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General RG Equations for Physical Neutrino Parameters and their Phenomenological Implications

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Abstract

The neutral leptonic sector of the Standard Model presumably consists of three neutrinos with non-zero Majorana masses with properties further determined by three mixing angles and three CP-violating phases. We derive the general renormalization group equations for these physical parameters and apply them to study the impact of radiative effects on neutrino physics. In particular, we examine the existing solutions to the solar and atmospheric neutrino problems, derive conclusions on their theoretical naturalness, and show how some of the measured neutrino parameters could be determined by purely radiative effects. For example, the mass splitting and mixing angle suggested by solar neutrino data could be entirely explained as a radiative effect if the small angle MSW solution is realized. On the other hand, the mass splitting required by atmospheric neutrino data is probably determined by unknown physics at a high energy scale. We also discuss the effect of non-zero CP-violating phases on radiative corrections.

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1 Introduction

There is mounting experimental evidence [1,2,3] that flavour is not conserved in the flux of atmospheric and solar neutrinos. The most plausible, simplest and best motivated interpretation for this phenomenon is that interaction eigenstate neutrinos mix in a non trivial way into neutrino mass-eigenstates with different non-zero masses, leading to flavour oscillations [4]. The theoretical scenario that results most economical in accounting for the observed neutrino anomalies[1] assumes that the known neutrinos of the Standard Model $(\nu_e, \nu_\mu, \nu_\tau)$ acquire Majorana masses through a dimension-5 operator [6], generated at some high energy scale Λ (the model can also be made supersymmetric).

The 3×3 neutrino mass matrix $\mathcal{M}_\nu$ is diagonalized according to

$$U^T \mathcal{M}_\nu U = \mathrm{diag}(m_1, m_2, m_3), \tag{1}$$

and we can choose $m_i \geq 0$. The MNS [7] unitary matrix U relates flavour and mass eigenstate neutrinos according to $\nu_\alpha = U_{\alpha i} \nu_i$. The 'CKM' matrix, V, is defined through

$$U = \mathrm{diag}(e^{i\alpha_e}, e^{i\alpha_\mu}, e^{i\alpha_\tau}) \cdot V \cdot \mathrm{diag}(e^{-i\phi/2}, e^{-i\phi'/2}, 1), \tag{2}$$

and we use the standard parametrization

$$V = R_{23}(\theta_1) \cdot \mathrm{diag}(e^{-i\delta/2}, 1, e^{i\delta/2}) \cdot R_{31}(\theta_2) \cdot \mathrm{diag}(e^{i\delta/2}, 1, e^{-i\delta/2}) \cdot R_{12}(\theta_3), \tag{3}$$

where $R_{ij}(\theta_k)$ is a rotation in the i-j plane by the mixing angle θ_k, that can be taken $0 \leq \theta_k \leq \pi/2$ without loss of generality. Explicitly,

$$V = \begin{pmatrix} c_2 c_3 & c_2 s_3 & s_2 e^{-i\delta} \\ -c_1 s_3 - s_1 s_2 c_3 e^{i\delta} & c_1 c_3 - s_1 s_2 s_3 e^{i\delta} & s_1 c_2 \\ s_1 s_3 - c_1 s_2 c_3 e^{i\delta} & -s_1 c_3 - c_1 s_2 s_3 e^{i\delta} & c_1 c_2 \end{pmatrix}. \tag{4}$$

where $s_i \equiv \sin\theta_i$, $c_i \equiv \cos\theta_i$.

For later use, it is convenient to define the intermediate matrix W as

$$W = V \cdot \mathrm{diag}(e^{-i\phi/2}, e^{-i\phi'/2}, 1)\ . \tag{5}$$

The phases $\alpha_e, \alpha_\mu, \alpha_\tau$, in eq. (2) are unphysical and may be rotated away by a redefinition of the flavour-eigenstate neutrinos ν_e, ν_μ, ν_τ, so that U coincides with W. It

[1]Leaving aside the, as yet unconfirmed, LSND anomaly [5].

is useful, however, to keep these phases for the discussions of the next sections. The phases δ, ϕ and ϕ' are physical and responsible for CP violation in the leptonic sector (although only δ can have an effect in neutrino oscillation experiments). The physical phases ϕ and ϕ' (usually kept out of the diagonalization matrix) are extracted from the equation

$$V^T \mathcal{M}_\nu V = \mathrm{diag}(\tilde{m}_1, \tilde{m}_2, \tilde{m}_3), \tag{6}$$

where $\tilde{m}_1 = m_1 e^{i\phi}$, $\tilde{m}_2 = m_2 e^{i\phi'}$, $\tilde{m}_3 = m_3$.

Let us now summarize the experimental information on the neutrino sector as setting the following bounds:

From SK + CHOOZ data [1,8] we learn that θ_2 is small, with

$$\sin^2\theta_2 < 0.1\ (0.2), \tag{7}$$

at 90% (99%) C.L., or $\sin^2 2\theta_2 < 0.36\,(0.64)$. The smallness of θ_2 implies that the oscillations of atmospheric and solar neutrinos are dominantly two-flavour oscillations, described by a single mixing angle θ_i (precise analysis of the data must also include the subdominant effects due to a non-zero θ_2 [9]).

In this approximation, oscillations of atmospheric neutrinos are dominantly $\nu_\mu \to \nu_\tau$ with [10]

$$\begin{aligned} 5\times 10^{-4}\ \mathrm{eV}^2 < \Delta m^2_{atm} &< 10^{-2}\ \mathrm{eV}^2\ , \\ \sin^2 2\theta_1 \quad &> \quad 0.82\ . \end{aligned} \tag{8}$$

Several parameter choices are possible to interpret the oscillations of solar neutrinos [3]:

The large angle MSW solution (LAMSW), with

$$\begin{aligned} 10^{-5}\ \mathrm{eV}^2 < \ \Delta m^2_{sol}\ &< 2\times 10^{-4}\ \mathrm{eV}^2, \\ 0.5 < \sin^2 2\theta_3 &< 1. \end{aligned} \tag{9}$$

The small angle MSW solution (SAMSW), with

$$\begin{aligned} 3\times 10^{-6}\ \mathrm{eV}^2 < \ \Delta m^2_{sol}\ &< 10^{-5}\ \mathrm{eV}^2, \\ 2\times 10^{-3} < \sin^2 2\theta_3 &< 2\times 10^{-2}. \end{aligned} \tag{10}$$

And the vacuum oscillation solution (VO), with

$$5 \times 10^{-11}\ \text{eV}^2 < \Delta m^2_{sol} < 1.1 \times 10^{-10}\ \text{eV}^2,$$
$$\sin^2 2\theta_3 \quad > \quad 0.67\,. \tag{11}$$

Other relevant experimental information concerns the non-observation of neutrinoless double β-decay, which requires the ee element of the $\mathcal{M}_\nu$ matrix to be bounded as [11]

$$\mathcal{M}_{ee} \equiv |m_1\, c_2^2 c_3^2 e^{i\phi} + m_2\, c_2^2 s_3^2 e^{i\phi'} + m_3\, s_2^2\, e^{i2\delta}| \lesssim 0.2\ \text{eV}. \tag{12}$$

In addition, Tritium β-decay experiments indicate $m_i < 2.5$ eV for any mass eigenstate with a significant ν_e component [12]. Finally, no experimental information is available yet on the CP-violating phases.

To write the previous bounds, we have followed the standard convention that orders the three mass eigenvalues with m_3 as the most split eigenvalue, *i.e.* $|\Delta m^2_{21}| < |\Delta m^2_{32}|, |\Delta m^2_{31}|$ (where $\Delta m^2_{ij} \equiv m_i^2 - m_j^2$) and identifies $|\Delta m^2_{21}| = \Delta m^2_{sol}$, $|\Delta m^2_{31}| = \Delta m^2_{atm}$. Note that observations require $|\Delta m^2_{21}| \ll |\Delta m^2_{32}| \sim |\Delta m^2_{31}|$ (see however [13]). We do not adopt any particular ordering for m_1, m_2. Nevertheless, note that although the interchange of m_1, m_2 leaves $\sin^2 2\theta_1$ and $\sin^2 2\theta_2$ unaffected, it flips the sign of $\cos 2\theta_3$, which is important for the MSW effect [14].

The relative wealth of experimental data has motivated efforts on the theoretical side to understand the possible patterns of neutrino masses and to follow the hints that such patterns offer on the fundamental symmetries underlying them. A more modest approach, perhaps more powerful at this given time, is to study Standard Model (SM) [or Minimal Supersymmetric Standard Model (MSSM)] radiative corrections (of well known origin) to neutrino parameters. Although neutrinos are famous for having extremely weak couplings to stable matter, this is not so for all the virtual particles that may appear in loops. Moreover, as we have seen, some of the mass splittings suggested by the data are very small and could be easily upset by radiative corrections of modest size. It turns out that, in many cases of interest, radiative effects have a very significant impact on neutrino physics.

Many papers have dealt recently with these issues [15,16,17,18,19,20,21,22,23]. The most often considered scenario is to assume that a (Majorana) effective mass operator for the three light neutrinos is generated by some unspecified mechanism at a high energy scale, Λ. If, apart from this mass operator, the effective theory below Λ is just the SM or the MSSM, the lowest dimension operator of this kind is

$$\delta\mathcal{L} = -\frac{1}{4}\kappa_{ij}(H\cdot L_i)(H\cdot L_j) + \text{h.c.}, \tag{13}$$

where H is the SM Higgs (or the MSSM Higgs with the appropriate hypercharge), L_i are the lepton doublets, and κ_{ij} is a (symmetric) matricial coupling. The neutrino mass matrix is then $\mathcal{M}_\nu = \kappa v^2/2$, with $v = 246$ GeV (with this definition κ and $\mathcal{M}_\nu$ obey the same RGE). Below the scale Λ, the most important radiative corrections to $\mathcal{M}_\nu$, which are proportional to $\log(\Lambda/M_Z)$ (where the Z boson mass, M_Z, sets the low-energy scale of interest), can be most conveniently computed using renormalization group (RG) techniques. In short, one has to run $\mathcal{M}_\nu$ from Λ down to M_Z with the appropriate RGE to obtain the radiatively corrected neutrino mass matrix. These corrections have interesting implications for the neutrino masses, the mixing angles and the CP-violating phases.

As we will see, and has been discussed extensively in the literature, even if one starts with degenerate neutrino masses at the scale Λ, the relevant radiative corrections (with a typical size controlled by the tau Yukawa coupling squared and a loop suppression factor) tend to induce mass splittings that can be too large for some of the proposed solutions to the observed neutrino problems. They are particularly dangerous for the VO solution to the solar neutrino problem, and just about right for the SAMSW solution, which is certainly suggestive. They tend to be too small to explain the atmospheric mass splitting, which presumably requires to be generated by effects of the physics beyond Λ. One interesting possibility that, beginning with degenerate neutrinos, can produce radiatively a mass pattern in agreement with experiment is a see-saw scenario, for which the RG analysis was performed in [15,16].

Degenerate (or nearly degenerate) neutrino mass eigenvalues, besides being a very symmetrical initial condition (as befits the physics at a more fundamental scale) offer two important bonuses: the mass splittings at low-energy would be entirely due to radiative corrections and the mixing angles can evolve quickly to some particular values. The reason for the latter is the following. In the presence of degenerate mass eigenvalues there is an ambiguity in the choice of the eigenvectors which translates to the definition of the mixing angles. When a perturbation is added that removes that degeneracy, the ambiguity is resolved in a well defined way (which depends on the perturbation) and a particular configuration of mixing angles is chosen. This is exactly what happens when a (sufficiently precise) initial mass degeneracy of two neutrino masses is lifted by radiative corrections. In RG language, the mixing angles can be driven quickly to infrared quasi-fixed points. Thus, RG effects could in principle be capable of explaining some of the measured values of the mixing angles.

In this paper we extend previous work of us [15,16,17] and others [18,19,20,21,22,23] along these lines. Assuming three flavours of light Majorana neutrinos, we derive the general renormalization group equations (RGEs) for their three masses, three mixing angles and three CP-violating phases (information alternatively encoded in the RGE for the mass eigenvalues and the complex mixing matrix U). This form of writing the RGEs represents an advantageous alternative to using the RGE for $\mathcal{M}_\nu$ and, after obtaining the radiatively corrected mass matrix, extracting from it the physical parameters in whatever parametrization it is chosen. These two alternative methods can be described in short as 'diagonalize and run' versus 'run and diagonalize'. One advantage of the method presented here is that it avoids the proliferation of unphysical parameters (notice that $\mathcal{M}_\nu$ has 12 degrees of freedom, but only 9 are physical), which allows to keep track of the physics in a more efficient way. Another one is that it permits to write the RGEs in a quite general form, without any reference to a particular scenario, such as the SM, the MSSM, see-saw, etc. This allows to appreciate interesting features, *e.g.* the mentioned presence of stable (pseudo infrared fixed-point) directions for mixing angles and CP phases, which are not consequence of a particular scenario. Finally, this method allows to determine the physical conditions for the validity of certain approximations, such as the two-flavour approximation, in a reliable way.

