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THE ABDUS SALAM INTERNATIONAL CENTRE FOR THEORETICAL PHYSICS

NUMERICAL SOLUTION OF DISCRETE-TIME ALGEBRAIC RICCATI EQUATION

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Abstract

In this paper, we present a naturally numerical method for finding the maximal hermitian solution X_+ of the Discrete-Time Algebraic Riccati Equation (DTARE) based on the convergence of a monotone sequence of hermitian matrices.

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1 Introduction

The paper is concerned with a new representation of the Discrete-Time Riccati Equation (DTARE)

$$X = Q + A^*XA - A^*XB(R + B^*XB)^{-1}B^*XA \tag{1}$$

where Q and R are hermitian matrices of sizes $n \times n$ and $m \times m$ respectively; the coefficients A and B are $n \times n$ and $n \times m$ respectively; $n \times n$ solution matrices X are to be found for which, of course, $(R + B * XB)$ is invertible, such that the solutions X are called *admissible.*
It is well known that the maximal hermite matrix plays a key role in the minimal factorization of *the realization*

$$\Psi(z) = B^*(z^{-1}I - A^*)^{-1}Q(zI - A)^{-1}B + R$$

In other words, we mention there exists a one-to-one correspondence between the admissible solutions of (1) and the set of all realizations $\Psi(z)$ with all their poles in $\mathcal{D} = \{\lambda : 0 < |\lambda| < 1\}$ and $\Psi(\infty) = I$, and with the property that

$$\Psi(z) = (\Phi(z^{-1}))^*D\Phi(z)$$

is a minimal factorization of $\Psi(z)$ for some matrix D, provided that (A, B) is controllable and $\sigma(A) \subset \mathcal{D}$. For more details, see [1], where the existence of the maximal hermite solution X_+ are also discussed.

Theorem 1.1 *Assume that $R > 0$, $Q \geq 0$, and (A, B) is stabilizable (i.e. there exists a matrix K such that $A + BK$ is stable), then X_+ exists and $X_+ \geq 0$.*

With the new representation of DTARE (1), the existence of its non-negatively-defined matrix is derived somehow relaxing the hypothesis on the stability of (A, B) (Corollary 3.7). Furthermore, it also yields a natural numerical algorithm for finding the solution X_+ of DTARE (Corollary 3.5).

2 About numerical algorithms

The numerical algorithms for solving the Riccati Equations can be roughly classified in two categories: invariant subspace and iterative methods.
The invariant (or deflating) subspace methods have a large scope of applicability. According to [2] the first invariant subspace method for the Continuous-Time Algebraic Riccati Equation (CTARE) was given in [3]. Extensions were reported in [4] and [5]. Numerical stable method for

computing bases of invariant subspaces via the ordered Schur form was initiated in [6] and [7]. This idea was applied to Riccati equations in [8], [9] and [10]. The most recent improvements were (based on singular value decomposition) [11], [12] as well as a provision of a symmetric representation of the Riccati solution. Also important is [13], where invariant space methods were applied even for "singular" problems.

The iterative methods produce a sequence of self-adjoint matrices that converge to the Riccati solution (without involving the symmetric representations of the Riccati solution). The best known example is, perhaps, Kleinman's algorithm [14] which is, in fact, a Newton-Raphson scheme, that works if $R > 0$ and $Q \geq 0$. A more general iterative method for the discrete time case was reported in [15]. Even though the applicability of iterative methods is rather limited, they are still considered for their numerical accuracy.

In this work, we present a new representation of the Discrete-Time Algebraic Riccati Equation, where the iterative entries X_k^ϵ are positive definite. The representation yields the monotonicity of the sequence $\{X_k^\epsilon\}$ in both variables $k \in \mathbb{N}$ and $\epsilon \in J \subset \mathbb{R}$.

3 Main results

We recall two important results of matrix inverse:

Lemma 3.1 (The matrix inverse lemma) *With the appropriate conditions on dimensions of matrices X, B, R, where X and R are invertible, we have*

$$(X^{-1} + BR^{-1}B^*)^{-1} = X - XB(R + B^*XB)^{-1}B^*X$$

Proof: See [16] problem 5.28, p.126 or [17] problem A4, p.668.

□

Lemma 3.2 *Assume that two matrices W, Z are hermitian and $W \geq Z > 0$ then $W^{-1} \leq Z^{-1}$*

Proof: See [18] p.92.

□

Proposition 3.3 *Assume X and R are invertible, then DTARE (1)*

$$X = Q + A^*XA - A^*XB(R + B^*XB)^{-1}B^*XA$$

is equivalent to

$$X = Q + A^*(X^{-1} + BR^{-1}B^*)^{-1}A \tag{2}$$

Proof: We proceed directly:

$$\begin{aligned} & Q + A^*XA - A^*XB(R + B^*XB)^{-1}B^*XA \\ = \; & Q + A^*[X - XB(R + B^*XB)^{-1}B^*X]A \quad \text{(from the lemma (3.1))} \\ = \; & Q + A^*(X^{-1} + BR^{-1}B^*)^{-1}A \end{aligned}$$

The proof is completed.

