Dirac-Kähler approach connected to quantum mechanics in Grassmann space

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Abstract

We compare the way one of us got spinors out of fields, which are a priori antisymmetric tensor fields, to the Dirac-Kähler rewriting. Since using our Grassmann formulation is simple it may be useful in describing the Dirac-Kähler formulation of spinors and in generalizing it to vector internal degrees of freedom and to charges. The "cheat" concerning the Lorentz transformations for spinors is the same in both cases and is put clearly forward in the Grassmann formulation. Also the generalizations are clearly pointed out. The discrete symmetries are discussed, in particular the appearance of two kinds of the time-reversal operators as well as the unavoidability of four families.

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I. INTRODUCTION

Kähler [1] has shown how to pack the Dirac wave function into the language of differential forms in the sense that the Dirac equation is an equation in which a linear operator acts on a linear combination u of p-forms (p=0, 1, ...,d; here d=dimension =4). This is the Dirac-Kähler formalism.

One of us [2] has long developed an a priori rather different formalism in an attempt to unify spin and charges. In this approach the spin degrees of freedom come out of canonically quantizing certain Grassmannian odd (position analogue in the sense of being on an analogue footing with x^a) variables θ^a. These variables are denoted by a vector index a, and there are at first to see no spinors at all!

One of the main purposes of the present article is to point out the analogy and nice relations between the two different ways of achieving the - almost miraculous - appearance of spin one half degrees of freedom in spite of starting from pure vectors and tensors.

Of course it is a priori impossible that vectorial and tensorial fields (or degrees of freedom) can be converted into spinorial ones without some "cheat". The "cheat" consists really in exchanging one set of Lorentz transformation generators by another set (which indeed means putting strongly to zero one type of Grassmann odd operators fulfilling the Clifford algebra and anticommuting with another type of Grassmann odd operators, which also fulfill the Clifford algebra [2]).

In fact one finds on page 512 in the Kähler's [1] article that there are two sets of rotation generators; one set for which the u field (in the Kähler's notation) transforms as a spinor field and another one for which it transforms as superpositions of vector and (antisymmetric) tensor fields. Analogously in the approach of one of us the apriori Lorentz transformation generators $S^{ab} = \tilde{S}^{ab} + \tilde{\tilde{S}}^{ab}$ have the wave function transform as vectors and antisymmetric tensors, while $\tilde{S}^{ab}$ ($= -i\frac{1}{4}[\tilde{a}^a, \tilde{a}^b]$) or $\tilde{\tilde{S}}^{ab}$ ($= -i\frac{1}{4}[\tilde{\tilde{a}}^a, \tilde{\tilde{a}}^b]$ and $[,]$ means the commutator) used alone are also possible Lorentz generators for which now the wave function transforms as a spinor wave function. By putting $\tilde{\tilde{a}}^a$ (which has the property that $[\tilde{S}^{ab}, \tilde{\tilde{a}}^c] = 0$) equal strongly to zero is the same as replacing $\mathcal{S}^{ab}$ by $\tilde{S}^{ab}$.

In both approaches to get spinors out of vectors and antisymmetric tensors, as start, you get not only one but several copies, families, of Dirac fields. This is a fundamental feature in as far as these different families are connected by the generator parts not used: If one for instance uses $\tilde{S}^{ab}$ as the Lorentz generator to get spinors, then the not used part $\tilde{\tilde{S}}^{ab}$ transforms the families (of the same Grassmann character) into each other.

It will be a major content of the present article to bring about a dictionary relating the two formalisms so that one can enjoy the simplicity of one also working on the other one. We also shall generalize the Kähler operators for $d = 4$, comment on the discrete symmetries, which in the one of us approach show up clearly and use the $d - 4$ dimensions to describe spins and charges [2].

In the following section we shall put forward the little part of the formalism of the work of one of us needed for the comparison with the Dirac-Kähler formalism.

In the next section again - section 3 - we shall then tell the (usual) Dirac-Kähler formalism as far as relevant.

The comparison which should now be rather obvious is performed in section 4.

In section 5 we shall analyse in the two approaches in parallel how the remarkable finding of the Dirac equation inside a purely tensorial-vectorial system of fields occurs.

In section 6 we shall comment on the evenness of the γ^a matrices, which have to transform Grassmann odd wave functions into Grassmann odd wave functions.

In section 7 we shall comment on discrete symmetries for either Kähler or the one of us approach, also discussing the realization of the discrete symmetries pointed up clearly by Weinberg in his book [8] on pages 100-105.

In section 8 we want to investigate how unavoidable the appearance of families is to this type of approaches.

In section 9 we shall look at how the ideas of one of us of extra dimensions generalizes the Kähler approach.

In section 10 we discuss the Nielsen and Ninomija [5] no go theorem for spinors on a lattice and a possible way out.

In section 11 we shall resume and deliver concluding remarks.

II. DIRAC EQUATIONS IN GRASSMANN SPACE

What we can call the Mankoč approach [2], and which is the work of one of us, is a rather ambitious model for going beyond the Standard Model with say 10 (or more) extra dimensions, but what we need for the present connection with the Dirac-Kähler [1] formalism is only the way in which the spin part of the Dirac particle fields comes about. The total number of dimensions in the model is (most hopefully) 13 +1 bosonic degrees of freedom, i.e. normal dimensions, and the same number of fermionic ones.

Let us call the dimension of space-time d and then the Dirac spinor degrees of freedom shall come from the odd Grassmannian variables $\theta^a, \quad a \in \{0, 1, 2, 3, 5, \cdot, d\}$.

In wanting to quantize or just to make Poisson brackets out of the d θ^a's we have two choices since we could either decide to make the different θ^a's their own conjugate, so that one only has $d/2$ degrees of freedom - this is the approach of Ravndal and DiVecchia [3] - or we could decide to consider the θ^a's configuration space variables only. In the latter case - which is the Mankoč case - we have then from the θ^a's different conjugate variables $p^{\theta a}$.

In this latter case we are entitled to write wave functions of the form

$$\psi(\theta^a) = \sum_{i=0,1,..,3,5,..,d} \quad \sum_{\{a_1<a_2<...<a_i\}\in\{0,1,..,3,5,..,d\}} \alpha_{a_1,a_2,...,a_i} \theta^{a_1}\theta^{a_2}\cdots\theta^{a_i}. \tag{1}$$

This is the only form a function of the odd Grassmannian variables θ^a can take. Thus the wave function space here has dimension 2^d. Completely analogously to usual quantum mechanics we have the operator for the conjugate variable θ^a to be

$$p^\theta_a := -i\overrightarrow{\frac{\partial}{\partial\theta^a}} := -i\overrightarrow{\partial}_a. \tag{2}$$

The right arrow here just tells, that the derivation has to be performed from the left hand side. These operators then obey the odd Heisenberg algebra, which written by means of the generalized commutators

$$\{A, B\} := AB - (-1)^{n_{AB}} BA \tag{3}$$

where

$$n_{AB} = \begin{cases} +1, & \text{if A and B have Grassmann odd character} \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

takes the form

$$\{p^{\theta a}, p^{\theta b}\} = 0 = \{\theta^a, \theta^b\}, \quad \{p^{\theta a}, \theta^b\} = -i\eta^{ab}. \tag{5}$$

Here η^{ab} is the flat metric $\eta = diag\{1, -1, -1, ...\}$.

For later use we shall define the operators

$$\tilde{a}^a := i(p^{\theta a} - i\theta^a), \quad \tilde{\tilde{a}}^a := -(p^{\theta a} + i\theta^a), \tag{6}$$

for which we can show that the $\tilde{a}^a$'s among themselves fulfill the Clifford algebra as do also the $\tilde{\tilde{a}}^a$'s, while they mutually anticommute:

$$\{\tilde{a}^a, \tilde{a}^b\} = 2\eta^{ab} = \{\tilde{\tilde{a}}^a, \tilde{\tilde{a}}^b\}, \quad \{\tilde{a}^a, \tilde{\tilde{a}}^b\} = 0. \tag{7}$$

Note that the linear combinations (6) presuppose a metric tensor, since otherwise only θ^a and $p^\theta{}_a$ but not θ_a and $p^{\theta\, a}$ are defined.

We could recognize formally

$$\text{either} \quad \tilde{a}^a p_a|\psi>= 0, \qquad \text{or} \quad \tilde{\tilde{a}}^a p_a|\psi>= 0 \tag{8}$$

as the Dirac-like equation, because of the above generalized commutation relations. Applying either the operator $\tilde{a}^a p_a$ or $\tilde{\tilde{a}}^a p_a$ on the two equations (Eqs.(8)) we get the Klein-Gordon equation $p^a p_a|\psi>= 0$. Here of course we defined

$$p_a = i\frac{\partial}{\partial x^a}. \tag{9}$$

However, it is rather obvious that these equations (8) are not Dirac equations in the sense of the wave function transforming as a spinor, w.r.t. to the generators for the Lorentz transformations, if taken as usual

$$\mathcal{S}^{ab} := \theta^a p^{\theta b} - \theta^b p^{\theta a}. \tag{10}$$

However it is easily seen that we can write these generators as the sum

$$\mathcal{S}^{ab} = \tilde{S}^{ab} + \tilde{\tilde{S}}^{ab}, \tag{11}$$

where we have defined

$$\tilde{S}^{ab} := -\frac{i}{4}[\tilde{a}^a, \tilde{a}^b], \quad \tilde{\tilde{S}}^{ab} := -\frac{i}{4}[\tilde{\tilde{a}}^a, \tilde{\tilde{a}}^b], \tag{12}$$

with $[A, B] := AB - BA$. One can now easily see that the solutions of the two equations (8) now transform as spinors with respect to either $\tilde{S}^{ab}$ or $\tilde{\tilde{S}}^{ab}$.