In section 2, these general RG formulas are extracted from the most general form of the RGE for the neutrino mass matrix $\mathcal{M}_\nu$. These can be particularized to any model of interest. We give as particular examples the SM, the MSSM and a see-saw scenario. Some conclusions that hold in general are also discussed. In section 3, using the equations derived in the previous section, we study the implications of radiative corrections for neutrino physics in the SM case (or the MSSM) neglecting the effect of CP violating phases, so that the mixing matrix U is real. Section 4 extends this analysis to the more general case of non-zero CP-violating phases. Section 5 is devoted to the conclusions and the Appendix contains some technical details regarding the stability conditions for the mixing matrix U in the general case, and how they are reached.

2 RGEs for physical parameters

The energy-scale evolution of the 3×3 neutrino mass matrix $\mathcal{M}_\nu$ is generically described by a RGE of the form ($t=\log\mu$):

$$\frac{d\mathcal{M}_\nu}{dt} = -(\kappa_U\mathcal{M}_\nu + \mathcal{M}_\nu P + P^T\mathcal{M}_\nu), \tag{14}$$

(in particular $\mathcal{M}_\nu^T = \mathcal{M}_\nu$ is preserved). For interesting frameworks, such as the SM, the MSSM or the see-saw mechanism (supersymmetric or not), κ_U and P are known explicitly [24,25,15]. In (14), the term $\kappa_U\mathcal{M}_\nu$ gives a family-universal scaling of $\mathcal{M}_\nu$ which does not affect its texture, while the non family-universal part (the most interesting effect) corresponds to the terms that involve the matrix P.

The goal of this section is to extract the RGEs for the physical neutrino parameters: the mass eigenvalues, the mixing angles and the CP phases. We find convenient to work out the RGEs for the completely general case first (*i.e.* without specifying the form of κ_U and P), and later specialize them for particular cases of interest. This allows to appreciate interesting features, *e.g.* the presence of stable (pseudo infrared fixed-point) directions for angles and phases, which are not a consequence of the particular framework chosen.

Using the parametrization and conventions of sect. 1, one gets from eqs. (1, 14), after some amusing algebra, the RGEs for the mass eigenvalues and the MNS matrix

$$\frac{dm_i}{dt} = -2m_i\hat{P}_{ii} - m_i Re(\kappa_U), \tag{15}$$

$$\frac{dU}{dt} = UT. \tag{16}$$

We have defined

$$\hat{P} \equiv \frac{1}{2}U^\dagger(P+P^\dagger)U, \tag{17}$$

while T is an anti-hermitian (so that the unitarity of U is preserved by the RG running) 3×3 matrix with

$$\begin{aligned}
T_{ii} &\equiv i\hat{Q}_{ii},\\
T_{ij} &\equiv \frac{1}{(m_i^2-m_j^2)}\left[(m_i^2+m_j^2)\hat{P}_{ij} + 2m_im_j\hat{P}^*_{ij}\right] + i\hat{Q}_{ij}\\
&= \nabla_{ij}Re(\hat{P}_{ij}) + i[\nabla_{ij}]^{-1}Im(\hat{P}_{ij}) + i\hat{Q}_{ij}, \qquad i\neq j,
\end{aligned} \tag{18}$$

where

$$\hat{Q} \equiv -\frac{i}{2} U^\dagger (P - P^\dagger) U, \tag{19}$$

and

$$\nabla_{ij} \equiv \frac{m_i + m_j}{m_i - m_j}. \tag{20}$$

With the above definitions we can write

$$U^\dagger P U = \hat{P} + i\hat{Q}, \tag{21}$$

with $\hat{P}$ and $\hat{Q}$ hermitian. Note that the RGE for U does not depend on the universal factor κ_U, as expected.

From eqs. (16–20), we can derive the general RGEs for the mixing angles, the CP-violating phases and the unphysical phases. For this task, it is useful to define

$$\tilde{T}_{21} = T_{21} e^{i(\phi - \phi')/2}, \quad \tilde{T}_{31} = T_{31} e^{i\phi/2}, \quad \tilde{T}_{32} = T_{32} e^{i\phi'/2}, \tag{22}$$

which allows to conveniently absorb[2] the phases ϕ and ϕ' everywhere. Then we find the following expressions for the RGEs for the mixing angles:

$$\frac{d\theta_1}{dt} = \frac{1}{c_2} \; Re\left[s_3 \tilde{T}_{31} - c_3 \tilde{T}_{32}\right], \tag{23}$$

$$\frac{d\theta_2}{dt} = \; -Re\left[\left(c_3 \tilde{T}_{31} + s_3 \tilde{T}_{32}\right) e^{-i\delta}\right], \tag{24}$$

$$\frac{d\theta_3}{dt} = -\frac{1}{c_2} \; Re\left[c_2 \tilde{T}_{21} + \left(s_3 \tilde{T}_{31} - c_3 \tilde{T}_{32}\right) s_2 e^{-i\delta}\right]; \tag{25}$$

the RGEs for the CP-violating phases:

$$\frac{d\delta}{dt} = Im\left[\frac{1}{s_3 c_3} \tilde{T}_{21} + \left(\frac{V_{21}}{c_1 c_2 s_2} e^{-i\delta} - \frac{c_3 V_{21}^*}{s_1 c_2 s_3}\right) \tilde{T}_{32} - \left(\frac{s_3 V_{22}^*}{s_1 c_2 c_3} + \frac{V_{22}}{c_1 c_2 s_2} e^{-i\delta}\right) \tilde{T}_{31}\right], \tag{26}$$

$$\frac{1}{2}\frac{d\phi}{dt} = Im\left[\frac{c_3}{s_3} \tilde{T}_{21} + \left(\frac{V_{32}^*}{c_1 c_2} - \frac{c_3 V_{21}^*}{s_1 c_2 s_3}\right) \tilde{T}_{32} + \left(\frac{V_{31}^*}{c_1 c_2} + \frac{V_{21}^*}{s_1 c_2}\right) \tilde{T}_{31}\right] + \hat{Q}_{33} - \hat{Q}_{11}, \tag{27}$$

$$\frac{1}{2}\frac{d\phi'}{dt} = Im\left[\frac{s_3}{c_3} \tilde{T}_{21} + \left(\frac{V_{32}^*}{c_1 c_2} + \frac{V_{22}^*}{s_1 c_2}\right) \tilde{T}_{32} + \left(\frac{V_{31}^*}{c_1 c_2} - \frac{s_3 V_{22}^*}{s_1 c_2 c_3}\right) \tilde{T}_{31}\right] + \hat{Q}_{33} - \hat{Q}_{22}; \tag{28}$$

[2]The formulas we have written do not depend on our convention $m_i \geq 0$, which is just a definition for ϕ and ϕ', and we could change at will the sign of m_i. The definition of the quantities $\tilde{T}_{ij}$ makes them independent of such choices of sign conventions.

and the RGEs for the unphysical phases:

$$\frac{d\alpha_e}{dt} = Im\left[\frac{1}{c_3 s_3}\tilde{T}_{21} + \left(\frac{V_{32}^*}{c_1 c_2} - \frac{c_3 V_{21}^*}{s_1 c_2 s_3}\right)\tilde{T}_{32} + \left(\frac{V_{31}^*}{c_1 c_2} - \frac{s_3 V_{22}^*}{s_1 c_2 c_3}\right)\tilde{T}_{31}\right] + \hat{Q}_{33}, \tag{29}$$

$$\frac{d\alpha_\mu}{dt} = \frac{1}{s_1 c_2} Im\left[V_{21}^* \tilde{T}_{31} + V_{22}^* \tilde{T}_{32}\right] + \hat{Q}_{33}, \tag{30}$$

$$\frac{d\alpha_\tau}{dt} = \frac{1}{c_1 c_2} Im\left[V_{31}^* \tilde{T}_{31} + V_{32}^* \tilde{T}_{32}\right] + \hat{Q}_{33} \ . \tag{31}$$

Some comments are in order at this point: 1) the RGE (16) for U is only satisfied if the unphysical phases are included [that is, in the general case W does not satisfy eq. (16)]. The reason for this is that the phases $\alpha_e, \alpha_\mu, \alpha_\tau$, that translate between U and the physical mixing matrix W, depend on the details in $\mathcal{M}_\nu$. When these details change through RG running the phases to be rotated away also change [thus eqs. (29–31)]. However, the explicit RGEs for the physical parameters m_i and δ, ϕ, ϕ' do not depend on $\alpha_e, \alpha_\mu, \alpha_\tau$ as it should be the case; 2) from (26–28) we also see that, if no phases are present in U originally, they are not generated by radiative corrections (unless P contains phases. If it does, one could generate radiatively small non-zero phases in U); 3) from the structure of T_{ij} (or ∇_{ij}) we see that large renormalization effects can be expected in two cases: a) if some couplings in P are large (not the case of the SM) or b) if there are (quasi)-degenerate mass eigenstates at tree level (causing $|\nabla_{ij}| \gg 1$).

An aspect of central importance in the discussion of the RG effects is that when the neutrino spectrum has (quasi)-degenerate eigenvalues, one expects large (even infinite for exact degeneracy) contributions to dU/dt. The reason is the following. When two mass eigenvalues, say i and j, are equal, there is an ambiguity in the choice of the associated eigenvectors, and thus in the definition of U. In particular, the columns i, j could be rotated at will, *i.e.* the matrix UR_{ij}, where R is an arbitrary rotation in the i-j plane, will also diagonalize the initial $\mathcal{M}_\nu$ matrix. When the perturbation due to RG running is added, the degeneracy $m_i = m_j$ will be normally lifted and a particular rotation of the columns i, j will be singled out: $U' = UR_{ij}$, giving $T'_{ij} \simeq 0$ and removing the singularity in eq. (16). The form of R is thus determined by the matrix P. In the Appendix we give the derivation of R for a general P. The matrix U' may be very different from the form of U originally chosen, thus the initial big jump in the evolution of U. It is easy to see, however, that the subsequent evolution of the matrix U will be smooth, even in the presence of a large $|\nabla_{ij}|$. If the initial degeneracy is not exact, but nearly so, then $|\nabla_{ij}| \gg 1$ and the initial U, whatever it is, will be rapidly driven by the RG-running close to the stable U' form. Thus, U' plays the role of an infrared pseudo-fixed point. Therefore, if some initial neutrino masses are (exactly or approximately) degenerate, we have the interesting possibility of predicting some low-energy mixing angles or CP phases just from radiative corrections. This possibility will be exploited in the next sections.

It is also useful to indicate that, in cases for which one particular $\tilde{T}_{ij}$ gives the dominant contribution to the RGEs, the following quantities are approximately conserved (here k can take any value):

$$|U_{ki}|^2 + |U_{kj}|^2 \ , \tag{32}$$

$$Im(U_{ki}^* U_{kj}) \quad , \tag{33}$$

$$[\nabla_{ij}]^{-1} Re(\hat{P}_{ij}) \ , \tag{34}$$

as can re readily proved by using (16) and keeping only terms $\sim T_{ij}$ (for the last conserved quantity we assume that dP/dt is never of the order of T_{ij}, which is quite reasonable to expect).

We turn now to particularly interesting scenarios. In the SM the expressions for κ_U and P are given by [24,25]

$$\kappa_U = \frac{1}{16\pi^2}[3g_2^2 - 2\lambda - 6h_t^2 - 2\mathrm{Tr}(\mathbf{Y_e^\dagger Y_e})], \tag{35}$$

where $g_2, \lambda, h_t, \mathbf{Y_e}$ are the $SU(2)$ gauge coupling, the quartic Higgs coupling, the top-Yukawa coupling and the matrix of Yukawa couplings for the charged leptons, respectively; and

$$P = \frac{1}{32\pi^2}\mathbf{Y_e^\dagger Y_e} \simeq \frac{h_\tau^2}{32\pi^2}\mathrm{diag}(0,0,1), \tag{36}$$

where h_τ is the tau-Yukawa coupling. In the following, we will work in the approximation of neglecting the electron and muon Yukawa couplings.

In the MSSM one has instead [24,25]

$$\kappa_U = \frac{1}{16\pi^2}\left[\frac{6}{5}g_1^2 + 6g_2^2 - 6\frac{h_t^2}{\sin^2\beta}\right], \tag{37}$$

where g_1 is the $U(1)$ gauge coupling, $\tan\beta$ is the supersymmetric parameter given by the ratio of the two Higgs vevs; and

$$P = -\frac{1}{16\pi^2}\frac{\mathbf{Y_e^\dagger Y_e}}{\cos^2\beta} \simeq -\frac{1}{16\pi^2}\frac{h_\tau^2}{\cos^2\beta}\mathrm{diag}(0,0,1). \tag{38}$$

Note that we could have written these equations in terms of the supersymmetric couplings $\tilde{h}_t = h_t/sin\beta$, $\tilde{h}_\tau = h_\tau/\cos\beta$ and $\tilde{\mathbf{Y}}_{\mathbf{e}} = \mathbf{Y}_{\mathbf{e}}/\cos\beta$.

