

CERN-TH/99-327
hep-th/nnmmmyy

Non-perturbative $\mathcal{N} = 1$ strings from geometric singularities

P. Mayr

TH Division, CERN, 1211 Geneva 23, Switzerland

Abstract. The study of curved D-brane geometries in type II strings implies a general relation between local singularities \mathcal{W} of Calabi–Yau manifolds and gravity free supersymmetric QFT's. The minimal supersymmetric case is described by F-theory compactifications on \mathcal{W} and can be used as a starting point to define minimal supersymmetric heterotic string compactifications on compact Calabi–Yau manifolds with holomorphic, stable gauge backgrounds. The geometric construction generalizes to non-perturbative vacua with five-branes and provides a framework to study non-perturbative dynamics of the heterotic theory.

To appear in the proceedings of the STRINGS '99 conference, Potsdam (Germany),
19--25 July 1999

CERN-TH/99-327

1. Introduction

Much of the work in string theory in the last years has been devoted to dualities between seemingly different physical theories. A remarkable aspect of these dualities is that they involve often a geometrization of physics, that is there is a correspondence of physical quantities with data of a geometric object. In extreme cases, one arrives at a duality between a physical theory and a geometric theory, with the physical data such as the moduli space, symmetries, spectrum and even correlation functions being in one-to-one correspondence with the quantities of a geometric theory. In particular, if the coupling constants of the physical theory are mapped to geometric parameters, the duality provides an extremely powerful tool to study quantum effects in the physical theory.

Generally, the most useful formulation of the geometric theory is in terms of complex geometries W , with the complex deformations of W being equivalent to the moduli space \mathcal{M}_{phys} of scalar vev's a_i of the physics theory. The geometry W can be defined as the zero locus of a holomorphic polynomial in some ambient space Y

$$W : p_W(x_i, u_\alpha) = 0, \tag{1}$$

where x_i are holomorphic coordinates on Y and u_α parametrize the complex structure of W . The idea is that there is a map $u_\alpha \rightarrow a_i$ which maps the complex structure moduli space of W to \mathcal{M}_{phys} . The prototype for the relation between a complex geometry W and a physical system is the case of $\mathcal{N} = 2$ $SU(2)$ Super-Yang-Mills theory in four-dimensions, whose one-dimensional moduli space on the Coulomb branch has been shown to be equivalent to the complex deformations of a torus [1]. However it was realized shortly later [2, 3] that this relation is only a special example of a very general connection between complex geometries of Calabi–Yau 3-fold singularities and $\mathcal{N} = 2$ supersymmetric gravity free quantum field theories in four dimensions. The link between geometry and physics is provided by D-brane geometries in type IIB strings in the following way[‡]. If the type IIB string is compactified on a Calabi–Yau 3-fold X with small 3-cycles C_i , there are extra light states from D3-branes wrapped on the C_i . In the limit of vanishing volume these states give rise to massless charged particles in uncompactified space-time [5], with their quantum numbers and interactions determined by the geometry of the cycle C_i embedded in X . Since the type II string coupling is in the hypermultiplet sector, which does not couple neutrally to the vector multiplets, the exact effective two derivative action of the vector multiplets on the Coulomb branch is given by the tree level type IIB physics, which in turn is determined by the classical complex geometry. In the limit of very small 3-cycles, which means we consider singular geometries, one can decouple gravity and most of the fundamental string spectrum and obtains an equivalence between the complex deformations of Calabi–Yau 3-fold singularities and the Coulomb branch of general gravity free $\mathcal{N} = 2$ QFT's in four dimensions. Apart from the exact non-perturbative effective action, there are other

[‡] For a review and more references, see [4].

important non-perturbative data of the field theory, such as the stable BPS spectrum [3] and S-duality symmetries [6], that are determined in the exact tree level type IIB theory and thus, the geometry of the singularity.

This relation between geometric Calabi–Yau 3-fold singularities and gravity free QFT’s can be extended [7]§ to theories with $\mathcal{N} = 1$ supersymmetries by replacing the type II string theory with F-theory [9]. In this case we obtain field theories that are embedded in minimal supersymmetric string theories in $d < 8$ dimensions, in particular $\mathcal{N} = 1$ supersymmetric string theories in four dimensions.