It is of great importance for the "trick" of manipulating what we shall consider to be the Lorentz transformations and thus to be able to make the "miraculous " shifts of Lorentz representation that is the somewhat remarkable characteristic of the Kähler type of shift in formulation interpretation, that both - untilded, the single tilded and the double tilded - $\mathcal{S}^{ab}$ obey the d-dimensional Lorentz generator algebra $\{M^{ab}, M^{cd}\} = -i(M^{ad}\eta^{bc} + M^{bc}\eta^{ad} - M^{ac}\eta^{bd} - M^{bd}\eta^{ac})$, when inserted for M^{ab}.

Really the "cheat" consist -as we shall return to - in replacing the Lorentz generators by the $\tilde{S}^{ab}$, say. This "cheat" means indeed that for this choice the operators $\tilde{\tilde{a}}^a$ have to be put strongly to zero in the generators of the Lorentz transformations (Eq.(10, 11, 12)) as well as in all the other operators, representing the physical quantities.

We shall present the one of us approach in further details in section 4 pointing out the similarities between this approach and the Kähler approach and generalizing the Kähler approach.

III. KÄHLER FORMULATION OF SPINORS

The Kähler formulation [1] takes its starting point by considering p-forms in the d-dimensional space, $d = 4$. Elegantly, the 1-forms say are defined as dual vectors to the (local) tangent spaces, and the higher p-forms can then be defined as antisymmetrized cartesian (exterior) products of the one-form spaces, and the 0-forms are the scalars; but we can perhaps more concretely think about the p-forms as formal linear combinations of the differentials of the coordinates dx^a : A general linear combination of forms is then written

$$u = u_0 + u_1 + ... + u_d \tag{13}$$

where the p-form is of the form

$$u_p = \frac{1}{p!} \sum_{i_1, i_2, \ldots, i_p} a_{i_1 i_2 \ldots i_p} \cdot dx^{i_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_p} =$$

$$\sum_{i_1 < i_2 \ldots < i_p} a_{i_1 i_2 \ldots i_p} \cdot dx^{i_1} \wedge dx^{i_2} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_p}. \tag{14}$$

Then one can define both the presumably most well known exterior algebra denoted by the exterior product $\wedge$ and the Clifford product $\vee$ among the forms. The wedge product $\wedge$ has the property of making the product of a p-form and a q-form be a (p+q)-form, if a p-form and a q-form have no common differentials. The Clifford product $dx^a \vee$ on a p-form is either a $p + 1$ form, if a p-form does not include a one form dx^a, or a $p - 1$ form, if a one form dx^a is included in a p-form.

Actually Kähler found how the Dirac equation could be written as an equation [1] (Eq. (26.6) in the Kähler's paper)

$$- i\delta u = (m + e \cdot \omega) \vee u \tag{15}$$

where the symbol u stands for a linear combination of p-forms (Eq.(13)) and $p \in \{0, 1, .., 3, 5, .., d\}$ with d being the dimension of space-time, namely d = 4 for the Kähler's case. Further in the notation of Kähler the symbol[1] δ denotes inner differentiation, which means the analogue of the exterior differential d but with the use of the Clifford product $\vee$ instead of the exterior product $\wedge$

$$\delta u = \sum_{i=1}^{3} dx^i \vee \frac{\partial u}{\partial x^i} - dt \vee \frac{\partial u}{\partial t} = du + \sum_{i=1}^{3} e_i \frac{\partial u}{\partial x^i} - e_i \frac{\partial u}{\partial t}. \tag{16}$$

The symbol $e \cdot \omega$ determines the coupling of the charge with the electromagnetic field $\omega = A_a dx^a, \quad a \in \{0,1,2,3\}$ and m means the electron mass, the symbol e_i transforms a p-form into $(p-1)$-form, if the p-form includes dx^i, otherwise it gives zero.

For a free massless particle living in a d dimensional space-time - this is what interests us in this paper since the mass term brings no new feature in the theory - Eq.(15) can be rewritten in the form

$$dx^a \vee p_a \;\; u = 0, \quad a = 0,1,2,3,5,...,d \tag{17}$$

where the symbol u stands again for a linear combination of p-forms (p = 0,1,2,3,5,...,d).

That is to say that the wave function describing the state of the spin one half particle is packed into the exterior algebra function u.

More about the Kähler's approach will come in section 4 giving the correspondence between that and the one with the Grassmann θ^a's, where we shall also give some generalizations.

IV. PARALLELISM BETWEEN THE TWO APPROACHES

We demonstrate the parallelism between the Kähler [1] and the one of us [2] approach in steps, paying attention on the Becher-Joos [4] paper as well. First we shall treat the

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[1] The notation in the Becher and Joos [4] paper is slightly different from the Kähler notation. The Becher and Joos paper uses $d = dx^a \frac{\partial}{\partial x^a}$, as Kähler does in his paper, but δ of Kähler is in the Becher and Joos paper replaced by $d - \delta$, which means that in their paper $\delta = -e^a \frac{\delta}{\delta x^a}$.

spin $\frac{1}{2}$ fields only, as Kähler did. We shall use the simple and transparent definition of the exterior and interior product in Grassmann space to generalize the Kähler approach to two kinds of δ (Eq.(16)) operators on the space of p-forms and then accordingly to three kinds of the generators of the Lorentz transformations, two of the spinorial and one of the vectorial character, the first kind transforming spinor $\frac{1}{2}$ fields, the second one transforming the vector fields. We comment on the Hodge star product for both approaches, define the scalar product of vectors in the vector space of either p-forms or of polynomials of θ^a's and comment on four replications of the Weyl bi-spinor. We also discuss briefly the vector representations in both approaches.

A. Dirac-Kähler equation and Dirac equation in Grassmann space for massless particles

We present here, side by side, the operators in the space of differential forms and in the space of polynomials of θ^a's. We present the exterior product

$$dx^a \ \wedge dx^b \ \wedge \cdots, \qquad \theta^a \ \theta^b \cdots, \tag{18}$$

the operator of "differentiation"

$$-\, ie^a, \qquad p^{\theta a} = -i \overrightarrow{\partial^a} = -i \overrightarrow{\frac{\partial}{\partial \theta_a}} \tag{19}$$

and the two superpositions of the above operators

$$dx^a \ \tilde{\vee} := dx^a \wedge + \, e^a, \qquad \tilde{a}^a := i(p^{\theta a} - i\theta^a),$$

$$dx^a \ \tilde{\tilde{\vee}} := i(dx^a \wedge - \, e^a), \qquad \tilde{\tilde{a}}^a := -(p^{\theta a} + i\theta^a). \tag{20}$$

Here $\tilde{\vee}$ stays instead of $\vee$ of Eq.(15), used by Kähler. Introducing the notation with $\tilde{}$ and $\tilde{\tilde{}}$ we not only point out the similarities between the two approaches but also the two possibilities for the Clifford product - only one of them used by Kähler. Both $\tilde{\vee}$ and $\tilde{\tilde{\vee}}$ are Clifford products on p-forms, while $\tilde{a}^a$ $\tilde{\tilde{a}}^a$ are the corresponding linear operators operating on the space of polynomials of θ^a's. One easily finds the commutation relations, if for both approaches the generalized form of commutators, presented in Eq.(4), are understood

$$\{dx^a\ \tilde{\vee},\ dx^b\ \tilde{\vee}\} = 2\eta^{ab}, \qquad \{\tilde{a}^a, \tilde{a}^b\} = 2\eta^{ab}$$

$$\{dx^a\ \tilde{\tilde{\vee}},\ dx^b\ \tilde{\tilde{\vee}}\} = 2\eta^{ab}, \qquad \{\tilde{\tilde{a}}^a, \tilde{\tilde{a}}^b\} = 2\eta^{ab}. \tag{21}$$

Here η^{ab} is the metric of space-time.

The vacuum state $|\ >$ is defined as

$$dx^a\ \tilde{\vee}\ |\ >= dx^a \wedge, \qquad \tilde{a}^a\ |\ >= \theta^a,$$

$$dx^a\ \tilde{\tilde{\vee}}\ |\ >= dx^a \wedge, \qquad \tilde{\tilde{a}}^a\ |\ >= \theta^a. \tag{22}$$

Now we can define the Dirac-like equations for both approaches:

$$dx^a\ \tilde{\vee}\ p_a\ u = 0, \qquad \tilde{a}^a\ p_a\ \psi(\theta^a) = 0,$$

$$dx^a\ \tilde{\tilde{\vee}}\ p_a\ u = 0, \qquad \tilde{\tilde{a}}^a\ p_a\ \psi(\theta^a) = 0. \tag{23}$$

Since $\{e^a, dx^b\ \wedge\} = \eta^{ab}$ and $\{e^a, e^b\} = 0 = \{dx^a\ \wedge, dx^b\ \wedge\} = 0$, while $\{-ip^{\theta a}, \theta^b\} = \eta^{ab}$ and $\{ip^{\theta a}, ip^{\theta b}\} = 0 = \{\theta^a, \theta^b\}$, it is obvious that e^a plays in the p-form formalism the role of the derivative with respect to a differential $1-$form, similarly as $ip^{\theta a}$ does with respect to a Grassmann coordinate.

Taking into account the above definitions, one easily finds that

$$dx^a\ \tilde{\vee}\ p_a\ \ dx^b\ \tilde{\vee}\ p_b\ u = p^a\ p_a\ u = 0, \qquad \tilde{a}^a\ p_a\ \ \tilde{a}^b\ p_b\ \psi(\theta^b) = p^a\ p_a\ \psi(\theta^b) = 0.$$

$$dx^a\ \tilde{\tilde{\vee}}\ p_a\ \ dx^b\ \tilde{\tilde{\vee}}\ p_b\ u = p^a\ p_a\ u = 0, \qquad \tilde{\tilde{a}}^a\ p_a\ \ \tilde{\tilde{a}}^b\ p_b\ \psi(\theta^b) = p^a\ p_a\ \psi(\theta^b) = 0. \tag{24}$$

Both vectors, the u, which are the superpositions of differential p-forms and the $\psi(\theta^a)$, which are polynomials in θ^a's are defined in a similar way (Eqs.(1 ,14)), as we shall point out in the following subsection.

