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Resummation of Singlet Parton Evolution at Small x

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Abstract

We propose an improvement of the splitting functions at small x which overcomes the apparent problems encountered by the BFKL approach. We obtain a stable expansion for the x -evolution function $\chi(M)$ near $M = 0$ by including in it a sequence of terms derived from the one- and two-loop anomalous dimension γ . The requirement of momentum conservation is always satisfied. The residual ambiguity on the splitting functions is effectively parameterized in terms of the value of λ , which fixes the small x asymptotic behaviour $x^{-\lambda}$ of the singlet parton distributions. We derive from this improved evolution function an expansion of the splitting function which leads to good apparent convergence, and to a description of scaling violations valid both at large and small x .

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1. The theory of scaling violations in deep inelastic scattering is one of the most solid consequences of asymptotic freedom and provides a set of fundamental tests of QCD. At large Q^2 and not too small but fixed x the QCD evolution equations for parton densities [1] provide the basic framework for the description of scaling violations. The complete splitting functions have been computed in perturbation theory at order α_s (LO approximation) and α_s^2 (NLO) [2]. For the first few moments the anomalous dimensions at order α_s^3 are also known [3].

At sufficiently small x the approximation of the splitting functions based on the first few terms in the expansion in powers of α_s is not in general a good approximation. If not for other reasons, as soon as x is small enough that $\alpha_s \xi \sim 1$, with $\xi = \log 1/x$, all terms of order $\alpha_s(\alpha_s \xi)^n$ and $\alpha_s^2(\alpha_s \xi)^n$ which are present [4] in the splitting functions must be considered in order to achieve an accuracy up to terms of order α_s^3 . In terms of the anomalous dimension $\gamma(N, \alpha_s)$, defined as the N -th Mellin moment of the singlet splitting function (actually the eigenvector with largest eigenvalue), these terms correspond to sequences of the form $(\alpha_s/N)^n$ or $\alpha_s(\alpha_s/N)^n$. In most of the kinematic region of HERA [5] the condition $\alpha_s \xi \sim 1$ is indeed true. Hence, in principle one could expect to see in the data indications of important corrections to the approximation [6, 7] of splitting functions computed only up to order α_s^2 and the corresponding small x behaviour. In reality this appears not to be the case: the data can be fitted quite well by the evolution equations in the NLO approximation [7, 8]. Of course it may be that some corrections exist but they are hidden in a redefinition of the gluon, which is the dominant parton density at small x . While the data do not support the presence of large corrections in the HERA kinematic region [9] the evaluation of the higher order corrections at small x to the singlet splitting function from the BFKL theory [10, 11, 12] appears to fail. The results of the recent calculation [13, 14, 15] of the NLO term χ_1 of the BFKL function $\chi = \alpha_s \chi_0 + \alpha_s^2 \chi_1 \dots$ show that the expansion is very badly behaved, with the non leading term completely overthrowing the main features of the leading term. Taken at face value, these results appear to hint at very large corrections to the singlet splitting function at small x in the region explored by HERA [16].

In this article we address this problem and propose a procedure to construct a meaningful improvement of the singlet splitting function at small x , using the information from the BFKL function. We start by defining an alternative expansion for the BFKL function $\chi(M)$ which, unlike the usual expansion, is well behaved and stable when going from LO to NLO, at least for values far from $M = 1$. This is obtained by adding suitable sequences of terms of the form $(\alpha_s/M)^n$ or $\alpha_s(\alpha_s/M)^n$ to $\alpha_s \chi_0$ or $\alpha_s^2 \chi_1$ respectively. The coefficients are determined by the known form of the singlet anomalous dimension at one and two loops. This amounts to a resummation [17] of $(\alpha_s \log Q^2/\mu^2)^n$ terms in the inverse M -Mellin transform space. This way of improving χ is completely analogous to the usual way of improving γ [18]. One important point, which is naturally reproduced with good accuracy by the above procedure, is the observation that the value of $\chi(M)$ at $M = 0$ is fixed by momentum conservation to be $\chi(0) = 1$. This observation plays a crucial role in formulating the novel expansion and explains why the normal BFKL expansion is so unstable near $M = 0$, with $\chi_0 \sim 1/M$, $\chi_1 \sim -1/M^2$ and so on. This rather model-independent step is already sufficient to show that no catastrophic deviations from the NLO approximation of the evolution equations are to be expected. The next step is to use this novel expansion of χ to determine small x resummation corrections to add to the LO and NLO anomalous dimensions γ . Defining λ as the minimum value of χ , $\chi(M_{min}) = \lambda$, and

using the results of ref. [19], a meaningful expansion for the improved anomalous dimension is written down in terms of χ_0 , χ_1 , and λ . The large negative correction to $\lambda_0/\alpha_s = \chi_0(1/2)$ induced by $\alpha_s\chi_1$, that is formally of order α_s but actually is of order one for the relevant values of α_s , suggests that λ should be reinterpreted as a nonperturbative parameter. We conclude by showing that the very good agreement of the data with the NLO evolution equation can be obtained by choosing a small value of λ , compatible with zero.

2. We consider the singlet parton density

$$G(\xi, t) = x[g(x, Q^2) + k_q \otimes q(x, Q^2)], \quad (1)$$

where $\xi = \log 1/x$, $t = \log Q^2/\mu^2$, $g(x, Q^2)$ and $q(x, Q^2)$ are the gluon and singlet quark parton densities, respectively, and k_q is such that, for each moment

$$G(N, t) = \int_0^\infty d\xi e^{-N\xi} G(\xi, t), \quad (2)$$

the associated anomalous dimension $\gamma(N, \alpha_s(t))$ corresponds to the largest eigenvalue in the singlet sector. At large t and fixed ξ the evolution equation in N -moment space is then

$$\frac{d}{dt}G(N, t) = \gamma(N, \alpha_s(t)) G(N, t), \quad (3)$$

where $\alpha_s(t)$ is the running coupling. The anomalous dimension is completely known at one and two loop level:

$$\gamma(N, \alpha_s) = \alpha_s\gamma_0(N) + \alpha_s^2\gamma_1(N) + \dots \quad (4)$$

As $\gamma(N, \alpha_s)$ is, for each N , the largest eigenvalue in the singlet sector, momentum conservation order by order in α_s implies that

$$\gamma(1, \alpha_s) = \gamma_0(1) = \gamma_1(1) = \dots = 0. \quad (5)$$

Similarly, at large ξ and fixed Q^2 , the following evolution equation for M moments is valid:

$$\frac{d}{d\xi}G(\xi, M) = \chi(M, \alpha_s) G(\xi, M), \quad (6)$$

where

$$G(\xi, M) = \int_{-\infty}^\infty dt e^{-Mt} G(\xi, t), \quad (7)$$

and $\chi(M, \alpha_s)$ is the BFKL function which is now known at NLO accuracy: ¹

$$\chi(M, \alpha_s) = \alpha_s\chi_0(M) + \alpha_s^2\chi_1(M) + \dots \quad (8)$$

In eq. (6) the coupling α_s is fixed. The inclusion of running effects in the BFKL theory is a delicate point. To next-to-leading order in α_s (i.e. to NLLx), running effects can be included [19, 20] by adding to χ_1 a term proportional to the first coefficient $\beta_0 = \frac{11}{3}n_c - \frac{2}{3}n_f$

¹Note that the normalization conventions for χ_0 and χ_1 used here are different from those used in either of refs.[13, 19].