In both cases, the previous formulas (15, 16) apply with [see (36–38)]

$$\hat{P}_{ii} = -\kappa_\tau |U_{3i}|^2, \tag{39}$$

$$T_{ii} \equiv i\hat{Q}_{ii} = 0, \tag{40}$$

$$T_{ij} = \kappa_\tau \left[\nabla_{ij} Re(U_{3i}^* U_{3j}) + i\nabla_{ij}^{-1} Im(U_{3i}^* U_{3j})\right], \tag{41}$$

where

$$\kappa_\tau = \frac{h_\tau^2}{32\pi^2}, \tag{42}$$

for the SM case, and

$$\kappa_\tau = -\frac{1}{16\pi^2}\frac{h_\tau^2}{\cos^2\beta}, \tag{43}$$

for the MSSM (note how κ_τ is enhanced for large $\tan\beta$). For future use we also define the related quantity

$$\epsilon_\tau \equiv \kappa_\tau \log(\Lambda/M_Z). \tag{44}$$

Hence, the RGEs for the neutrino masses given by eq. (15), as well as the RGEs for the mixing angles, the CP-violating phases and the unphysical phases given by eqs. (23–31), hold with the substitutions of eqs. (39, 41).

One can apply the general results to study other cases of interest beyond the SM or the MSSM. One particularly relevant case is the see-saw scenario [26]. The model includes three heavy right-handed neutrinos N_i with Majorana masses given by a 3×3 matrix $\mathcal{M}$ with overall scale $M \gg M_Z$ such that the see-saw mechanism is implemented. Above M, the effective neutrino mass matrix is given by

$$\mathcal{M}_\nu = m_D^T \mathcal{M}^{-1} m_D, \tag{45}$$

where m_D is an ordinary Dirac mass matrix coming from the conventional Yukawa couplings between the left-handed neutrinos, ν_e, ν_μ, ν_τ, and the right-handed ones. The running of $\mathcal{M}$ and m_D is of the general form

$$\frac{d\mathcal{M}}{dt} = \mathcal{M} P_M + P_M^T \mathcal{M}, \tag{46}$$

and

$$\frac{dm_D}{dt} = m_D(\kappa_U' I_3 + P_D), \tag{47}$$

where κ'_U gives a family universal contribution. Combining (46) and (47) with (45) one gets an RGE for $\mathcal{M}_\nu$ above the Majorana scale M of the form (14) with

$$P = m_D^{-1} P_M m_D - P_D, \tag{48}$$

where we have assumed that m_D has an inverse. Applying the general formulas to this particular case one can then easily reproduce previous results derived in the literature for this kind of scenarios [15,16].

3 Implications (real case)

The case in which CP-violating phases are zero, and thus the 'CKM' matrix V is real, is the one most extensively studied in the literature, perhaps because there is not any information yet about leptonic CP violation. This section is devoted to the study of this particularly interesting scenario, assuming that below the high-energy scale, Λ, the effective theory is just the SM or the MSSM, with an effective Majorana mass matrix for the neutrinos. It is convenient in this real case to work with the mass eigenvalues $\tilde{m}_i$ defined in eq. (6)

$$\tilde{m}_1 = m_1 e^{i\phi}, \qquad \tilde{m}_2 = m_2 e^{i\phi'}, \qquad \tilde{m}_3 = m_3, \tag{49}$$

where $\phi, \phi' = 0$ or π. All $\tilde{m}_i$'s are now real, but $\tilde{m}_1$ and $\tilde{m}_2$ can be negative. We also define the quantities $\tilde{\nabla}_{ij}$ as

$$\tilde{\nabla}_{ij} \equiv \frac{\tilde{m}_i + \tilde{m}_j}{\tilde{m}_i - \tilde{m}_j}. \tag{50}$$

Then, neglecting all the charged-lepton Yukawa couplings except h_τ, the RG equations for the mass eigenvalues and the matrix V read

$$\frac{d\tilde{m}_i}{dt} = -2\kappa_\tau \tilde{m}_i V_{3i}^2 - \tilde{m}_i \kappa_U, \tag{51}$$

$$\frac{dV}{dt} = VT, \tag{52}$$

where κ_U is given in eqs. (35, 37), κ_τ is given in eqs. (42,43) and

$$\begin{aligned} &T_{ii} = 0, \\ &T_{ij} = \kappa_\tau \tilde{\nabla}_{ij} V_{3i} V_{3j} \ \ (i \neq j) \ . \end{aligned} \tag{53}$$

The RGEs for the mixing angles are

$$\frac{d\theta_1}{dt} = -\kappa_\tau c_1(-s_3 V_{31}\tilde{\nabla}_{31} + c_3 V_{32}\tilde{\nabla}_{32}), \tag{54}$$

$$\frac{d\theta_2}{dt} = -\kappa_\tau c_1 c_2(c_3 V_{31}\tilde{\nabla}_{31} + s_3 V_{32}\tilde{\nabla}_{32}), \tag{55}$$

$$\frac{d\theta_3}{dt} = -\kappa_\tau(c_1 s_2 s_3 V_{31}\tilde{\nabla}_{31} - c_1 s_2 c_3 V_{32}\tilde{\nabla}_{32} + V_{31}V_{32}\tilde{\nabla}_{21})\ . \tag{56}$$

From eqs. (52, 53), the first conclusion is that the radiative corrections to the matrix V will be very small unless $|\kappa_\tau\tilde{\nabla}_{ij}| \gtrsim 1$ for some i,j. This generically requires mass degeneracy (both in the absolute value and the sign), except for the supersymmetric case with very large $\tan\beta$, and thus large κ_τ. The interesting cases then are a completely degenerate spectrum $m_1^2 \simeq m_2^2 \simeq m_3^2$ or an intermediate one with $m_1^2 \simeq m_2^2 \not\simeq m_3^2$.

The initial degree of degeneracy of two given neutrino masses, say i,j, plays a crucial role for the subsequent RG evolution of V. In particular, it is important how small the initial mass splitting $\Delta\tilde{m}_{ij}^{(0)}$ is compared with the typical one generated by RG evolution $|\delta_{RGE}\Delta\tilde{m}_{ij}| \sim |\epsilon_\tau| m_0$, where m_0 is the average mass scale of the two neutrinos. There are four possibilities: 1) If $\Delta\tilde{m}_{ij}^{(0)} = 0$, there is exact degeneracy and V jumps immediately to V' (note that we could have redefined the initial V to coincide exactly with the stable matrix V' to start with, and then the evolution is smooth). 2) If $|\Delta\tilde{m}_{ij}^{(0)}| \lesssim 2|\kappa_\tau| m_0$ (*i.e.* $|\kappa_\tau\tilde{\nabla}_{ij}| \gtrsim 1$), V will quickly reach V' (we will be more precise in the next subsection). 3) If $2|\kappa_\tau| m_0 \lesssim |\Delta\tilde{m}_{ij}^{(0)}| \lesssim 2|\epsilon_\tau| m_0$ (*i.e.* , $[\log(\Lambda/M_Z)]^{-1} \lesssim |\kappa_\tau\tilde{\nabla}_{ij}| \lesssim 1$), V will tend to approach V' but may not reach it if the total amount of running is not long enough (other possible effects in this case will be discussed later on). 4) Finally, if $|\Delta\tilde{m}_{ij}^{(0)}| \gtrsim 2|\epsilon_\tau| m_0$, no dramatic effects are expected from RG corrections to V (even if still the masses can be considered degenerate in the sense $|\Delta\tilde{m}_{ij}^{(0)}|/m_0 \ll 1$).

For the discussion of cases of physical relevance, we will assume that the solar neutrino problem is solved by one of the standard solutions, so that $|\Delta m_{21}^2| \ll |\Delta m_{32}^2| \sim |\Delta m_{31}^2|$. In some of the cases that we will analyze (in fact, in the most interesting cases) the initial values for some of the m_i's are degenerate, so there exists an ambiguity in the labelling, which generically is removed after RG running. If the initial m_i's are only approximately degenerate (which we call quasi-degenerate), in principle there is no such ambiguity, but the RG running may alter the relative size of the Δm_{ij}^2 splittings, thus requiring a relabelling of the m_3 eigenvalue at low energy. Let us study in turn the impact (and physical implications) of the RG running on the neutrino masses and the mixing angles.

3.1 Mixing angles

As we have discussed, in cases with some (sufficiently) degenerate neutrino masses, large RG corrections to the matrix U have the effect of driving it close to a stable form (an infrared pseudo-fixed point). This rises the interesting possibility of predicting (at least partially) low-energy mixing angles just from radiative effects. In the event of $m_i \simeq m_j$, the stable form of V, say V', is characterized by the condition $T'_{ij} \simeq 0$, which removes potential singularities in eq. (52), and requires $V'_{3i} = 0$ or $V'_{3j} = 0$. The sign of the initial $\kappa_\tau \tilde{\nabla}_{ij}$ will determine which one is realized, as we will see shortly.

We can estimate more precisely how V evolves to V' in the following way. Suppose there are two nearly degenerate neutrinos with $|\epsilon_\tau \tilde{\nabla}_{ij}| \gtrsim 1$, so that this term dominates the RGEs of V_{3i}, V_{3j} and $\tilde{\nabla}_{ij}$ itself:

$$\frac{dV_{3i}^2}{dt} \simeq -\frac{dV_{3j}^2}{dt} = -2\kappa_\tau V_{3i}^2 V_{3j}^2 \tilde{\nabla}_{ij} \ , \tag{57}$$

$$\frac{d\tilde{\nabla}_{ij}^{-1}}{dt} \simeq \kappa_\tau \left(V_{3j}^2 - V_{3i}^2 \right) . \tag{58}$$

Eq. (57) implies that, in the infrared, $V_{3j} \to 0$ ($V_{3i} \to 0$) for positive (negative) initial $\kappa_\tau \tilde{\nabla}_{ij}$. Without loss of generality, we can choose here the labels i, j so that $\kappa_\tau \tilde{\nabla}_{ij}^{(0)} > 0$, where the superscript (0) denotes the initial (high-energy) value. Hence, $V_{3j} \to 0$. Eqs. (57, 58) imply

$$\frac{d}{dt}\left[\frac{V_{3i}V_{3j}}{\tilde{\nabla}_{ij}}\right] \simeq 0. \tag{59}$$

The conserved quantity $V_{3i}V_{3j}/\tilde{\nabla}_{ij}$ is the SM real version of the general conserved quantity given by eq. (34). Notice that $[\tilde{\nabla}_{ij}]^{-1} = (\tilde{m}_i^2 - \tilde{m}_j^2)/(\tilde{m}_i + \tilde{m}_j)^2$ is a measure of the relative splitting of $\tilde{m}_i$ and $\tilde{m}_j$. For quasi-degenerate masses, $[\tilde{\nabla}_{ij}]^{-1} \simeq \Delta m_{ij}^2/4m^2$.

From (58) we find

$$[\nabla_{ij}]^{-1} = [\nabla_{ij}^{(0)}]^{-1} - \epsilon_\tau \langle V_{3j}^2 - V_{3i}^2 \rangle, \tag{60}$$

where by $\langle V_{3j}^2 - V_{3i}^2 \rangle$ we mean the average value in the interval of running. When $V \to V'$ quickly, we can estimate that, with our choice of indexes

$$\langle V_{3j}^2 - V_{3i}^2 \rangle \sim ({V'}_{3j}^2 - {V'}_{3i}^2) \sim -\frac{1}{2}, \tag{61}$$

(note that $V'_{3j} \to 0$ by the running, while $V'^2_{3i} \sim 1/2$ is required to agree with experimental indications).

Using (60) in the conservation equation (59) we end up with the useful relation

$$\frac{V_{3i}V_{3j}}{V_{3i}^{(0)}V_{3j}^{(0)}} = \left[1 - \epsilon_\tau \langle V_{3j}^2 - V_{3i}^2 \rangle \tilde{\nabla}_{ij}^{(0)}\right]^{-1}. \tag{62}$$

This equation is interesting since it gives a measure of the change in $V_{3i}V_{3j}$, and thus in V, as a function of the initial $\tilde{\nabla}_{ij}$ and the amount of running. The stable form of V is achieved for $V_{3i}V_{3j} \to 0$. From eqs. (61,62) we can evaluate the initial $\tilde{\nabla}_{ij}$ to get $V_{3i}V_{3j} = V_{3i}^{(0)}V_{3j}^{(0)}/F$, where F is an arbitrary factor

$$V_{3i}V_{3j} = V_{3i}^{(0)}V_{3j}^{(0)}/F \;\Rightarrow\; \kappa_\tau \tilde{\nabla}_{ij}^{(0)} \simeq \frac{2(F-1)}{\log \frac{\Lambda}{M_Z}}. \tag{63}$$

If V finishes close to the stable form, that means $F \gg 1$. For example, for $F > 10$ and $\Lambda = 10^{10}$ GeV we need $\kappa_\tau \tilde{\nabla}_{ij}^{(0)} > 1$.

