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THE RELATIVE PLURICANONICAL STABILITY FOR 3-FOLDS OF GENERAL TYPE

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Abstract

The aim of this paper is to improve a theorem of János Kollár using a different method. For a given smooth Complex projective threefold X of general type, suppose the plurigenus $P_k(X) \geq 2$, Kollár proved that the $(11k+5)$-canonical map is birational. Here we show that either the $(7k+3)$-canonical map or the $(7k+5)$-canonical map is birational and the $(13k+6)$-canonical map is stably birational onto its image. If $P_k(X) \geq 3$, then the m-canonical map is birational for $m \geq 10k+8$. In particular, ϕ_{12} is birational when $p_g(X) \geq 2$ and ϕ_{11} is birational when $p_g(X) \geq 3$.

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Introduction

Let X be a smooth projective 3-fold of general type defined over $\mathbb{C}$ and denote by ϕ_m the m-canonical map of X, which is the rational map associated with the linear system $|mK_X|$. Let $P_k(X) := h^0(X, \mathcal{O}_X(kK_X))$ for any positive integer k, we usually call $P_k(X)$ *the k-th plurigenus* of X which is a birational invariant. For a given positive integer m_0, we say that ϕ_{m_0} is *stably birational* if ϕ_m is birational onto its image for all $m \geq m_0$. Since the Kodaira dimension $\mathrm{kod}(X) = 3$, ϕ_m is birational for $m \gg 0$. In this paper, we consider the following

Problem. *Suppose $P_k(X) \geq 2$, for which value $m_0(k)$, does $|m_0(k)K_X|$ define a stably birational map onto its image?*

In 1986, Kollár ([5, Corollary 4.8]) first gave an effective result and proved that the $(11k+5)$-canonical map is birational if $P_k(X) \geq 2$. However, his method cannot tell whether ϕ_m is still birational for all $m > 11k+5$. On the other hand, it seems to us that the number $11k+5$ is not the optimal one. This paper aims to present a better result as the following

Main Theorem. *Let X be a nonsingular projective threefold of general type and suppose $P_k(X) \geq 2$, then*

(i) either ϕ_{7k+3} or ϕ_{7k+5} is birational onto its image;

(ii) ϕ_{13k+6} is stably birational onto its image;

(iii) ϕ_{10k+8} is stably birational providing that $P_k(X) \geq 3$.

In particular, if $p_g(X) \geq 2$, then ϕ_m is birational for all $m \geq 12$; if $p_g(X) \geq 3$, then ϕ_m is birational for all $m \geq 11$.

Noting that the main obstacle which prevents Kollár's method from getting a better bound is the case when X admits a rational pencil of certain surfaces of general type, we mainly study this situation in an alternative way. First we build some birationality criteria for adjoint systems on a surface of general type, then we reduce the problem to the surface case while finding suitable divisors on the threefold whose restrictions to the surface satisfy those criteria. The Kawamata-Viehweg vanishing theorem plays a key role throughout our argument.

Definition. Let X be a normal projective variety and D be a Weil divisor on X. Denote by $\Phi_{|D|}$ the natural rational map defined by the linear system $|D|$. $|D|$ is called *base point free* if it has neither fixed components nor base points.

If $|L|$ is a linear system on X without fixed components and $h^0(X, L) \geq 2$, we mean *a general irreducible element S* of $|L|$ as follows:

(1) if $\dim\Phi_{|L|}(X) \geq 2$, then S is a general member of $|L|$.

(2) if $\dim\Phi_{|L|}(X) = 1$, then L is linearly equivalent to a union of distinct reduced irreducible divisors of the same type. Explicitly, $L \sim_{\mathrm{lin}} \sum S_i$. We mean S a general S_i.

X is called *minimal* if the canonical divisor K_X is nef, i.e. $K_X \cdot C \geq 0$ for all proper curve $C \subset X$.

X is said to *be of general type* if the Kodaira dimension $\mathrm{kod}(X) = \dim(X)$.

X is said to *have only terminal singularities* according to Reid ([7]) if the following two conditions hold:

(i) for some integer $r \geq 1$, rK_X is Cartier;

(ii) for some resolution $f: Y \longrightarrow X$, $K_Y = f^*(K_X) + \sum a_i E_i$ for $0 < a_i \in \mathbb{Q}$ for all i, where the E_i vary all the exceptional divisors on Y.

1. Preparation

Throughout our argument, the Kawamata-Viehweg vanishing theorem is always employed as a much more effective tool. We use it in the following form.

Vanishing Theorem. *([3] or [10]) Let X be a nonsingular complete variety, $D \in \mathrm{Div}(X)\otimes \mathbb{Q}$. Assume the following two conditions:*

(1) D is nef and big;

(2) the fractional part of D has supports with only normal crossings.

Then $H^i(X, \mathcal{O}_X(\ulcorner D \urcorner + K_X)) = 0$ for $i > 0$, where $\ulcorner D \urcorner$ is the round-up of D, i.e. the minimum integral divisor with $\ulcorner D \urcorner - D \geq 0$.