We see that either $dx^a\ \tilde{\vee}\ p_a = 0$ or $dx^a\ \tilde{\tilde{\vee}}\ p_a = 0$, similarly as either $\tilde{a}^a\ p_a = 0$ or $\tilde{\tilde{a}}^a\ p_a = 0$ can represent the Dirac-like equation.

B. Vector space of two approaches

The superpositions of p-forms on which the Dirac-Kähler equation is defined and the superpositions of polynomials in Grassmann space, on which the Dirac-like equations are defined, are

$$u = \sum_{i=0,1,2,3,5,..,d} \quad \sum_{a_1<a_2<\ldots<a_i\in\{0,1,2,3,5,..,d\}} a_{a_1 a_2 \ldots a_p} \cdot dx^{a_1} \wedge dx^{a_2} \wedge dx^{a_3} \wedge \cdots \wedge dx^{a_i},$$

$$\psi(\theta^a) = \sum_{i=0,d} \quad \sum_{a_1<a_2<\ldots<a_i\in\{0,1,2,3,5,\ldots,d\}} \alpha_{a_1 a_2 \ldots a_i} \cdot \theta^{a_1} \theta^{a_2} \cdots \theta^{a_i}. \tag{25}$$

The coefficients $\alpha_{a_1,a_2,..,a_i}$ depend on coordinates x^a in both cases and are antisymmetric tensors of the rank i with respect to indices $a_k \in \{a_1, .., a_i\}$. The vector space is in both cases 16 dimensional.

C. Dirac γ^a -like operators

Both, $dx^a\ \tilde{\vee}$ and $dx^a\ \tilde{\tilde{\vee}}$ define the algebra of the γ^a matrices and so they do both $\tilde{a}^a$ and $\tilde{\tilde{a}}^a$. One would thus be tempted to identify

$$\gamma^a_{\text{naive}} := dx^a\ \tilde{\vee}, \qquad \text{or} \qquad \gamma^a_{\text{naive}} := \tilde{a}^a. \tag{26}$$

But there is a large freedom in defining what to identify with the gamma-matrices, because except when using γ^0 as a parity operation you have an even number of gamma matrices occurring in the physical applications. Then you may multiply all the gamma matrices by some factor provided it does not disturb their algebra nor their even products. We shall comment this point in section 6.

1. Problem of statistics of gamma-matrices

This freedom might be used to solve, what seems a problem:

Having an odd Grassmann character, neither $\tilde{a}^a$ nor $\tilde{\tilde{a}}^a$ should be recognized as the Dirac γ^a operators, since they would change, when operating on polynomials of θ^a, polynomials of an odd Grassmann character to polynomials of an even Grassmann character. One would, however, expect - since Grassmann odd fields second quantize to fermions, while Grassmann even fields second quantize to bosons - that the γ^a operators do not change the Grassmann character of wave functions. One can notice, that similarly to the Grassmann case, also the two types of the Clifford products defined on p-forms, change the oddness or the evenness of the p-forms: an even p-form, $p = 2n$, is changed by either $dx^a \ \tilde{\vee}$ or $dx^a \ \tilde{\tilde{\vee}}$ to an odd q-form, with either $q = p + 1$, if dx^a is not included in a p-form, or $q = p - 1$, if dx^a is included in a p-form, while an odd p-form, $p = 2n + 1$, is changed to an even $p + 1$-form or $p - 1$-form.

2. First solution to gamma-matrix statistics problem

We shall later therefore propose that accordingly

$$\text{either} \quad \tilde{\gamma}^a := i \ dx^0 \ \tilde{\tilde{\vee}} \ dx^a \ \tilde{\vee}, \quad \text{or} \qquad \tilde{\gamma}^a = i \ \tilde{\tilde{a}}^0 \ \tilde{a}^a \tag{27}$$

are recognized as the Dirac γ^a operators operating on the space of p-forms or polynomials of θ^a's, respectively, since they both have an even Grassmann character and they both fulfill the Clifford algebra

$$\{\tilde{\gamma}^a, \tilde{\gamma}^b\} = 2\eta^{ab}. \tag{28}$$

Of course, the role of ($\tilde{}$) and ($\tilde{\tilde{}}$) can in either the Kähler case or the case of polynomials in Grassmann space, be exchanged.

Whether we define the gamma-matrices by (27) or (26) makes only a difference for an odd products of gamma-matrices, but for applications such as construction of currents $\bar{\psi}\gamma^a\psi$ or for the Lorentz generators on the spinors $-i\frac{1}{4} \ [\gamma^a, \gamma^b]$ only products of even numbers of gamma-matrices occur, except for the parity representation on the Dirac fields, where the γ^0-matrix is used alone. This γ^0-matrix has to simulate the parity reflection which is either

$$\vec{dx} \rightarrow -\vec{dx}, \quad \text{or} \quad \vec{\theta} \rightarrow -\vec{\theta}. \tag{29}$$

The "ugly" gamma-matrix identifications (27) indeed perform this operation. And as long as the physical applications are the ones just mentioned - and that should be sufficient - the choice (27) is satisfactory: Living in the Grassmann odd part of the Hilbert space, we don't move into the Grassmann even part of it.

The canonical quantization of Grassmann odd fields, that is the procedure with the Hamiltonian and the Poisson brackets, then automatically assures the anticommuting relations between the operators of the fermionic fields.

3. Solution by redefinition of oddness

The simplest solution to the problem with the evenness and oddness is to use the "naive" gamma-matrix identifications (26) and simply ignore that the even-odd-ness does not match. This is what Kähler did, we can say, in as far as he did not really identify the even-odd-ness of the p-forms with the statistics of Dirac fields. If one - along the lines of the Becher's and Joos's paper ([4]) - will make a second quantized theory based on the Kähler trick one does not proceed by insisting on taking p-forms to be fermionic only when p is odd. Becher and Joos take all the forms as fermion fields and assume then anticommuting relations for operators of fields. This simplest solution can thus be claimed to be the one applied by Kähler and used by Becher and Joos: They simply do not dream about in advance postulating that the p-forms should necessarily be taken to be boson or fermion fields depending on whether p is even or odd.

It is only when one as one of us in her model has the requirement of canonical quantization saying that the θ^a's should be Grassmann odd objects, which indeed they are, that the problem occurs.

D. Generators of Lorentz transformations

Again, we are presenting the generators of the Lorentz transformations of spinors for both approaches

$$M^{ab} = L^{ab} + \mathcal{S}^{ab}, \qquad L^{ab} = x^a p^b - x^b p^a, \tag{30}$$

differing among themselves in the definition of $\mathcal{S}^{ab}$ only, which define the generators of the Lorentz transformations in the internal space, that is in the space of p-forms or polynomials of θ^a's, respectively. While Kähler suggested the definition

$$\mathcal{S}^{ab} = dx^a \wedge dx^b, \quad \mathcal{S}^{ab} u = \frac{1}{2}((dx^a \wedge dx^b) \vee u - u \vee (dx^a \wedge dx^b)), \tag{31}$$

in the Grassmann case [2] the operator $\mathcal{S}^{ab}$ is one of the two generators defined above (Eq. (12)), that is

$$\text{either} \quad \mathcal{S}^{ab} = \tilde{S}^{ab} = -\frac{i}{4}[\tilde{a}^a, \tilde{a}^b] = -\frac{i}{4}[\tilde{\gamma}^a, \tilde{\gamma}^b], \quad \text{or} \qquad \mathcal{S}^{ab} = \tilde{\tilde{S}}^{ab} = -\frac{i}{4}[\tilde{\tilde{a}}^a, \tilde{\tilde{a}}^b]. \tag{32}$$

One further finds

$$[\tilde{S}^{ab}, \tilde{a}^c] = i(\eta^{ac}\tilde{a}^b - \eta^{bc}\tilde{a}^a), \qquad [\tilde{\tilde{S}}^{ab}, \tilde{\tilde{a}}^c] = i(\eta^{ac}\tilde{\tilde{a}}^b - \eta^{bc}\tilde{\tilde{a}}^a), \quad \text{while} \quad [\tilde{S}^{ab}, \tilde{\tilde{a}}^c] = 0 = [\tilde{\tilde{S}}^{ab}, \tilde{a}^c]. \tag{33}$$

One can also in the Kähler case define two kinds of the Lorentz generators, which operate on the internal space of p-forms, according to two kinds of the Clifford products, presented above. Following the definitions in the one of us [2] approach, one can write the $\mathcal{S}^{ab}$ for the Kähler case

$$\text{either} \;\; \tilde{\mathcal{S}}^{ab} = -\frac{i}{4}[dx^a \;\wedge + \, e^a, \; dx^b \;\wedge + \, e^b] = -\frac{i}{4}[\tilde{\gamma}^a, \tilde{\gamma}^b], \quad \text{or} \quad \tilde{\tilde{\mathcal{S}}}^{ab} = \frac{i}{4}[dx^a \;\wedge - \, e^a, \; dx^b \;\wedge - \, e^b]. \tag{34}$$

Not only are in this case the similarities between the two approaches more transparent, also the definition of the generators of the Lorentz transformations in the space of p-forms simplifies very much.

One further finds for the spinorial case

$$[M^{ab}, \tilde{\gamma}^a \;\; p_a] = 0, \quad \text{for} \quad M^{ab} = L^{ab} + \tilde{S}^{ab}, \tag{35}$$

which demonstrates that the total angular momentum for a free massless particle is conserved. The above equation is true for both approaches and the generators of the Lorentz transformations M^{ab} fulfill the Lorentz algebra in both cases.