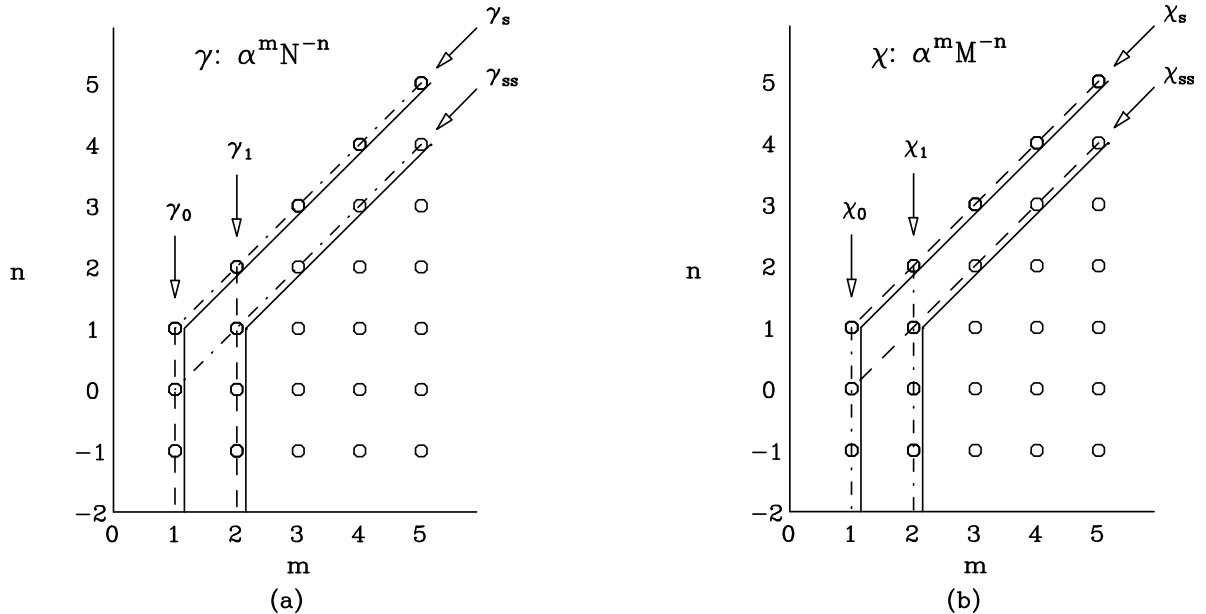


Figure 1: Graphical representation of different expansions of (a) γ and (b) χ in powers of α_s and $1/N$ (a) and $1/M$ (b) to order m and n respectively, and the different relations between these expansions. Vertical lines correspond to terms of the same fixed order in α_s : for example the one loop anomalous dimension γ_0 contains terms with $m = 1, n = 1, 0, -1, -2, \dots$. Diagonal lines correspond to terms of the same order in α_s at fixed $\frac{\alpha_s}{N}$ (a) or $\frac{\alpha_s}{M}$ (b): for example $\gamma_s(\alpha_s/N)$ contains terms with $m = n = 1, 2, 3, \dots$. The sum of terms in a vertical line of the γ plot is related by duality eqs. (10,11) to the sum of terms in a diagonal line in the χ plot and conversely (marked by the same line style). The solid lines denote terms of the same order in the “envelope” or “double leading” expansion discussed in the text.

of the β -function. Since furthermore the extra term depends on the definition of the gluon density, it is also necessary to specify the choice of factorization scheme: here we choose the $\overline{\text{MS}}$ scheme, so that the χ_1 that we will consider in the sequel is given by [19]

$$\chi_1(M) = \frac{1}{4\pi^2} n_c^2 \tilde{\delta}(M) + \frac{1}{8\pi^2} \beta_0 n_c ((2\psi'(1) - \psi'(M) - \psi'(1-M)) + \frac{1}{4n_c^2} \chi_0(M)^2), \quad (9)$$

where the function $\tilde{\delta}$ is defined in the first of ref. [13].

In the region where Q^2 and $1/x$ are both large the t and ξ evolution equations, *i.e.* eqs.(3,6), are simultaneously valid, and their mutual consistency requires the validity of the “duality” relation [11, 21]:

$$\chi(\gamma(N, \alpha_s), \alpha_s) = N, \quad (10)$$

and its inverse

$$\gamma(\chi(M, \alpha_s), \alpha_s) = M. \quad (11)$$

Using eq. (10), knowledge of the expansion eq. (8) of $\chi(M, \alpha_s)$ to LO and NLO in α_s at fixed M determines the coefficients of the expansion of $\gamma(N, \alpha_s)$ in powers of α_s at fixed $\frac{\alpha_s}{N}$:

$$\gamma(N, \alpha_s) = \gamma_s\left(\frac{\alpha_s}{N}\right) + \alpha_s \gamma_{ss}\left(\frac{\alpha_s}{N}\right) + \dots, \quad (12)$$

where γ_s and γ_{ss} contain respectively sums of all the leading and subleading singularities of γ (see fig. 1),

$$\chi_0(\gamma_s(\frac{\alpha_s}{N})) = \frac{N}{\alpha_s}, \quad (13)$$

$$\gamma_{ss}(\frac{\alpha_s}{N}) = -\frac{\chi_1(\gamma_s(\frac{\alpha_s}{N}))}{\chi_0'(\gamma_s(\frac{\alpha_s}{N}))}. \quad (14)$$

This corresponds to an expansion of the splitting function in logarithms of x : if for example we write

$$\gamma_s(\frac{\alpha_s}{N}) = \sum_{k=1}^{\infty} g_k^{(s)} \left(\frac{\alpha_s}{N}\right)^k \quad (15)$$

(where $g_1^{(s)} = n_c/\pi$, $g_2^{(s)} = g_3^{(s)} = 0$, $g_4^{(s)} = 2\zeta(3)n_c/\pi, \dots$), then the associated splitting function

$$P_s(\alpha_s\xi) \equiv \int_{c-i\infty}^{c+i\infty} \frac{dN}{2\pi i \alpha_s} e^{N\xi} \gamma_s(\frac{\alpha_s}{N}) = \sum_{k=1}^{\infty} \frac{g_k^{(s)}}{(k-1)!} (\alpha_s\xi)^{(k-1)}, \quad (16)$$

and similarly for the subleading singularities $P_{ss}(\alpha_s\xi)$, etc.