Notice that the condition to get $F \gg 1$, from eq. (63), can be restated as $[\Delta m_{ij}^2]^{(0)} \ll 2\epsilon_\tau m_0^2 \sim \delta_{RGE} \Delta m_{ij}^2$. This means that the final value for Δm_{ij}^2 must be essentially given by $\delta_{RGE} \Delta m_{ij}^2$, that is

$$\Delta m_{ij}^2 \simeq 2\kappa_\tau m^2 \log \frac{\Lambda}{M_Z}. \tag{64}$$

This is interesting since it implies that if some $|\tilde{\nabla}_{ij}|$ is initially large enough to drive the matrix V into a stable form, not only the final mixing angles, but also the final Δm_{ij}^2 splitting will be determined just by the radiative corrections. This result will be useful for the analysis of cases of physical interest.

If V does not finish close to the stable form, $|V_{3i}V_{3j}|$ may increase instead of going to zero (thus the value of F could be less than 1). For this to happen, one must start with $\kappa_\tau V_{3i}^2 < \kappa_\tau V_{3j}^2$. Then $[\nabla_{ij}]^{-1}$ decreases (as the mass eigenvalues get closer) and by eq. (59), $|V_{3i}V_{3j}|$ initially grows. If the running stops before the eventual decreasing of $|V_{3i}V_{3j}|$ below its initial value, then $\epsilon_\tau \langle V_{3j}^2 - V_{3i}^2 \rangle > 0$ in eq. (62). Notice that a large increasing in $V_{3i}V_{3j}$ needs a cancellation in that equation, namely $\frac{1}{2}|\kappa_\tau| \log \frac{\Lambda}{M_Z} \simeq (1-F) \left|\tilde{\nabla}_{ij}^{(0)}\right|^{-1}$, which implies a certain amount of fine-tuning (the initial mass splitting must be of the order of the one generated by running, which implies a correlation between two quantities of totally different physical origins).

In the rest of the section we will consider the physics of the different types of spectrum which are relevant for the evolution of V (and thus of the mixing angles). They are the following:

$$\begin{aligned}
&(i) &&\tilde{m}_a \simeq \tilde{m}_b \simeq \tilde{m}_c \Rightarrow |\tilde{\nabla}_{ab}|, |\tilde{\nabla}_{bc}|, |\tilde{\nabla}_{ac}| \gg 1, \\
&(ii) &&-\tilde{m}_a \simeq -\tilde{m}_b \simeq \tilde{m}_c \Rightarrow |\tilde{\nabla}_{ab}| \gg 1, \\
&(iii) &&\tilde{m}_a \simeq \tilde{m}_b \not\simeq \pm\tilde{m}_c \Rightarrow |\tilde{\nabla}_{ab}| \gg 1, \\
&(iv) &&-\tilde{m}_a \simeq \tilde{m}_b \not\simeq \pm\tilde{m}_c.
\end{aligned} \tag{65}$$

We leave the indices unspecified as they will be determined only after RG evolution (recall that m_3 is defined as the most split eigenvalue). From the point of view of the running of the matrix V, what matters are the sizable $|\tilde{\nabla}_{ij}|$. So, case *(iv)* is trivial (the matrix V changes little) and cases *(ii)–(iii)* can be analyzed simultaneously.

Before embarking in the discussion of these cases, note that experimental observations require

$$V_{3i} \simeq (s_3/\sqrt{2}, -c_3/\sqrt{2}, \pm 1/\sqrt{2}), \tag{66}$$

with different values for θ_3 depending on the choice of solution to the solar neutrino problem.

Case (i)

In case *(i)*, if $|\kappa_\tau \tilde{\nabla}_{ij}| > 1$ for all i,j pairs, then $V \to V'$ with $V'_{3i}V'_{3j} \to 0$, and this is obviously incompatible with the desired form (66). Possible ways out would require starting with nearly degenerate masses but not so close as to drive V to its infrared fixed-point form. Such scenarios really belong in one of the next cases.

Cases (ii)– (iii)

Since m_3 is by definition the most split eigenvalue, it is clear that in case *(iii)* the label c corresponds to 3. This may not be so for case *(ii)*, although we will see that a correct phenomenology requires it too. For the moment we will maintain the a, b, c labels.

In these cases the RGEs for the matrix V are dominated by the T_{ab} terms. Explicitly,

$$\begin{aligned}
\frac{dV_{3a}}{dt} &\simeq -\kappa_\tau V_{3b}^2 V_{3a} \tilde{\nabla}_{ab}, \\
\frac{dV_{3b}}{dt} &\simeq \kappa_\tau V_{3a}^2 V_{3b} \tilde{\nabla}_{ab}, \\
\frac{dV_{3c}}{dt} &\simeq 0.
\end{aligned} \tag{67}$$

If initially $\kappa_\tau \tilde{\nabla}_{ab} > 0$ ($\kappa_\tau \tilde{\nabla}_{ab} < 0$), then $V_{3b} \to 0$ ($V_{3a} \to 0$) in the infrared. From (66) we see that only V_{31} or V_{32} can be accepted to vanish [actually, this is guaranteed for case *(iii)*, since $c = 3$]. As we are free to interchange the labels 1,2, we choose $V_{31} \to 1$. On the other hand, the radiatively corrected masses are given by the expression (we absorb the universal scaling factor in a multiplicative redefinition of $\tilde{m}_i$)

$$\tilde{m}_i \to \tilde{m}_i (1 + 2\epsilon_\tau \langle V_{3i}^2 \rangle), \tag{68}$$

where, as usual, $\langle V_{3i}^2 \rangle$ is the average value of V_{3i}^2 in the interval of running. If $|\kappa_\tau \tilde{\nabla}_{ij}| > 1$ then $V \to V'$ quickly and we can set $\langle V_{3i}^2 \rangle \simeq {V'}_{3i}^2$.

Eq. (66) and $V_{31} \to 0$ imply that ${V'}_{32}^2 \simeq {V'}_{33}^2 \simeq 1/2$. We have then, from eq. (68),

$$\begin{aligned} \tilde{m}_1 &\to \tilde{m}_1, \\ \tilde{m}_2 &\to \tilde{m}_2 (1 + \epsilon_\tau), \\ \tilde{m}_3 &\to \tilde{m}_3 (1 + \epsilon_\tau). \end{aligned} \tag{69}$$

The conventional labelling requires $|\Delta m_{32}^2|, |\Delta m_{31}^2| \gg |\Delta m_{21}^2|$, so $\tilde{m}_2^2$ and $\tilde{m}_3^2$ must have an initial splitting large enough $\sim \Delta m_{atm}^2$ from the beginning (it will not be generated by the running)[3]. From eq. (65) we conclude that also in case *(ii)* c must be 3, in order to guarantee a correct phenomenology. On the other hand, as was discussed before, if initially $|\kappa_\tau \tilde{\nabla}_{21}| > 1$, so that V reaches the stable form V', then the low-energy mass splitting is essentially the one generated by the running:

$$\Delta m_{21}^2 \simeq 2 |\epsilon_\tau| m_0^2. \tag{70}$$

Interestingly enough, this can be naturally of the right size for the SAMSW solution (3×10^{-6} eV$^2 < \Delta m_{sol}^2 < 10^{-5}$ eV^2). For example, in the intermediate cases *(iii)* this is achieved with sizeable values of the cut-off ($\Lambda \gtrsim 10^{12}$ GeV) and/or working in the supersymmetric scenario.

The final (stable) form of V in this case is

$$V' \simeq \begin{pmatrix} V'_{11} & V'_{12} & V_{13}^{(0)} \\ V'_{21} & V'_{22} & V_{23}^{(0)} \\ 0 & V'_{32} & V_{33}^{(0)} \end{pmatrix} , \tag{71}$$

[3]Conversely, if we do not start with such a splitting, eq. (69) indicates that m_1 becomes the most split eigenvalue, so it should be relabelled as m_3, leading to $V_{33} \to 0$, which is not acceptable.

where $V_{ij}^{(0)}$ denote the initial V_{ij} values. The values of the remaining V'_{ij} can be straightforwardly written in terms of the former using the unitarity of the matrix V' and the condition $V'_{31} = 0$. Moreover, we know that at low energy $s_2 \simeq 0$, $s_1 \simeq \pm 1/\sqrt{2}$, which allows to fill the third column of eq. (71), and thus the complete final matrix V'

$$V' \simeq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & \mp 1/\sqrt{2} & \pm 1/\sqrt{2} \end{pmatrix} , \tag{72}$$

This implies $\sin^2 2\theta_3 \simeq 0$, which is also consistent with the SAMSW solution. It is worth-stressing that the value of $\sin^2 2\theta_3$ is obtained just from the RG-running, independently of its initial value[4]. This was noted in ref. [17] for the (intermediate) case *(iii)* , but clearly it also works for the completely degenerate case *(ii)*. Let us also notice that the neutrinoless double β-decay constraint implies that case *(ii)* is only consistent if $m \lesssim 0.2$ eV, see eq. (12).

3.2 Neutrino masses

The first consequence from eq. (51) is that neutrino mass differences get small modifications, unless the scenario is supersymmetric and $\tan\beta \gg 1$, so that $\kappa_\tau \gtrsim \mathcal{O}(1)$. This means in particular that if the neutrino spectrum is hierarchical, *i.e.* $m_1^2 < m_2^2 \ll m_3^2$, satisfying $\Delta m_{ij}^2 \sim \max\{m_i^2, m_j^2\}$, then radiative effects are not going to appreciably change the spectrum and the mass splittings: $|\delta_{RGE}\Delta m_{ij}^2| \ll |\Delta m_{ij}^2|$.

However, in a completely or partially degenerate neutrino scenario, *i.e.* $m_1^2 \simeq m_2^2 \simeq m_3^2$ or $m_1^2 \simeq m_2^2 \not\simeq m_3^2$ respectively, the shifts induced by the RGEs may be of the order or larger than the initial ones. They could also be larger than those required to explain solar neutrino oscillations. As for the mixing angles, the relevant cases in the study of the RGEs for the masses involve some complete or partial degeneracy. There is an important difference however. We learnt from the previous subsection that the mixing angles can only appreciably change if some $|\kappa_\tau \tilde{\nabla}_{ij}| \gtrsim 1$. For the mass splittings, there can be cases where, even for much smaller $|\kappa_\tau \tilde{\nabla}_{ij}|$ the RG effects are physically relevant. Actually, they can be very important too if the signs of the masses are opposite, $\tilde{m}_i \simeq -\tilde{m}_j$, in clear contrast with the mixing angles (notice that in this case $|\tilde{\nabla}_{ij}| \ll 1$). In particular, starting from a high-energy scale, Λ, down to M_Z the

[4]Note, however, that the value of V_{33} in this case is not affected by the RG running and thus has to be fixed by physics beyond Λ.

"solar" splitting Δm^2_{21} can get a RG correction

$$\delta_{RGE}\Delta m^2_{21} \sim 4m^2\kappa_\tau\langle V^2_{32}-V^2_{31}\rangle \log\frac{\Lambda}{M_Z}, \tag{73}$$

where $\langle V^2_{32}-V^2_{31}\rangle$ is to be understood as an average value in the interval of running. In degenerate scenarios this is typically much larger than the splitting required for the VO solution, unless $\langle V^2_{32}-V^2_{31}\rangle \simeq 0$, *i.e.* $(s_1s_3-c_1s_2c_3)^2 \simeq (-s_1c_3-c_1s_2s_3)^2$ or, equivalently [17,20],

$$\tan 2\theta_3 \simeq \frac{\cos^2\theta_1\sin^2\theta_2-\sin^2\theta_1}{\sin\theta_2\sin 2\theta_1}. \tag{74}$$

This can be most naturally achieved with $s_2 \simeq 0$, $\sin^2 2\theta_3 \simeq 1$, which amounts to a scenario close to bimaximal mixing, although it is not the only possibility. On the other hand, this way-out for the VO solution requires that $V^2_{32}-V^2_{31}$ must be stable along the RG running. This implies in turn that both V_{32} and V_{31} must be stable. To see this notice that if $\tilde\nabla_{21}$ is dominant, then $V^2_{32}+V^2_{31} \simeq$ const., while if $\tilde\nabla_{32}$ is dominant, then $V^2_{31} = 1-V^2_{32}-V^2_{33} \simeq$ const. As discussed in the previous subsection, and it is apparent from eq. (52), V will change little, unless some $|\kappa_\tau\tilde\nabla_{ij}| \gtrsim 1$, which corresponds precisely to a (at least partially) degenerate case in which the $\tilde m_i$ and $\tilde m_j$ masses have equal signs ($|\kappa_\tau\tilde\nabla_{ij}|$ could also be sizeable if the scenario is supersymmetric with very large $\tan\beta$). Let us analyze all the possibilities.

If $\tilde m_1 \simeq \tilde m_2$, *i.e.* both have the same sign, then $|\tilde\nabla_{21}|$ is the largest $|\tilde\nabla_{ij}|$. $|\tilde\nabla_{21}|$ should not be large enough to lead the matrix V to its stable form, as that would imply either $V_{31}\to 0$ or $V_{32}\to 0$. But this means that Δm^2_{21} cannot be too small initially (or along the running). More precisely, it must be $\Delta m^2_{21} \gtrsim 2|\epsilon_\tau|m^2$. In the case $m^2_1 \simeq m^2_2 \gg m^2_3$ this is too large for the VO solution since m^2 must be at least $\sim \Delta m^2_{atm}$. The opposite case, with $m^2_1 \simeq m^2_2 \ll m^2_3$, can be viable but only if the degenerate pair has very small masses, $m^2 \lesssim 10^{-6}\,\mathrm{eV}^2$.