In addition, the operators of the Lorentz transformations with the vectorial character can also be defined for both approaches in an equivalent way, that is as a sum of the two operators of the spinorial character

$$\mathcal{S}^{ab} = \tilde{S}^{ab} + \tilde{\tilde{S}}^{ab} = -i(dx^a \wedge e^b - dx^b \wedge e^a), \qquad \mathcal{S}^{ab} = \tilde{S}^{ab} + \tilde{\tilde{S}}^{ab} = \theta^a p^{\theta b} - \theta^b p^{\theta a}, \tag{36}$$

which again fulfill the Lorentz algebra. The operator $\mathcal{S}^{ab} = -i(dx^a \wedge e^b - dx^b \wedge e^a)$, if being applied on differential p-forms, transforms vectors into vectors, correspondingly $\mathcal{S}^{ab} = \theta^a p^{\theta b} - \theta^b p^{\theta a}$, if being applied to polynomial of θ^a's transforms vectors into vectors [2].

Elements of the Lorentz group can be written for both approaches, for either spinorial or vectorial kind of the generators as

$$U = e^{-\omega_{ab} M^{ab}}, \tag{37}$$

where ω_{ab} are parameters of the group. If M^{ab} are equal to either $L^{ab} + \tilde{S}^{ab}$ or $L^{ab} + \tilde{\tilde{S}}^{ab}$, the period of transformations is 4π either in the space of differential forms or in the Grassmann space, demonstrating the spinorial character of the operator. If M^{ab} is the sum of L^{ab} and $\tilde{S}^{ab} + \tilde{\tilde{S}}^{ab}$, the period of transformation is 2π, manifesting the vectorial character of the operator.

E. Hodge star product

In the way how we have defined the operators in the space of p-form, the definition of the "Hodge star" operator, defined by Kähler working in the space of p-forms and the space of θ^a polynomials, will be respectively

$$\tilde{\Gamma} = \quad i \prod_{a=0,1,2,3,5,..,d} \tilde{\gamma}^a, \tag{38}$$

with $\tilde{\gamma}^a$ equal to either $dx^0 \ \tilde{\tilde{\vee}} \ dx^a \ \tilde{\vee}$ in the Kähler case, or to $i \ \tilde{\tilde{a}}^0 \ \tilde{a}^a$ in the one of us [2] approach. For an even d the factor with double tilde ($\tilde{\tilde{}}$) can be in both cases omitted ($\tilde{\Gamma} =$ either $i \ \prod_a \ \tilde{a}^a$ or $i \ \prod_a \ dx^a \ \tilde{\vee}$, Again we could distinguish the operators $\tilde{\Gamma}$ and $\tilde{\tilde{\Gamma}}$ in both cases, according to the elements, which define the Casimir). It follows that

$$\frac{1}{2}(1 \pm \tilde{\Gamma}) \tag{39}$$

are the two operators, which when being applied on wave functions defined either on p-forms or on polynomials in Grassmann space, project out the left or right handed component, respectively.

One easily recognizes that when being applied on a vacuum state $| >$, the operator $\tilde{\Gamma}$ behaves as a "hodge star" product, since one finds for d even

$$-i\tilde{\Gamma} \ | \ >= dx^0 \wedge dx^1 \wedge .... \wedge dx^d, \qquad -i\tilde{\Gamma} \ | \ >= \theta^0\theta^1 ...\theta^d. \tag{40}$$

F. Scalar product

In the Mankoč's approach [2] the scalar product between the two functions $\psi^{(1)}(\theta^a)$ and $\psi^{(2)}(\theta^a)$ is defined as follows

$$< \psi^{(1)}|\psi^{(2)} >= \int d^d\theta \ \ (\omega \ \ \psi^{(1)}(\theta^a))^* \ \ \psi^{(2)}(\theta^a). \tag{41}$$

Here ω is the weight function

$$\omega = \prod_{i=0,1,2,3,5,..,d} (\theta^i + \overrightarrow{\partial^i}) \tag{42}$$

which operates on the first function, $\psi^{(1)}$, only while

$$\int d\theta^a = 0, \quad \int d\theta^a\theta^a = 1, \quad a = 0,1,2,3,5,..,d, \tag{43}$$

no summation over repeated index is meant and

$$\int d^d\theta \ \theta^0\theta^1\theta^2\theta^3\theta^5 ...\theta^d = 1, \quad d^d\theta = d\theta^d ...d\theta^5 d\theta^3 d\theta^2 d\theta^1 d\theta^0. \tag{44}$$

Since $\theta^{a*} = \theta^a$, * means the complex conjugation and $^+$ means the hermitian conjugation, then with respect to the above defined scalar product the operator $\theta^{a+} = -\eta^{aa}\overrightarrow{\partial}^a$, $\overrightarrow{\partial}^{a+} = -\theta^a\eta^{aa}$, while $\tilde{a}^{a+} = -\eta^{aa}\tilde{a}^a$ and $\tilde{\tilde{a}}^{a+} = -\eta^{aa}\tilde{\tilde{a}}^a$. Again no summation over repeated index is performed. Accordingly the operators of the Lorentz transformations of spinorial character are self-adjoint (if $a \neq 0$ and $b \neq 0$) or anti-self-adjoint (if a = 0 or b = 0).

According to Eqs.(41, 25) the scalar product of two functions $\psi^{(1)}(\theta^a)$ and $\psi^{(2)}(\theta^a)$) can be written as follows

$$< \psi^{(1)}|\psi^{(2)} > = \sum_{0,d} \sum_{\alpha_1<\alpha_2<..<\alpha_i} \alpha^{(1)*}_{\alpha_1\alpha_2..\alpha_i} \cdot \alpha^{(2)}_{\alpha_1\alpha_2..\alpha_i} \tag{45}$$

in complete analogy with the usual definition of scalar products in ordinary space. Kähler [1] defined in Eq. (15.11) and on page 519 the scalar product of two superpositions of p-forms $u^{(1)}$ and $u^{(2)}$ as follows

$$< u^{(1)}|u^{(2)} > = \sum_{0,d} \sum_{\alpha_1<\alpha_2<..<\alpha_i} \alpha^{(1)}_{\alpha_1\alpha_2..\alpha_i} \cdot \alpha^{(2)}_{\alpha_1\alpha_2..\alpha_i}, \tag{46}$$

which (for real coefficients $\alpha^{(k)}_{\alpha_1\alpha_2..\alpha_i}$, $k = 1, 2$) agrees with Eq.(45).

G. Four families of solutions in Kähler or in approach in Grassmann space

We shall limit ourselves in $d = 4$ and in spinorial case (as indeed Kähler did). The representations for higher d, analyzed with respect to the groups $SO(1,3)\times SU(3)\times SU(2)\times U(1)$, and some other groups, in Grassmann space are (only for Grassmann even part of the space belonging to the groups which does not include $SO(1,3)$) presented in ref. [2].

In the case of $d = 4$ one may arrange the space of 2^d vectors into four times two Weyl spinors, one left ($< \tilde{\Gamma}^{(4)} >= -1$) and one right ($< \tilde{\Gamma}^{(4)} >= 1$) handed. We are presenting this vectors, which are at the same time the eigenvectors of $\tilde{S}^{12}$ and $\tilde{S}^{03}$, as polynomials of θ^m's, $m \in (0, 1, 2, 3)$. The two Weyl vectors are connected by the operation of $\tilde{\gamma}^m$ operators (Eq.(27)).

Taking into account that $\tilde{a}^a| \;>= \theta^a$, where $| \;>$ is the vacuum state (Eq.(22)), we find

a	i	${}^a\psi_i(\theta)$	$\tilde{S}^{12}$	$\tilde{S}^{03}$	$\tilde{\Gamma}^{(4)}$
1	1	$\frac{1}{2}(\tilde{a}^1 - i\tilde{a}^2)(\tilde{a}^0 - \tilde{a}^3)$	$-\frac{1}{2}$	$\frac{i}{2}$	-1
1	2	$-\frac{1}{2}(1 + i\tilde{a}^1\tilde{a}^2)(1 - \tilde{a}^0\tilde{a}^3)$	$\frac{1}{2}$	$-\frac{i}{2}$	-1
2	1	$\frac{1}{2}(\tilde{a}^1 - i\tilde{a}^2)(\tilde{a}^0 + \tilde{a}^3)$	$-\frac{1}{2}$	$-\frac{i}{2}$	1
2	2	$-\frac{1}{2}(1 + i\tilde{a}^1\tilde{a}^2)(1 + \tilde{a}^0\tilde{a}^3)$	$\frac{1}{2}$	$\frac{i}{2}$	1
3	1	$\frac{1}{2}(\tilde{a}^1 - i\tilde{a}^2)(1 - \tilde{a}^0\tilde{a}^3)$	$-\frac{1}{2}$	$-\frac{i}{2}$	1
3	2	$-\frac{1}{2}(1 + i\tilde{a}^1\tilde{a}^2)(\tilde{a}^0 - \tilde{a}^3)$	$\frac{1}{2}$	$\frac{i}{2}$	1
4	1	$\frac{1}{2}(\tilde{a}^1 - i\tilde{a}^2)(1 + \tilde{a}^0\tilde{a}^3)$	$-\frac{1}{2}$	$\frac{i}{2}$	-1
4	2	$-\frac{1}{2}(1 + i\tilde{a}^1\tilde{a}^2)(\tilde{a}^0 + \tilde{a}^3)$	$\frac{1}{2}$	$-\frac{i}{2}$	-1
5	1	$\frac{1}{2}(1 - i\tilde{a}^1\tilde{a}^2)(\tilde{a}^0 - \tilde{a}^3)$	$-\frac{1}{2}$	$\frac{i}{2}$	-1
5	2	$-\frac{1}{2}(\tilde{a}^1 + i\tilde{a}^2)(1 - \tilde{a}^0\tilde{a}^3)$	$\frac{1}{2}$	$-\frac{i}{2}$	-1
6	1	$\frac{1}{2}(1 - i\tilde{a}^1\tilde{a}^2)(\tilde{a}^0 + \tilde{a}^3)$	$-\frac{1}{2}$	$-\frac{i}{2}$	1
6	2	$-\frac{1}{2}(\tilde{a}^1 + i\tilde{a}^2)(1 + \tilde{a}^0\tilde{a}^3)$	$\frac{1}{2}$	$\frac{i}{2}$	1
7	1	$\frac{1}{2}(1 - i\tilde{a}^1\tilde{a}^2)(1 - \tilde{a}^0\tilde{a}^3)$	$-\frac{1}{2}$	$-\frac{i}{2}$	1
7	2	$-\frac{1}{2}(\tilde{a}^1 + i\tilde{a}^2)(\tilde{a}^0 - \tilde{a}^3)$	$\frac{1}{2}$	$\frac{i}{2}$	1
8	1	$\frac{1}{2}(1 - i\tilde{a}^1\tilde{a}^2)(1 + \tilde{a}^0\tilde{a}^3)$	$-\frac{1}{2}$	$\frac{i}{2}$	-1
8	2	$-\frac{1}{2}(\tilde{a}^1 + i\tilde{a}^2)(\tilde{a}^0 + \tilde{a}^3)$	$\frac{1}{2}$	$-\frac{i}{2}$	-1