Another important principle that is tacitly used throughout the text is due to Tankeev ([9]). Explicitly, on a smooth projective variety X, if we have a base point free system $|M|$ and an effective divisor D, we want to study the birationality of the map $\Phi_{|D+M|}$. Now let S be a general irreducible element of $|M|$, then S is a smooth divisor on X by Bertini's theorem. Suppose we have known that $\Phi_{|D+M|}$ can distinguish general irreducible elements and that $\Phi_{|D+M|}\big|_S$ is birational, then Tankeev's principle implies the birationality of $\Phi_{|D+M|}$.

Lemma 1.1. *([8, Corollary 2]) Let S be a nonsingular algebraic surface, L be a nef divisor on S, $L^2 \geq 10$ and let ϕ be a map defined by $|L+K_S|$. If ϕ is not birational, then S contains a base point free pencil E' with $L \cdot E' = 1$ or $L \cdot E' = 2$.*

Lemma 1.2. *Let S be a nonsingular projective surface of general type, suppose L is a divisor with $h^0(S, L) \geq 2$, then $h^0(S, K_S + L) \geq 2$. In particular, if $\chi(\mathcal{O}_S) \geq 3$, then $h^0(S, K_S + L) \geq 4$.*

Proof. Taking a general irreducible element C in the moving part of $|L|$, then C is a nef divisor, $C \leq L$ and C is a curve of genus ≥ 2. By R-R on the surface S, we have

$$h^0(S, K_S + L) \geq h^0(S, K_S + C) \geq \frac{1}{2}(K_S \cdot C + C^2) + \chi(\mathcal{O}_S).$$

It is easy to get the result. $\square$

Lemma 1.3. *Let S be a nonsingular projective surface of general type, L be a nef divisor, $L^2 \geq 3$ and $\dim\Phi_{|L|}(S) = 2$, then $|K_S + 2L|$ gives a birational map.*

Proof. We have $(2L)^2 \geq 12$. If $\Phi_{|K_S+2L|}$ is not birational, then according to Lemma 1.1, there is a base point free pencil E' such that $2L \cdot E' \leq 2$, i.e. $L \cdot E' = 1$. Since $\dim\Phi_{|L|}(S) = 2$ and E' is a curve of genus ≥ 2, we see that $L \cdot E' \geq 2$, a contradiction. $\square$

Lemma 1.4. *Let S be a nonsingular projective surface of general type, L_i is a divisor on S such that $\dim\Phi_{|L_i|}(S) \geq i$ for $i = 1,\ 2$, then $|K_S + 2L_2 + L_1|$ gives a birational map.*

Proof. Modulo blowing-ups, we can suppose that the $|L_i|$ be base point free for $i = 1,\ 2$. This means that L_2 is nef and big and that L_1 is nef.

If the system $|L_2|$ gives a birational map, then so does $|K_S + 2L_2 + L_1|$, because $K_S + L_1$ is effective by Lemma 1.2.

Otherwise, we have $L_2^2 \geq 2$. Now we have $(2L_2 + L_1)^2 \geq 12$. If $|K_S + 2L_2 + L_1|$ does not give a birational map, then, by Lemma 1.1, there is a free pencil E' on S such that

$$(2L_2 + L_1) \cdot E' \leq 2.$$

This means $L_2 \cdot E' = 1$. Note that E' is a curve of genus ≥ 2 and $|L_2|$ gives a generically finite map. The Riemann-Roch theorem on the curve E' derives that $\deg(L_2|_{E'}) \geq 2$. We have derived a contradiction. $\square$

Lemma 1.5. *Let X be a nonsingular projective 3-fold of general type. Suppose L_i is a divisor on X such that* $\dim\Phi_{|L_i|}(X) \geq i$ *for $i = 1,\ 2,\ 3$, then $|K_X + 2L_3 + L_2 + L_1|$ gives a birational map.*

Proof. Take a birational modification $\pi : X' \longrightarrow X$, according to Hironaka, such that the $|\pi^*(L_i)|$ are all base point free for $i > 0$. On X', we can study the system $|K_{X'} + 2\pi^*(L_3) + \pi^*(L_2) + \pi^*(L_1)|$. Let M_i be the moving part of $|\pi^*(L_i)|$, we have

$$|K_{X'} + 2M_3 + M_2 + M_1| \subset |K_{X'} + 2\pi^*(L_3) + \pi^*(L_2) + \pi^*(L_1)|.$$

Therefore, for simplicity, we can suppose from the beginning that the $|L_i|$ are base point free on X. So L_3 is nef and big under this assumption.

Step 1. Verifying that $K_X + 2L_3 + L_2$ is effective.