An especially interesting class of the latter consists of heterotic string compactifications on Calabi–Yau 3-folds Z , which are the most promising perturbative string theories from the phenomenological point of view. However an aspect of these so-called $(0, 2)$ vacua that has turned out to be difficult to study, is the specification of a suitable gauge background, which takes the form of a holomorphic stable vector bundle (or sheaf) on Z . It turns out [7] that these bundles appear as the moduli space of a certain class of field theories that are ”dual” to Calabi–Yau 4-fold singularities in the previous sense and can be constructed and studied using the geometric duality, very much as in the case of $\mathcal{N} = 2$ supersymmetric QFT’s. Again we obtain non-perturbative information from the geometric perspective and in particular the geometric construction generalizes to non-perturbative vacua of the heterotic string, including those with background 5-branes [10] and non-perturbative gauge symmetries.

2. Singularities for $\mathcal{N} = 1$ supersymmetric theories

We have mentioned already that the crucial link between the geometric singularities and the physical system associated to it is provided by the curved geometry of D-branes of type II strings wrapped on vanishing cycles. However, to reduce supersymmetry, we need to compactify F-theory on the singularity, rather than type II strings. In general, F-theory has a very different way to generate massless states, in particular from open strings stretching between background D7 branes. However note that in the duality between F-theory compactified on a Calabi–Yau n -fold W_n times a torus T^2 and type IIA on W_n , the radii of T^2 are mapped to geometric moduli of the type IIA theory on W_n . We can therefore describe the minimal supersymmetric theory from F-theory on W_n in terms of a type IIA compactification on the F-theory-limit of W_n , defined as the patch in the geometric moduli of type IIA on W_n that corresponds to the decompactification of the T^2 factor in the dual F-theory compactification. As a matter of fact this allows to adopt many of the methods developed in the $\mathcal{N} = 2$ context to the minimal supersymmetric case. In particular one can establish in this way the following geometric duality:

Similarly as in the $\mathcal{N} = 2$ context, one needs only type IIA (and F-theory) physics to establish the above equivalence. A necessary assumption that is needed is that Z_n is elliptically fibered. The equivalence holds for any structure group H of the bundle.

§ See ref. [8] for an introduction to the subject.

Calabi–Yau Geometry \mathcal{W}_{n+1}	$\mathcal{N} = 1$ Field theory (Z_n, V)
Local singularities \mathcal{W}_{n+1} in $n + 1$ -fold fibrations of elliptically fibered ALE spaces	Compactification of $\mathcal{N} = 1$, 10d SYM on ell. fibered Calabi–Yau Z_n ; Holomorphic stable vector bundle V on Z_n

Note that the moduli space of the $\mathcal{N} = 1$ field theory appearing on the right hand side of the above table is precisely the same as that of the gauge background of a heterotic compactification on Z_n in the point particle limit, under the condition that the structure H fits in the perturbative heterotic gauge symmetry G_0^{het} , e.g. $E_8 \times E_8$. Since one arrives at the above equivalence using only type IIA/F-theory physics, one can *derive* type IIA/heterotic and F-theory/heterotic duality in the point particle limit. Note that the equivalence is true for *any* H ; for $H \subset G_0^{het}$ it becomes equivalent to a duality with the heterotic string in the point particle limit.

Since the field theory involved in the geometric duality is so closely related to a string theory compactification, about which we do not have a good control, it makes sense to consider the reverse process of decoupling gravity and find *global* embeddings of the singularities \mathcal{W}_{n+1} into compact Calabi–Yau manifolds W_{n+1} to describe dual pairs of the full F-theory/heterotic and IIA/heterotic dualities away from the point particle limit. Interestingly one finds that global embeddings exist precisely if $H \in G_0^{het}$. While it is expected from the physics point of view that we recover the known consistent string theories in this way, the result is non-trivial from the geometric point of view. The global embeddings define dual pairs of string duality (as opposed to the relation between singularity and field theory), namely a compact Calabi–Yau $n + 1$ -fold W_{n+1} on which F-theory is compactified and a compact Calabi–Yau Z_n together with a family of holomorphic stable vector bundles V on it that describe the dual heterotic data. Moreover one obtains a precise map of the moduli spaces of the two theories in the point particle limit. We can then *use* the duality $W_{n+1} \leftrightarrow (Z, V)$ away from this limit to study non-perturbative properties of the minimal supersymmetric heterotic string vacua.