Table I.-Irreducible representations of the two subgroups $SU(2) \times SU(2)$ of the group $SO(1,3)$ as defined by the generators of the spinorial character $\tilde{S}^{12}, \tilde{S}^{03}$ and the operator of handedness $\tilde{\Gamma}^{(4)}$. The four copies of the Weyl bispinors have either an odd or an even Grassmann character. The generators $\tilde{\tilde{S}}^{mn}$, $m, n \in (0,1,2,3)$, transform the two copies of the same Grassmann character one into another.

Similarly also the Kähler spinors can be arranged into four copies. We find them by only replacing in Table I. $\tilde{a}^a$ by $dx^a \tilde{\vee}$. We shall discuss this point also in the next section.

H. Vector representations of group $SO(1,3)$

Analysing the irreducible representations of the group $SO(1,3)$ in analogy with the spinor case but taking into account the generator of the Lorentz transformations of the vector type (Eqs.(10, 36)) one finds [2] for d = 4 two scalars (a scalar and a pseudo scalar), two three vectors (in the $SU(2) \times SU(2)$ representation of $SO(1,3)$ usually denoted by $(1,0)$ and $(0,1)$ representation, respectively, with $< \Gamma^{(4)} >$ equal to ± 1) and two four vectors (in the $SU(2) \times SU(2)$ representation of $SO(1,3)$ both denoted by $(\frac{1}{2}, \frac{1}{2})$ and differing among themselves in the Grassmann character) all of which are eigenvectors of $S^{(4)2} = \frac{1}{2} S^{ab} S_{ab}$, $\Gamma^{(4)} = i \frac{(-2i)^2}{4!} \epsilon_{abcd} \mathcal{S}^{ab} \mathcal{S}^{cd}$, $\mathcal{S}^{12}$ and S^{03}. Using Eq.(36) and analyzing the vector space of p-forms in an analogous way as the space of the Grassmann polynomials, one finds the same kind of representations also in the Kähler case.

Both, in the spinor case and in the vector case one has 2^4 dimensional vector space.

V. APPEARANCE OF SPINORS

One may quite strongly wonder about how it is at all possible that there appear the Dirac equation - usually being an equation for a *spinor* field - out of models with only scalar, vector and tensor objects! Immediately one would say that it is of course sheer impossible to construct spinors such as Dirac fields out of the integer spin objects such as the differential one forms and their external products or of the θ^a's and their products $\theta^a \theta^b \cdots \theta^c$.

Let us say immediately that it also only can be done by a "cheat". This "cheat" really consists in *replacing* the Lorentz transformation concept (including rotation concept) by exchanging the Lorentz generators $\mathcal{S}^{ab}$ by the $\tilde{S}^{ab}$ say (or the $\tilde{\tilde{S}}^{ab}$ if we choose them instead), see equations (32, 34). This indeed means that one of the two kinds of operators fulfilling the Clifford algebra and anticommuting with the other kind - it has been made a choice of $dx^a \tilde{\tilde{\nabla}}$ in the Kähler case and $\tilde{\tilde{a}}^a$ in the approach of one of us - are put to zero in the operators of

Lorentz transformations; as well as in all the operators representing the physical quantities. The use of $dx^0 \tilde{\tilde{\nabla}}$ or $\tilde{\tilde{a}}^0$ in the operator $\tilde{\gamma}^0$ is the exception only used to simulate the Grassmann even parity operation $\vec{dx}^a \rightarrow -\vec{dx}^a$ and $\vec{\theta} \rightarrow -\vec{\theta}$, respectively.

The assumption, which we call "cheat" was made in the Kähler approach [1] and in its lattice version [4], as well as in the approach of one of us [2].

In ref. [2] the $\tilde{\tilde{a}}^a$'s are argued away on the ground that with a certain single particle action

$$I = \int d\tau d\xi \quad L(x, \theta, \tau, \xi), \tag{47}$$

(with x^a being ordinary coordinates, θ^a Grassmann coordinates, $a \in \{0, 1, .., d\}$, τ an ordinary time parameter and ξ an anticommuting time parameter and assuming $X^a = x^a + \epsilon \xi \theta^a$ and making a choice for ϵ) with which we shall not go in details here, the $\tilde{\tilde{a}}^a$ appear to be zero as one of the constraints. This constraint has been used to put $\tilde{\tilde{a}}^a$'s equal zero in the further calculations in this reference and it was used as argument for dropping the $\tilde{\tilde{S}}^{ab}$-part of the Lorentz generator $\mathcal{S}^{ab}$. [2] Let us stress that once the $\tilde{\tilde{a}}^a$ or $dx^a \tilde{\tilde{\nabla}}$ is dropped and accordingly *the* $\tilde{\tilde{S}}^{ab}$ *is dropped* - for whatever reason - *one is no longer asking for the representation under the same Lorentz transformations (including rotations) and one shall not expect to find say integer spin even if the field considered is purely constructed from scalars, vectors and tensors* !

Let us point out further that what happens is that as well the θ^a polynomials of one of us as the linear combinations of p-forms in the Kähler approach can be formulated as *double spinors*, i.e. expressions with two (Dirac) spinor indices, α and β say, and that the "cheat" consists in *dropping* from the concept of Lorentz transformations the transformations in *one of these indices.* In fact we can rewrite:

---

[2] We point out in ref.([2]) that this constraint, when once being taken into account by putting it zero in all the physical operators, was not further treated as a weak equation. Furthermore such a weak equation - $\tilde{\tilde{a}}^a$ is an odd Grassmann operator - can not at all be fulfilled.

For the even d case one has

$$\text{either} \quad \psi_{\alpha\beta}(\{\theta^a\}) := \sum_{i=0}^{d} (\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_i})_{\alpha\beta}\, \theta^{a_1}\theta^{a_2}\cdots\theta^{a_i},$$

$$\text{or} \quad \psi_{\alpha\beta}(\{dx^a\}) := \sum_{i=0}^{d} (\gamma_{a_1}\gamma_{a_2}\cdots\gamma_{a_i})_{\alpha\beta} dx^{a_1}\wedge dx^{a_2}\wedge\cdots dx^{a_i}\wedge, \tag{48}$$

while for the odd d case one has:

$$\text{either} \quad \psi_{\alpha\beta\Gamma}(\{\theta^a\}) := \sum_{i=0}^{d} (\gamma_{(\Gamma)a_1}\gamma_{(\Gamma)a_2}\cdots\gamma_{(\Gamma)a_i})_{\alpha\beta}\, \theta^{a_1}\theta^{a_2}\cdots\theta^{a_i},$$

$$\text{or} \quad \psi_{\alpha\beta\Gamma}(\{\theta^a\}) := \sum_{i=0}^{d} (\gamma_{(\Gamma)a_1}\gamma_{(\Gamma)a_2}\cdots\gamma_{(\Gamma)a_i})_{\alpha\beta}\, dx^{a_1}\ \wedge dx^{a_2}\ \wedge\cdots dx^{a_i}\ \wedge, \tag{49}$$

with the convention $a_1 < a_2 < a_3 < ... < a_i$. Here the sums run over the number i of factors in the products of $dx^a\wedge$ or θ^a coordinates, a number, which is the same as the number of gamma-matrix factors and it should be remarked that we include the possibility $i = 0$ which means no factors and is taken to mean that the product of zero $dx^a\wedge$ or θ^a-factors is unity and the product of zero gamma matrices is the unit matrix. The indices α, β are the spinor indices and taking the product of gamma-matrices conceived of as matrices the symbol $(...)_{\alpha\beta}$ stands for an element in the α-th row and in the β-th column. There is an understood Einstein convention summation over the contracted vector indices a_k, k=1,2,...,i. The gamma-matrices are in the even dimension case $2^{d/2}$ by $2^{d/2}$ matrices and in the odd dimension case $2^{(d-1)/2}$ by $2^{(d-1)/2}$ matrices. In the odd case we have worked with two (slightly) different gamma-matrix choices - and thus have written the gamma-matrices as depending on the sign Γ as $\gamma(\Gamma)_{a_k}$ - namely gamma matrix choices obeying

$$\Gamma = \gamma_1\gamma_2\cdots\gamma_d. \tag{50}$$

The γ_a matrices should be constructed of course so that they obey the Clifford algebra