With that caveat in mind, if $m^2_1 \sim m^2_2$, the VO scenario requires $\tilde m_1 \simeq -\tilde m_2$, to avoid a disastrously large $|\tilde\nabla_{21}|$. In that case, $|\tilde\nabla_{21}| \ll 1$ and $\tilde\nabla_{32}$ (or equivalently $\tilde\nabla_{31}$ if it is $\tilde m_1$ the one with the same sign as $\tilde m_3$) will be dominant. Still we need from eq. (73) $\langle V^2_{32}-V^2_{31}\rangle \simeq 0$ with high accuracy, so one has to guarantee that the matrix V does not change much, to avoid dangerous changes in $V^2_{32}-V^2_{31}$. In particular, to avoid running into the stable form of V we require again $\Delta m^2_{32} \gtrsim 2m^2\kappa_\tau \log\frac{\Lambda}{M_Z}$. In the present case, this condition is easily satisfied for realistic Δm^2_{32}, so it is natural to get a quite stable matrix V, as required. But still, $V_{32}^2 - V_{31}^2$ will (slightly) change according to

$$\frac{d(V_{32}^2 - V_{31}^2)}{dt} \simeq 2\kappa_\tau V_{33}^2 V_{32}^2 \tilde{\nabla}_{32}. \tag{75}$$

Then, we can integrate eq. (75) in leading log approximation and substitute in eq. (73) to obtain

$$\delta_{RGE}\Delta m_{21}^2 \sim -4m^2\kappa_\tau^2 V_{33}^2 V_{32}^2 \tilde{\nabla}_{32} \log^2\frac{\Lambda}{M_Z} \simeq -\frac{1}{2}m^2\kappa_\tau^2\tilde{\nabla}_{32}\log^2\frac{\Lambda}{M_Z}\ , \tag{76}$$

(we have used $\langle V_{33}^2 V_{32}^2\rangle \simeq 1/16$) which implies

$$\Delta m_{21}^2 \equiv \Delta m_{sol}^2 > 2\frac{m^4}{\Delta m_{atm}^2}\kappa_\tau^2 \log^2\frac{\Lambda}{M_Z}. \tag{77}$$

This condition can be satisfied, but is barely compatible with a relevant cosmological role for neutrino masses and the VO solution (for the MSSM this conclusion is stronger as κ_τ is larger). On the other hand, condition (77) is easily fulfilled in a partially degenerate scenario (the so-called pseudo-Dirac or intermediate case, $m_1^2 \sim m_2^2 \gg m_3^2$), where $m^2 \sim \Delta m_{atm}^2$. Let us also remark that the splitting induced by the RGEs is potentially so large as compared to that required for the VO solution, that the constraint (74) must be satisfied with enormous accuracy [if some symmetry is invoked [20] to justify (74), it must be either exact or minutely broken].

The previous paragraphs summarize and extend to the general case the results from refs.[15,16,17] concerning the mass splittings induced by radiative corrections, especially in relation to the viability of the VO solution.

3.3 The two-flavour case

It has been claimed [18,19,25] that, working in the two flavour (ν_μ, ν_τ) approximation, the RG running could drive the atmospheric angle $\sin^2 2\theta_1$ from a nearly minimal value at high energy to nearly maximal at low energy. This is an interesting possibility, since $\sin^2 2\theta_1$ is known to be nearly maximal. This subsection is devoted to this particular issue on the light of the previous discussions in this section.

The analysis of refs.[18,19,25] was based on the RG equation for $\sin^2 2\theta_1$ [25]

$$\frac{d}{dt}\sin^2 2\theta_1 = 2\kappa_\tau\ \sin^2 2\theta_1 \cos^2 2\theta_1 \frac{\mathcal{M}_{33}+\mathcal{M}_{22}}{\mathcal{M}_{33}-\mathcal{M}_{22}}. \tag{78}$$

The observation was that if the diagonal elements of the mass matrix are close enough, $\sin^2 2\theta_1$ could undergo a large increment.

Now, starting from the general three flavour case, the two flavour approximation concerning RG running (not to be confused with the two-flavour approximation for oscillations) will be exact if $s_2 = s_3 = 0$, which is a stable condition. The matrix V is then simply given by

$$V \simeq \begin{pmatrix} 1 & & \\ & c_1 & s_1 \\ & -s_1 & c_1 \end{pmatrix} , \tag{79}$$

There is a physically interesting instance, namely the SAMSW solution, which essentially corresponds to this scenario. Then, from (54), the RGE for θ_1 is

$$\frac{d\theta_1}{dt} = \kappa_\tau \sin\theta_1 \cos\theta_1 \tilde{\nabla}_{32} = \kappa_\tau \sin\theta_1 \cos\theta_1 \frac{\tilde{m}_3 + \tilde{m}_2}{\tilde{m}_3 - \tilde{m}_2} . \tag{80}$$

The corresponding RG equation for $\sin^2 2\theta_1$ is then given by

$$\frac{d}{dt}\sin^2 2\theta_1 = 2\kappa_\tau \sin^2 2\theta_1 \cos 2\theta_1 \frac{\tilde{m}_3 + \tilde{m}_2}{\tilde{m}_3 - \tilde{m}_2} . \tag{81}$$

It is easy to check that eq. (81) is indeed equivalent to eq. (78), but it is more practical, as it is written in terms of physical parameters. In any case, eq. (80) is the more useful equation for the analysis[5]. Clearly, we can only expect a large modification in the angle if $|\kappa_\tau \tilde{\nabla}_{32}| \gtrsim 1$. This requires $\tilde{m}_3 \sim \tilde{m}_2$, unless the scenario is supersymmetric with very large $\tan\beta$ ($\gtrsim 100$). So, we must be in case *(ii)* (note that case *(iii)* has $c = 3$ by definition). However, our conclusions were that case *(ii)* can only be acceptable if labels a, b correspond to 1, 2, and thus c to 3, which is inconsistent with $\tilde{m}_3 \sim \tilde{m}_2$. The solution to this apparent contradiction is that the viability of this $\tilde{m}_3 \sim \tilde{m}_2$ scenario requires the running to stop before reaching the stability form of V, while in the analysis of subsection 3.1 we assumed that the stable form was reached. Let us see this in closer detail.

If we start for example with $\theta_1 \simeq 0$ (but not exactly $\theta_1 = 0$, which is a stable value) and negative $\kappa_\tau \tilde{\nabla}_{32}$, the RGE (80) will drive $\theta_1 \to \pi/2$. Schematically,

$$\begin{pmatrix} 1 & & \\ & 1 & 0 \\ & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & & \\ & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \to \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix} \tag{82}$$

[5] Incidentally, $\sin^2 2\theta_1 = 1$, *i.e.* maximal, is not a stable RG point, as has been argued in the literature [25]. From the general equations (54-56) we can see that this is also the generic case in the 3-flavour scenario, although one can easily determine particular conditions for stability. *E.g.* if $\tilde{\nabla}_{32}$ is the dominant term and $c_3 = 0$, then θ_1 is stable for any value.

(changing the sign of $\kappa_\tau \tilde{\nabla}_{32}$ would reverse the direction of the arrows). Notice that this is in agreement with our conclusions for case *(ii)*: if $|\kappa_\tau \tilde{\nabla}_{32}| \gtrsim 1$, $V_{33} \to 0$, which is inconsistent with experiment. However, there is an intermediate scale at which the mixing must be maximal, $\sin^2 2\theta_1 = 1$ (of course this does not occur if we start already near the stable $\theta_1 = \pi/2$ value). This critical scale may be at low energy, and this is the possibility exploited in ref.[18,19,25]. This of course requires some tuning of the initial parameters, and we will be more explicit about this shortly.

Comparison of eq. (80) with the general equation for θ_1, eq. (54), shows that there could be other ways in which the two-flavour approximation for the running is correct. Namely, eq. (54) reduces to (80) if one of the following possibilities occur

$$
\begin{array}{ll}
(a) & s_3 = 0, \\
(b) & \tilde{\nabla}_{31} \simeq \tilde{\nabla}_{32}.
\end{array}
$$

These cases were not apparent at all from eq. (78) used in previous analyses. Now, scenario *(a)* is only acceptable if the $s_3 = 0$ condition is stable along the running, while s_1 is substantially changing. It can be checked from (54–56) that this occurs only if $s_2 = 0$. Therefore, scenario *(a)* can only work in the case previously commented, *i.e.* eq. (79). On the other hand, scenario *(b)* can work in many cases.

Let us now estimate the amount of fine-tuning that these two-flavour scenarios require for the the atmospheric angle to be driven to maximal values at M_Z thanks to the RG running. In particular, for case *(a)*, we notice that the product $V_{32}V_{33} \propto \sin 2\theta_1$ must grow ($V_{32}V_{33} \to V_{32}V_{33}/F$) in a factor $1/F$, with $F \ll 1$. From the discussion in subsection 3.1, we know that this requires a suitable cancellation in eq. (62) between the initial mass splitting Δm^2 and that induced by the running $2\epsilon_\tau m^2$. Assuming for example a supersymmetric scenario (thus $\kappa_\tau < 0$) with $\tilde{\nabla}_{ij} > 0$, we find

$$
\left| \frac{\Delta m^2 - 2\epsilon_\tau m^2}{\Delta m^2} \right| < F \ll 1, \tag{83}
$$

(we have made the estimate $\langle V_{33}^2 - V_{32}^2 \rangle \sim 1/2$, which works well for small values of the initial mixing, as we have checked numerically). This shows that the fine-tuning between the two terms in the numerator of (83), which have completely different origins, is of one part in $1/F$. Therefore, the more important the increase in $\sin^2 2\theta_1$, the greater the fine-tuning. Figure 1 illustrates well the behaviour of the running $\sin^2 2\theta_1$ and the

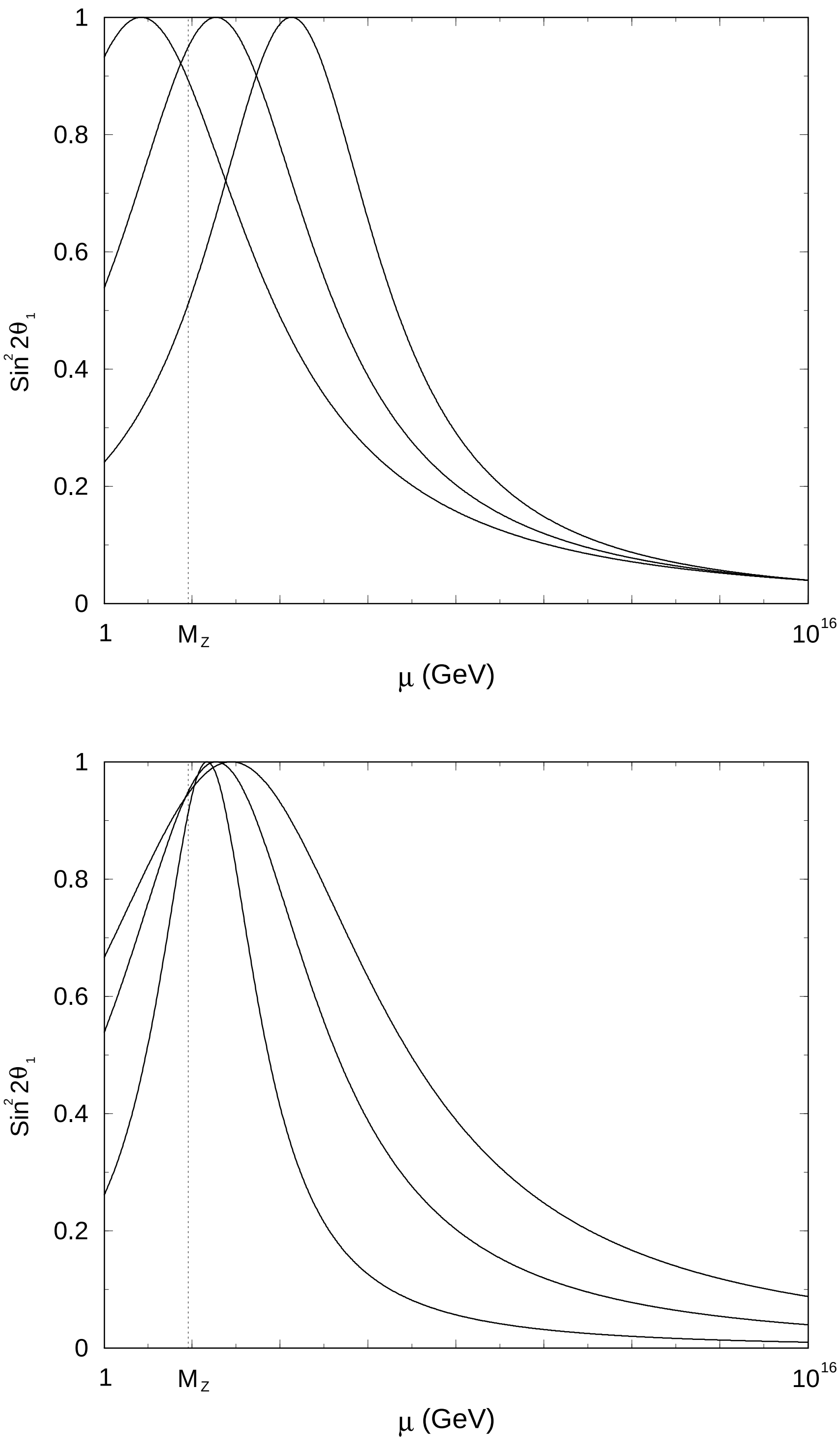

Figure 1: Running of $\sin^2 2\theta_1$ in the two-flavour approximation. In the upper plot, we fix the initial value of the mixing angle but vary the initial mass splitting $\Delta m = 0.00393$ eV in ± 0.0005 eV. In the lower plot, we vary instead the initial value of the angle. We have chosen $m = 1$ eV and $\kappa_\tau = 6.2 \times 10^{-5}$ to get maximal mixing near M_Z.

