directions in $W$.

Note also the fact that $v$ may be a $(H_1, ..., H_m)$-vertex of $D$ simultaneously for distinct choices of $(H_1, ..., H_m)$.

We develop now an idea of decomposition of sets of vectors, to be used in the division of cases in sections 10 and 11.

**Construction 9.4** Suppose that $A$ is a finite set of non-zero, pair-wise non-parallel vectors in $\mathbb{R}^m$ and suppose $A$ spans $\mathbb{R}^m$. The example we have in mind is of $\mathcal{N}(\Lambda)$, the set of normals.

A *decomposition* of $A$ is a partition $A = A_1 \cup A_2$ such that $V_1 \cap V_2 = 0$ where each $V_j$ is the space spanned by $A_j$, $j = 1, 2$. $A$ is *indecomposible* if no such decomposition is possible. It is not hard to show that every set $A$ has a unique partition into indecomposible subsets.

Suppose that $B \subset A$ is a basis for $\mathbb{R}^m$. Then, by requiring $B$ to an orthonormal basis, we define an inner product which we write as square brackets $[.,.]_B$. Define a finite graph $G(B; A)$ with vertices $B$ and an edge from $x$ to $y$ whenever $x \neq y$ and there is a $z \in A \setminus B$ such that $[x, z]_B \neq 0$ and $[y, z]_B \neq 0$.

**Lemma 9.5** *The following are equivalent.*

*i/ $A$ is indecomposible;*

*ii/ for all bases $B \subset A$, $G(B; A)$ is connected;*

*iii/ for some basis $B \subset A$, $G(B; A)$ is connected.* •

**Remark 9.6** Note that if $\phi\colon \mathbb{R}^m \longrightarrow \mathbb{R}^m$ is a linear bijection, then $A$ is indecomposible if and only if $\phi(A)$ is indecomposible.

**Proposition 9.7** *Suppose that* $\dim W > 1$*, that* $0 \notin A \subset W$ *spans* $W$ *and that* $A$ *has no parallel elements. Suppose that* $B \subset A$ *is a basis for* $W$ *and that* $G(B; A)$ *is connected. Then there is a closed convex polytope,* $C$*, of* $W$*, with interior, such that*

*i/ each $b \in B$ is the normal of exactly one face in $C$;*

*ii/ the normal of each face of $C$ is an element of $A$;*

*iii/ there is a vertex $v$ of $C$ in the mutual intersection of the faces normal to some element of $B$;*

*iv/ and we can find a $t_o > 0$ so that the points $v + \sum_{a \in A \setminus B} t_a a$ are in the interior of $C$ for all $0 < t_a < t_o$, $a \in A \setminus B$.*

*(All normals are taken with respect to the inner product $[.,.]_B$.)*

**Proof** We prove this by induction on $|A|$. If $|A| \leqslant 2$, then graph connectedness implies that $|A| = |B| = 1$ which is excluded by assumption on the dimension of $W$.

Consider the case $|A| = 3$; this implies that $|B| = 2$. Graph connectedness also implies that $a$ has non-zero components in each $B$ coordinate direction. Thus we can construct a triangle $C$ in $W$ with the required properties.

For larger values of $|A|$, graph connectedness again shows that we can find $a \in A \setminus B$. Consider the graph $G(B; A \setminus a)$, a subgraph of $G(B; A)$ with the same vertex set. Write $G_1, ..., G_k$ for the connected components of $G(B; A \setminus a)$; this defines a partition of the vertices $B = B_1 \cup B_2 ... \cup B_k$. Let $W_j$ be the space spanned by $B_j$; then $G_j = G(B_j; A_j)$ where $A_j = (A \setminus a) \cap W_j$. Let $\pi_j$ be the orthonormal projection of $W$ onto $W_j$. Since the reinstatement of $a$ reconnects the graph, we have $\pi_j(a) \neq 0$ for all $1 \leqslant j \leqslant k$.

If $\dim W_j \geqslant 2$, then, since $|\pi_j(A \setminus a) \setminus \{0\}| < |A|$, induction shows that we can construct a closed convex set $C_j$ in $W_j$, with interior and with properties i/ to iv/ above with respect to $B_j$ and $A_j$. Let $v_j$ be the vertex distinguished in property iii/.

On the other hand, if $\dim W_j = 1$, then define $C_j$ to be the closed interval with end points 0 and $\pi_j(a)(\neq 0)$. Let $v_j = 0$.

Consider $C' = \{\sum c_j : c_j \in C_j, 1 \leqslant j \leqslant k\}$, where we have decomposed $W = \oplus W_j$. Then $C'$ is a closed convex polytope with interior, and the point $v = v_1 \oplus \cdots \oplus v_k$ is a vertex of $C'$ contained in the intersection of faces normal to elements of $B$. Moreover, every face of $C'$ is normal to some element of $A \setminus a$. Property iv/ is also quickly confirmed.