Likewise, the inverse duality eq. (11) relates the fixed order expansion eq. (4) of $\gamma(N, \alpha_s)$ to an expansion of $\chi(M, \alpha_s)$ in powers of α_s with $\frac{\alpha_s}{M}$ fixed: if

$$\chi(M, \alpha_s) = \chi_s(\frac{\alpha_s}{M}) + \alpha_s \chi_{ss}(\frac{\alpha_s}{M}) + \dots, \quad (17)$$

where now $\chi_s(\frac{\alpha_s}{M})$ and $\chi_{ss}(\frac{\alpha_s}{M})$ contain the leading and subleading singularities respectively of $\chi(M, \alpha_s)$, then

$$\gamma_0(\chi_s(\frac{\alpha_s}{M})) = \frac{M}{\alpha_s}, \quad (18)$$

$$\chi_{ss}(\frac{\alpha_s}{M}) = -\frac{\gamma_1(\chi_s(\frac{\alpha_s}{M}))}{\gamma_0'(\chi_s(\frac{\alpha_s}{M}))}. \quad (19)$$

In principle, since χ_0 and χ_1 are known, they can be used to construct an improvement of the splitting function which includes a summation of leading and subleading logarithms of x . However, as is now well known, the calculation [13, 14, 15] of χ_1 has shown that this procedure is confronted with serious problems. The fixed order expansion eq. (8) is very badly behaved: at relevant values of α_s the NLO term completely overwhelms the LO term. In particular, near $M = 0$, the behaviour is unstable, with $\chi_0 \sim 1/M$, $\chi_1 \sim -1/M^2$. Also, the value of χ near the minimum is subject to a large negative NLO correction, which turns the minimum into a maximum and can even reverse the sign of χ at the minimum. Finally, if one considers the resulting γ_0 and γ_1 or their Mellin transforms $P_0(x)$ and $P_1(x)$ one finds that the NLO terms become much larger than the LO terms and negative in the region of relevance for the HERA data [16]. We now discuss our proposals to deal with all these problems.

Our first observation is that a much more stable expansion for $\chi(M)$ can be obtained if we make appropriate use of the additional information which is contained in the one and two loop anomalous dimensions γ_0 and γ_1 . Instead of trying to improve the fixed order expansion eq. (4)

of γ by all order summation of singularities deduced from the fixed order expansion eq. (8) of χ , we attempt the converse: we improve $\chi_0(M)$ by adding to it the all order summation of singularities χ_s eq. (18) deduced from γ_0 , $\chi_1(M)$ by adding to it χ_{ss} deduced from γ_1 eq. (19), and so on. It can then be seen that the instability at $M = 0$ of the usual fixed order expansion of χ was inevitable: momentum conservation for the anomalous dimension, eq. (5), implies, given the duality relation, that the value of $\chi(M)$ at $M = 0$ is fixed to unity, since from eq. (10) we see that at $N = 1$

$$\chi(\gamma(1, \alpha_s), \alpha_s) = \chi(0, \alpha_s) = 1. \quad (20)$$

It follows that the fixed order expansion of χ must be poorly behaved near $M = 0$: a simple model of this behaviour is to think of replacing α_s/M with $\alpha_s/(M + \alpha_s) = \alpha_s/M - \alpha_s^2/M^2 + \dots$ in order to satisfy the momentum conservation constraint. We thus propose a reorganization of the expansion of χ into a “double leading” (DL) expansion, organized in terms of “envelopes” of the contributions summarized in fig. 1b: each order contains a “vertical” sequence of terms of fixed order in α_s , supplemented by a “diagonal” resummation of singular terms of the same order in α_s if α_s/M is considered fixed. To NLO the new expansion is thus

$$\begin{aligned} \chi(M, \alpha_s) &= \left[\alpha_s \chi_0(M) + \chi_s \left(\frac{\alpha_s}{M} \right) - \frac{n_c \alpha_s}{\pi M} \right] \\ &+ \alpha_s \left[\alpha_s \chi_1(M) + \chi_{ss} \left(\frac{\alpha_s}{M} \right) - \alpha_s \left(\frac{f_2}{M^2} + \frac{f_1}{M} \right) - f_0 \right] + \dots \end{aligned} \quad (21)$$

where the LO and NLO terms are contained in the respective square brackets. Thus the LO term contains three contributions: $\chi_0(M)$ is the leading BFKL function eq. (8), $\chi_s(\alpha_s/M)$ eq. (18) are resummed leading singularities deduced from the one loop anomalous dimension, and $n_c \alpha_s / (\pi M)$ is subtracted to avoid double counting. At LO the momentum conservation constraint eq. (20) is satisfied exactly because $\gamma_0(1) = 0$ and $[\chi_0(M) - \frac{n_c}{\pi M}] \sim M^2$ near $M = 0$. At NLO there are again three types of contributions: $\chi_1(M)$ from the NLO fixed order calculation (eq. (9)), the resummed subleading singularities $\chi_{ss}(\alpha_s/M)$ deduced from the two loop anomalous dimension, and three double counting terms, $f_0 = 0$, $f_1 = -n_f(13 + 10n_c^2)/(36\pi^2 n_c^3)$ and $f_2 = n_c^2(11 + 2n_f/n_c^3)/(12\pi^2)$ (corresponding to those terms with $(m, n) = (1, 0), (2, 1), (2, 2)$ respectively in fig. 1b). Note that at the next-to-leading level the momentum conservation constraint is not exactly satisfied because the constant contribution to χ_1 does not vanish in $\overline{\text{MS}}$, even though it is numerically very small (see fig. 2). It could be made exactly zero by a refinement of the double counting subtraction but we leave further discussion of this point for later.

Plots of the various LO and NLO approximations to χ are shown in fig. 2. In this and other plots in this paper we take $\alpha_s = 0.2$, which is a typical value in the HERA region, and the number of active flavours $n_f = 4$. We see that, as discussed above, the usual fixed order expansion eq. (8) in terms of χ_0 and χ_1 is very unstable. However, the new expansion eq. (21) is stable up to $M \lesssim 0.3 - 0.4$. Furthermore, in this region, χ evaluated in the double leading expansion (21) is very close to the resummations of leading and subleading singularities eq. (17) obtained by duality eq. (18,19) from the one and two loop anomalous dimensions. This shows that in this region the dominant contribution to χ , and thus to γ , comes from the resummation of logarithms of Q^2/μ^2 with $Q^2 \gg \mu^2$.