fine-tuning involved. We show $\sin^2 2\theta_1$ as a function of the energy scale μ, from $\Lambda = 10^{16}$ GeV down to M_Z (we let the angle run below M_Z, where it should really stop, just for illustrative purposes) according to eq. (80). The parameters have been chosen to reach maximal mixing near M_Z starting from a small mixing angle at Λ. It is shown how, after reaching that maximal value, $\sin^2 2\theta_1$ goes down again, seeking the stable 0 value [in accordance with the expected behaviour, see (82)]. In the upper plot we demonstrate the dependence of the effect on the initial mass splitting: when it is varied slightly (leaving the initial angle fixed), the scale at which maximal mixing is achieved moves quite rapidly: an initial mass splitting slightly off and the mixing at M_Z drops below the experimental lower bound. The lower plot shows instead the dependence with the initial condition for the mixing. When that initial value increases (decreases), the fine-tuning required is smaller, and this is reflected in a wider (thinner) half-width of the curve.

Still within case *(a)*, using again the conservation of $V_{33}V_{32}/\tilde{\nabla}_{32}$, we can easily evaluate the relation between the initial and final mass splittings as

$$\Delta m^2(M_Z) \equiv \Delta m^2_{atm} \simeq F\ \Delta m^2(\Lambda). \tag{84}$$

Using eq. (83) we can obtain what is the value of κ_τ required as

$$|\kappa_\tau| \sim \frac{1 \pm F}{2F} \frac{\Delta m^2_{atm}}{m^2 \log \frac{\Lambda}{M_Z}}. \tag{85}$$

For example, if $F \sim 1/5$, which implies an important increase in $\sin^2 2\theta_1$ with a moderate fine-tuning, and using 10^{-3} eV$^2 \leq m^2 \leq 4$ eV^2, $\Delta m^2_{atm} = 10^{-3}$ eV^2 and 10^5 GeV $\leq \Lambda \leq 10^{16}$ GeV, we obtain $1.5 \times 10^{-5} \lesssim |\kappa_\tau| \lesssim 0.28$, which means that the scenario must be supersymmetric with $\tan\beta > 5$.

On the other hand, since V_{32} and V_{33} are not stable, we can evaluate from eq. (73) the increment in the "solar" neutrino mass splitting Δm^2_{21}. Using now $\langle V_{32}^2 - V_{31}^2 \rangle \sim F/2$ and (85), we get

$$\delta_{RGE} \Delta m^2_{21} \sim 4 \Delta m^2_{atm} \tag{86}$$

which is too large. This is not surprising, as m_1^2 is stable but m_2^2 experiments a change of order Δm^2_{atm}. Of course, one could still get Δm^2_{21} of the right size tuning the value of $\tilde{m}_1$ so as to compensate the large RG correction (86) but that would be rather unnatural. So, we conclude that the scenario *(a)*, or equivalently eq. (79), cannot implement a substantial increase of $\sin^2 2\theta_1$ in practice.

Let us consider now scenario *(b)*. In principle, it can be acomplished in many cases. Namely, whenever $m_1^2 \lesssim m_2^2 \ll m_3^2$ (hierarchical spectrum); $m_1^2 \simeq m_2^2 \gg m_3^2$ (intermediate or pseudo-Dirac spectrum); or $m_1^2 \simeq m_2^2 \simeq m_3^2$ (degenerate spectrum) with $\tilde{m}_1/\tilde{m}_2 > 0$ and $\Delta m_{21}^2 \ll \Delta m_{32}^2$. In the first two cases, $\tilde{\nabla}_{32} \simeq -1$. Therefore, if one demands a significant increase of $\sin^2 2\theta_1$ in eq. (80), then one needs $|\kappa_\tau| \log \frac{\Lambda}{M_Z} \gtrsim 1$, which signals the breakdown of perturbation theory. With regard to the degenerate case, since $\tilde{m}_1$ and $\tilde{m}_2$ have equal signs, the neutrinoless double β-decay constraint implies that $m \lesssim 0.2$ eV, see eq. (12). This means that $|\tilde{\nabla}_{32}|$ cannot be very large. Taking $\Delta m_{atm}^2 > 10^{-3}$ eV^2, we get $|\tilde{\nabla}_{32}| = 4m^2/\Delta m_{atm}^2 \lesssim 160$. Therefore, an appreciable variation of $\sin^2 2\theta_1$ requires $|\kappa_\tau| \log \frac{\Lambda}{M_Z} \sim \Delta m_{atm}^2/m^2 \gtrsim 1/40$, which means $\tan\beta > 50$, quite a large value. Anyway, in this degenerate case, $|\tilde{\nabla}_{21}| \gg |\tilde{\nabla}_{32}|$. Consequently, if $|\tilde{\nabla}_{32}|$ is large enough to modify the V_{3i} entries (which is our assumption), $\tilde{\nabla}_{21}$ will lead to $V_{31} \to 0$ (or equivalently $V_{32} \to 0$) rapidly, which requires an SAMSW scenario [see discussion after eq. (71)]. But this is disastrous: from eq. (73), noting that $\langle V_{32}^2 - V_{31}^2 \rangle$ is necessarily sizeable, $|\delta_{RGE} \Delta m_{21}^2| = \mathcal{O}(\Delta m_{atm}^2)$, in disagreement with observations (again barring an unnatural cancellation between the initial and the RG-induced mass splitting).

To summarize, the two-flavour approximation for the running of the θ_1 angle can be a good one if the previous (a) or (b) conditions are fulfilled. Then the size of $\kappa_\tau \tilde{\nabla}_{32}$ will determine whether $\sin^2 2\theta_1$ can get a substantial increase or not. If it can, then Δm_{21}^2 will get too large along the running.

The possibility of obtaining an enhancement of $\sin^2 2\theta_1$ by the RG running seems so attractive that one may wonder if it could be realized in a 3-flavour scenario. Unfortunately, things do not seem to improve much. From the previous discussion, it is clear that the only possibility to consider here is $m_1^2 \simeq m_2^2 \simeq m_3^2$ (degenerate spectrum) with $\tilde{m}_1/\tilde{m}_2 < 0$ and $\Delta m_{21}^2 \ll \Delta m_{32}^2$. Otherwise, the scenario is equivalent to the two-flavour case, which we know does not work, or it has too small $\kappa_\tau \tilde{\nabla}_{ij}$ to produce appreciable modifications in the angles. In the scenario selected only $\kappa_\tau \tilde{\nabla}_{32}$ can be relevant. The effect of the running on the Δm_{21}^2 splitting is still given by eq. (73), which means that we will get $|\delta_{RGE} \Delta m_{21}^2| = \mathcal{O}(\Delta m_{atm}^2)$ (unacceptable in any scenario) unless $V_{32}^2 - V_{31}^2 \sim 0$. This condition can be certainly arranged at low energy (*i.e.* at the final point of the running) using $s_2 = 0$, $\sin^2 2\theta_3 = 1$ (which implies a nearly bimaximal scenario). However, if the running is going to modify appreciably the matrix V (which is mandatory to get an important enhancement of $\sin^2 2\theta_1$) this condition will not be stable. Notice that the dominance of $\kappa_\tau \tilde{\nabla}_{32}$ imply that $V_{31} \simeq$ const., while V_{32}, V_{33} should vary substantially. In consequence the final Δm^2_{21} will be again too large.

Schematically, the modification of the matrix U (starting with low $\sin^2 2\theta_1$) will be

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{2} & \frac{1}{2\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \end{pmatrix} \to \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} \to \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{2} & \frac{2}{\sqrt{6}} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}. \tag{87}$$

(This example has $\cos\theta_1 \sim 0$, but the conclusions are the same for examples with $\sin\theta_1 \sim 0$). In the initial part of the running $\sin^2 2\theta_1$ experiments a large enhancement, so the parameters ($\kappa_\tau \tilde{\nabla}_{32}$ and Λ) should be tuned to have the running finishing at the intermediate point. But in any case, it is apparent that $V^2_{32} - V^2_{31} \neq 0$ along the running, thus yielding too large Δm^2_{21}.

4 Implications (general complex case)

Non-zero CP-violating phases can have a non-trivial effect on the RG evolution of neutrino mixing angles and masses. Such effects have been considered previously in a 2 generation case [21]. With the formalism presented in section 2 we can undertake easily the more realistic analysis for 3 generations.

As in the previous section, we will assume that below the high-energy scale Λ, the effective theory is just the SM or the MSSM, plus the effective Majorana mass matrix for the neutrinos. As shown in sect. 2, the RGEs for the neutrino masses and the matrix U are given by eqs. (15, 16), with the appropriate substitutions. Explicitly,

$$\frac{dm_i}{dt} = -2m_i|U_{3i}|^2 - m_i\kappa_U, \tag{88}$$

$$\frac{dU}{dt} = UT, \tag{89}$$

where κ_U gives a universal effect [see eqs. (35, 37)] and

$$\begin{aligned} &T_{ii} = 0 \\ &T_{ij} = \kappa_\tau \left[\nabla_{ij} Re(U^*_{3i}U_{3j}) + i\nabla^{-1}_{ij} Im(U^*_{3i}U_{3j})\right],\ (i \neq j). \end{aligned} \tag{90}$$

The explicit RGEs for the mixing angles, the CP-violating phases and the unphysical phases are given by eqs. (23–31), using everywhere the substitutions of eqs. (39, 41).

As in the real case, large effects in the radiative corrections are expected when the neutrino spectrum has (quasi)-degenerate eigenvalues. When the degeneracy is sufficiently good (see previous section), U changes rapidly from its unperturbed $t = 0$ form to a stable form that has non-singular dU/dt. To be more precise, if $m_i \simeq m_j$, then $|\nabla_{ij}| \gg 1$, and U is rapidly driven to a form that ensures $T_{ij} \simeq 0$. Since near $t = 0$, one has $T_{ij} \sim \nabla_{ij} Re(U^*_{3i} U_{3j})$, the stable form, say U', must satisfy

$$Re({U'}^*_{3i} U'_{3j}) = 0 \tag{91}$$

for any pair i, j with $m_i \simeq m_j$. Since the unphysical phase α_τ, cancels out in the previous equation, we can also write

$$Re({W'}^*_{3i} {W'}_{3j}) = 0, \tag{92}$$

where the matrix W was defined in eq. (5).

If all masses are quite different, then no dramatic RG effects appear and the discussion is similar to the corresponding one in the real case (see previous section). In the following subsections we then consider the two interesting cases of two-fold or three-fold degeneracy. The following general analysis contains as a particular case the real scenario considered in detail in the previous section. Note that now, two real scenarios with degenerate masses of either equal or opposite signs are treated together and correspond to choices of ϕ (or ϕ') equal to 0 or π, respectively. Note that, although the case with opposite signs has now a very large $|\nabla_{ij}|$, having $\phi - \phi' = \pi$ gives a T_{ij} under control, and the case is, of course, stable.

As in the real case, for the discussion of cases of physical relevance, we will assume that the solar neutrino problem is solved by one of the standard solutions, with $|\Delta m^2_{21}| \ll |\Delta m^2_{32}| \sim |\Delta m^2_{31}|$.