It is possible though that property i/ does not hold for $C'$; this is where we use the extra element $a$ chosen at the beginning. Let $H$ be the hyperplane orthogonal to $a$.

Suppose that $F, F'$ are two faces of $C'$ orthogonal to $b \in B$ and with $v \in F$. By construction, $F'$ has the form $F + \sum_{j \in J} \pi_j(a)$ where $J$ is a subset of those indices $j$ for which $\dim W_j = 1$. However, we know that $\pi_j(a) \neq 0$ for each $j \in J$ and also, by property iv/, that $v_j + t\pi_j(a) \in C_j \setminus v_j$ for all $t > 0$ sufficiently small and for all $j$. Therefore, for all $t > 0$ sufficiently small, $ta + v + H$ separates $v$ from $F'$.

Since there are a finite number of combinations of faces and $j$ to consider, we can choose $t > 0$ such that $ta + v + H$ separates $v$ from every face of $C'$ orthonormal to some element of $B$ which does not actually contain $v$. Let $H^+$ be the closed half space which contains $v$ and which has boundary $ta + v + H$. Then $C = C' \cap H^+$ is a set which obeys all the conditions required by the lemma. •

**Theorem 9.8** *Suppose that $\mathcal{N}(\Lambda)$ is indecomposible. Suppose that $H_i, 1 \leqslant i \leqslant m$ is a collection of elements of $\Lambda$ which intersect at a point $\{v\} = \cap H_i$. Then there is a convex $\Lambda$-tope, $C$, for which $v$ is the unique $(H_1, ..., H_m)$-vertex in $C$.*

**Proof** Let $B' = \{\lambda(H_i) : 1 \leqslant i \leqslant m\}$, a basis for $W$. Let $\phi : W \longrightarrow W$ be a vector space automorphism which sends $B'$ to an orthonormal unit basis of $W$. For each $\lambda \in \mathcal{N}$ let $a(\lambda)$ be the unit vector normal to $\phi(\lambda^\perp)$ such that $[a(\lambda), \phi(\lambda)] > 0$. Let $A = \{a(\lambda) : \lambda \in \mathcal{N}\}$ and $B = \{a(\lambda) : \lambda \in B'\}$. Note that, by construction, $a(\lambda) = \phi(\lambda)$ for each $\lambda \in B$. Therefore the inner product $[.,.]_B = 0$ used to define the graph $G(B; A)$ above is precisely the canonical inner product, $\langle .,. \rangle$, in $W$.

By remark 9.6, $A$ is indecomposable. Thus the graph $G(B;A)$ is connected by 9.5, and so we may form by 9.7 a convex set, $C_o$, in $W$ with the properties outlined in 9.7. The set $C_1 = \phi^{-1}(C_o)$ is then a closed convex polytope in $W$, with interior, and such that

i/ each $\lambda \in B'$ is the normal of exactly one face in $C_1$,

ii/ the normal of each face of $C_1$ is an element of $\mathcal{N}$, and

iii/ there is a vertex, $v_1$, of $C_1$ in the mutual intersection of the faces normal to some element of $B'$.

However, we know that the orbit of an element $H$ of $\Lambda$ is dense in $W$ in the sense that for every hyperplane, $H'$, of $W$, parallel to $H$, and every $\epsilon > 0$, there is an $H'' \in [H]$ such that $H'$ and $H''$ are separated by a vector of length at most $\epsilon$. Therefore we may adjust $C_1$ slightly without disturbing the combinatorial properties of its faces to form a $\Lambda$-tope with the same properties. The vertex $v_1$ is disturbed to the new vertex, $v \in \mathcal{P}$, with the required properties. •

## §10 Theorem 8.9: The indecomposible case

We pursue the abstract construction of section 9 a little further.

Suppose that $v$ is a $(H_1, ..., H_m)$ vertex of $D$, a $\Lambda$-tope. A sufficiently small spherical neighbourhood of $v$, $U$ say, is covered by $2^m$ conical regions, with interior, bounded by the hyperplanes $H_i$. By construction, $D \cap U$ is a (uniquely defined) union of some of these regions.

**Definition 10.1** With the data above, we define $j'(v, (H_1, ..., H_m), D)$ equal to the number, mod 2, of the conical regions, found above, which unite to form $D \cap U$. Note that this number is independent of the choice of $U$ sufficiently small.

The following is clear by construction.