Beyond $M \sim 0.4$, the size of the contributions from collinear singular and nonsingular terms becomes comparable (after all here $Q^2 \sim \mu^2$), but the calculation of the latter (from the fixed

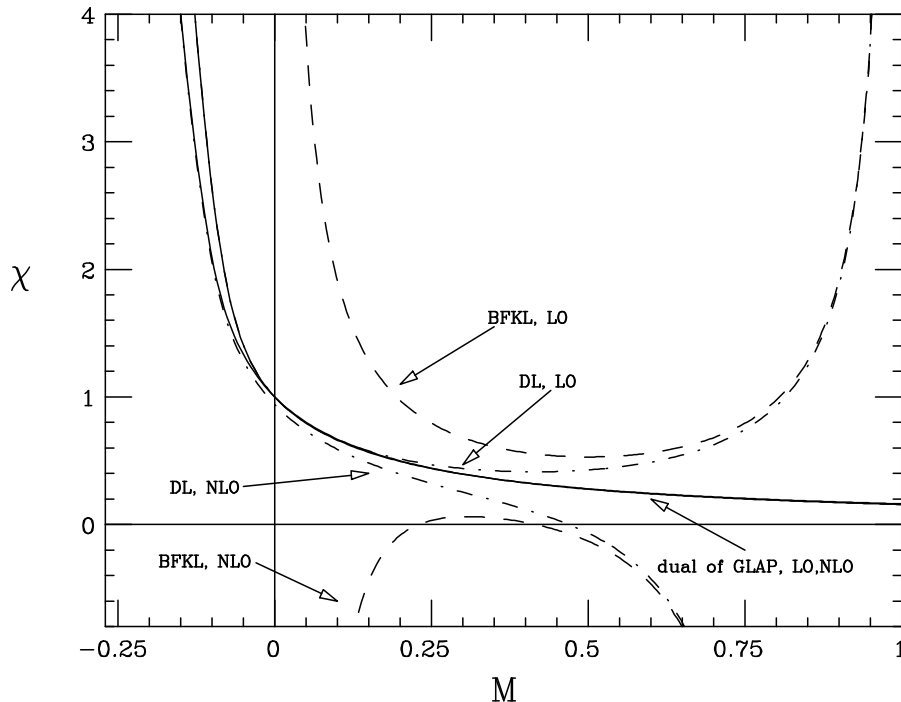


Figure 2: Plots of different approximations to the χ function discussed in the text: the BFKL leading and next-to-leading order functions eq. (8), $\alpha_s \chi_0$ and $\alpha_s \chi_0 + \alpha_s^2 \chi_1$ (dashed); the LO and NLO dual $\alpha_s \chi_s$ and $\alpha_s \chi_s + \alpha_s^2 \chi_{ss}$ of the one and two loop anomalous dimensions (solid), and the double-leading functions at LO and NLO defined in eq. (21) (dotdashed). Note that the double-leading curves coincide with the resummed ones at small M , and with the fixed order ones at large M .

expansion eq. (8)) has become unstable due to the influence of the singularities at $M = 1$. No complete and reliable description of χ seems possible without some sort of stabilization of these singularities. However, since they correspond to infrared singularities of the BFKL kernel (specifically logarithms of Q^2/μ^2 with $Q^2 \ll \mu^2$) this would necessarily be model dependent. In particular, such a stabilization cannot be easily deduced from the resummation of the $M = 0$ singularities: the original symmetry of the gluon-gluon amplitude at large s is spoiled by running coupling effects and by unknown effects from the coefficient function through which it is related to the deep-inelastic structure functions, in a way which is very difficult to control near the photoproduction limit $M \rightarrow 1$. We thus prefer not to enter into this problem: rather we will discuss later a practical procedure to bypass it.

The results summarized in fig. 2 clearly illustrate the superiority of the new double leading expansion of χ over the fixed order expansion, and already indicate that the complete χ function could after all lead to only small departures from ordinary two loop evolution.

3. Having constructed a more satisfactory expansion eq. (21) of the kernel χ , we now derive from it an improved form of the anomalous dimension γ to be used in the evolution eq. (3), in order to achieve a more complete description of scaling violations valid both at large and small x . In principle, this can be done by using the duality relation eq. (10), which simply gives the

function γ as the inverse of the function χ . However, in order to derive an analytic expression for $\gamma(N, \alpha_s)$ which also allows us to clarify the relation to previous attempts we start from the naive double-leading expansion of γ [18] in which terms are organized into “envelopes” of the contributions summarized in fig. 1a in an analogous way to the double leading expansion (21) of χ :

$$\begin{aligned} \gamma(N, \alpha_s) &= \left[\alpha_s \gamma_0(N) + \gamma_s \left(\frac{\alpha_s}{N} \right) - \frac{n_c \alpha_s}{\pi N} \right] \\ &\quad + \alpha_s \left[\alpha_s \gamma_1(N) + \gamma_{ss} \left(\frac{\alpha_s}{N} \right) - \alpha_s \left(\frac{e_2}{N^2} + \frac{e_1}{N} \right) - e_0 \right] + \dots, \end{aligned} \quad (22)$$

where now $e_2 = g_2^{(s)} = 0$, $e_1 = g_1^{(ss)} = n_f n_c (5 + 13/(2n_c^2))/(18\pi^2)$ and $e_0 = -(\frac{11}{2}n_c^3 + n_f)/(6\pi n_c^2)$. In this equation, the leading and subleading singularities γ_s and γ_{ss} are obtained using duality eq. (10) from χ_0 and χ_1 , and summed up to give expressions which are exact at NLLx. These are then added to the usual one and two loop contributions, and the subtractions take care of the double counting of singular terms.

It can be shown that the dual of the double leading expansion of χ eq. (21) coincides with this double leading expansion of γ eq. (22) order by order in perturbation theory, up to terms which are higher order in the sense of the double leading expansions. However, it is clear that these additional subleading terms must be numerically important. Indeed, it is well known that at small N the anomalous dimension in the small- x expansion eq. (12) is completely dominated by $\gamma_{ss}(\alpha_s/N)$ which grows very large and negative, leading to completely unphysical results in the HERA region [16]. It is clear that this perturbative instability will also be a problem in the double leading expansion eq. (22). On the other hand, we know from fig. 2 that the exact dual of χ in double leading expansion is stable, and not too far from the usual two loop result. The origin of this instability problem, and a suitable reorganization of the perturbative expansion which allows the resummation of the dominant part of the subleading terms have been discussed in ref. [19]. After this resummation, the resulting expression for γ in double leading expansion will be very close to the exact dual of the corresponding expansion of χ .