3. Vector bundles and type II strings

At first sight it is not obvious that F-theory compactified on the geometric singularities \mathcal{W}_{n+1} describes holomorphic stable vector bundles on compact Calabi–Yau manifolds Z_n of one complex dimension less. The relation between the geometric singularity and vector bundles can be traced back to a symmetry of type IIA on $K3 \times T^2$ [6] that is closely related to mirror symmetry on K3. Consider a type IIA string compactified on a K3 surface. This gives an $\mathcal{N} = 2$ supersymmetric theory in six dimensions. From dimensional reduction we obtain twenty $\mathcal{N} = 2$ vector multiplets which contain as their bosonic degrees of freedom each one vector and four real scalars. The latter transform as a $\underline{4}$ under an $SO(4)_R$ R-symmetry and parametrize the 58 metric moduli of K3 plus 22 values of the B -fields on $H_2(K3)$.

In a given algebraic realization M_2 of K3, the 80 moduli of K3 split in complexified Kähler and complex structure moduli \parallel . The $SO(4)_R$ rotations acting on the scalars in $\underline{4}$ do not preserve this split in general. In particular there is an $SO(4)_R$ rotation that exchanges Kähler and complex structure moduli spaces \mathcal{M}_{KM} and \mathcal{M}_{CS} , respectively. Such a transformation is usually known as *mirror symmetry* of K3 manifolds, which says that a type IIA string compactified on M_2 describes the same physics as a type IIA string compactification on a "different" \P K3 manifold W_2 , with the Kähler and complex structure moduli exchanged. The new algebraic K3 surface W_2 is called the mirror manifold of M_2 . We have the following identifications under mirror symmetry:

$$M_2 \leftrightarrow W_2, \quad (2)$$

$$\mathcal{M}_{CS}(M_2) \leftrightarrow \mathcal{M}_{KM}(W_2), \quad \mathcal{M}_{KM}(M_2) \leftrightarrow \mathcal{M}_{CS}(W_2).$$

There is an intriguing generalization of this symmetry upon compactification on a further T^2 to four dimensions. In this case we obtain an $\mathcal{N} = 4$ supersymmetric string theory. The $\mathcal{N} = 4$ vector multiplet now contains 6 scalars which transform in a $\underline{6}$ of an $SO(6)_R$ symmetry. The two extra scalars are the internal components A_i of the six-dimensional gauge fields on the T^2 , call it E , and thus parametrize the stringy moduli space of Wilson lines on the elliptic curve, $\mathcal{M}_E^{string}(H)$. Here H denotes the structure group of the Wilson line background. Again the $SO(6)_R$ rotations provide identifications within the type IIA moduli space $\mathcal{M}_{IIA}(K3 \times T^2)$:

$$\mathcal{M}_{KM}(M_2) \leftrightarrow \mathcal{M}_{CS}(W_2) \leftrightarrow \mathcal{M}_E^{string}(H). \quad (3)$$

In particular there is now an element of $SO(6)_R$, corresponding to the last arrow, which provides a relation between the moduli space of stringy Wilson lines on E with the Kähler moduli space of an algebraic K3 manifold.