$$\{\gamma_a, \gamma_b\} = 2\eta^{ab} \tag{51}$$

and we could e.g. choose

$$
\begin{array}{lllllllll}
\gamma_1 & := & i\sigma_2^1 & \times & \sigma_3^2 & \times & \sigma_3^3 & \times \cdots \times & \sigma_3^n \\
\gamma_2 & := & -i\sigma_1^1 & \times & \sigma_3^2 & \times & \sigma_3^3 & \times \cdots \times & \sigma_3^n \\
\gamma_3 & := & iI^1 & \times & \sigma_2^2 & \times & \sigma_3^3 & \times \cdots \times & \sigma_3^n \\
\gamma_4 & := & iI^1 & \times & (-)\sigma_1^2 & \times & \sigma_3^3 & \times \cdots \times & \sigma_3^n \\
\gamma_5 & := & iI^1 & \times & I^2 & \times & \sigma_2^3 & \times \cdots \times & \sigma_3^n \\
\vdots & & \vdots & \vdots & \vdots & \ddots & \vdots & & \\
\gamma_{2n-1} & := & iI^1 & \times & I^2 & \times & I^3 & \times \cdots \times & \sigma_2^n \\
\gamma_{2n} & := & iI^1 & \times & I^2 & \times & I^3 & \times \cdots \times & (-)\sigma_1^n
\end{array}
$$

for an even dimension $d = 2n$, while for an odd dimension $d = 2n + 1$ the gamma matrix γ^{2n+1} has to be included

$$\gamma_{2n+1} := i\Gamma\sigma_3^1 \times \sigma_3^2 \times \sigma_3^3 \times \cdots \times \sigma_3^n,$$

with $\Gamma = \prod_a^{2n+1} \gamma_a$. (see e.g. ([7])). The above metric is supposed to be Euclidean. For the Minkowski metric $\gamma_1 \to -i\gamma_1$ has to be taken, if the index 1 is recognized as the "time" index. We shall make use of the Minkowski metric, counting the γ^a from $0, 1, 2, 3, 5, ..d$, and assuming the metric $\eta^{ab} = diag(1, -1, -1, ..., -1)$.

In this notation we can see that for fixed values of the index β we obtain one of the four bispinors in Table I. conceived of as a spinor in the index α and with the understanding that the $\tilde{a}^a$ in the table lead to the corresponding θ^a, when acting on the vacuum state. The equivalent table for the Kähler approach follows by replacing θ^a by $dx^a\wedge$.

It is our main point to show that the action by the operators $dx^a \vee$ or $\tilde{a}^a$ and $\tilde{\tilde{\vee}}$ or $\tilde{\tilde{a}}^a$ in the representation based on the basis $\psi_{\alpha\beta}(\{dx^a\})$ or $\psi_{\alpha\beta}(\{\theta^a\})$ with $\alpha, \beta \in \{1, 2, ..., \}$, $\left\{ \begin{array}{ccc} 2^{(d-1)/2} & \text{for} & d \quad \text{odd} \\ 2^{d/2} & \text{for} & d \quad \text{even} \end{array} \right\}$ transforms the index α and β, respectively, of the basis $\psi_{\alpha\beta}(\{dx^a\})$ or equivalently $\psi_{\alpha\beta}(\{\theta^a\})$ as follows:

$$\text{either} \quad dx^a \,\tilde{\vee}\, \psi_{\alpha\beta(\Gamma)}(\{dx^a\}) \;\propto\; \gamma^a_{\alpha\gamma}\; \psi_{\gamma\beta(\Gamma)}(\{dx^a\}),$$

$$\text{corresponding to} \quad \tilde{a}^a\; \psi_{\alpha\beta(\Gamma)}(\{\theta^a\}) \;\propto\; \gamma^a_{\alpha\gamma}\; \psi_{\gamma\beta(\Gamma)}(\{\theta^a\}),$$

$$\text{or} \quad dx^a\ \tilde{\tilde{\nabla}}\ \psi_{\alpha\beta(\Gamma)}(\{dx^a\})\ \propto\ \psi_{\alpha\gamma(-\Gamma)}(\{dx^a\})\ \gamma^a_{\gamma\beta},$$

$$\text{corresponding to} \quad \tilde{\tilde{a}}^a\ \psi_{\alpha\beta(\Gamma)}(\{\theta^a\})\ \propto\ \psi_{\alpha\gamma(-\Gamma)}(\{\theta^a\})\ \gamma^a_{\gamma\beta}, \tag{52}$$

which demonstrates the similarities between the spinors of the one of us approach and the Kähler approach: The operators $dx^a\ \tilde{\nabla}$ and $\tilde{a}^a$ transform the left index of the basis $\psi_{\alpha\beta(\Gamma)}(\{dx^a\})$, or correspondingly of the basis $\psi_{\alpha\beta(\Gamma)}(\{\theta^a\})$, while keeping the right index fixed and the operators $dx^a\tilde{\tilde{\nabla}}$ and $\tilde{\tilde{a}}^a$ transform the right index of the basis $\psi_{\alpha\beta(\Gamma)}(\{dx^a\})$, or correspondingly of the basis $\psi_{\alpha\beta(\Gamma)}(\{\theta^a\})$ and keep the left index fixed. Under the action of either $dx^a\tilde{\nabla}$ and $\tilde{a}^a$ or $dx^a\tilde{\tilde{\nabla}}$ and $\tilde{\tilde{a}}^a$ the basic functions transform as spinors. The index in parentheses (Γ) is defined for only odd d. We can count that the number of spinors is 2^d either in the Mankoč's approach or in the Kähler's approach; the d dimensional Grassmann space or the space of p-forms has 2^d basic functions.

We shall prove the above formulas for action of the $\tilde{a}^a$ and $\tilde{\tilde{a}}^a$. The proof is also valid for the Kähler case if $\tilde{a}^a$ is replaced by $dx^a\ \tilde{\nabla}$ and $\tilde{\tilde{a}}^a$ by $dx^a\tilde{\tilde{\nabla}}$.

A. Proof of our formula for action of $\tilde{a}^a$ and $\tilde{\tilde{a}}^a$

Let us first introduce the notation

$$\gamma^A :=\ \gamma^a\gamma^b\cdots\gamma^c, \qquad \gamma^{\overline{A}} :=\ \gamma^c\gamma^b\cdots\gamma^a, \tag{53}$$

with $a < b < \cdots < c \in A$. We recognize that

$$\text{Trace}\ (\gamma_A\gamma^{\overline{B}}) = \text{Trace(I)}\ \delta_A{}^B, \quad \sum_A(\gamma_A)_{\alpha\beta}(\gamma^{\overline{A}})_{\gamma\delta} = \text{Trace(I)}\ \delta_{\alpha\gamma}\ \delta_{\beta\delta} \tag{54}$$

and

$$\sum_i(\gamma_{A_i})_{\alpha\beta}(\gamma^c\gamma^{\overline{A}_i})_{\gamma\delta} = \text{Trace(I)}\ (\gamma^c)_{\alpha\delta}\delta_{\beta\gamma}, \quad \sum_i(\gamma_{A_i})_{\alpha\beta}(\gamma^{\overline{A}_i}(-1)^i\gamma^c)_{\gamma\delta} = \text{Trace(I)}\ (\gamma^c)_{\gamma\beta}\delta_{\delta\alpha}. \tag{55}$$

Using the first equation we find

$$\theta^A = \frac{1}{\mathrm{Trace(I)}}\ (\gamma^{\overline{A}})_{\alpha\beta}\ \psi_{\beta\alpha(\Gamma)}(\{\theta^a\}). \tag{56}$$

The index (Γ) has the meaning for only an odd d. That is why we put it in parenthesis. We may accordingly write

$$\psi_{\alpha\beta(\Gamma)}(\{\theta^a\}) := \sum_{i=0} \frac{1}{\mathrm{Trace(I)}}\ (\gamma_{A_i})_{\alpha\beta}(\gamma^{\overline{A}_i})_{\gamma\delta}\psi_{\delta\gamma(\Gamma)}(\{\theta^a\}), \tag{57}$$

with $a_1 < a_2, \cdots, < a_i \in A_i$ in ascending order and with $\overline{A}_i$ in descending order.

Then we find, taking into account that $\tilde{a}^a|\ >= \theta^a$, $\tilde{\tilde{a}}^a|\ >= -i\theta^a$, where $|\ >$ is a vacuum state and Eq.(7)

$$\tilde{a}^c\ \psi_{\alpha\beta(\Gamma)}(\{\theta^a\}) :=\ \sum_i (\gamma_{A_i})_{\alpha\beta}\tilde{a}^c\theta^{A_i} = \sum_i (\gamma_{A_i})_{\alpha\beta}\tilde{a}^c\tilde{a}^{A_i}|\ >=$$

$$\sum_i \frac{1}{\mathrm{Trace(I)}}\ (\gamma_{A_i})_{\alpha\beta}(\gamma^c\gamma^{\overline{A}_i})_{\gamma\delta}\psi_{\delta\gamma(\Gamma)}(\{\theta^a\}).$$

Using the above relations we further find

$$\tilde{a}^c\ \psi_{\alpha\beta(\Gamma)}(\{\theta^a\}) := (-1)^{\tilde{f}(d,c)}(\gamma^c)_{\alpha\gamma}\psi_{\gamma\beta(\Gamma)}(\{\theta^a\}), \tag{58}$$

where $(-1)^{\tilde{f}(d,c)}$ is ± 1, which depends on the operator $\tilde{a}^c$ and the dimension of the space.

We find in a similar way

$$\tilde{\tilde{a}}^c\ \psi_{\alpha\beta(\Gamma)}(\{\theta^a\}) :=\ \sum_i (\gamma_{A_i})_{\alpha\beta}\tilde{\tilde{a}}^c\tilde{a}^{A_i}|\ > =\ \sum_i (-1)^i(\gamma_{A_i})_{\alpha\beta}\tilde{a}^{A_i}\tilde{\tilde{a}}^c|\ > =$$

$$\sum_i (-1)^i(\gamma_{A_i})_{\alpha\beta}\tilde{a}^{A_i}\tilde{\tilde{a}}^c|\ > =\ \sum_i \frac{(-1)^i}{\mathrm{Trace(I)}}\ (\gamma_{A_i})_{\alpha\beta}(\gamma^{\overline{A}_i}\gamma^c)_{\gamma\delta}\ \psi_{\delta\gamma(\Gamma)}(\{\theta^a\}),$$

which finally gives

$$\tilde{\tilde{a}}^c\ \psi_{\alpha\beta(\Gamma)}(\{\theta^a\}) :=\ (-1)^{\tilde{\tilde{f}}(d,c)}\psi_{\alpha\gamma(-\Gamma)}(\{\theta^a\})(\gamma^c)_{\gamma\beta}, \tag{59}$$

with the signum $(-1)^{\tilde{\tilde{f}}(d,c)}$ depending on the dimension of the space and the operator $\tilde{\tilde{a}}^c$.