4.1 Three-fold degeneracy

In the case $m_1 \simeq m_2 \simeq m_3 \simeq m$, with $|\kappa_\tau \nabla_{ij}| \gg 1$ for all i, j, the matrix U (or equivalently W) quickly reaches a stable value U' (W') with

$$Re({W'}^*_{31} {W'}_{32}) = Re({W'}^*_{32} {W'}_{33}) = Re({W'}^*_{31} {W'}_{33}) = 0. \tag{93}$$

Using the fact that W'_{33} is real in the parametrization (4), the last two equalities in (93) imply

$$W'_{33} = 0 \quad \text{or} \quad Re(W'_{31}) = Re(W'_{32}) = 0. \tag{94}$$

If $W'_{33} \neq 0$, eq. (94) and the first equality in (93) imply

$$W'_{31} = 0, \quad \text{or} \quad W'_{32} = 0. \tag{95}$$

In other words, there must be some $W'_{3i} = 0$ (the ambiguity in the labelling of the three original eigenvalues is reflected in the ambiguity in the i-label of $W'_{3i} = 0$). Then, by unitarity, the values of $|W'_{3j}|^2 = |U'_{3j}|^2$ are $\{0, x, 1-x\}$, with $0 \leq x \leq 1$ (the value of x is determined by the initial form of the matrix W). So, from the expressions for dm_i/dt we obtain the following low-energy shifts in the mass eigenvalues:

$$\Delta m_i \simeq m\epsilon_\tau \{0, x, 1-x\}, \tag{96}$$

where m is essentially the initial value of the three masses at the scale Λ.

Therefore, for a given value of x we know how the degeneracy in the mass spectrum is lifted and then the labelling of the three eigenvalues is determined according to our convention (thus U' is fixed). It is then easy to show that no value of x gives a stable U' in agreement with the available experimental information on the neutrino mixing angles. For $x < 0.25$ (or > 0.75) one has $U'_{31} \to 0$ and $|U_{33}|^2 > 0.75$, in conflict with the limit $c_1^2 c_2^2 < 0.71$ which follows from the experimental bound (8). For $0.25 < x < 0.75$, $U'_{33} \to 0$ instead, and this is also incompatible with experimental limits: it implies either $c_2^2 \to 0$ [in conflict with (7)], or $c_1^2 \to 0$ [in conflict with (8)].

In summary, the case of three-fold degeneracy of initial neutrino masses leads to a stable U' which does not accomodate the values of θ_i and Δm^2_{ij} suggested by experiment. Note that this analysis contains and expands the subcases *(i)–(ii)* for real U analyzed in the previous section. This negative result has two obvious way-outs. If the spectrum is almost degenerate, but all $|\kappa_\tau \nabla_{ij}| \ll 1$, then the matrix U is very slightly renormalized, so it never reaches the stable (disastrous) form. In this case the RG corrections to mixing angles and phases are basically irrelevant. This extreme situation is difficult to implement in the VO scenario, since $|\kappa_\tau \nabla_{21}|$ is necessarily sizeable. The second way-out is that $|\kappa_\tau \nabla_{32}| \ll 1$ (even though $|\nabla_{32}| \gg 1$), while $|\kappa_\tau \nabla_{21}|$ can be significant. This will occur if the Δm^2_{32} splitting is arranged to sensible ("atmospheric") values from the beginning. Then, the analysis for the mixing angles and CP phases is identical to a case with two-fold degeneracy, which is analyzed in the next subsection.

4.2 Two-fold degeneracy

We assume here that the initial $\Delta m^2_{32}, \Delta m^2_{31}$ splittings are large enough to keep m_3 as the most split eigenvalue after the RG effects, while $m_1 \simeq m_2$. In this case, only $|\kappa_\tau \nabla_{21}| \gtrsim 1$. This drives the original matrix U to a stable form U' given by (see Appendix)

$$U' = U R_{12}(\Gamma), \tag{97}$$

where $R_{12}(\Gamma)$ is a rotation in the 1-2 plane by an angle Γ, to be computed below. We see that the third column of U is not changed appreciably, in accordance with the fact that dU_{i3}/dt does not depend on ∇_{21}. The rotation angle Γ is determined by the condition (91)

$$Re(U'^*_{31} U'_{32}) = 0, \tag{98}$$

which gives

$$\tan 2\Gamma = \frac{2Re(U^*_{31} U_{32})}{|U_{32}|^2 - |U_{31}|^2}. \tag{99}$$

This allows to determine U' completely and thus the physical parameters as a function of the initial conditions. Let us examine this issue in closer detail.

With regard to the mixing angles, from the RGEs (23–25) one immediately sees that θ_1 and θ_2 are not renormalized strongly ($d\theta_{1,2}/dt$ do not depend on T_{12}) while θ_3 is driven to a stable value, $\theta_3^{(f)}$. More precisely,

$$\begin{aligned} \theta_1^{(f)} &\simeq \theta_1 \\ \theta_2^{(f)} &\simeq \theta_2 \\ \sin^2 2\theta_3^{(f)} &= p^2 + q^2, \end{aligned} \tag{100}$$

where

$$p = \sin 2\Gamma \sin[(\phi - \phi')/2], \tag{101}$$

and

$$q = \sin 2\Gamma \cos 2\theta_3 \cos[(\phi - \phi')/2] + \sin 2\theta_3 \cos 2\Gamma. \tag{102}$$

The CP-violating phases are also driven to particular values $\delta^{(f)}, \phi^{(f)}, \phi'^{(f)}$ which are complicated functions of the initial values of the parameters in U:

$$\tan\delta^{(f)} = \frac{q\tan\delta - p}{q + p\tan\delta}\ , \tag{103}$$

$$\tan[(\phi^{(f)} - \phi'^{(f)})/2] = \frac{\sin 2\theta_3 \sin[(\phi-\phi')/2]}{\cos 2\Gamma \sin 2\theta_3 \cos[(\phi-\phi')/2] + \sin 2\Gamma \cos 2\theta_3}\ , \tag{104}$$

and

$$\phi^{(f)} + \phi'^{(f)} - 2\delta^{(f)} = \phi + \phi' - 2\delta. \tag{105}$$

The last equation follows from $d(\phi + \phi' - 2\delta)/dt \simeq 0$, which can be checked directly from (26)-(28).

As explained in section 2, see (32-34), other quantities of interest which are conserved in this case are $Im(U^*_{k1}U_{k2})$. For $k = 1$, we obtain the conserved quantity

$$\sin 2\theta_3 \sin[(\phi-\phi')/2], \tag{106}$$

and, for $k = 2, 3$,

$$\sin[(\phi-\phi')/2]\cos\delta\cos 2\theta_3 - \sin\delta\cos[(\phi-\phi')/2]. \tag{107}$$

These expressions can be sometimes conveniently used instead of the explicit formulas (100)-(104).

For physical applications it is interesting to use for θ_1 and θ_2 the values suggested by experiment[6], $\sin^2 2\theta_1 \sim 1$ and $\theta_2 \sim 0$, which permits to obtain more explicit expressions for θ_3 and the CP-violating phases at low energy. In doing this, it is important to keep corrections due to a small (but non-zero) value of θ_2, as they can become dominant for some choices of the parameters. For the angle $\theta_3^{(f)}$, we obtain the remarkably simple expression

$$\sin^2 2\theta_3^{(f)} = \sin^2 2\theta_3 \sin^2[(\phi-\phi')/2] + \mathcal{O}(s_2^2), \tag{108}$$

which gives the final $\theta_3^{(f)}$ as a function of the initial θ_3 and $\phi - \phi'$. We see that, for the SAMSW solution of the solar neutrino problem, we must have either $\sin^2 2\theta_3 \sim 0$ or $\sin^2[(\phi-\phi')/2] \sim 0$ at the scale Λ. On the other hand, if the solar mixing is also

[6]Due to the stability of θ_1 and θ_2, this initial condition at Λ is approximately maintained by RG evolution down to M_Z. The origin of such initial condition could be some flavour symmetry or perhaps a fixed point in the RG running above Λ.

maximal, $\sin^2 2\theta_3^{(f)} \sim 1$, then both $\sin^2 2\theta_3$ and $\sin^2[(\phi - \phi')/2]$ must be of order 1 originally.

A word of caution should be said about the higher order term in (108). The neglected term, calculated exactly, is [we introduce the short-hand notation $\varphi \equiv (\phi - \phi')/2$ and $\sigma \equiv \sin 2\theta_1 = \pm 1$]:

$$4s_2^2 \frac{N_1^2}{D_1^2 + D_2^2}, \tag{109}$$

with

$$\begin{aligned} N_1 &= c_\varphi c_\delta + s_\varphi s_\delta c_{2\theta_3}, \\ D_1 &= -c_\varphi c_2^2 s_{2\theta_3} + 2\sigma s_2 (c_\varphi c_\delta c_{2\theta_3} + s_\varphi s_\delta), \\ D_2 &= c_2^2 c_{2\theta_3} + 2\sigma s_2 c_\delta s_{2\theta_3} \ , \end{aligned} \tag{110}$$

where all the quantities are understood at the initial (high energy) scale. For generic values of the parameters, the term (109) will be negligible as it is formally $\mathcal{O}(s_2^2)$. However, for some particular choices of the parameters the denominator in (109) can be also $\mathcal{O}(s_2^2)$ resulting in a non-suppressed correction in (108). This occurs, for example, if $c_{2\theta_3} \simeq 0$ and $c_\varphi \simeq 0$ simultaneously.

Expressions (108) and (109) for $\sin^2 2\theta_3^{(f)}$, in conjunction with the conserved quantities (105–107), are enough to determine exactly the rest of parameters at low-energy. For example, we obtain

$$\tan^2[(\phi^{(f)} - \phi'^{(f)})/2] = \frac{\sin^2 2\theta_3 \sin^2 \varphi (D_1^2 + D_2^2)}{4 \sin^2 \theta_2 N_1^2}, \tag{111}$$

Note the $1/s_2^2$ dependence that would generically drive $\phi^{(f)} - \phi'^{(f)} \to \pm\pi$. However, this is not necessarily the case because the smallness of s_2 can be compensated by a small numerator in (111), as commented above.

If $\phi = \phi'$ initially, it is easy to see that the only physical parameter that undergoes a sizable change is θ_3, with $\theta_3^{(f)} = \theta_3 + \Gamma$. This follows directly from $U' = U R_{12}(\Gamma)$ because now $R_{12}(\Gamma)$ commutes with $\mathrm{diag}(e^{i\phi}, e^{i\phi}, 1)$. Moreover, eq. (108) gives $\theta_3^{(f)} \to \mathcal{O}(s_2) \sim 0$ irrespective of the initial value of θ_3. This is just right for the SAMSW solution of the solar neutrino problem and reproduces the result discussed in the previous section for the real case. We see however that this appealing possibility dissappears for generic values of ϕ and ϕ'.

If $\phi - \phi' = \pm\pi$ instead, one gets $\tan 2\Gamma \sim s_2 \sin\delta$. For $\theta_2 = 0$ or $\delta = 0$ this implies a stable matrix U from the beginning, see eq. (97). This is in correspondence with the real case $\phi = 0, \phi' = \pi$ discussed in the previous section. For non-zero values of δ and/or θ_2, however, the matrix U is no longer stable, although typically Γ, and thus the corrections to U, will be small. We conclude that, the presence of a non-zero phase $\delta = 0$ tends to destabilize this scenario, but the effect is small for small θ_2.

Another interesting result that we can extract from eq. (103) is the possiblility of generating a non-zero phase $\delta^{(f)}$ even if $\delta = 0$ originally, provided the phases ϕ, ϕ' are non zero and satisfy $\sin[2(\phi - \phi')] \neq 0$. The resulting $\delta^{(f)}$, when $\delta = 0$, is given by

$$\tan\delta^{(f)} = -\frac{p}{q}, \tag{112}$$

or, explicitly,

$$\tan\delta^{(f)} = \frac{-s_\varphi c_\varphi (c_2^2 \sin 2\theta_3 - 2\sigma s_2 \cos 2\theta_3)}{c_2^2 s_\varphi^2 \sin 2\theta_3 \cos 2\theta_3 + 2\sigma s_2 (1 - s_\varphi^2 \cos^2 2\theta_3)} \tag{113}$$

This shows that one can generate radiatively even a maximal value for the CP-violating phase.

4.3 Neutrino masses

In the complex case, the running of the neutrino mass eigenvalues is governed by

$$\frac{dm_i}{dt} = -2\kappa_\tau m_i |U_{3i}|^2 - \kappa_U m_i, \tag{114}$$

and many of the generic results obtained in the real case go over to the complex one. Some of the significant differences are discussed in this subsection.

From (114) we see that the phases ϕ and ϕ' do not affect directly the running of the masses. However, they have an important influence in the stability of a given choice of the matrix U, and this in turn will influence the evolution of the masses. For example, in the degenerate case $m_1 \simeq m_2 = m_0$, the initial choices $\phi - \phi' = 0$ and $\phi - \phi' = \pi$ (which correspond to $\tilde{m}_1 \simeq \pm\tilde{m}_2$, already analyzed in the real case) give identical dm_i/dt at $t = 0$. However, in the first case U changes abruptly, while in the second it evolves smoothly, and this has an evident effect on the subsequent running of $m_{1,2}$. The phase δ, on the other hand, enters U_{31} and U_{32}, and thus has a direct effect on the size of dm_i/dt, although its importance is somewhat reduced by the smallness of θ_2.