Let us illustrate the symmetry (3) of the type IIA compactification on $K3 \times T^2$ in the dual heterotic picture where it is very natural; however we stress that we do not need the heterotic dual to derive the symmetry. Fig. 1 shows the type IIA compactification and the corresponding dual heterotic string compactified on $T_1^2 \times T_2^2 \times T_3^2$. The

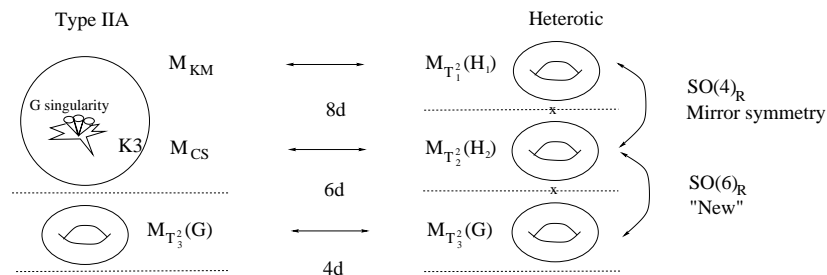


Figure 1. The symmetry (3) from a heterotic point of view.

factorization of T^6 on the heterotic side is related to *i*) the elliptic fibration and *ii*) the

\parallel Subscripts, as in M_2 , denote the complex dimension of a geometry.

\P Since the K3 surface is unique, mirror symmetry acts as a discrete identification in the moduli space of K3 surfaces.

factorization of the gauge background on $K3 \times T^2$ in the type IIA theory. Under these circumstances, the Kähler moduli of $K3$ can be identified with moduli space $\mathcal{M}_{T_1^2}(H_1)$ of H_1 Wilson lines on the first T_1^2 and similarly complex structure moduli with Wilson lines in the second factor $\mathcal{M}_{T_2^2}(H_2)$. The non-abelian gauge symmetry in six dimensions is G which on the type IIA side arises from the G singularity and on the heterotic side is the commutant of $H_1 \times H_2$ in G_0^{het} . Upon further compactification to four dimensions on a torus T_3^2 , we obtain another factor $\mathcal{M}_{T_3^2}(G)$ on both sides.

The heterotic theory has an obvious symmetry under permutation of the three T^2 factors. The exchange of the first two tori amounts to mirror symmetry in the type IIA theory. On the other hand an exchange that involves the third torus gives a new symmetry of the moduli space of the type IIA compactification that identifies Wilson lines with structure group G on T^2 with either Kähler or complex deformations of a G singularity of $K3$. This is the new relation described in (3).

Finally we can decouple most of the fundamental string states and gravity in the limit of a singular geometry of the $K3$, or in terms of physics, in the field theory limit. The elliptically fibered 2-fold singularities \mathcal{M}_2 that describe the local patch in M_2 (and similarly a local geometry \mathcal{W}_2 for the complex geometry side) obtained in this limit describe field theory Wilson lines on a torus E :

$$M_{KM}(\mathcal{M}_2) \cong M_{CS}(\mathcal{W}_2) \cong M_E^{FT}(H). \quad (4)$$

Note that this relation will be more general as the one obtained in the global $K3$ context. In the latter, $r = \text{rank } H$ is bounded from above by the dimension of $H_2(K3)$, whereas in (4) we can actually consider local singularities of ALE spaces of arbitrary rank.

The above field theory limit is compatible with taking the F-theory limit of the type IIA theory. The F-theory compactification on \mathcal{W}_2 gives then the deliberate geometric dual we are looking for.

4. Holomorphic stable bundles on Calabi–Yau n -folds

In the previous section we have seen how holomorphic (semi-)stable bundles on a Calabi–Yau 1-fold, or simply Wilson lines on T^2 , are related to F-theory compactified on a 2 complex dimensional Calabi–Yau singularity \mathcal{W}_2 . To obtain holomorphic stable bundles on Calabi–Yau n -folds, we consider holomorphic fibrations of \mathcal{W}_2 over some $n-1$ complex dimensional base B_{n-1} . The total space \mathcal{W}_{n+1} , which is required to satisfy the Calabi–Yau condition in order to be a valid type IIA and F-theory background, is an $n+1$ dimensional non-compact singularity with an elliptic fibration inherited from that of \mathcal{W}_2 .