We have therefore proven the two equations which determine the action of the operators $\tilde{a}^a$ and $\tilde{\tilde{a}}^a$ on the basic function $\psi_{\alpha\gamma(-\Gamma)}(\{\theta^a\})$.

VI. GETTING AN EVEN GAMMA MATRIX

According to the Eqs.(58, 59) it is obvious that the γ^a **matrices, entering into the Dirac-Kähler approach or one of us approach for spinors, have an odd Grassmann character** since both, $dx^a\ \tilde{\nabla}$ and $\tilde{a}^a$ as well as $dx^a\ \tilde{\tilde{\nabla}}$ and $\tilde{\tilde{a}}^a$, have an odd Grassmann character. They therefore transform a Grassmann odd basic function into a Grassmann even basic function changing fermion fields into boson fields. It is clear that such γ^a matrices are not appropriate to enter into the equations of motion and Lagrangeans for spinors.

There are several possibilities to avoid this trouble [2]. One of them was presented in section 4. If working with $dx^a\ \tilde{\nabla}$ or $\tilde{a}^a$ alone, putting $dx^a\ \tilde{\tilde{\nabla}}$ or $\tilde{\tilde{a}}^a$ in the Hamiltonian, Lagrangean and all the operators equal to zero, the $\tilde{\gamma}^a$ matrices of an even Grassmann character can be defined as proposed in Eq.(27) $\tilde{\gamma}^a := idx^0\ \tilde{\tilde{\nabla}}\ dx^a\tilde{\nabla}$ or $\tilde{\gamma}^a := i\tilde{\tilde{a}}^0\tilde{a}^a$, which fulfill the Clifford algebra $\{\tilde{\gamma}^a, \tilde{\gamma}^b\} = 2\eta^{ab}$, while as we already have said $\tilde{\mathcal{S}}^{ab} = -\frac{i}{4}[\tilde{\gamma}^a, \tilde{\gamma}^b]$. We then have

$$\tilde{\gamma}^a \psi_{\alpha\beta(\Gamma)}(\{\}) = \gamma^a_{\alpha\gamma}\psi_{\gamma\delta(-\Gamma)}(\{\theta^a\})\gamma^0_{\delta\beta}. \tag{60}$$

One can check that $\tilde{\gamma}^a$ have all the properties of the Dirac γ^a matrices.

(Exchanging $dx^a\ \tilde{\nabla}$ or $\tilde{a}^a$ by $dx^a\ \tilde{\tilde{\nabla}}$ or $\tilde{\tilde{a}}^a$, respectively, the gamma-matrices defined as $\tilde{\tilde{\gamma}}^a :=$ either $i\ dx^0\ \tilde{\nabla}\ dx^a\ \tilde{\tilde{\nabla}}$ or $i\ \tilde{a}^0\tilde{\tilde{a}}^a$ have again all the properties of the Dirac γ^a matrices.)

VII. DISCRETE SYMMETRIES

We shall comment in this section the discrete symmetries of spinors and vectors in the Hilbert space spanned over either the Grassmann coordinate space or the space of differential forms from the point of view of the one particle states of massless Dirac (that is the Weyl) particles.

In oder to define the discrete symmetries of the Lorentz group we introduce the space inversion P and the time inversion T operator in ordinary space-time in the usual way. We shall assume the case $d = 4$.

$$P x^a P^{-1} = x_a, \quad T x^a T^{-1} = -x_a \tag{61}$$

with the metric $\eta^{ab}, x^a = \eta^{ab} x_b$ already defined in section 2. Since one wants the time reversal operator to leave p^0, that is the zero component of the ordinary space-time momentum operator (p^a), unchanged ($p^0 \to p^0$), while the space component $\vec{p}$ should change sign ($\vec{p} \to -\vec{p}$), one also requires

$$T i T^{-1} = -i, \quad \text{leading} \quad \text{to} \quad T p^a T^{-1} = p_a. \tag{62}$$

We first shall treat spinors. Having the representation of spinors expressed in terms of polynomials of θ^a's in Table I., which also represents the corresponding superpositions of p-forms if θ^a is accordingly substituted by $dx^a \wedge$, we expect each of the four copies of Dirac massless spinors to transform under discrete symmetries of the Lorentz transformations in an usual way.

The parity operator P should transform left handed spinors with $< \Gamma^{(4)} >= -1$ to right handed spinors with $< \Gamma^{(4)} >= 1$, without changing the spin of the spinors. This is what $\tilde{\gamma}^0$ (Eq.(27)) does for any of four copies of the Dirac massless spinors, which are the Weyl bispinors of Table I., separately.

The time reversal operator T should transform left handed spinors with $< \Gamma^{(4)} >= -1$ and spin $\frac{1}{2}$ to left handed spinors with $< \Gamma^{(4)} >= -1$ and spin $-\frac{1}{2}$, what the operator

$$T = i \tau_{int} \cdot \tau_x K, \quad \tau_{int} = \tilde{\gamma}^1 \tilde{\gamma}^3, \quad \tau_x = diag(-1,1,1,1), \quad \text{and} \quad K i K^{-1} = -i \tag{63}$$

does when applied to any of four copies of the Dirac spinors of Table I. This transformation involves only members of the same copy of the Dirac bispinor. The operators $\tilde{\gamma}^a$ which are defined in Eq.(27), have due to the appropriate choice of phases of the spinors of Table I, the usual chiral matrix representation (for both approaches - the Kähler and the one of us).

One would, however, expect that the time and the space reversal operators should work in both spaces - that is in the ordinary space-time and in the space of either Grassmann polynomials or in the space of p-forms - in an equivalent way

$$Px^a P^{-1} = x_a, \quad Tx^a T^{-1} = -x_a,$$

$$P\theta^a P^{-1} = \theta_a, \quad \text{or correspondingly} \quad Pdx^a \wedge P^{-1} = dx_a \wedge,$$

$$\text{and} \quad T\theta^a T^{-1} = -\theta_a, \quad \text{or correspondingly} \quad Tdx^a \wedge T^{-1} = -dx_a \wedge,$$

$$TiT^{-1} = -i, \quad \text{leading to} \quad Tp^a T^{-1} = p_a \tag{64}$$

and changing equivalently the momenta conjugate to coordinates in either the one of us or the Kähler approach.

Applying the transformation P of Eq.(64) on any of four copies of the Dirac bispinors of Table I., one obtains the same result as in the above, that is the standard definition of the space-reversal operation. Applying the transformation T of Eq.(64) on, let us say, the first spinor of the first copy of the Dirac bispinors of Table I. (that is on ${}^1\psi(\theta)_1$), one obtains the last spinor of the last copy (that is ${}^8\psi(\theta)_2$). The left handed spinor with spin $-\frac{1}{2}$ transformed to the left handed spinor of spin $\frac{1}{2}$, just as it did under the usual time-reversal transformation, except that in this case the copy of spinors has been changed.

One can write down the matrix representation for this second kind of the time-reversal transformation. If we choose for the basis the first copy of bispinors of Table I. and the fourth copy of bispinors of Table I, we obtain the matrix:

$$T' = \begin{pmatrix} 0 & 0 & i\sigma_2 \exp i\varphi K_\varphi & 0 \\ 0 & 0 & 0 & i\sigma_2 \exp i\varphi K_\varphi \\ -i\sigma_2 \exp -i\varphi K_\varphi & 0 & 0 & 0 \\ 0 & -i\sigma_2 \exp -i\varphi K_\varphi & 0 & 0 \end{pmatrix}, \tag{65}$$

where the $\exp i\varphi = 1$, due to the choice of the phase of the spinors in Table I., and K_φ means that the complex conjugation has to be performed on the phase coefficients only, which in our case have again been chosen to be one.

This is the time-reversal operation discussed by Weinberg [8] in Appendix C of the Weinberg's book[3].

When vectors and scalars are treated in the similar way for either of the two approaches, it turns out that the time-reversal operators do not transform one copy into another one.

We payed attention in this section on only spin degrees of freedom. The complex conjugation affects, of course, the higher part of the internal space as well, affecting the charges of spinors, vectors and tensors, if one thinks of the extension [2] as discussed in section 9.

VIII. UNAVOIDABILITY OF FAMILIES

We want to look at the funny shift of the spin compared to the a priori spin for a field by shifting a priori generators $M^{ab} = L^{ab} + \mathcal{S}^{ab}$ out by anther set $\tilde{M}^{ab} = L^{ab} + \tilde{S}^{ab}$ as a general nice idea. A prerequisite for that working is that the difference between the two proposals for Lorentz generators

$$\tilde{\tilde{M}}^{ab} := M^{ab} - \tilde{M}^{ab} \tag{66}$$

is also a conserved set of quantities. In the notation above of course we find

$$\tilde{\tilde{M}}^{ab} = \tilde{\tilde{S}}^{ab}. \tag{67}$$

Assuming that there is indeed such two Lorentz generator symmetries in a model, we can ask for the representation under both for a given set of fields, and we can even ask for representation under the difference algebra $\tilde{\tilde{M}}^{ab}$. In order to shift in going from M^{ab}

[3]The two kinds of the time reversal operators has already been discussed in refs.([2]). The appearance of the second kind of the time reversal operator in the Weinberg's book as well as in the Wigner's book [9] was pointed out [10] to the authors in the Workshop "What comes beyond the Standard model" at Bled 1999, when it was suggested that the second kind could generate states, which may be used to describe the sterile neutrinos.

to $\tilde{M}^{ab}$ from integer spin to half integer spin the representation for the fields in question must at least be spin 1/2 for $\tilde{\tilde{M}}^{ab}$. Actually in the cases we discussed the $\tilde{\tilde{M}}^{ab}$ were in the Dirac spinor representation. But that means that the representation of the fields which shift representation going from M^{ab} to $\tilde{M}^{ab}$ have to belong under $\tilde{\tilde{M}}^{ab}$ to at least a spin 1/2 which means at least the Weyl spin representation of the Lorentz group, and that has $2^{(d/2-1)}$ dimensions. But that means then that a given representation of the final $\tilde{M}^{ab}$ Lorentz group always must occur in at least $2^{(d/2-1)}$ families.