**Lemma 10.2** *Suppose that $D$ and $B$ are two $\Lambda$-topes with disjoint interior and that $v \in P$ is a $(H_1, ..., H_m)$-vertex of $D$ and of $B$, for some choice of hyperplanes $H_i$. Then either*

*i/ $v$ is not a $(H_1, ..., H_m)$-vertex of $D \cup B$ and*

$$0 = j'(v, (H_1, ..., H_m), D) + j'(v, (H_1, ..., H_m), B) \qquad mod\ 2$$

*or*

*ii/ $v$ is $(H_1, ..., H_m)$-vertex of $D \cup B$ and*

$$j'(v, (H_1, ..., H_m), D \cup B) = j'(v, (H_1, ..., H_m), D) + j'(v, (H_1, ..., H_m), B) \qquad mod\ 2$$

•

We now define a map.

**Definition 10.3** Consider the two index sets

$$\mathcal{J}_o = \{(p, (H_1, ..., H_m)) \in \mathcal{P} \times \Lambda^m : \{p\} = \cap\{H_i : 1 \leqslant i \leqslant m\}\ \}$$

$$\mathcal{J} = \{([p], ([H_1], ..., [H_m])) \in \Omega(\mathcal{P}) \times \Omega(\Lambda)^m : (p, (H_1, ..., H_m)) \in \mathcal{J}_o\}$$

By construction, $\Gamma$ acts naturally on $\mathcal{J}_o$ and $\mathcal{J} = \mathcal{J}_o/\Gamma$. Also $\mathcal{J}$ is infinite if and only if $\Omega(\mathcal{P})$ is infinite.

Consider the group $\oplus_{\mathcal{J}_o}(\mathbb{Z}/2)$ as a $\Gamma$-module with $\Gamma$ acting coordinate-wise. Therefore, $H_0(\Gamma, \oplus_{\mathcal{J}_o}(\mathbb{Z}/2)) = \oplus_{\mathcal{J}}(\mathbb{Z}/2)$.

Given $D$, a $\Lambda$-tope, with a vertex $v$, define $j_o(v, D)$ to be the element of $\oplus_{\mathcal{J}_o}(\mathbb{Z}/2)$ which is zero everywhere except on the coordinates

$$\{(v, (H_1, ..., H_m)) : \ v \text{ is a } (H_1, ..., H_m)\text{-vertex of } D\} \quad (\subset \mathcal{J}_o)$$

where we dictate the value $j'(v, (H_1, ..., H_m), D)$.

Likewise, define $j(v, D) \in \oplus_{\mathcal{J}}(\mathbb{Z}/2)$ as the element which is zero everywhere except on the coordinates $\{([v], ([H_1], ..., [H_m])) : \ v$ is a $(H_1, ..., H_m)$-vertex of $D\}$ where we dictate the value $j'(v, (H_1, ..., H_m), D)$.

Let $\xi_o(D) = \sum j_o(v, D)$ and $\xi(D) = \sum j(v, D)$ where each sum is over all vertices of $D$.

**Proposition 10.4** *The map $\xi$ defined in this way is $\Gamma$ invariant and additive in $\mathcal{A}_\Lambda$. Thus we define a map $\xi_o : CW \longrightarrow \oplus_{\mathcal{J}_o}(\mathbb{Z}/2)$ which quotients through to $\xi_* : H_0(\Gamma; CW) \longrightarrow \oplus_{\mathcal{J}}(\mathbb{Z}/2)$.*

**Proof** It is clear that $\xi$ is $\Gamma$ invariant since $[v] = [v + \gamma]$ and $j'(v, (H_1, ..., H_m), D) = j'(v + \gamma, (H_1 + \gamma, ..., H_m + \gamma), D + \gamma)$ by construction. We need to check now that this extends additively.

Suppose $D$ is an $\Lambda$-tope and we can write $D$ as the disjoint union of two $\Lambda$-topes, $D_j$, $j = 1, 2$. To show additivity of $\xi$ it will suffice to show that in this case $\xi_o(D) = \xi_o(D_1) + \xi_o(D_2)$. To do this we need only appeal to Lemma 10.2 above.

The quotienting through to homology is immediate. •

**Corollary 10.5** *Suppose that $\mathcal{N}(\Lambda)$ is indecomposible and $\mathcal{P}$ is infinitely generated, then $H_0(\Gamma, CW) \otimes \mathbb{Q}$ is infinite dimensional.*

**Proof** If $\mathcal{P}$ is infinitely generated, then a cardinality argument finds classes $[H_1], ..., [H_m]$ such that $\cap H_i$ is a singleton, and such that the set $\mathcal{P}' = \{v : \cap K_i = \{v\}, K_i \in [H_i]\}$ has an infinite number of disjoint $\Gamma$ orbits.