The dual heterotic picture is the following [11]: the data of H Wilson lines on T^2 are fibered over the same base B_{n-1} , in virtue of the adiabatic principle [12]. The T^2 fibration over B_{n-1} results in a compact Calabi–Yau n -fold Z_n on which the heterotic string is compactified and the gauge background on T^2 extends to a family of holomorphic stable bundles on Z_n .

Let us now sketch the general form of the dual singularities \mathcal{W}_{n+1} which have a nice geometric and group theoretical interpretation. Let $(y, x, z; v)$ denote a special set of coordinates on the elliptically fibered ALE space \mathcal{W}_2 ; in particular v is a coordinate on the base \mathbf{P}^1 of the elliptic fibration of \mathcal{W}_2 . Moreover the x_i are coordinates on the base B_{n-1} of the \mathcal{W}_2 fibration of \mathcal{W}_{n+1} . The singularity \mathcal{W}_{n+1} is defined as the vanishing of the polynomial p :

$$p = p_0 + p_+ = p_0(y, x, z|x_i) + \sum_i v^i p_+^i(y, x, z|x_i) = 0. \quad (5)$$

We have divided the polynomial p in a v independent part p_0 and a v dependent piece p_+ . They describe the geometry Z_n and the gauge bundle V on Z_n , respectively. In fact p_0 is the defining equation for the elliptically fibered, compact Calabi–Yau Z_n in Weierstrass form. The polynomial p_+ that describes the bundle has a very nice group theoretical structure. If we consider the extended Dynkin diagram of the structure group H of V , then each node of index s_i contributes precisely one monomial $p_+^{s_i}$ to p_+ . E.g., for $SU(N)$ we have N nodes of index one and the equation for p_+ becomes (for N even)

$$p_+ = v^1 \cdot (z^N a_N(x_i) + xz^{N-2} a_{N-2}(x_i) + \dots + x^{N/2} a_0(x_i)). \quad (6)$$

Since in this case v appears only linearly in p , it can be integrated out as far as variation of the complex structure is concerned. This yields the two separate conditions $p_0 = 0$ and $p_+ = 0$. The first equation defines Z_n . The N roots of the second equation determine N points on the elliptic fiber E of Z_n , with their position on E varying with the values of the base variables x_i . We thus recover the spectral cover construction of $SU(N)$ bundles on elliptic fibrations described in [11, 13].

More generally, we obtain a similar geometric representation of families of holomorphic stable bundles on Z_n for any H . The actual construction of the geometries \mathcal{W}_{n+1} can be phrased best in the framework of toric geometry. This approach has the advantage of being very general, in particular in treating general Calabi–Yau manifolds Z_n with non-smooth elliptic fibrations, as well as studying singularities in both, the geometric and gauge data. Moreover one can define a toric map that identifies the dual objects

$$f : \mathcal{W}_{n+1} \leftrightarrow (Z_n, V), \quad (7)$$

with important physical data of the heterotic side, such as the geometry of the manifold Z_n , the family of bundles on it, non-perturbative 5-branes or gauge symmetries and the stability of the bundle having a very simple and systematic representation in terms of the toric geometry of the singularity \mathcal{W}_{n+1} on the left side [7, 14]. This allows for a simple construction of deliberate families of stable bundles V with arbitrary structure group H on any toric Calabi–Yau in terms of the singularity \mathcal{W}_{n+1} .

Moreover, if $H \in G_0^{het}$ one can easily determine the possible global embeddings of the local singularity $\mathcal{W}_{n+1} \hookrightarrow W_{n+1}$, where W_{n+1} is now the global Calabi–Yau $n+1$ -fold which gives the dual *string* theory when used as an F-theory compactification. The map $W_{n+1} \leftrightarrow (Z_n, V)$ can then be used to study non-perturbative properties of the minimal supersymmetric heterotic string theory on Z_n , as is illustrated in the next section.

5. Some results on (non-perturbative) $\mathcal{N} = 1$ heterotic strings

Let us finally sketch some results on (non-perturbative) heterotic physics which have been obtained in [7, 14] using the above method.