The procedure of ref. [19] can be interpreted in a simple way whenever the all-order “true” function $\chi(M, \alpha_s)$ possesses a minimum at a real value of M , M_{min} , with $0 < M_{min} < 1$ (although the final result for the anomalous dimension will retain its validity even in the absence of such minimum). Using λ to denote this minimum value of χ ,

$$\lambda \equiv \chi(M_{min}, \alpha_s) = \lambda_0 + \Delta\lambda, \quad \lambda_0 \equiv \alpha_s \chi_0\left(\frac{1}{2}\right) = \frac{4n_c}{\pi} \alpha_s \ln 2. \quad (23)$$

The instability turns out to be due to the fact that higher order contributions to γ must change the asymptotic small x behaviour from $x^{-\lambda_0}$ to $x^{-\lambda}$. The starting point of the proposed procedure consists of absorbing the value of the correction to the value of χ at the minimum into the leading order term in the expansion of χ :

$$\begin{aligned} \chi(M, \alpha_s) &= \alpha_s \chi_0(M) + \alpha_s^2 \chi_1(M) + \dots \\ &= (\alpha_s \chi_0(M) + \Delta\lambda) + \alpha_s^2 \tilde{\chi}_1(M) + \dots, \end{aligned} \quad (24)$$

where $\tilde{\chi}_n(M) \equiv \chi_n(M) - c_n$, with c_n chosen so that $\tilde{\chi}_n(M)$ no longer leads to an $O(\alpha_s^n)$ shift in the minimum. Since the position M_{min} of the all-order minimum is not known, one must in practice expand it in powers of α_s around the leading order value $M = \frac{1}{2}$, so at higher

orders the expressions for the subtraction constants c_n can become quite complicated functions of χ_i and their derivatives at $M = \frac{1}{2}$ [19]. However at NLO we have simply $c_1 = \chi_1(\frac{1}{2})$, so $\Delta\lambda = \alpha_s^2 \chi_1(\frac{1}{2}) + \dots$.

A stable expansion of γ in resummed leading and subleading singularities can now be obtained from the duality eqs.(13,14,...) by treating $\chi_0 + \Delta\lambda$ as the LO contribution to χ , and the subsequent terms $\tilde{\chi}_i$ as perturbative corrections to it. Of course, since the reorganization eq. (24) amounts to a reshuffling of perturbative orders, to any finite order the anomalous dimension obtained in this way will be equal to the old one up to formally subleading corrections. Explicitly, we find in place of the previous expansion in sums of singularities eq. (12) the resummed expansion

$$\gamma(N, \alpha_s) = \gamma_s \left(\frac{\alpha_s}{N-\Delta\lambda} \right) + \alpha_s \tilde{\gamma}_{ss} \left(\frac{\alpha_s}{N-\Delta\lambda} \right) + \dots, \quad (25)$$

where

$$\tilde{\gamma}_{ss} \left(\frac{\alpha_s}{N-\Delta\lambda} \right) \equiv \gamma_{ss} \left(\frac{\alpha_s}{N-\Delta\lambda} \right) - \frac{\chi_1(\frac{1}{2})}{\chi_0' \left(\gamma_s \left(\frac{\alpha_s}{N-\Delta\lambda} \right) \right)}. \quad (26)$$

In terms of splitting functions this resummed expansion is simply

$$\begin{aligned} xP(x, \alpha_s) &= \alpha_s e^{\xi\Delta\lambda} [P_s(\alpha_s\xi) + \alpha_s \tilde{P}_{ss}(\alpha_s\xi) + \dots] \\ &= \alpha_s e^{\xi\Delta\lambda} [P_s(\alpha_s\xi) + \alpha_s \tilde{P}_{ss}(\alpha_s\xi) - \xi\Delta\lambda P_s(\alpha_s\xi) + \dots]. \end{aligned} \quad (27)$$

The expansion is now stable [19], in the sense that it may be shown that $\tilde{P}_{ss}(\alpha_s\xi)/P_s(\alpha_s\xi)$ remains bounded as $\xi \rightarrow \infty$: subleading corrections will then be small provided only that α_s is sufficiently small. This result may be shown to be true to all orders in perturbation theory, using an inductive argument.

We can thus replace the unresummed singularities γ_s and γ_{ss} in eq. (22) with the resummed singularities eq. (25) to obtain a double leading expansion with stable small x behaviour:

$$\begin{aligned} \gamma(N, \alpha_s) &= \left[\alpha_s \gamma_0(N) + \gamma_s \left(\frac{\alpha_s}{N-\Delta\lambda} \right) - \alpha_s \frac{n_c}{\pi N} \right] \\ &+ \alpha_s \left[\alpha_s \gamma_1(N) + \tilde{\gamma}_{ss} \left(\frac{\alpha_s}{N-\Delta\lambda} \right) - \alpha_s \left(\frac{\epsilon_2}{N^2} + \frac{\epsilon_1}{N} \right) - e_0 \right] + \dots. \end{aligned} \quad (28)$$

Momentum conservation is violated by the resummation because γ_s and γ_{ss} and the subtraction terms do not vanish at $N = 1$. It can be restored by simply adding to the constant e_0 a further series of constant terms beginning at $O(\alpha_s^2)$: these are all formally subleading in the double leading expansion. This constant shift in γ is precisely analogous to the shift made on χ in eq. (23) which generated the resummation.

It is important to recognize that there is inevitably an ambiguity in the double counting subtraction terms in eq. (28). For example, at the leading order of the double leading expansion instead of subtracting $\frac{n_c \alpha_s}{\pi N}$ we could have subtracted $\frac{n_c \alpha_s}{\pi(N-\Delta\lambda)}$, since this differs only by formally subleading terms: $\Delta\lambda = O(\alpha_s^2)$, so

$$\frac{\alpha_s}{N} = \frac{\alpha_s}{N-\Delta\lambda} \left(1 - \frac{\Delta\lambda}{N-\Delta\lambda} + \dots \right). \quad (29)$$

Following the same type of subtraction at NLO, the resummed double leading anomalous dimension may thus be written as

$$\begin{aligned} \gamma(N, \alpha_s) = & \left[\alpha_s \gamma_0(N) + \gamma_s \left(\frac{\alpha_s}{N - \Delta\lambda} \right) - \alpha_s \frac{n_c}{\pi(N - \Delta\lambda)} \right] \\ & + \alpha_s \left[\alpha_s \gamma_1(N) + \tilde{\gamma}_{ss} \left(\frac{\alpha_s}{N - \Delta\lambda} \right) + \frac{n_c \Delta\lambda}{\pi(N - \Delta\lambda)^2} - \alpha_s \left(\frac{\epsilon_2}{(N - \Delta\lambda)^2} + \frac{\epsilon_1}{N - \Delta\lambda} \right) - \epsilon_0 \right] + \dots \end{aligned} \quad (30)$$

The extra term at NLO comes from the first correction in eq. (29), which is of order $\frac{\alpha_s^3}{N^2}$, and thus a subleading singularity. The characteristic feature of this alternative resummation is that the fixed order anomalous dimensions γ_0 , γ_1 are preserved in their entirety, including the position of their singularities. As with the previous expansion eq. (28) momentum conservation may be imposed by adding to ϵ_0 a series of terms constant in N and starting at $O(\alpha_s^2)$.