We have $\dim\Phi_{|L_2|}(X) \geq 2$. So a general member $S \in |L_2|$ is a nonsingular projective surface of general type. Using the vanishing theorem to the exact sequence

$$0 \longrightarrow \mathcal{O}_X(K_X + 2L_3) \longrightarrow \mathcal{O}_X(K_X + 2L_3 + S) \longrightarrow \mathcal{O}_S(K_S + 2L_3|_S) \longrightarrow 0,$$

we get the surjective map

$$H^0(X, K_X + 2L_3 + S) \longrightarrow H^0(S, K_S + 2L_3|_S) \longrightarrow 0.$$

From Lemma 1.2, we know $K_S + 2L_3|_S$ is effective, so is $K_X + 2L_3 + L_2$.

Step 2. Reduction to surface case.

Taking a 1-dimensional sub-system of $|L_1|$, then this system defines a rational map onto $\mathbb{P}^1$. Taking further blowing-up if necessary, we can also suppose that this system defines a morphism $f : X \longrightarrow \mathbb{P}^1$. Taking the Stein factorization of f, one obtains a derived fibration $g : X \longrightarrow C$. A general fibre of f can be written as a disjoint union $\sum F_i$. Let F be a general fibre of g, then it is a nonsingular projective surface of general type and we have $F \leq L_1$. Now considering the system $|K_X + 2L_3 + L_2 + \sum F_i|$, it can distinguish general fibres of g because of $K_X + 2L_3 + L_2$ is effective and $2L_3 + L_2$ is nef and big. Using the vanishing theorem again, we have

$$|K_X + 2L_3 + L_2 + \sum F_i|\ \big|_F = |K_F + 2L_3' + L_2'|,$$

where $L_3' := L_3|_F$ and $L_2' := L_2|_F$. Lemma 1.4 shows that the right system gives a birational map, so does $|K_X + 2L_3 + L_2 + L_1|$. The proof is completed. $\square$

Lemma 1.6. *Let X be a nonsingular variety of dimension n, $D \in \mathrm{Div}(X) \otimes \mathbb{Q}$ be a $\mathbb{Q}$-divisor on X. Then we have the following:*

(i) if S is a smooth irreducible divisor on X, then $\ulcorner D \urcorner|_S \geq \ulcorner D|_S \urcorner$;

(ii) if $\pi : X' \longrightarrow X$ is a birational morphism, then $\pi^*(\ulcorner D \urcorner) \geq \ulcorner \pi^*(D) \urcorner$.

Proof. We can write D as $G + \sum_{i=1}^t a_i E_i$, where G is a divisor, the E_i are effective divisors for each i and $0 < a_i < 1$, $\forall\ i$. So we only have to prove the lemma for effective $\mathbb{Q}$-divisors. That is easy to check. $\square$

Lemma 1.7. *Let X be a nonsingular projective threefold of general type. Let D be a divisor on X with $h^0(X,D)\geq 2$ and suppose $|D|$ has no fixed components. Denote by F a general irreducible element of $|D|$. If L is another divisor such that $\dim\Phi_{|L|}(F)\geq 1$, then mK_X+L+D is effective and $\dim\Phi_{|mK_X+L+D|}(F)\geq 1$ for all $m\geq 2$.*

Proof. According to the 3-dimensional MMP ([4] and [6]), X has a minimal model X_0 which is normal projective with only $\mathbb{Q}$-factorial terminal singularities. Let $\alpha : X \dashrightarrow X_0$ be the contraction which is a rational map. Take a common resolution X' with $\pi' : X' \longrightarrow X$ and $\pi : X' \longrightarrow X_0$ such that $\pi=\alpha\circ\pi'$ and that

(1) both $|\pi'^*(L)|$ and $|\pi'^*(D)|$ have no base points (they may have fixed components);

(2) $\pi^*(K_{X_0})$ has supports with only normal crossings.

This is possible because of Hironaka's big theorem. Since $\pi'^*(mK_X+L+D)\leq mK_{X'}+\pi'^*(L)+\pi'^*(D)$ and

$$\pi'_*\mathcal{O}_{X'}(mK_{X'}+\pi'^*(L)+\pi'^*(D))=\mathcal{O}_X(mK_X+L+D)=\pi'_*\pi'^*\mathcal{O}_X(mK_X+L+D),$$

then $h^0\big(X',\pi'^*(mK_X+L+D)\big)=h^0\big(X',mK_{X'}+\pi'^*(L)+\pi'^*(D)\big)$, so

$$\Phi_{|\pi'^*(mK_X+L+D)|}\ \text{and}\ \Phi_{|mK_{X'}+\pi'^*(L)+\pi'^*(D)|}$$

have the same behavior. Let S be a general irreducible element of the moving part of $|\pi'^*(D)|$, then $\dim\Phi_{|\pi'^*(L)|}(S)\geq 1$ by assumption. Therefore it is sufficient to show

$$\dim\Phi_{|mK_{X'}+\pi'^*(L)+\pi'^*(D)|}(S)\geq 1$$

for $m\geq 2$. Let H be the moving part of $|\pi'^*(L)|$, then H is nef since $|H|$ is base point free. We have

$$|K_{X'}+\ulcorner(m-1)\pi^*K_{X_0}\urcorner+H+S|\subset|mK_{X'}+\pi'^*(L)+\pi'^*(D)|.$$