Select from $\mathcal{P}'$ an infinite sequence, $\{v_n\}$, transverse to the orbit structure, and for each $v_n$ construct, by Theorem 9.8 and Definition 10.3 above, a $\Lambda$-tope, $A_n$, which has $\xi(A_n)$ equal to 1 on the coordinate $([v_n],([H_1],...,[H_m]))$. Since only a finite number of coordinates are indicated in any particular element, $\xi(A_n)$, we may assume, passing to a subsequence if necessary, that for $n' < n$, we have $\xi(A_{n'})$ equal to 0 on the coordinate $([v_n],([H_1],...,[H_m]))$. Therefore, these $\xi(A_n)$ are independent by construction, and so the map $\xi$ has infinitely generated image. We are done by 8.10. •

## §11 The Proof of Theorem 8.9 completed

Once again we exploit the generalities of section 9.

**Construction 11.1** Suppose that $\mathcal{N}(\Lambda)$ is decomposible and that $\mathcal{N}_1, \mathcal{N}_2, ..., \mathcal{N}_k$ are its components. Let $W_j = (\cup_{i\neq j}\mathcal{N}_i)^{\perp} = \cap_{i\neq j}\mathcal{N}_i^{\perp}$ so that $W = \oplus_j W_j$. Let $\pi_j$ be the skew projection onto $W_j$ with kernel $\underset{i\neq j}{\oplus} W_i$.

**Lemma 11.2** *For each $j$, the map $\pi_j$ is open and sends $W$ onto $W_j$, indeed $\pi_j^{-1}(W_j) = W$. Therefore, we have $\overline{W} = \Pi\overline{W}_j$ and $CW = \otimes_j CW_j$.*

*The group $\Gamma$ acts naturally on $W_j$ via the isomorphism $W/W_j^{\perp} \equiv W_j$ induced by the adjoint of $\pi_j$. In the product representation above, $\Gamma$ acts by the diagonal action.* •

The space $\overline{W}_j$ has an alternative construction analogous to that of $\overline{W}$ (9.2). There is a collection of cutting hyperplanes, $\Lambda_j$, so that the clopen sets in $\overline{W}_j$ are the $\Lambda_j$-topes. In fact, we have by construction, $\Lambda_j = \{H \cap W_j : H \in \Lambda, \lambda(H) \in \mathcal{N}_j\}$ and $\mathcal{N}(\Lambda_j) = \{\lambda \in \mathcal{N} : \lambda \in W_j\} = \mathcal{N}_j$ , so only those elements of $\Lambda$ whose normal is parallel to $W_j$ influence the topology on $\overline{W}_j$.

In particular, we gather the following facts.

**Lemma 11.3** *Given the construction above,*

*i/ $\mathcal{N}(\Lambda_j)$ is indecomposable;*

*ii/ if $H \in \Lambda_j$, then there is a unique $H' \in \Lambda$ (equal to $\pi_j^{-1}(H)$) such that $H = H' \cap W_j$, and $\lambda(H') = \lambda(H)$; and*

*iii/ $\mathcal{P}_j$, the point set defined using $\Lambda_j$, is equal to $\pi_j(\mathcal{P})$. In the product in 11.2, $\Pi_j \mathcal{P}_j = \mathcal{P}$.* •

We are now in a position to prove a more general form of Theorem 9.8. Recall the indexing sets, $\mathcal{J}$ and $\mathcal{J}_o$ from 10.3.

The following theorem improves the argument of 10.5 to the decomposible case. This involves taking independent constructions of suitable $\Lambda$-topes in section 9 and combining their properties with respect to a product of maps, $\xi$, constructed in section 10 for the indecomposible case. An additional subtlety has to be met if we have 1 dimensional parts to our decomposition.

**Theorem 11.4** *Given the data above, there is a map $\xi_*: H_0(\Gamma; CW) \longrightarrow \oplus_{\mathcal{J}}\mathbb{Z}/2$ such that for all $v \in \mathcal{P}$, there are $H_i \in \Lambda : 1 \leqslant i \leqslant m$ so that $([v], ([H_1], ..., [H_m])) \in \mathcal{J}$, and an element $f_v \in H_0(\Gamma, CW)$ such that $\xi_*(f_v)$ has value 1 at coordinate $([v], ([H_1], ..., [H_m]))$.*

**Proof** We prove the conclusion with some superstructure.

**Claim:** *There are maps, $\xi_o$ and $\xi_*$ so that the following diagram commutes.*

$$\begin{array}{ccc} CW & \xrightarrow{\xi_o} & \oplus_{\mathcal{J}_o}\mathbb{Z}/2 \\ \Big\downarrow q & & \Big\downarrow q \\ H_0(\Gamma; CW) & \xrightarrow{\xi_*} & \oplus_{\mathcal{J}}\mathbb{Z}/2. \end{array}$$

*Moreover, for each $v \in \mathcal{P}$, there are $H_i : 1 \leqslant i \leqslant m$ so that $([v], ([H_1], ..., [H_m])) \in \mathcal{J}$, and an element $f_v \in CW$ such that $\xi_* q(f_v)$ has value 1 at coordinate $([v], ([H_1], ..., [H_m]))$.*

The previous work establishes this claim in the indecomposable case with the sole exception of when $|\mathcal{N}| = 1$, which we tidy up now.