This completes our procedure of inclusion of the most important part of the subleading corrections, as we shall see shortly by a direct comparison of the resummed expansions eq. (28) and eq. (30) with the exact dual of χ evaluated according to eq. (21). In the sequel we will discuss the phenomenology based on the two resummed expansions eq. (28) and eq. (30) on an equal footing, taking the spread of the results as an indication of the residual ambiguity due to subleading terms. Although formally the differences between the two expansions are subleading, we will find that in practice they may be quite substantial, because $\Delta\lambda$ may be large.

4. So far we have constructed resummations of the anomalous dimension and splitting function which satisfy the elementary requirements of perturbative stability and momentum conservation. This construction relies necessarily on the value λ of χ near its minimum, since it is this which determines the small x behaviour of successive approximations to the splitting function. In order to obtain a formulation that can be of practical use for actual phenomenology, we will need however to improve the description of $\chi(M)$ in the “central region” near its minimum M_{min} , since as we already observed, we cannot reliably determine the position and value of the minimum of χ without a stabilization of the $M = 1$ singularity. Indeed, we can see from fig. 2 that in the central region χ evaluated in the double leading expansion is dominated by the presumably unphysical $M = 1$ poles of χ , and at NLO this means that it actually has no minimum, becoming rapidly negative. However, one can use the value λ of the true χ at the minimum as a useful parameter for an effective description of the χ function around $M = 1/2$. Indeed, $\Delta\lambda$ as estimated from its next-to-leading order value $\alpha_s^2 \chi_1(1/2)$ turns out to be of the same order as λ_0 for plausible values of α_s , a feature which can be also directly seen from fig. 2. This supports the idea that λ and $\Delta\lambda$ are not truly perturbative quantities: in general we expect that the overall shift of the minimum will still be of the order of λ_0 and negative. It is this order transmutation that makes the impact of the resummations eq. (28,30), and the differences between them, quite substantial.

In fig. 3 and fig. 4 we display the results for the resummed anomalous dimensions in the two different expansions eq. (28) and eq. (30) respectively, each computed at next-to-leading order. In both figures we show for comparison the fixed order anomalous dimension $\alpha_s \gamma_0(N) + \alpha_s^2 \gamma_1(N)$ eq. (4). Also for comparison, we show the exact dual of χ computed at NLO in the double leading expansion eq. (21), obtained from eq. (10) by exact numerical inversion. This curve is thus simply the inverse of the corresponding curve already shown in fig. 2.

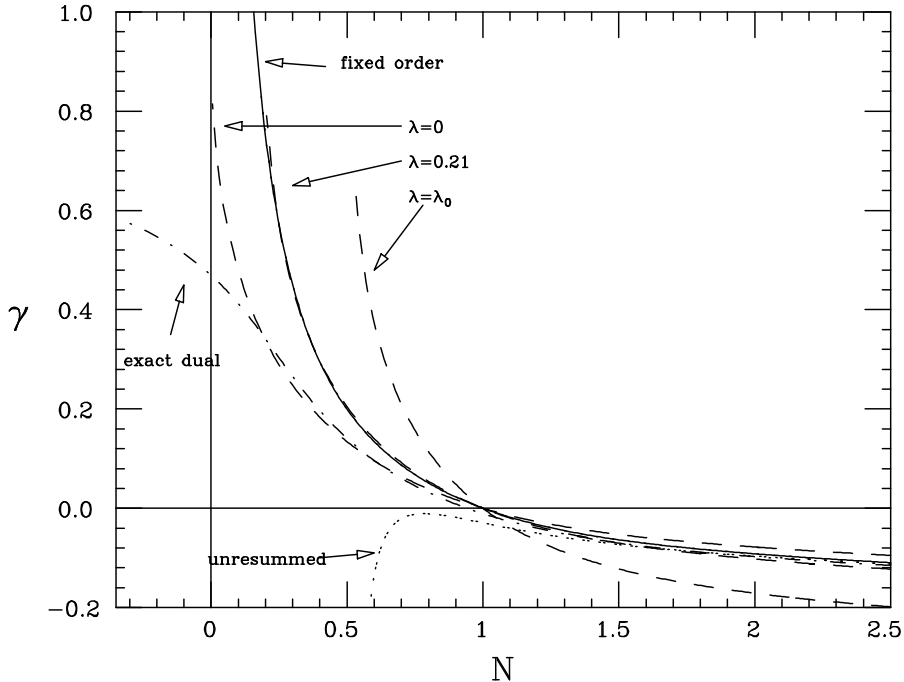


Figure 3: Comparison of the anomalous dimension γ evaluated at NLO in the resummed expansion eq. (28) for three different values of λ (dashed) with the usual fixed order perturbative anomalous dimension (also at NLO) eq. (4) (dotted) and that obtained by exact duality from χ at NLO in the expansion eq. (21) as displayed in fig. 2 (solid). The unresummed γ eq. (22) is also shown at NLO. Notice that the $\lambda = 0.21$ curve is very close to the two loop anomalous dimension down to the branch point at $N = \lambda$.

In fig. 3 we show the anomalous dimension computed at NLO using the resummation eq. (28), for $\lambda = \lambda_0$ and $\lambda = 0$. The first value corresponds to the LO approximation to λ , while the second value is close to the NLO approximation when α_s is in the region $\alpha_s \sim 0.1 - 0.2$. We might expect the value of λ as determined by the actual all-order minimum of χ to lie within this range. Note that, in general, the resummed anomalous dimension has a cut starting at $N = \lambda$, which corresponds to the $x^{-\lambda}$ power rise; for this reason our plots stop at this value of N . The $\lambda = 0$ curve, corresponding to the next-to-leading order approximation to λ , is seen to be very close to the exact dual of χ at NLO in the expansion eq. (21), as already anticipated. This is to be contrasted with the corresponding unresummed anomalous dimension eq. (22), which is also displayed in fig. 3, and is characterized by the rapid fall at small N discussed already in ref. [16, 19]. This comparison demonstrates that indeed the perturbative reorganization eliminates this pathological step decrease. The resummed curve with $\lambda = 0$ and the exact dual of χ become rather different for small $N \lesssim 0.2$. However, this is precisely the range of N which corresponds to the central region of M where we cannot trust the next-to-leading order determination of χ . Finally, we show that we can choose a value of $\lambda \simeq 0.21$ such that the resummed anomalous dimension closely reproduces the two loop result down to the branch point at $N = \lambda$. This shows that the absence of visible deviations from the usual two loop evolution can be accommodated by the resummed anomalous dimension. However this is not