Stability of the bundle V

We complained already that it is very hard to find explicit solutions to the stability equations for V and in the sequel we claimed that we can construct a large class of such stable bundles seemingly without effort using the geometric singularities \mathcal{W}_{n+1} . Obviously, there has to be quite a good reason for this simplification in the geometric construction. Indeed there is one important property of the geometric singularities \mathcal{W}_{n+1} that we did not interpret so far in the heterotic duals: the Calabi–Yau condition that ensures that the type IIA, or F-theory, compactified on \mathcal{W}_{n+1} is consistent. Indeed one can show [14] that the Calabi–Yau property of \mathcal{W}_{n+1} translates to a stability condition on the background V on Z_n , expressed in terms of a bound on the first Chern class $\eta = c_1(\mathcal{N})$ of a line bundle \mathcal{N} that is an important characteristic data of the vector bundle V on Z [11]. One finds that

$$\nu(G) c_1(\mathcal{L}) \leq \eta (\leq 12c_1(\mathcal{L})), \quad (8)$$

where G is the singularity type of the fiber geometry W_2 and $\nu(G)$ is a certain characteristic number defined in [14]. Explicit expressions for \mathcal{N} and the bounds on η for toric bases, such as \mathbf{P}^2 and \mathbf{F}_n and blow ups thereof, can be found in [7]⁺.

The answer to why it was so easy to get stable solutions from the geometric construction is thus that we have replaced the complicated condition for stability in the heterotic string, about which we have poor control, with the simple Calabi–Yau condition on the dual type IIA/F-theory side, which is much easier to deal with.

Standard embedding

A simple solution to solve the background equations for V is to set the gauge connection equal to the spin connection of the manifold, $V = TZ$. This is the so-called standard embedding. Though this configuration is not too appealing for phenomenology - in four dimensions, the heterotic gauge group is $E_6 \times E_8$ and the matter spectrum is tied to the Euler number of Z - it has been studied extensively in the past because it was one of the few known solutions. The structure group of V is $SU(n)$ for a Calabi–Yau n -fold.

The global geometry W_{n+1} corresponding to the F-theory dual can be determined in the following way. From the fact that the bundle in the second factor is trivial it follows that the line bundle \mathcal{N} is trivial in the second E_8 factor. The Weierstrass form for W_{n+1} and Z_n takes the following form:

$$p_{Z_n} = y^2 + x^3 + x\tilde{z}^4 f + \tilde{z}^6 g,$$

⁺ A bound on η derived in [15] is different from those in [7] and [14].

$$p_{W_n} = y^2 + x^3 + x(\hat{z}zw)^4 f + (\hat{z}zw)^6 g + \hat{z}^6 z^5 w^7 \Delta + \hat{z}^6 z^7 w^5. \quad (9)$$

Here (z, w) denote the variables parametrizing the base \mathbf{P}^1 of the elliptically fibered K3 fiber W_2 of the fibration $\pi_F : W_{n+1} \rightarrow B_{n-1}$ and $\Delta = 4f^3 + 27g^2$ is the discriminant of the elliptic fibration of Z_n . This generalizes the six-dimensional result obtained in [16] to four and lower dimensions.

Non-perturbative gauge symmetries

In the F-theory picture, gauge symmetries correspond to singularities in the elliptic fibration. Those located over the base B_{n-1} have a perturbative interpretation in the heterotic dual, whereas non-perturbative ones are located over curves in B_{n-1} [17]. Using the map $W_{n+1} \rightarrow (Z_n, V)$ we can describe the heterotic compactification that leads to the same dynamics non-perturbatively. In the six-dimensional case we find:

(*) *Consider the $E_8 \times E_8$ string compactified on an elliptically fibered K3 with a singularity of type G at a point $s = 0$ and a special gauge background \hat{V} . If the restriction of the spectral cover* of $\hat{V}|_E$ to the fiber E at $s = 0$ is sufficiently trivial, the heterotic string acquires a non-perturbative gauge symmetry $G_{np} \supset G$.*

The triviality condition can be made precise by specifying the behavior of V near $s = 0$ [7]. Similar results hold for four-dimensional compactifications.