If $|\mathcal{N}| = 1$ then $\dim W = 1$, and so, by section 7, $H_0(\Gamma; CW) = \mathbb{Z}^{M+k-1}$ where $M = rank(\Gamma) > 1$ and $k$ is the number of orbits in $\mathcal{P}$. We see that $\mathcal{J}$ is in natural correspondence with $\Omega(\mathcal{P})$ in this case, and so $k$ is also the cardinality of $\mathcal{J}$.

Take, therefore, a surjection $s: H_0(\Gamma; CW) \longrightarrow \oplus_{\mathcal{J}}\mathbb{Z}/2$. With the canonical map $q: \oplus_{\mathcal{J}_o}\mathbb{Z}/2 \longrightarrow \oplus_{\mathcal{J}}\mathbb{Z}/2$ construct the surjection

$$s - q: H_0(\Gamma; CW) \oplus (\oplus_{\mathcal{J}_o}\mathbb{Z}/2) \longrightarrow \oplus_{\mathcal{J}}\mathbb{Z}/2.$$

Since $CW \equiv \mathbb{Z}^\infty$ (i.e. the infinite direct sum of $\mathbb{Z}$), we have complete freedom to find a map $*$ which makes the following sequence exact at the middle term

$$\mathbb{Z}^\infty \equiv CW \xrightarrow{q \oplus *} H_0(\Gamma; CW) \oplus (\oplus_{\mathcal{J}_o}\mathbb{Z}/2) \xrightarrow{s-q} \oplus_{\mathcal{J}}\mathbb{Z}/2.$$

The exactness gives the commuting square property. By surjectivity, for each $[v] \in \Omega(\mathcal{P})$ we define $f_v \in CW$ quite arbitrarily so that $sq(f_v)$ has value 1 on the coordinate $[v] \in \mathcal{J}$. This completes the indecomposable case.

Now suppose that $\mathcal{N} = \cup \mathcal{N}_j$ is a partition into indecomposibles. Form the spaces $W_j$, of dimension $m_j = \dim W_j$, etc. as above. $\Gamma$ acts by translation by elements $\pi_j(\Gamma)$

which may or may not be free. If we must consider only free actions, then we should consider rather the action of $\pi_j(\Gamma)$, which is also represented by a free subaction by a complimented subgroup of $\Gamma$. Lemma 4.4 gives a natural equation between $H_0(\Gamma, CW_j)$ and $H_0(\pi_j(\Gamma), CW_j)$ therefore, so no complication arises if we stick with the $\Gamma$ action.

Note also that $rank(\pi_j\Gamma) > 1$ (since $\pi_j\Gamma$ is dense in $W_j$) so, should $m_j = 1$, the argument for $\dim W = 1$ above continues to apply.

Consider the index sets $\mathcal{J}_{jo}$ and $\mathcal{J}_j$ formed from $\mathcal{P}_j$ and $\Lambda_j$ in $W_j$, as for $\mathcal{P}$ and $\Lambda$ before. By (11.2) we can write each element of $\mathcal{J}_{jo}$ in the form $(\pi_j(v), (\pi_j(H_1), ..., \pi_j(H_{m_j})))$ for some $v \in \mathcal{P}$ and some uniquely determined collection, $H_1, ..., H_{m_j}$, of elements of $\Lambda$ whose intersection is an affine translate of the space $\pi_j^{-1}(0) = \bigoplus_{i \neq j} W_i$. Therefore the map $\mathcal{J}_o \longrightarrow \Pi_j \mathcal{J}_{jo}$ which sends $(v, (H_1, ..., H_m))$ to $(\pi_j(v), (\pi_j(H_1), ..., \pi_j(H_m))^*)$ in the $j$th coordinate (the star indicates that we drop any 0 entries from the list) is well-defined and, in fact, a bijection.

Consider the group $\Gamma' = \cap\{\pi_j^{-1}(\pi_j(\Gamma)) : 1 \leqslant j \leqslant k\}$ which contains $\Gamma$ automatically. By construction, we have $\mathcal{J}_o/\Gamma' = \Pi_j(\mathcal{J}_{jo}/\pi_j(\Gamma)) = \Pi_j\mathcal{J}_j$. Thus we have a sequence of maps

$$\Pi_j \mathcal{J}_{jo} = \mathcal{J}_o \xrightarrow{q} \mathcal{J}_o/\Gamma \xrightarrow{q'} \mathcal{J}_o/\Gamma' = \Pi_j \mathcal{J}_j$$

which will be used in the sequel.