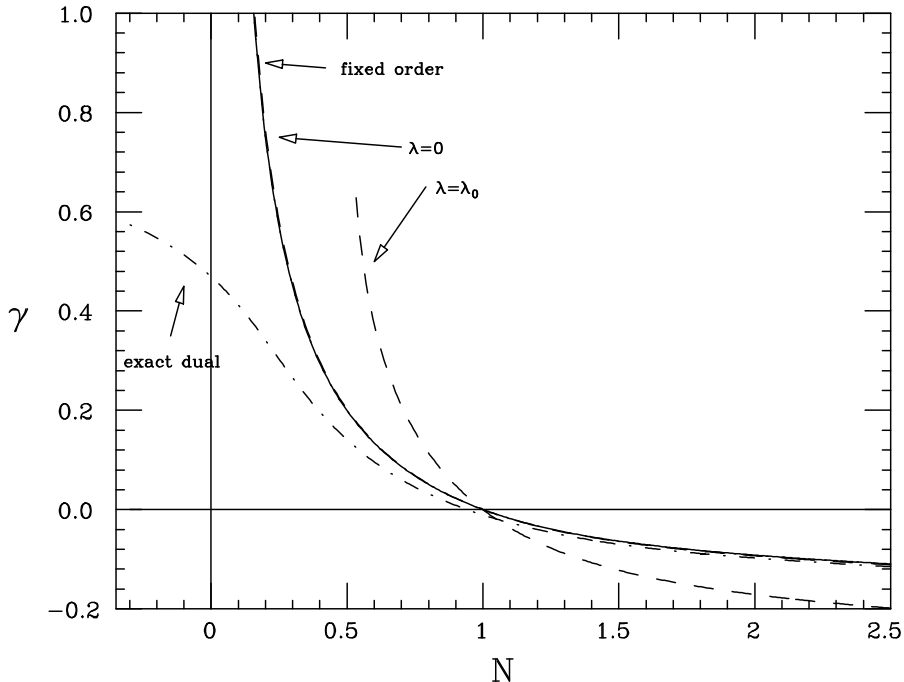


Figure 4: Comparison of the anomalous dimension γ evaluated at NLO in the resummed expansion eq. (30) for two different values of λ (dashed) with the usual fixed order perturbative anomalous dimension (also at NLO) eq. (4) (solid) and that obtained by exact duality from χ at NLO as in fig. 3 (dotdash). Notice that the $\lambda = 0$ curve is virtually indistinguishable from the fixed order anomalous dimension for all values of N .

necessarily the best option phenomenologically: perhaps the data could be better fitted by a different value of λ if a suitable modification of the input parton distributions is introduced. It is nevertheless clear that large values of λ such as $\lambda \approx \lambda_0$ can be easily excluded within the framework of this resummation, since they would lead to sizeable deviations from the standard two loop scaling violations in the medium and large x region.

The splitting functions corresponding to the anomalous dimensions of fig. 3 are displayed in fig. 5. The basic qualitative features are of course preserved: in particular, the curves with small values of $\lambda = 0$ and $\lambda = 0.21$ are closest to the two loop result. However, on a more quantitative level, it is clear that anomalous dimensions which coincide in a certain range of N , but differ in other regions (such as very small N) may lead to splitting functions which differ over a considerable region in x . In particular, the $\lambda = 0.21$ curve displays the predicted $x^{-\lambda}$ growth at sufficiently large ξ ($x \lesssim 10^{-4}$). The dip seen in the figure for intermediate values of ξ is necessary in order to compensate this growth in such a way that the moments for moderate values of N remain unchanged. Note that the $x^{-\lambda}$ behaviour of the splitting function at small x is corrected by logs [19]: $P_s \underset{\xi \rightarrow \infty}{\sim} \xi^{-3/2} x^{-\lambda}$. If $\lambda = 0$ this logarithmic drop provides the dominant large ξ behaviour which appears in the figure.

If the anomalous dimensions are instead resummed as in eq. (30), the results are as shown in fig. 4, again for the two very different values of λ , $\lambda = 0$ and $\lambda = \lambda_0$. When $\lambda = 0$ the resummed anomalous dimension is now essentially indistinguishable from the two loop result.

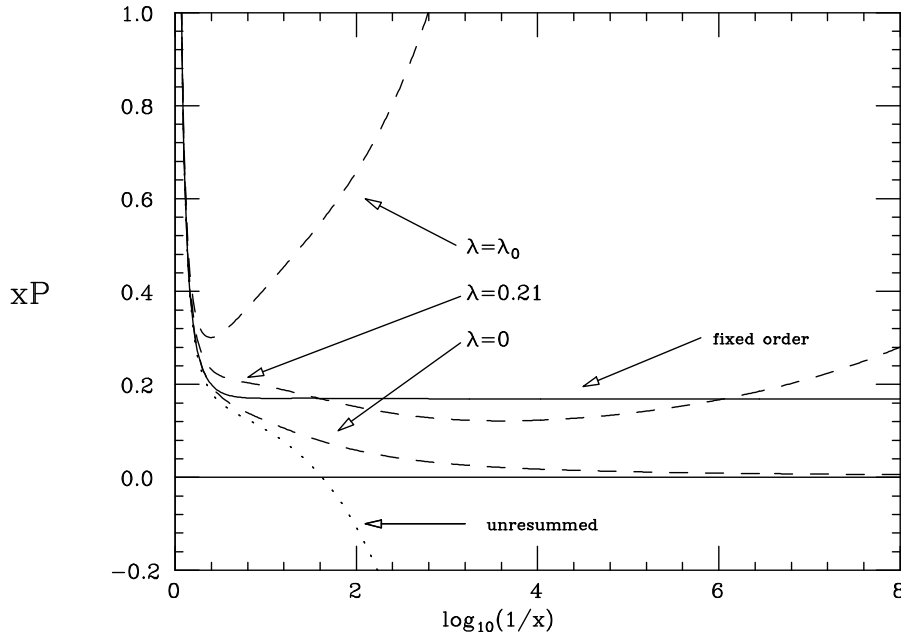


Figure 5: The splitting functions corresponding to the anomalous dimensions of fig. 3.