□

Now we turn to investigate a class of hermitian matrix sequence with one parameter $\epsilon \in J \subset \mathbb{R}$.

Let us denote $J = \{\epsilon \in \mathbb{R} : Q^\epsilon := Q + \epsilon I > 0\}$ and obviously remark that the set J is not empty. Set

$$X_1^\epsilon := Q^\epsilon + A^*(BR^{-1}B^*)^{-1}A \tag{3}$$

$$X_{k+1}^\epsilon := Q^\epsilon + A^*((X_k^\epsilon)^{-1} + BR^{-1}B^*)^{-1}A \qquad k \geq 1 \tag{4}$$

Theorem 3.4 *Assume that Q^ϵ and R are positive-definite hermite matrices with a fixed ϵ, then the sequence (3), (4) defines a non-increasing positive-definite hermite matrix, which converges to the maximal hermite solution X_+^ϵ of DTARE*

$$X = Q^\epsilon + A^*XA - A^*XB(R + B^*XB)^{-1}B^*XA \tag{5}$$

Proof: The non-increasing property of X_k^ϵ is proved by induction.

First notice that $X_1^\epsilon \geq Q^\epsilon > 0$ by definition (3) and so $(X_1^\epsilon)^{-1}$ is positive-definite. This implies $X_2^\epsilon = Q^\epsilon + A^*(X_1^{-1} + BR^{-1}B^*)^{-1}A \geq Q^\epsilon > 0$.

Applying Proposition 3.2 to the case of $W := (X_1^\epsilon)^{-1} + BR^{-1}B^*$ and $Z := BR^{-1}B^*$, we have

$$((X_1^\epsilon)^{-1} + BR^{-1}B^*)^{-1} \leq (BR^{-1}B^*)^{-1}$$

and

$$X_2^\epsilon - X_1^\epsilon = A^*[((X_1^\epsilon)^{-1} + BR^{-1}B^*)^{-1} - (BR^{-1}B^*)^{-1}]A \leq 0$$

Hence $X_1^\epsilon \geq X_2^\epsilon \geq Q^\epsilon > 0$.

Now assume that $X_1^\epsilon \geq X_2^\epsilon \geq ... \geq X_k^\epsilon \geq Q^\epsilon > 0$.

With the same argument, we have

$$X_{k+1}^\epsilon = Q^\epsilon + A^*((X_k^\epsilon)^{-1} + BR^{-1}B^*)^{-1}A \geq Q^\epsilon > 0$$

Then, applying Proposition 3.2 to the case of $W := X^\epsilon_{k-1}; Z := X^\epsilon_k$, we get

$$(X^\epsilon_{k-1})^{-1} \leq (X^\epsilon_k)^{-1}$$

This implies $0 < (X^\epsilon_{k-1})^{-1} + BR^{-1}B^* \leq (X^\epsilon_k)^{-1} + BR^{-1}B^*$.
Proposition 3.2 is applied again to yield

$$((X^\epsilon_{k-1})^{-1} + BR^{-1}B^*)^{-1} \geq ((X^\epsilon_k)^{-1} + BR^{-1}B^*)^{-1}$$

and

$$X^\epsilon_{k+1} - X^\epsilon_k = A^*[((X^\epsilon_k)^{-1} + BR^{-1}B^*)^{-1} - ((X^\epsilon_{k-1})^{-1} + BR^{-1}B^*)^{-1}]A \leq 0$$

Hence with a given $\epsilon \in J$, $\{X^\epsilon_k\}$ is a non-increasing, positive-definite sequence in terms of k; thereby completing the induction argument.

We also conclude that the sequence $\{X^\epsilon_k\}$ converges to a positive-definite hermite matrix and the limit

$$X^\epsilon_\infty = \lim_{k\to\infty} X^\epsilon_k \geq Q^\epsilon$$

The fact that X^ϵ_∞ is a solution, DTARE (5) is easily derived when applying Proposition 3.3 to the right-hand side of (4) and then letting k tend to infinity.