Having proved the claim in the indecomposable case, we may suppose, for each $1 \leqslant j \leqslant k$, that we have commuting squares

$$\begin{array}{ccc} CW_j & \xrightarrow{\xi_{jo}} & \oplus_{\mathcal{J}_{jo}} \mathbb{Z}/2 \\ \Big\downarrow q_j & & \Big\downarrow q_j \\ H_0(\Gamma; CW_j) & \xrightarrow{\xi_{j*}} & \oplus_{\mathcal{J}_j} \mathbb{Z}/2 \end{array}$$

and for each $v \in \mathcal{P}$, we find elements $f_{v,j} \in CW_j$ defined from $\pi_j(v) \in \mathcal{P}_j$ (Lemma 11.3), so that $\xi_{j*}q_j(f_{v,j})$ has value 1 in the $([\pi_j(v)], [H_{1,j}], \ldots, [H_{m_j,j}]))$ coordinate.

The equation $CW = \otimes_j CW_j$ allows us to build the map

$$\otimes\xi_{jo}: CW \longrightarrow \otimes_j \oplus_{\mathcal{J}_{jo}} \mathbb{Z}/2$$

and the range is naturally equal to $\oplus_{\mathcal{J}_o} \otimes_j \mathbb{Z}/2 = \oplus_{\mathcal{J}_o}\mathbb{Z}/2$ using the equations between index sets noted before. The construction is clearly $\Gamma$ equivariant and so we deduce a quotiented map: $\xi_*: H_0(\Gamma; CW) \longrightarrow \oplus_{\mathcal{J}}\mathbb{Z}/2$ which completes a square as required, with $q: \oplus_{\mathcal{J}_o}\mathbb{Z}/2 \longrightarrow \oplus_{\mathcal{J}}\mathbb{Z}/2$ defined from the corresponding map on the index sets, described above.

Likewise, the index map $q' : \mathcal{J} \longrightarrow \Pi_j \mathcal{J}_j$ describes the quotient of $\oplus_{\mathcal{J}} \mathbb{Z}/2$ by the $\Gamma'$ action, giving the map, for which we use the same letter,

$$q' : \oplus_{\mathcal{J}} \mathbb{Z}/2 \longrightarrow \oplus_{\Pi_j \mathcal{J}_j} \mathbb{Z}/2 = \otimes_j \oplus_{\mathcal{J}_j} \mathbb{Z}/2.$$

The composition $q'q$ is therefore defined

$$\otimes_j \oplus_{\mathcal{J}_{jo}} \mathbb{Z}/2 = \oplus_{\mathcal{J}_o} \mathbb{Z}/2 \xrightarrow{q'q} \otimes_j \oplus_{\mathcal{J}_j} \mathbb{Z}/2.$$

Given $v$, the element $f_v = \otimes_j f_{v,j} \in CW$ is, by assumption, mapped by

$$\otimes_j q_j \xi_{j*} : CW \longrightarrow \otimes_j \oplus_{\mathcal{J}_j} \mathbb{Z}/2 = \oplus_{\Pi_j \mathcal{J}_j} \mathbb{Z}/2$$

to an element with value 1 at the $\Pi_j([\pi_j(v)], [H_{1,j}], ..., [H_{m_j,j}])$ coordinate. So therefore $\otimes_j q_j \xi_{j*} = q'q\xi_o$. Keeping track of $\Gamma$-orbits in the index sets, we find $q\xi_o(f_v)$ with value 1 in the $([v], [H'_1], ..., [H'_m])$ coordinate, with $H'_i$ chosen in $\Lambda$ so that $(\pi_j(H'_1), ..., \pi_j(H'_m))^* = (H_{1,j}, ..., H_{m_j,j})$. So the decomposible case follows. •

**Proof of Theorem 8.9** We now assume the construction above for $W = V$, $\Lambda = \Lambda_u$ and $\Gamma = \Gamma_{\mathcal{T}}$. The result follows exactly as for Corollary 10.5, using the more general result above. •

## §12 Corollaries of Theorem 8.9

To apply Theorem 8.9 we must be able to count the orbits in $\mathcal{P}$. This is a geometric exercise, and each case will have its own peculiarities. However, we present in this section two elementary general conditions which are sufficient to give infinite orbits in $\mathcal{P}$.

Recall the general set-up from section 9, and the construction of $\mathcal{P}$ as points which are the proper intersection of $m$ hyperplanes picked from $\Lambda$.