The Kawamata-Viehweg vanishing theorem gives

$$\begin{aligned}&|K_{X'}+\ulcorner(m-1)\pi^*K_{X_0}\urcorner+H+S|\ \big|_S\\ &=\big|\ K_S+\ulcorner(m-1)\pi^*K_{X_0}\urcorner|_S+M\ \big|\supset|K_S+\ulcorner B\urcorner+M|,\end{aligned}$$

where $B:=(m-1)\pi^*K_{X_0}|_S$ is nef and big on S and $M:=H|_S$. From the assumption, we have $h^0(S,M)\geq 2$. Choosing a 1-dimensional sub-system $|C|$ in $|M|$, modulo blowing-ups, we can suppose $|C|$ be base point free. Also from the vanishing theorem, we have

$$|K_S+\ulcorner B\urcorner+C|\ \big|_C=|K_C+D|,$$

where $D:=\ulcorner B\urcorner|_C$ is a divisor on the curve C with positive degree since $D\geq\ulcorner B|_C\urcorner$ by Lemma 1.6(i). Because $g(C)\geq 2$, we have $h^0(K_C+D)\geq 2$. This means $|K_C+D|$ gives a generically finite map and

$$\dim\Phi_{|K_S+\ulcorner B\urcorner+C|}(C)=1$$

thus $K_{X'}+\ulcorner(m-1)\pi^*K_{X_0}\urcorner+\pi'^*(L)+\pi'^*(D)$ is effective and the image of S through the map defined by this divisor is at least 1. The proof is completed. $\square$

2. Proof of the main theorem

2.1 Basic formula. Let X be a nonsingular projective threefold, $f: X \longrightarrow C$ be a fibration onto a nonsingular curve C. From the spectral sequence:

$$E_2^{p,q} := H^p(C, R^q f_* \omega_X) \Rightarrow E^n := H^n(X, \omega_X),$$

we get by direct calculation that

$$h^2(X, \mathcal{O}_X) = h^1(C, f_*\omega_X) + h^0(C, R^1 f_* \omega_X),$$

$$q(X) := h^1(X, \mathcal{O}_X) = b + h^1(C, R^1 f_* \omega_X),$$

where b denotes the genus of C.

2.2 Review of Kollár's technique. Let X be a smooth projective 3-fold of general type and suppose $P_k(X) \geq 2$. Choose a 1-dimensional sub-system of $|kK_X|$ and replace X by a birational model X' where this pencil defines a morphism $g: X' \longrightarrow \mathbb{P}^1$. (For simplicity, we can suppose $X' = X$.) Let S be a general irreducible element of this pencil, then a general fibre of g is a disjoint union of some surfaces with the same type as S and S is a smooth projective surface of general type. Let $t = k(2p+1) + p$. Then $H^0(\omega_X^t) = H^0(\mathbb{P}^1, g_*\omega_X^t)$ and we have an injection $\mathcal{O}(1) \hookrightarrow g_*\omega_X^k$, and hence an injection $\mathcal{O}(2p+1) \hookrightarrow g_*\omega_X^{k(2p+1)}$. This gives an injection

$$\mathcal{O}(2p+1) \otimes g_*\omega_X^p \hookrightarrow g_*\omega_X^t,$$

where $\mathcal{O}(2p+1) \otimes g_*\omega_X^p = \mathcal{O}(1) \otimes g_*\omega_{X/\mathbb{P}^1}^p$. Now it is well-known that $g_*\omega_{X/\mathbb{P}^1}^p$ is a sum of line bundles of non-negative degree on $\mathbb{P}^1$. If $p \geq 5$, the local sections of $g_*\omega_X^p$ give a birational map for S, and all these extend to global sections of $\mathcal{O}(2p+1) \otimes g_*\omega_X^p$. Moreover its sections separate the fibres from each other, hence ϕ_t is a birational map for X.

From the above method, according to [1] and [11], we have

(1) ϕ_{5k+2} is generically finite for X if S is not a surface with $p_g(S) = q(S) = 0$ and $K_{S_0}^2 = 1$, where S_0 is the minimal model of S. Otherwise, we have at least $\dim\phi_{5k+2}(X) \geq 2$;

(2) ϕ_{7k+3} is birational for X if S is not a surface with

$$(K_{S_0}^2, p_g(S)) = (1, 2) \text{ or } (2, 3).$$

2.3 Proof of the main theorem. According to the 3-dimensional MMP, we can suppose X is a minimal model with at worst $\mathbb{Q}$-factorial terminal singularities. This means that K_X is a nef and big $\mathbb{Q}$-divisor. We begin from a minimal model in order to make use of the Kawamata-Viehweg vanishing theorem.

Theorem 2.3.1. *Let X be a nonsingular projective 3-fold of general type and suppose $P_k(X) \geq 2$, then either ϕ_{7k+3} or ϕ_{7k+5} is birational.*

Proof. Suppose X is a minimal model with at worst $\mathbb{Q}$-factorial terminal singularities. Choose a 1-dimensional sub-system Λ of $|kK_X|$ and take a birational modification $\pi: X' \longrightarrow X$ such that

(i) X' is nonsingular;

(ii) $\pi^*\Lambda$ gives a morphism;

(iii) the fractional part of $\pi^*(K_X)$ has supports with only normal crossings.