We point out that the above triviality condition on the spectral cover does not imply that the field strength of V , which measures the behavior of V near the singularity, is trivial. In fact it has been shown recently in [18] that if $F = 0$ on the singularity, the conformal field theory of the heterotic string is well behaved and there is no non-perturbative gauge symmetry.

Non-perturbative dualities

The map $f : W_{n+1} \leftrightarrow (Z_n, V)$ can be ambiguous in the sense that there are two (or even more) ways to associate a pair (Z_n, V) to W_{n+1} . If the two maps are compatible with the same elliptic fibration, we obtain a non-perturbative duality of two heterotic string theories

$$(Z_n, V) \sim (Z'_n, V'). \quad (10)$$

The conditions under which this duality exists, can be formulated in simple properties of the toric polyhedron $\Delta_{W_{n+1}}^*$ associated to the Calabi–Yau W_{n+1} in its construction as a toric hypersurface [7]. The generic form of the duality is the following:

(**) *Let the heterotic string be compactified on a Calabi–Yau three-fold with G' singularity and with a certain gauge background with structure group H such that the toric data $\Delta_{W_{n+1}}^*$ fulfil the above mentioned condition. Then there exists a non-perturbatively equivalent compactification on a Calabi–Yau manifold with G singularity and with a specific gauge background with structure group H' .*

* For $G \neq SU(N)$ we use the generalization of the spectral cover in terms of the sections of a certain weighted projective bundle over B_{n-1} [11].

Here H (H') is the commutant of G (G') in $E_8 \times E_8$. Note that the duality exchanges the groups of the geometric singularity and the gauge bundle: the singularity in the dual manifold is the commutant of the structure group and *vice versa*. Let us give an extreme example of such an duality: the heterotic string with a trivial gauge bundle on a smooth K3 has a perturbative $E_8 \times E_8$ gauge symmetry and in addition $n'_T = 24$ non-perturbative tensor multiplets from the 24 five-branes required to satisfy anomaly cancellation in six dimensions. The dual theory is a heterotic theory with $E_8 \times E_8$ gauge bundle on a K3 with $E_8 \times E_8$ singularity. The perturbative gauge symmetry is trivial, while non-perturbative dynamics associated to the singularity produce both, the $E_8 \times E_8$ gauge symmetry as well as 24 extra tensor multiplets.

This duality reminds very much of a known duality in the linear sigma model formulation of heterotic strings, namely a symmetry under the exchange of the data defining the manifold and the data defining the bundle [19]. However note that in our case this duality is in general *non-perturbative*.

Mirror symmetry of F-theory

Consider a six-dimensional F-theory compactification on W_3 and the associated heterotic dual (Z_2, V) as defined above. If the mirror manifold M_3 of W_3 is also elliptically fibered, there is also a heterotic theory (Z'_2, V') associated to it and one can ask the question of how the two theories are related. The answer is that after compactification on a three-torus to three dimensions, the two become equivalent in virtue of mirror symmetry of type II strings [20].

A proposal for the relation between (Z_2, V) and (Z'_2, V') has been made in [21] based on a comparison of hodge numbers and gauge symmetries in some cases. Using the toric map $W_{n+1} \rightarrow (Z_n, V)$ one can determine the two theories; in fact one finds a subtle realization of Higgs and Coulomb branches in toric geometry as expected from the action of three-dimensional mirror symmetry [7]. The general relation is of a similar form as in (**) above: a heterotic theory with a G bundle compactified on a K3 manifold with \hat{H} singularity gets mapped to a heterotic theory with a bundle with structure group \hat{H} compactified on a manifold with G singularity. Note that we have an explicit map of the moduli spaces of the two theories. It would be interesting to formulate the associated three-dimensional dual pairs associated to this relation, as in [22].