**Definition 12.1** Suppose that $H_1, ..., H_m$ is a set of hyperplanes chosen from $\Lambda$, intersecting in a single point. Let $\mathcal{P}^*$ be the collection of points in $\mathcal{P}$ which can be defined as the intersection of $H'_1, \dots, H'_m$ with each $H'_i$ in the same $\Gamma$-orbit as $H_i$, $1 \leqslant i \leqslant m$.

For each $i \in \{1, 2, \dots, m\}$, define $\pi_i : W \longrightarrow \cap_{j \neq i} H_j$ as the surjective idempotent with kernel $H_i$.

We can define a product structure to $W$ via the isomorphism $W \xrightarrow{\oplus_i \pi_i} \oplus_i H_i$.

Given $i \in \{1, 2, \dots, m\}$, let $\mathcal{P}_i = \pi_i(\mathcal{P}^*)$, and $\Gamma_i = \pi_i(\Gamma)$.

**Proposition 12.2** *In the product description above, $\oplus \mathcal{P}_i = \mathcal{P}^*$ and $\oplus \Gamma_i \supset \Gamma$.*

*If $\sum_i rank\Gamma_i > rank\Gamma$, then the number of $\Gamma$ orbits in $\mathcal{P}^*$ is infinite.*

**Proof** Each $\Gamma_i$ acts naturally on $\cap_{j\neq i}H_j$ by translation. By construction $\mathcal{P}^*$ is the free $\oplus\Gamma_i$ orbit of a single point, namely the point at the intersection $\cap H_i$; from this the first equality follows immediately. The second containment is straightforward. Moreover, there is a 1-1 correspondence between the $\Gamma$ orbits of $\mathcal{P}^*$ and the $\Gamma$ cosets in $\oplus\Gamma_i$, from which the third statement is immediate. •

**Corollary 12.3** *Suppose that* $rank\Gamma < 2\dim W$*, then* $\mathcal{P}$ *is infinitely generated.*

**Proof** Recall that $\Gamma$ is a dense subgroup of $W$ implying that $rank\Gamma_i \geqslant 2$ for each $i$. The inequality of Proposition 12.2 above gives the result therefore (recall that $\mathcal{P}^* \subset \mathcal{P}$ by definition). •

**Corollary 12.4** *Suppose that* $\mathcal{T}_u$ *is the canonical projection method tiling in* $\mathbb{R}^d$*, with data* $(E, u)$*, and suppose that* $E \cap \mathbb{Z}^N = 0$*. If* $rank\Delta + 2d < N$ *then* $H_0(\mathcal{G}\mathcal{T}_u) \otimes \mathbb{Q}$ *is infinite dimensional, and so* $\mathcal{T}_u$ *is not a substitution tiling.*

**Proof** With the same correspondences as noted in the proof of Theorem 8.9 at the end of section 11, note that $rank\Gamma_{\mathcal{T}} = N - rank\Delta$ and that $m = \dim V = N - d - rank\Delta$. Now use the last corollary. •

**Examples 12.5** The examples of the Octagonal tiling (see eg. [**Been**] [**Soc**] [**Be**]), a substitution tiling with $N = 4$, $\Delta = 0$ and $d = 2$, shows that the inequality of Corollary 12.4 should be strict. And the Penrose tiling (see eg. [**Soc**] [**S**]), a substitution tiling with $N = 5$, $\Delta \equiv \mathbb{Z}$ and $d = 2$, shows that all components of the inequality are important.

We can pursue the construction above a little further, by considering higher dimensional intersections of transverse elements of $\Lambda$.

**Definition 12.6** Suppose that $H_1, \ldots, H_m$ are elements of $\Lambda$ with single point mutual intersection $\{p\} = \cap H_j$.

Write $\Gamma(H_1, \ldots, H_k)$ for the stabiliser in $\Gamma$ of the intersection $L = \cap_{1\leqslant j\leqslant k}H_j$, i.e. the complemented subgroup, $\{\gamma \in \Gamma \colon L + \gamma = L\}$.

Let $\Gamma'(H_1, \ldots, H_k : H_{k+1}, \ldots, H_m) = \{\alpha \in \mathbb{R}^N : \{\alpha + p\} = L \cap (L' + \gamma), \gamma \in \Gamma\}$, where $L' = \cap_{k+1\leqslant j\leqslant m}H_j$.

We think of $\Gamma'(H_1, \ldots, H_k : H_{k+1}, \ldots, H_m)$ as the projection of $\Gamma$ onto $L$ collapsing in the $L'$ direction.

The following is straightforward from the definitions.

**Lemma 12.7** *With the notation above* *i/* $\Gamma'(H_1,\ldots,H_k : H_{k+1},\ldots,H_m)$ *is a group with* $\Gamma(H_1,\ldots,H_k)$ *as a subgroup.*

*ii/* $\Gamma'(H_1,\ldots,H_k : H_{k+1},\ldots,H_m) + p = \mathcal{P} \cap L$.