IX. GENERALIZATION TO EXTRA DIMENSIONS

We have discussed the connection between the Grassmann θ^a formulation and the Kähler formalism for general dimension d and thus we could apply it simply in the d=4 case, or we could use it in extended models with extra dimensions. One should note that the connection between the spinor and the forms is such that for each extra two dimensions the number of components of a Dirac-spinor goes up by a factor 2, and at the same time the number of families also doubles. This agrees with that adding one extra θ^a doubles the number of terms in the θ^a polynomials and thus adding two would make this number four times as big.

Let us now study the application of the extra degrees of freedom which consists in supposing the Kähler degrees of freedom or equivalently the Grassmann θ^a's we discussed to the case where the d dimensional space is used in a Kaluza-Klein type model. That is to say we here look at a Kaluza-Klein model extended with θ^a's or the forms, much more rich than usual Kaluza-Klein. It has long been suggested [2] that special kinds of rotations of the spins especially in the extra $(d-4)$ dimensions manifest themselves as generators for charges observable at the end for the four dimensional particles. Since both the extra dimension spin degrees of freedom and the ordinary spin degrees of freedom originate from the θ^a's or the forms we have a unification of these internal degrees of freedom. We can say then that the generators rotating these degrees of freedom, namely the just mentioned charges acting as hinger dimensional spins (at high energy) and the 4-dimensional spin, are unified.

Such rotations of the internal spin degrees of freedom would in order to correspond to a Kaluza-Klein gauge fields with massless gauge bosons have to represent full symmetries of the vacuum state, i.e. they should as in usual Kaluza-Klein correspond to Killing-vectors, but with the further degrees of freedom also corresponding to symmetry for the latter. So at the end we may consider also the charges associated with the internal spin as ordinary Klauza-Klein charges, of course in the sense of being for the very rich model considered here. But of course unless we have the θ^a or forms degrees of freedom one could risk that the gauge field from such symmetry could be practically decoupled.

Let us now look at what the "families" found in the Dirac-Kähler will develop into in case we use it for a Kaluza-Klein type model, as just proposed: Usually the number of surviving massless fermions into the (3+1) space consists only of those which are connected with "zero-modes". This is to be understood so that we imagine Weyl particles in the high (d) dimensional space because of an Atiah-Singer theorem in $(d-4)$ dimensional "staying compactified" space ensures some modes with the extra dimension part of the Dirac operator gets zero for some number of modes - for each d-dimensional family.

If the model had a strength for compact space Atiah-Singer theorem $''A.S.strenght''$ and if the dimension of the full space, the number of θ^a's, is d, so that the number of families at the d-dimensional level becomes $2^{d/2}$, the total number of at low energy observable "families " should be

$$\#families ='' A.S.strength'' * 2^{d/2}. \tag{68}$$

As an example take the model [2] which has $d = 14$ and at first - at the high energy level - $SO(1,13)$ Lorentz group, but which should be broken to (in two steps) to first $SO(1,7) \times SO(6)$ and then to $SO(1,3) \times SU(3) \times SU(2)$.

X. DISCUSSION OF SPECIES DOUBLING PROBLEM

We may see the appearance of equally many (namely $2^{d/2-1}$) right handed and left handed "flavours " in the Kähler model as an expression for the no go theorem [5] for putting chiral charge conserving fermions on the lattice in as far as we could make attempts to make lattice fermions along the lines of Becher and Joos [4] . In fact it would of course have been a counterexample to the no go [5] theorem if there had been a different number of right and of left Weyl particle species in the Becher-Joos model, because in the free model the number of particles functions as a conserved charge.

As is very well known the Becher-Joos model really is just the Kogut-Susskind [11] lattice fermion model, it is also well known that it does not violate the no go theorem [5] and this is because there is this species doubling, which can be interpreted as the flavours.

Becher and Joos show that the Kogut-Susskind lattice description of Dirac fields is equivalent to the lattice approximation of the Dirac-Kähler equation. (see page 344 in the Becher-Joos [4] article).

This Kogut-Susskind model is one that gives us Dirac particles, but we can seek to get to Weyl particles in a naive $\Gamma^{(4)}$ (or $\tilde{\Gamma}^{(4)}$ or γ^5 in the usual notation) projecting way, but of course now such a projection would have to be translated into the language with the vector and scalar fields in the Kähler's formulation, and it is rather easy to see [6] that requiring only one $\Gamma^{(4)}$ projection implies that the coefficient to one p-form say dx^H should relate (just by a sign $\times i$) to that of the by the Hodge star $*$ associated $*dx^H$ (See subsection 4.5). Actually we easily see that requiring the restriction that

$$(1+\Gamma^{(4)})\psi = 0 \tag{69}$$

in the language of Kähler becomes

$$(1+i*)u = 0. \tag{70}$$

If we want like Joos and Becher to put the theory on the lattice there is a difficulty in just imposing this constraint, because the natural relation imposed by the Hodge star $*$ on the lattice would go from lattice to the dual lattice and we could not identify without a somewhat ambiguous choice the ∗dual of a given lattice element, so as to impose the "self duality" condition.

Could we possibly invent a way to circumvent the no go theorem [5] for chirality conserving fermions on the lattice by making the species doublers bosons instead of fermions, both having though spin 1/2 ?

In the formulation by one of us which we have related to the Kähler formulation there is (naturally) assigned different Grassmanian character to different components of the wave function. In fact the wave function with coefficients to monomial terms that are products of different sets of (mutually different) θ^a-variables - in the sense of course that a polynomial is given by its coefficients -, and thus the coefficients to the products with an even number of factors have different Grassmannian character from those of the odd number of factors. That actually is in the theory of one of us somewhat of an embarrassing reason for a super selection rule, which though may be overcome by taking into account the charges related to extra dimensions appearing in that model. But here we now want to point out the hope that these very Grassmann character problems may be used as a new idea to circumvent the no go theorem. In fact we could hope for that spin 1/2 and say left handed flavour appear with fermionic statistics (the Grassmann odd character) while spin 1/2 flavour with bosonic statistics would appear as right handed, and that even on the lattice.

XI. CONCLUDING REMARKS

The way that Mankoč [2] chooses to quantize the system, that is a particle moving in ordinary and Grassmann coordinate space, is to let the wave function be allowed to be any function of the d Gassmann variables θ^a, so that any such function represents a state of the system. But in this quantization the $\tilde{\tilde{a}}^a$'s can not be put weakly to zero. In other words that quantization turned out not to obey the equation expected from expression for the canonical momentum $p^{\theta a}$, being proportional to the coordinate θ^a as derived from the Lagrangian. If, however, in the operators such as the Hamiltonian and the Lorentz transformation operators $\tilde{\tilde{a}}^a$'s are just put strongly to zero, so that all the operators only depend on $\tilde{a}^a$, while either $\tilde{a}^a$ or $\tilde{\tilde{a}}^b$ fulfill the Clifford algebra: $\{\tilde{a}^a, \tilde{\tilde{a}}^b\} = 0$ and $\{\tilde{a}^a, \tilde{a}^b\} = 2\eta^{ab} = \{\tilde{\tilde{a}}^a, \tilde{\tilde{a}}^b\}$, the expressions obtained after having put the $\tilde{\tilde{a}}^a$'s to zero describe spinor degrees of freedom. In particular, only the operators $\tilde{S}^{ab}$ are used as the Lorentz generator. One has accordingly the new Lorentz transformations instead of the a priori one in the wave function on Grassmann space quantization used. In that case the argument for only having integer spin breaks down, what the calculations indeed confirm to happen.

We should now attempt to get an understanding of what goes on here by using a basis inspired from the Dirac-Kähler-construction, which is a way often used on lattices to implement fermions on the lattice. The Dirac-Kähler construction starts from a field theory with a series of fields which are 0-form, 1-form, 2-form, ...,d-form. They can be thought of as being expanded on a basis of all the wedge product combinations of the basis dx^1, dx^2, ...,dx^d for the one-forms, including wedge products from zero factors to d factors. In the Dirac-Kähler construction one succeeds in constructing out of these "all types of forms" $2^{d/2}$ Dirac spinor fields. This construction is without a "cheat" impossible in much the same way as Mankoč's approach ought to be.

We have pointed out clearly in this paper how this "cheat" occurs in both approaches, showing up all the similarities of the two approaches and using the simple presentation of the quantum mechanics in Grassmann space to not only simplify the Dirac-Kähler approach but also to generalize it. We have shown in particular that in both approaches besides the (two kinds of) generators for the Lorentz transformations for spinors also the generators for vectors and tensors exist. There are four copies of the Weyl bispinors. One kind of the spinorial type of the Lorentz transformations defines the Weyl spinors, another kind transforms one copy of Weyl spinors into another of the same Grassmann character. We also have shown the two kinds of the time reversal operators, as well as the fact that in Grassmann space or space of differential forms of d dimensions, $d > 4$, spins and charges unify. We pointed out the necessity of defining the gamma-matrices of an even Grassmann character.

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