6. Outlook

The geometric construction of minimal supersymmetric heterotic string vacua in four dimensions and their F-theory duals opens an interesting way to study phenomenologically relevant string compactifications. The identifications between W_{n+1} and (Z_n, V) may serve as a starting point to study more refined non-perturbative relations between

the dual F- and heterotic string theory. In particular we would like to formulate both perturbative and non-perturbative quantities of the heterotic string, e.g. gauge and Yukawa couplings as well as superpotential of the $(0, 2)$ vacua, in terms of geometric quantities on W_{n+1} . A concrete example are the correlation functions determined by holomorphic prepotentials, considered in [23]‡. Also for the superpotential, a geometrical approach exists and has been studied in [25, 23, 26, 27, 28]. In the ideal (and supposedly too optimistic) case, we can hope to obtain exact results for at least some of the holomorphic physical quantities in the $\mathcal{N} = 1$ theory from geometric information on W_{n+1} , similarly as it happened to work in the case of $\mathcal{N} = 2$ supersymmetry.

Acknowledgements

I would like to thank Per Berglund for the collaboration on this work and the organizers of Strings '99 for the opportunity to present it and for their hospitality.

- [1] N. Seiberg and E. Witten, Nucl. Phys. B426 (1994) 19, erratum: ibid 430 (1994) 396;
- [2] S. Kachru, A. Klemm, W. Lerche, P. Mayr and C. Vafa, Nucl. Phys. B459 (1996) 537.
- [3] A. Klemm, W. Lerche, P. Mayr, C. Vafa, N. Warner, Nucl. Phys. B477 (1996) 746.
- [4] P. Mayr, Fortsch. Phys. 47 (1999) 39.
- [5] A. Strominger, Nucl. Phys. B451 (1995) 96.
- [6] S. Katz, P. Mayr and C. Vafa, Adv. Theor. Math. Phys. 1 (1998) 53.
- [7] P. Berglund and P. Mayr, Adv. Theor. Math. Phys. 2 (1998) 1307.
- [8] P. Mayr, $N = 1$ heterotic vacua from mirror symmetry, hep-th/9904115;
D-branes and F-theory, proceedings of the Trieste spring school 1999.
- [9] C. Vafa, Nucl. Phys. B469 (1996) 403.
- [10] M.J. Duff, R. Minasian and E. Witten, Nucl. Phys. B465 (1996) 413.
- [11] R. Friedman, J.W. Morgan and E. Witten, Comm. Math. Phys. 187 (1997) 679.
- [12] C. Vafa and E. Witten, Nucl. Phys. Proc. Suppl. 46 (1996) 225.
- [13] M. Bershadsky, A. Johansen, T. Pantev and V. Sadov, Nucl. Phys. B505 (1997) 165.
- [14] P. Berglund and P. Mayr, *Stability of vector bundles from F theory*, hep-th/9904114.
- [15] G. Rajesh, J. High Energy Phys. 12 (1998) 18.
- [16] P.S. Aspinwall and R.Y. Donagi, Adv. Theor. Math. Phys. 2 (1998) 1041.
- [17] D. Morrison and C. Vafa, Nucl. Phys. B473 (1996) 74; Nucl. Phys. B476 (1996) 437.
- [18] E. Witten, *Heterotic string conformal field theory and A-D-E singularities*, hep-th/9909229.
- [19] J. Distler and S. Kachru, Nucl. Phys. B442 (1995) 64.
- [20] K. Intriligator and N. Seiberg, Phys. Lett. B387 (1996) 513.
- [21] E. Perevalov and G. Rajesh, Phys. Rev. Lett. 79 (1997) 2931.
- [22] K. Hori, H. Ooguri and C. Vafa, Nucl. Phys. B504 (1997) 147.
- [23] P. Mayr, Nucl. Phys. B494 (1997) 489.
- [24] W. Lerche and S. Stieberger, Adv. Theor. Math. Phys. 2 (1998) 1105.
- [25] S. Kachru and E. Silverstein, Nucl. Phys. B482 (1996) 92.
- [26] S. Katz and C. Vafa, Nucl. Phys. B497 (1997) 196.
- [27] C. Vafa, Adv. Theor. Math. Phys. 2 (1998) 497.
- [28] D.-E. Diaconescu and S. Gukov, Nucl. Phys. B535 (1998) 171.

‡ See also [24] for an interesting study of correlation functions in the eight-dimensional duality.