This is possible because of Hironaka's big theorem. Set $g_1 := \Phi_\Lambda \circ \pi$ and let $X' \xrightarrow{f_1} W_1 \xrightarrow{s_1} \mathbb{P}^1$ be the Stein factorization of g_1. Denote $b := g(W_1)$, the geometric genus of the curve W_1.

If $b > 0$, then the moving part of Λ is base point free. Let $\sum S_i$ be the moving part of Λ, then $\sum S_i \le kK_X$ and a general S_i is a smooth projective surface of general type, since the singularities on X are isolated. Using Kawamata's vanishing theorem ([4]) to $\mathbb{Q}$-Cartier Weil divisors on minimal threefold X, we see that $|(a+1)K_X + \sum S_i|$ can distinguish general S_i for $a > 0$ and

$$H^0(X, (a+1)K_X + \sum S_i) \longrightarrow \oplus H^0(S_i, (a+1)K_{S_i})$$

is surjective. Therefore it is obvious that ϕ_m is effective whenever $m \ge k+2$, generically finite whenever $m \ge 2k+2$, birational whenever $m \ge 2k+4$.

So, from now on, we can suppose that $b = 0$. We have a fibration $f_1 : X' \longrightarrow \mathbb{P}^1$. Let F be a general fibre of f_1. By virtue of 2.2(2), we can suppose that F is a surface with invariants $(K_{F_0}^2, p_g(F)) = (1,2)$ or $(2,3)$, where F_0 is the minimal model of F. F is the moving part of $\pi^*\Lambda$ and $F \le_{\mathbb{Q}} \pi^*(kK_X)$. We automatically have $q(F) = 0$. First we study the system $|K_{X'} + \ulcorner k\pi^*(K_X)\urcorner + F|$. For a general fibre F, the vanishing theorem gives that

$$|K_{X'} + \ulcorner k\pi^*(K_X)\urcorner + F| \,\big|_F = \big| \, K_F + \ulcorner k\pi^*(K_X)\urcorner|_F \, \big|,$$

where $\ulcorner k\pi^*(K_X)\urcorner|_F$ is effective. This means that $(2k+1)K_{X'}$ is effective and $\dim\phi_{2k+1}(F) \ge 1$. By Lemma 1.7, we see that $mK_{X'}$ is effective and $\dim\phi_m(F) \ge 1$ for $m \ge 3k+3$.

Actually, we have $\dim\phi_{3k+2}(F) = 2$. In fact, we have

$$|K_{X'} + \ulcorner (2k+1)\pi^*(K_X)\urcorner + F| \,\big|_F \supset \big| \, K_F + M_{2k+1}|_F \, \big|,$$

where M_{2k+1} is the moving part of $|\ulcorner (2k+1)\pi^*K_X\urcorner|$. It is easy to check that $\big| \, K_F + M_{2k+1}|_F \, \big|$ gives a generically finite map because $q(F) = 0$ and $p_g(F) > 0$. Thus

$$\dim\Phi_{|K_{X'} + \ulcorner (2k+1)\pi^*(K_X)\urcorner + F|}(F) \ge 2.$$

We have $|K_{X'} + \ulcorner 2(3k+2)\pi^*(K_X)\urcorner + F| \subset |(7k+5)K_{X'}|$. $K_{X'} + \ulcorner 2(3k+2)\pi^*(K_X)\urcorner$ is effective by the above argument. So $|K_{X'} + \ulcorner 2(3k+2)\pi^*(K_X)\urcorner + F|$ can distinguish general fibre F. On the other hand, the Kawamata-Viehweg vanishing theorem gives

$$\begin{aligned} |K_{X'} + \ulcorner 2(3k+2)\pi^*(K_X)\urcorner + F| \,\big|_F &= \big| \, K_F + \ulcorner 2(3k+2)\pi^*(K_X)\urcorner|_F \, \big| \\ &\supset |K_F + 2L_{3k+2}|, \end{aligned}$$

where $L_{3k+2} := M_{3k+2}|_F$. It is sufficient to show that $|K_F + 2L_{3k+2}|$ gives a birational map for F. We have already known that $|L_{3k+2}|$ gives a generically finite map for F. Excluding the fixed components of $|L_{3k+2}|$, we can suppose that $|L_{3k+2}|$ are moving on the surface F. So L_{3k+2} is nef. If $|L_{3k+2}|$ gives a birational map, then so does $|K_F + 2L_{3k+2}|$. Otherwise,