Now we prove that X^ϵ_∞ is the maximal hermitian solution of DTARE (5).
First, by the maximality of X^ϵ_+ we have $X^\epsilon_+ \geq X^\epsilon_\infty > 0$.
The inverse inequality is derived by induction. At first, we have

$$\begin{aligned} X^\epsilon_+ &= Q^\epsilon + A^*X^\epsilon_+A - A^*X^\epsilon_+B(R + B^*X^\epsilon_+B)^{-1}B^*X^\epsilon_+A \\ &= Q^\epsilon + A^*((X^\epsilon_+)^{-1} + BR^{-1}B^*)^{-1}A \\ &\leq Q^\epsilon + A^*(BR^{-1}B^*)^{-1}A \\ &= X^\epsilon_1 \end{aligned}$$

and now assume that $X^\epsilon_+ \leq X^\epsilon_k$, then

$$\begin{aligned} X^\epsilon_+ &= Q^\epsilon + A^*((X^\epsilon_+)^{-1} + BR^{-1}B^*)^{-1}A \\ &\leq Q^\epsilon + A^*((X^\epsilon_k)^{-1} + BR^{-1}B^*)^{-1}A \\ &= X^\epsilon_{k+1} \end{aligned}$$

Hence $X^\epsilon_+ \leq X^\epsilon_k$ for all $k \in I\!N$ by induction argument. This implies $X^\epsilon_+ \leq X^\epsilon_\infty$. The equality $X^\epsilon_+ = X^\epsilon_\infty$ is proved and this completes the proof.

□

When Q is positive-definite and $\epsilon = 0$, the matrix Q^ϵ and DTARE (5) concide with Q and DTARE (1), respectively. Then, the following corollary is considered as a result of Theorem 3.4

Corollary 3.5 *Assume that Q, R are positive-definite hermite matrices, then the following matrix sequence*

$$\begin{cases} X_1 & = Q + A^*(BR^{-1}B^*)^{-1}A \\ X_{k+1} & = Q + A^*(X_k^{-1} + BR^{-1}B^*)^{-1}A \qquad k \geq 1 \end{cases}$$

decreasingly converges to the positive-definite maximal hermite solution X_+ of DTARE (1).

The following theorem mentions the monotonicity with respect to the parameter ϵ of the maximal solutions X_+^ϵ.

Theorem 3.6 *X_k^ϵ and X_+^ϵ are non-decreasing functions on J for every $k \in I\!N$.*

Proof: The theorem is proved by induction.
Let $\epsilon_1 \leq \epsilon_2$ for $\epsilon_1, \epsilon_2 \in J$, we have $Q^{\epsilon_1} \leq Q^{\epsilon_2}$ and then

$$0 \leq X_1^{\epsilon_1} = Q^{\epsilon_1} + A^*(BR^{-1}B^*)^{-1}A \leq Q^{\epsilon_2} + A^*(BR^{-1}B^*)^{-1}A = X_1^\epsilon$$

Now suppose that $0 < X_k^{\epsilon_1} \leq X_k^{\epsilon_2}$, Proposition (3.2) is applied to have

$$\begin{aligned} 0 < X_{k+1}^{\epsilon_1} &= Q^{\epsilon_1} + A^*((X_k^{\epsilon_1})^{-1} + BR^{-1}B^*)^{-1}A \\ &\leq Q^{\epsilon_2} + A^*((X_k^{\epsilon_2})^{-1} + BR^{-1}B^*)^{-1}A \\ &= X_{k+1}^{\epsilon_2} \end{aligned}$$

Hence $X_k^{\epsilon_1} \leq X_k^{\epsilon_2}$ for all $k \in I\!N$ and the induction argument is completed.
Let k tend to infinity. Then we have $X_+^{\epsilon_1} \leq X_+^{\epsilon_2}$ and complete the proof.

Corollary 3.7 (The existence of non-negative definite solution of DTARE (1)) *Assume that $Q \geq 0$ and $R > 0$, then the set of non-negative-definite solutions of DTARE (1) is not empty.*

Proof: Since $Q \geq 0$ then $Q^\epsilon = Q + \epsilon I > 0, \quad \forall \epsilon > 0$. Morever, the positive-definite maximal solutions X_+^ϵ exist and are non-decreasing on the interval $(0, \infty)$.
Hence the limit

$$X^{0^+} = \lim_{\epsilon \downarrow 0^+} X_+^\epsilon$$

exists. We shall prove that X^{0^+} is a non-negative-definite solution of DTARE (1).
From (4)

$$X_{k+1}^\epsilon = Q^\epsilon + A^*X_k^\epsilon A - A^*X_k^\epsilon B(R + B^*X_k^\epsilon B)^{-1}B^*X_k^\epsilon A$$

Let k tend to infinity

$$X_+^\epsilon = Q^\epsilon + A^*X_+^\epsilon A - A^*X_+^\epsilon B(R + B^*X_+^\epsilon B)^{-1}B^*X_+^\epsilon A$$

and then $\epsilon \downarrow 0^+$

$$X^{0^+} = Q^\epsilon + A^*X^{0^+}A - A^*X^{0^+}B(R + B^*X^{0^+}B)^{-1}B^*X^{0^+}A$$

Hence X^{0^+} is a solution of DTARE (1). This solution is non-negative-definite hermite matrix because X_+^ϵ are. Therefore, the set of non-negative-definite hermite solutions is not empty.

□

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