This is due to the fact that the simple poles at $N = 0$ which are now retained in γ_0 and γ_1 provide the dominant small N behaviour. The branch point at $N = \lambda$ in γ_s and γ_{ss} is then relatively subdominant. This remains of course true for all $\lambda \leq 0$, and in practice also for small values of λ such as $\lambda \lesssim 0.1$. When instead $\lambda = \lambda_0$ the result does not differ appreciably from the resummed anomalous dimension shown in fig. 3, since now the dominant small N behaviour is given by the branch point at $N = \lambda_0$, which is not affected by changes in the double counting prescription. Summarizing, the peculiar feature of the resummation eq. (30) is that it leads to results which are extremely close to usual two loops for any value of $\lambda \leq 0$, without the need for a fine-tuning of λ .

Finally, in fig. 6 we display the splitting functions obtained from the resummed anomalous dimensions of fig. 4. The $\lambda = \lambda_0$ case is, as expected, very close to the corresponding curve in fig. 5. However the $\lambda = 0$ curve is now in significantly better agreement with the two loop result than any of the resummed splitting functions of fig. 5, even that computed with the optimized value $\lambda = 0.21$. Moreover, this agreement now holds in the entire range of ξ . This is due to the fact that the corresponding anomalous dimension is now very close to the fixed order one for all $N > 0$, and not only for $N > \lambda = 0.21$. This demonstrates explicitly that one cannot exclude the possibility that the known small x corrections to splitting functions resum to a result which is essentially indistinguishable from the two-loop one. This however is only possible if $\lambda \lesssim 0$.

To summarise, we find that the known success of perturbative evolution, and in particular double asymptotic scaling at HERA can be accommodated within two distinct possibilities, both of which are compatible with our current knowledge of anomalous dimensions at small x , and in particular with the inclusion of corrections derived from the BFKL equation to usual

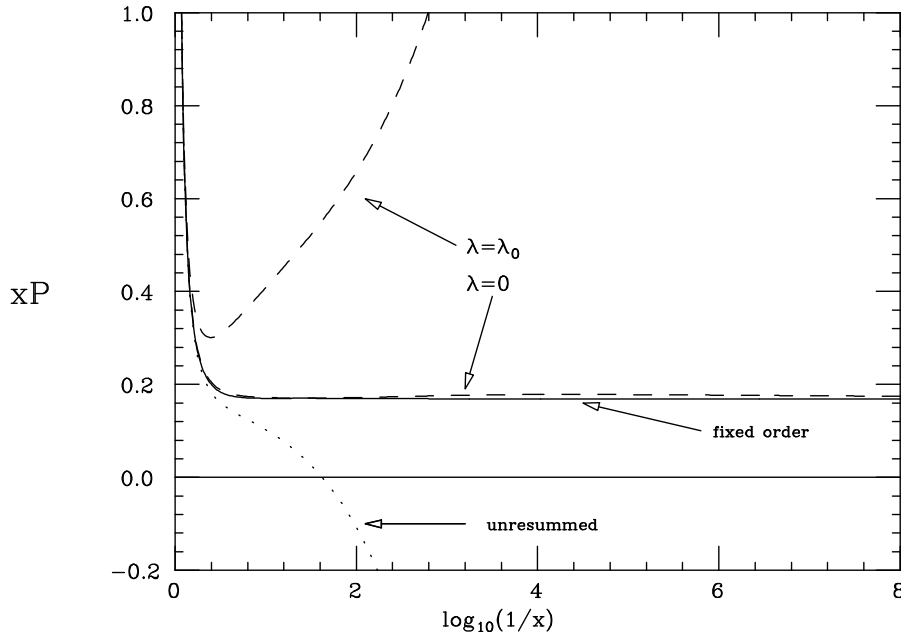


Figure 6: The splitting functions corresponding to the anomalous dimensions of fig. 4.

perturbative evolution. One possibility, embodied by the resummed anomalous dimension eq. (30) with $\lambda \lesssim 0$, is that double scaling remains a very good approximation to perturbative evolution even if the $x \rightarrow 0$ limit is taken at finite Q^2 . The other option, corresponding to the resummation eq. (28) with a small value of λ , is that double scaling is a good approximation in a wide region at small x , including the HERA region, but eventually substantial deviations from it will show up at sufficiently small x . In the latter case, the best-fit parton distributions might be significantly differ from those determined at two loops even at the edge of the HERA kinematic region. Both resummations are however fully compatible with a smooth matching to Regge theory in the low Q^2 region [22].

5. In conclusion, we have presented a procedure for the systematic improvement of the splitting functions at small x which overcomes the difficulties of a straightforward implementation of the BFKL approach. The basic ingredients of our approach are the following. First, we achieve a stabilization of the perturbative expansion of the function χ near $M = 0$ through the resummation of all the LO and NLO collinear singularities derived from the known one- and two-loop anomalous dimensions. The resulting χ function is regular at $M = 0$, and in fact, to a good accuracy, satisfies the requirement imposed by momentum conservation via duality. Then, we acknowledge that without a similar stabilization of the $M = 1$ singularity it is not possible to obtain an reliable determination of χ in the central region $M \sim 1/2$. However, we do not have an equally model-independent prescription to achieve this stabilization at $M = 1$. Nevertheless, the behaviour of χ in the central region can be effectively parameterized in terms of a single parameter λ which fixes the asymptotic small x behaviour of the singlet parton distribution. This enables us to arrive at an analytic expression for the improved splitting function, which is valid both at small and large x and is free of perturbative instabilities.

This formulation can be directly confronted with the data, which ultimately will provide a determination of λ along with α_s and the input parton densities. The well known agreement of the small x data with the usual Q^2 evolution equations suggests that the optimal value of λ will turn out to be small, and possibly even negative for the relevant value of α_s . Such a value of λ is theoretically attractive, because it corresponds to a structure function whose leading-twist component does not grow as a power of x in the Regge limit: it would thus be compatible with unitarity constraints, and with an extension of the region of applicability of perturbation theory towards this limit.

Several alternative approaches to deal with the same problem through the resummation of various classes of formally subleading contributions have been recently presented in the literature. Specific proposals are based on making a particular choice of the renormalization scale [23], or on a different identification of the large logs which are resummed by the ξ evolution equation (6), either by a function of ξ itself [24], or by a function of Q^2 [17, 25], or both [26]. The main shortcoming of these approaches is their model dependence. For instance, in ref. [25] the value of λ is calculated, and χ is supposedly determined for all $0 \leq M \leq 1$. This however requires the introduction of a symmetrization of χ , which we consider to be strongly model dependent: indeed, in ref. [25] it is recognized that their value of λ only signals the limit of applicability of their computation. We contrast this situation with the approach to resummation presented here, which makes maximal use of all the available model-independent information, with a realistic parameterization of the remaining uncertainties. We expect further progress to be possible only on the basis of either genuinely nonperturbative input, or through a substantial extension of the standard perturbative domain.

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