$$L_{3k+2}^2 \ge 2(h^0(F, L_{3k+2}) - 2).$$

Considering the following three natural maps

$$H^0(X', M_{3k+2}) \xrightarrow{\alpha} H^0(F, L_{3k+2})$$
$$H^0(X', K_{X'} + \ulcorner(2k+1)\pi^*(K_X)\urcorner + F) \xrightarrow{\beta} H^0(F, K_F + \ulcorner(2k+1)\pi^*(K_X)\urcorner|_F) \longrightarrow 0$$
$$H^0\big(X', (3k+2)K_{X'}\big) \xrightarrow{\gamma} H^0\big(F, (3k+2)K_F\big)$$

where β is surjective by the Kawamata-Viehweg vanishing theorem. We see that

$$\dim_{\mathbb{C}}\big(\mathrm{im}(\alpha)\big) = \dim_{\mathbb{C}}\big(\mathrm{im}(\gamma)\big) \geq \dim_{\mathbb{C}}\big(\mathrm{im}(\beta)\big) = h^0(F, K_F + D_{2k+1})$$

where $D_{2k+1} := \ulcorner(2k+1)\pi^*(K_X)\urcorner|_F$ and $h^0(F, D_{2k+1}) \geq 2$. So $h^0(F, K_F + D_{2k+1}) \geq 4$, according to Lemma 1.2, because we have $\chi(\mathcal{O}_F) \geq 3$ in this case. Thus

$$L^2_{3k+2} \geq 2\big(h^0(F, L_{3k+2}) - 2\big) \geq 2\Big(\dim_{\mathbb{C}}\big(\mathrm{im}(\alpha)\big) - 2\Big) \geq 4$$

and then $|K_F + 2L_{3k+2}|$ gives a birational map by Lemma 1.3. So ϕ_{7k+5} is birational.

Finally, for all $m \geq 10k+7$, set $t := m - 7k - 5 \geq 3k+2$, then $\dim\phi_t(F) \geq 1$. In particular, $tK_{X'}$ is effective. So ϕ_m is birational for all $m \geq 10k+7$ in this case. $\square$

Corollary 2.3.1. *Let X be an irregular nonsingular 3-fold of general type, suppose $P_k(X) \geq 2$, then ϕ_{7k+3} is birational. Therefore at least ϕ_{143} is birational according to Kollár and Fletcher.*

Proof. In the proof of the last theorem, if $b > 0$, then ϕ_m is birational for $m \geq 2k+4$. If $b = 0$, we can use the formula of $q(X)$ to the fibration $f_1 : X' \longrightarrow \mathbb{P}^1$. When $q(X) > 0$, then we must have $q(F) > 0$. Then $\Phi_{|3K_F|}$ is birational for the fibre F, so is $\Phi_{|(7k+3)K_X|}$ by 2.2(2). Moreover, we have $P_{20}(X) \geq 2$ for any irregular 3-fold of general type according to Kollár ([5]) and Fletcher ([2]). Thus ϕ_{143} is birational. $\square$

Theorem 2.3.2. *Let X be a nonsingular projective threefold of general type and suppose $P_k(X) \geq 2$, then ϕ_m is birational for $m \geq 13k+6$.*

Proof. Suppose X be a minimal model with at worst $\mathbb{Q}$-factorial terminal singularities. Make a birational modification $\pi : X' \longrightarrow X$ such that:

(i) X' is nonsingular;

(ii) $|kK_{X'}|$ gives a morphism;

(iii) the fractional part of $\pi^*(K_X)$ has supports with only normal crossings.

Set $g := \Phi_{|kK_X|} \circ \pi$ and $W' := \overline{\Phi_{|kK_X|}(X)}$. Let $X' \xrightarrow{f} W \xrightarrow{s} W'$ be the Stein factorization of g.

We would like to formulate our proof through two steps as follows.

Case 1. $\dim\phi_k(X) \geq 2$.

Set $kK_{X'} \sim_{\mathrm{lin}} M_k + Z_k$, where M_k is the moving part and Z_k is the fixed part. Then a general member $S \in |M_k|$ is an irreducible nonsingular projective surface of general type. Write $K_{X'} = \pi^*(K_X) + \sum a_i E_i$, where the E_i are exceptional divisors for π, $0 < a_i \in \mathbb{Q}$ for each i. Obviously, $\ulcorner\pi^*(K_X)\urcorner \leq K_{X'}$. Because $h^0(X', \ulcorner\pi^*(kK_X)\urcorner) = h^0(X', kK_{X'})$, we can see that M_k is actually also the moving part of $|\ulcorner\pi^*(kK_X)\urcorner|$. Thus we have

$$\pi^*(kK_X) \geq_{\mathbb{Q}} M_k + \sum b_i E_i,$$

where $0 \le b_i \in \mathbb{Q}$ for each i.