*iii/ If* $q \in \mathcal{P} \cap L$, *then* $(\Gamma + q) \cap L = \Gamma(H_1,\ldots,H_k) + q$. •

This gives immediately an easy way to determine whether we have an infinite number of orbits in $\mathcal{P}$.

**Lemma 12.8** *If, for some choice of* $H_1,\ldots,H_m$, *the group* $\Gamma(H_1,\ldots,H_k)$ *has infinite index in* $\Gamma'(H_1,\ldots,H_k : H_{k+1},\ldots,H_m)$ *(equivalently, if the rank of* $\Gamma(H_1,\ldots,H_k)$ *is strictly smaller than the rank of* $\Gamma'(H_1,\ldots,H_k : H_{k+1},\ldots,H_m)$*), then* $\mathcal{P}$ *contains an infinite number of* $\Gamma$ *orbits.*

**Proof** By Lemma 12.7 ii/ and iii/ the orbits in $\mathcal{P}$ which intersect $L$ are enumerated precisely by the cosets of $\Gamma(H_1,\ldots,H_k)$ in $\Gamma'(H_1,\ldots,H_k : H_{k+1},\ldots,H_m)$. This is infinite by assumption. •

**Definition 12.9** Given a transverse collection, $\mathcal{H} = H_1,\ldots,H_m$, of elements of $\Lambda$ and $J \subset \{1,2,\ldots,m\}$, write $\Gamma(\mathcal{H},J) = \Gamma(H_j : j \in J)$ and $\Gamma'(\mathcal{H},J) = \Gamma'(H_j : j \in J | H_j : j \notin J)$.

The techniques proving the last two lemmas, give the following.

**Lemma 12.10** *For every choice of* $\mathcal{H} = (H_1,\ldots,H_m)$ *as above and for every pair of sets,* $J_1, J_2$ *such that* $J_1 \cup J_2 = \{1,2,...,m\}$, $\Gamma'(\mathcal{H},J_1) + \Gamma'(\mathcal{H},J_2)$ *and* $\Gamma(\mathcal{H},J_1) + \Gamma(\mathcal{H},J_2)$ *are both direct sums.*

*If* $\mathcal{P}$ *has a finite number of* $\Gamma$ *orbits, then*

$$\Gamma(\mathcal{H},J_1) + \Gamma(\mathcal{H},J_2) \subset \Gamma(\mathcal{H},J_1 \cap J_2) \subset \Gamma'(\mathcal{H},J_1 \cap J_2) \subset \Gamma'(\mathcal{H},J_1) + \Gamma'(\mathcal{H},J_2)$$

*is a sequence of subgroups all of mutual finite index.* •

From which we deduce

**Theorem 12.11** *Suppose* $\mathcal{P}$ *has a finite number of* $\Gamma$ *orbits and that* $\mathcal{H} = (H_1,\ldots,H_m)$ *is a collection of transverse elements of* $\Lambda$. *Then for all* $J_1, J_2 \subset \{1,2,..,m\}$ *such that* $|J_2| = m-1$, *we have* $rank\Gamma(\mathcal{H},J_1) = (m-|J_1|)rank\Gamma(\mathcal{H},J_2)$.

*In particular, putting* $J_1 = \emptyset$, *we have* $rank\Gamma = m\, rank\Gamma(\mathcal{H},J_2)$.

**Corollary 12.12** *If* $\mathcal{T}_u$ *is a canonical projection method tiling with finite rationalised* $H_0$, *then there is an integer,* $g$ *say, such that* $(g-1)N' = gd$, *where* $N' = N - rank\Delta$.

**Examples 12.13** For Penrose tilings, octagonal tilings and dodecagonal tilings (the undecorated versions described in [**Soc**]) we have always $N' = 4$ and $d = 2$, hence $g = 2$. From the above it is clear that $g$ must be 2 in $d = 2$ if the rationalized homology is finitely generated. And indeed, all three tilings above are substitutional. It is not difficult to construct examples in which $d = 4$ and $g = 3$, for example.

It is clear also that in the generic placement of planes projections of $\Gamma$ onto lines will have higher rank than intersections and so the $\Gamma$ groups of 12.6 will be of strictly smaller rank than $\Gamma'$. Thus, from 12.8 and 8.9, we deduce

**Theorem 12.14** *Suppose that $\mathcal{T}$ is a canonical projection method tiling and that $E$ is in generic position, then $H_0(\mathcal{G}\mathcal{T}) \otimes \mathbb{Q}$ is infinite dimensional, and $\mathcal{T}$ is not a substitution.* $\bullet$

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