We claim that $mK_{X'}$ is always effective for $m \ge 2k+1$. In fact, for any $t \in \mathbb{Z}^+$, we consider the system

$$|K_{X'} + \ulcorner \pi^*((t+k)K_X)\urcorner + S|.$$

It is a sub-system of $|(2k+t+1)K_{X'}|$. By the Kawamata-Viehweg vanishing theorem, we have a surjective map

$$H^0(X', K_{X'} + \ulcorner \pi^*((t+k)K_X)\urcorner + S) \longrightarrow H^0(S, K_S + \ulcorner \pi^*((t+k)K_X)\urcorner|_S) \longrightarrow 0.$$

Noting that $\ulcorner \pi^*((t+k)K_X)\urcorner \ge \ulcorner \pi^*(tK_X)\urcorner + M_k$, also by Lemma 1.6(i), it is sufficient to show that $K_S + \ulcorner \pi^*(tK_X)|_S\urcorner + M_k|_S$ is effective. When $t = 0$, then $h^0(S, K_S + M_k|_S) \ge 2$ by Lemma 1.2, because $h^0(S, M_k|_S) \ge 2$. When $t > 0$, choose a 1-dimensional sub-system $|C|$ in the moving part of $\mid M_k|_S \mid$. Modulo blowing-ups, we can suppose $|C|$ is free from base points and then C is nef and $C \le M_k|_S$. We have $g(C) \ge 2$. Because $\pi^*(tK_X)|_S$ is a nef and big $\mathbb{Q}$-divisor on S, by the Kawamata-Viehweg vanishing theorem, we also get a surjective map

$$H^0(S, K_S + \ulcorner \pi^*(tK_X)|_S\urcorner + C) \longrightarrow H^0(C, K_C + D) \longrightarrow 0,$$

where $D := \ulcorner \pi^*(tK_X)|_S\urcorner \big|_C$ is a divisor on C with positive degree. Thus $h^0(C, K_C+D) \ge 2$. This leads to the effectiveness of $(2k+t+1)K_{X'}$. Moreover, actually we have proved that $\dim\phi_m(S) \ge 1$ for $m \ge 2k+1$.

Now we prove that ϕ_{3k+1} is generically finite. Considering the system

$$|K_{X'} + \ulcorner 2k\pi^*(K_X)\urcorner + M_k|,$$

as we have shown in the above that $(2k+1)K_{X'}$ is effective, so $|K_{X'} + \ulcorner 2k\pi^*(K_X)\urcorner + M_k|$ can distinguish general S. By the Kawamata-Viehweg vanishing theorem, we have

$$|K_{X'} + \ulcorner 2k\pi^*(K_X)\urcorner + S| \,\big|_S = \big| \, K_S + \ulcorner 2k\pi^*(K_X)\urcorner|_S \, \big| \, .$$

We have

$$\big| \, K_S + \ulcorner 2k\pi^*(K_X)\urcorner|_S \, \big| \supset \big| \, K_S + \ulcorner k\pi^*(K_X)|_S\urcorner + M_k|_S \, \big| \, .$$

Noting that $h^0(S, M_k|_S) \ge 2$, $K_S + \ulcorner k\pi^*(K_X)|_S\urcorner \ge K_S + M_k|_S$, which is also effective by Lemma 1.2, and $k\pi^*(K_X)|_S$ is a nef and big $\mathbb{Q}$-divisor on S, it is easy to verify that $\big| \, K_S + \ulcorner k\pi^*(K_X)|_S\urcorner + M_k|_S \, \big|$ gives a generically finite map. In fact, choose a 1-dimensional sub-system $|C|$ in the moving part of $\big| \, M_k|_S \, \big|$. For the same reason, we can suppose $|C|$ is free from base points. $\big| \, K_S + \ulcorner k\pi^*(K_X)|_S\urcorner + C \, \big|$ can distinguish general C, and we have

$$|K_S + \ulcorner k\pi^*(K_X)|_S\urcorner + C| \,\big|_C = |K_C + D|,$$

where D is a divisor on C with positive degree. Because $g(C) \ge 2$, thus $h^0(K_C + D) \ge 2$ and $|K_C + D|$ gives a generically finite map.

Finally, we want to show that ϕ_m is birational for $m \ge 9k+4$. Let $t := m - 7k - 3$, then $t \ge 2k+1$. Denote by M_{3k+1} the moving part of $|(3k+1)K_{X'}|$ and by M_t the moving part of $|tK_{X'}|$. We have

$$|K_{X'} + \ulcorner (t+6k+2)\pi^*(K_X)\urcorner + M_k| \subset |mK_{X'}|.$$

Because $t+6k+3>2k+1$, $K_{X'}+\ulcorner(t+6k+2)\pi^*(K_X)\urcorner$ is effective, thus the left system in the above can distinguish general S. Furthermore, the vanishing theorem gives

$$|K_{X'}+\ulcorner(t+6k+2)\pi^*(K_X)\urcorner+M_k|\,\Big|_S=|K_S+L|,$$

where $L:=\ulcorner(t+6k+2)\pi^*(K_X)\urcorner\,\Big|_S\geq 2M_{3k+1}|_S+M_t|_S$. By Lemma 1.4, $|K_S+L|$ gives a birational map, so does $|mK_{X'}|$.

Case 2. $\dim\phi_k(X)=1$.

In this case, W is a nonsingular curve of genus b. Let F be a general fibre of f, then F is an irreducible smooth projective surface of general type. We have $M_k\sim_{\text{lin}}\sum F_i$, where the F_i are fibres of f for each i.

By a parallel argument as in the proof of Theorem 2.3.1, we see that ϕ_m is birational for $m\geq 2k+4$ if $b>0$. And if $b=0$ while F is a surface with the invariants $\big(K^2_{F_0},p_g(F)\big)=(1,2)$ or $(2,3)$, then ϕ_m is birational for $m\geq 10k+7$.

Otherwise, we use Kollár's method. From 2.2, we know that ϕ_{7k+3} is birational and $\dim\phi_{5k+2}(X)\geq 2$. Thus, by Lemma 1.7, $mK_{X'}$ is effective for $m\geq 6k+4$. Since we have $|K_{X'}+\ulcorner(5k+2)\pi^*(K_X)\urcorner+F|\,\Big|_F=|K_F+D|$ where $D:=\ulcorner(5k+2)\pi^*(K_X)\urcorner\,\Big|_F$ is effective and $h^0(F,D)\geq 2$, we see that K_F+D is effective and thus $(6k+3)K_{X'}$ is effective. So ϕ_m is birational for $m\geq 13k+6$, which means that ϕ_{13k+6} is stably birational. $\square$

Theorem 2.3.3. *Let X be a nonsingular projective threefold of general type and suppose $P_k(X)\geq 3$, then ϕ_m is birational for all $m\geq 10k+8$.*

Proof. When $\dim\phi_k(X)\geq 2$, we know from Case 1 of Theorem 2.3.2 that ϕ_m is birational for $m\geq 9k+4$. When $|kK_X|$ is composed of a pencil, from the proof of Theorem 2.3.1, we see that ϕ_k will derive a fibration $f:X'\longrightarrow W$ onto a nonsingular curve. If $b:=g(W)>0$, then ϕ_m is birational for $m\geq 2k+4$.

The remained case is the one when $b=0$. We have an injection $\mathcal{O}(2)\hookrightarrow f_*\omega^k_{X'}$. So, for each $p>0$, we have

$$\mathcal{O}(1)\otimes f_*\omega^p_{X'/\mathbb{P}^1}=\mathcal{O}(2p+1)\otimes f_*\omega^p_{X'}\hookrightarrow f_*\omega^{k(p+1)+p}_{X'}.$$

Thus Kollár's method implies that ϕ_{6k+5} is birational, ϕ_{4k+3} is generically finite and that $\dim\phi_{3k+2}(X)\geq 2$. Now using our method, we can see that $mK_{X'}$ is effective for $m\geq 4k+4$ by Lemma 1.7. Since $(4k+3)K_{X'}$ is also effective, thus ϕ_m is birational for $m\geq 10k+8$. $\square$

Corollary 2.3.2. *Let X be a nonsingular projective threefold of general type and suppose $p_g(X)\geq 3$, then ϕ_m is birational for $m\geq 11$.*

Proof. Keep the same notations as in the proof of Theorem 2.3.2. When $\dim\phi_1(X)\geq 2$, we set $L_3:=4K_{X'}$, $L_2=L_1:=K_{X'}$. Then $|L_3|$ gives a generically finite map by virtue of Case 1, Theorem 2.3.2. Using Lemma 1.5, we see that $|K_{X'}+2L_3+L_2+L_1|$ gives a birational map. Thus ϕ_{11} is birational.

When $\dim\phi_1(X)=1$, we see from the proof of Theorem 2.3.3 that ϕ_{11} is also birational. $\square$

Theorem 2.3.1, Theorem 2.3.2, Theorem 2.3.3 and Corollary 2.3.2 imply the main theorem.

3. Open problems

3.1. Let X be a nonsingular projective variety of general type of dimension n. We define

$k_0(X) := min\{k|\ P_k(X) \geq 2\}$;

$k_s(X) := min\{k|\ \phi_m$ is birational for $m \geq k\}$;

$\mu_s(X) := \frac{k_s(X)}{k_0(X)}$, which is called *the relative pluricanonical stability* of X. Obviously, $\mu_s(X)$ is a birational invariant.

$\mu_s(n) := \sup\{\mu_s(X)|\ X$ is a n-fold of general type$\}$, which is called the n-th *relative pluricanonical stability.*

It is well-known that $\mu_s(1) = 3$ and $\mu_s(2) = 5$ ([1]). From the main theorem, we have $\mu_s(3) \leq 16$. What is the exact value of $\mu_s(3)$? It is also interesting to study $\mu_s(n)$ for $n \geq 4$, even if we don't know whether we should have $\mu_s(n) < +\infty$.

3.2. We would like to ask a very natural question which never happens in surface case.

Question. *Does there exist a smooth projective threefold X of general type and two positive integers $k_1 < k_2$ such that ϕ_{k_1} is birational while ϕ_{k_2} is not birational?*

Of course, it may happen for some threefold that $P_{k_1} > P_{k_2}$ even if $k_1 < k_2$. But we have not found any counter example yet to the above question.

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