One important instance in which the stability of U is crucial for the viability of the scenario is the VO solution to the solar neutrino problem. As discussed in the real

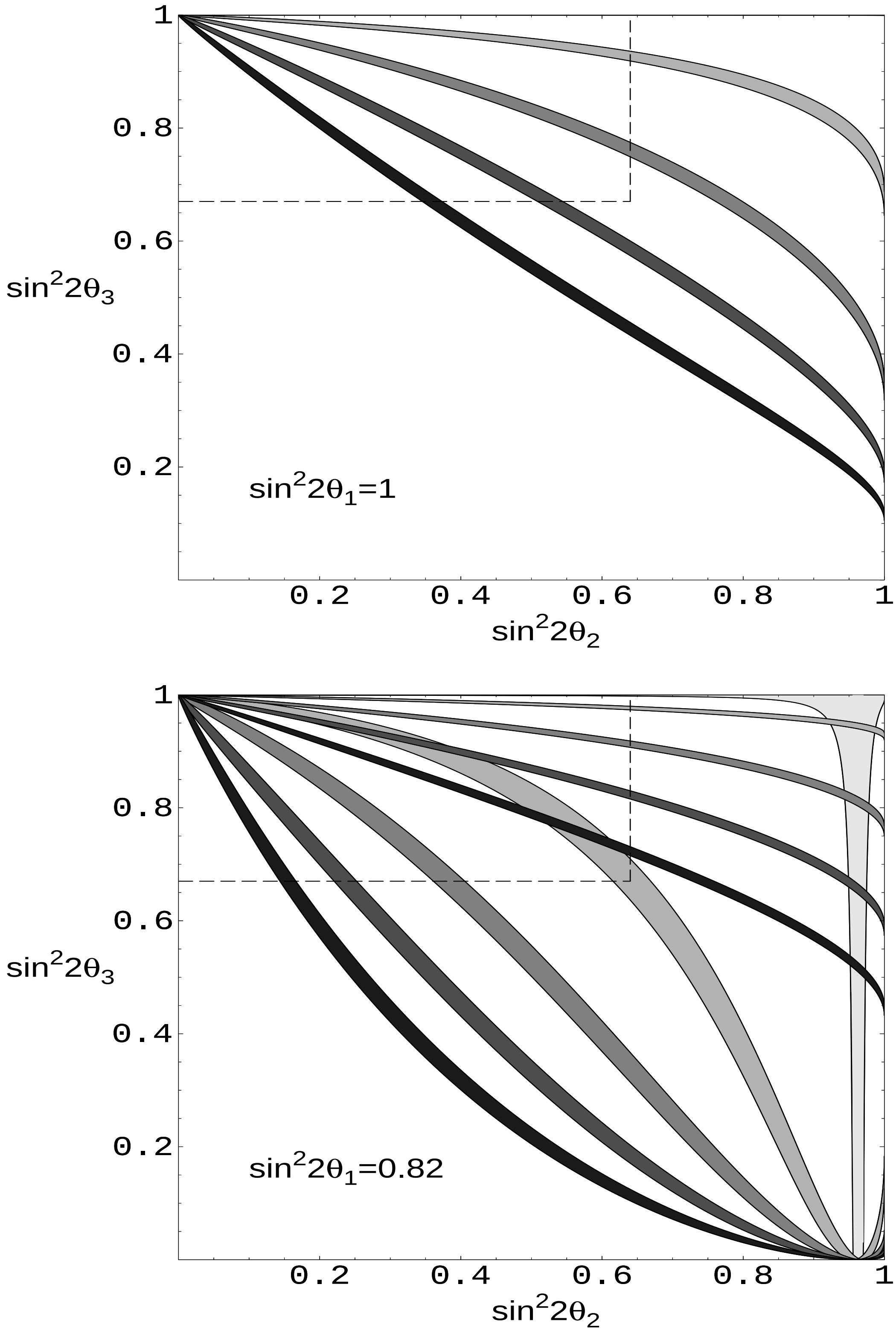

Figure 2: Allowed regions (in which $\Delta m^2_{21} \leq 1.1 \times 10^{-10}\,\mathrm{eV}^2$) in the plane $(\sin^2 2\theta_2, \sin^2 2\theta_3)$ for the SM case for $\sin^2 2\theta_1 = 1$ (upper plot) and 0.82 (lower) and a cut-off scale $\Lambda = 10^{12}\,\mathrm{GeV}$. The different strips correspond to different values of the phase δ, from 0 (darker color) to $\pi/2$ (lighter).

case, in a scenario with total or partial degeneracy of neutrino masses, the final Δm^2_{21} has to be much smaller than the typical size of the RG shift in the masses squared $\sim 2m_0^2\epsilon_\tau$. Hence, starting with $m_1 \simeq m_2 \gg m_3$ at the high energy scale Λ, to keep Δm^2_{21} under control requires an exquisite cancellation in the running of m_1 and m_2. From (114), this cancellation is

$$|(|U'_{31}|^2 - |U'_{32}|^2)| \lesssim \frac{\Delta m^2_{sol}}{4m_0^2\epsilon_\tau}\ . \tag{115}$$

In this condition, we have written U' and not U, because if the degeneracy is good enough U jumps to U' almost immediately away from Λ and remains close to it during the rest of the running down to M_Z. By definition, U' satisfies the stability condition

$$Re(U'^*_{31}U'_{32}) = 0. \tag{116}$$

From (115) we find that, to keep the solar mass splitting under control, the mixing angles (at low energy) must be correlated according to

$$\tan 2\theta_3 \simeq \frac{\cos^2\theta_1\sin^2\theta_2 - \sin^2\theta_1}{\sin\theta_2\sin 2\theta_1\cos\delta}\ . \tag{117}$$

This equation generalizes (74) to the complex case. Eqs. (116, 117) must be satisfied simultaneously for the viability of the VO scenario in totally or partially degenerate scenarios.

From eq. (117) we can see that a non-zero δ will influence the correlations that must occur. The effect is discussed in figure 1, where, for two different values of $\sin^2 2\theta_1$ (the maximal one, 1, and the experimental lower bound, 0.82) we plot the region of the plane $(\sin^2 2\theta_2, \sin^2 2\theta_3)$ where the cancellation (115) takes place. The different strips[7] correspond to different values of $\cos\delta$, from 1 (darker color) to 0 (lighter), in 1/4 steps. We set the high energy scale Λ at the typical see-saw value 10^{12} GeV and require $\Delta m^2_{12} \leq 1.1\times 10^{-10}\,\mathrm{eV}^2$ with the initial neutrino mass $m_0^2 = \Delta m^2_{atm} = 5\times 10^{-4}\,\mathrm{eV}^2$. For much smaller Λ the strips in figure 1 get thicker [it is easier to satisfy (115) because the size of radiative corrections is smaller]. Note that $\delta = \pi/2$ is special because, in such case, eq. (117) has in principle two solutions: *a)* $\sin^2 2\theta_3 = 1$ and *b)* $s_2^2 = \tan^2\theta_1$. The region satisfying (115) will consist of two strips centered around those two lines.

[7]For a fixed value of $\sin^2 2\theta_2$ and $\sin^2 2\theta_1$ there are in principle 4 values of $\sin^2 2\theta_3$ satisfying (117), corresponding to the fact that the interchange $\cos\theta \to \sin\theta$ leaves $\sin^2 2\theta$ invariant. We do not plot the two solutions with $s_2 > c_2$, which would be in conflict with CHOOZ+SK data. Note that, for $\sin^2 2\theta_1 = 1$ the two remaining solutions merge in one.

In the case $\sin^2 2\theta_1 = 1$, the solution *b)* gives $s_2^2 = 1$, which is beyond experimental bounds and only solution *a)*, $\sin^2 2\theta_3 = 1$, remains (the region around that line has zero width and cannot be seen in the plot). For $\sin^2 2\theta_3 < 1$, solution *b)* can have $s_2 < c_2$ and we include it in the figure. The corresponding region is T-shaped like the one shown in figure 1, lower plot.

On the other hand, the condition (116) for stability can be satisfied for any choice of mixing angles and δ by adjusting $\phi - \phi'$ to the apropriate value, which is given by

$$\tan[(\phi - \phi')/2] = \frac{\sin 2\theta_3(\sin^2\theta_1 - \cos^2\theta_1 \sin^2\theta_2) - \sin 2\theta_1 \cos 2\theta_3 \sin\theta_2 \cos\delta}{\sin 2\theta_1 \sin\theta_2 \sin\delta} \ . \quad (118)$$

This would be the complex version of the requirement of opposite signs for the two degenerate eigenvalues in the real case. Using (117), the previous result simplifies further to

$$\tan[(\phi - \phi')/2] = -\frac{1}{\tan\delta \cos 2\theta_3}. \quad (119)$$

Unlike the real case, to impose condition (118) for generic values of the parameters, seems hard to justify from some underlying symmetry.

5 Conclusions

Assuming three flavours of light Majorana neutrinos, with no reference to any particular scenario, we have derived the general renormalization group equations (RGEs) for the physical neutrino parameters: three masses, three mixing angles and three CP-violating phases [information alternatively encoded in the RGE for the masses and the complex mixing (MNS) matrix U]. This form of writing the RGEs represents an advantageous alternative to using the RGE for $\mathcal{M}_\nu$. It avoids the proliferation of unphysical parameters, which allows to keep track of the physics in a more efficient way. It also permits to appreciate interesting features, *e.g.* presence of stable (pseudo infrared fixed-point) directions for mixing angles and phases, which are not consequence of a particular scenario.

We have then particularized the RGEs for relevant scenarios. Namely, when the effective theory below Λ (the scale at which the neutrino mass operator is generated), is given by the SM or MSSM; or when $\mathcal{M}_\nu$ is generated by a see-saw mechanism. In the first two scenarios we have analyzed in detail the physical implications of the RGEs, separating the case where U is real (*i.e.* no CP phases) and the general complex case.

For the real case, we have noticed that if one starts with two masses suficiently degenerate (in absolute value and sign), say $m_i \simeq m_j$, the U matrix is driven to a stable (infrared pseudo-fixed point) form, providing net predictions for the mixing angles. Whenever this happens the corresponding mass splitting, Δm^2_{ij}, is entirely determined at low energy by the RG running. This amounts to a very predictive scenario. Depending on what are the initial (quasi) degenerate neutrinos, the scenario can be realistic or not. In particular, starting with $m_1 \simeq m_2$ and an initial $\Delta m^2_{32} \sim \Delta m^2_{atm}$, we finish at low energy with a small "solar" θ_3 angle and a Δm^2_{21} splitting just of the right size for the SAMSW solution to the solar neutrino problem, which is certainly remarkable. On the other hand, the radiative corrections to the mass splittings are potentially dangerous for the VO scenario, which becomes unviable if $m_1^2 \simeq m_2^2 \gg m_3^2$, unless $m_1 \simeq -m_2$, the "solar" θ_3 angle is close to maximal and the common mass is below the range of cosmological relevance.

We have also shown that previous claims (realized in the two-flavour approximation) in the sense that the RG running could provide a substantial enhancement of the atmospheric mixing, $\sin^2 2\theta_1$ cannot work in practice, since the mechanism leads to an unacceptably large "solar" splitting, Δm^2_{21}. We have shown that, unfortunately, this is also the case in the more general 3-flavour scenario.

For the general complex case, most of the conclusions are similar, but there are interesting new effects. In particular, the previously considered "radiative" SAMSW scenario requires $\phi - \phi' \simeq 0$ (unless the "solar" θ_3 angle is set by hand at a small value from the beginning). On the other hand, if $\phi - \phi'$ is different from zero and the neutrino spectrum has a two-fold degeneracy, the RGEs will generically drive its value to $\pm\pi$. It is also worth-noticing that the CP phase δ can be driven to maximal values by RG corrections. On the other hand, if $\cos\delta \neq 0$, the viability of the VO scenario when $m_1^2 \simeq m_2^2$ requires, besides the previous conditions, a delicate cancellation involving the three CP phases, which makes the scenario more unnatural.

Addendum

Shortly after the completion of this work, there appeared a paper by P.H. Chankowski, W. Królikowski and S. Pokorski [27] dealing with similar subjects. More precisely, they work out the RGEs for mass eigenvalues and mixing angles below the high energy scale Λ, assuming that the effective theory below Λ is the SM or MSSM and a real MNS mixing matrix, U. Therefore, their work corresponds to the issues studied here in sect. 3. As far as we have checked, their results are in agreement with ours. They also notice and stress the existence of stable (infrared pseudo-fixed points) in the evolution of the mixings.

There is a point however in which we disagree. They work out the 2-flavour approximation, reaching an RGE for the (atmospheric) angle θ_1 which coincides with our equation (80) [or equivalently eq. (81)], but they argue that the equation normally used in the literature, eq. (78), is incorrect and inconsistent with (80, 81), which is not the case (once the difference between mass matrix entries and mass eigenvalues is taken into account).

On the other hand they argue, correctly, that the usual claim that a maximal angle is RG stable is not correct. Actually, we have shown that this is also the case in a 3-flavour scenario, giving the more general conditions which would guarantee stability.

Finally, let us remark that in our paper we study also the general complex case, *i.e.* when CP phases are present. In addition, in sect. 2 we work out the RGEs for a completely general case (with no reference to *e.g.* SM or MSSM), showing explicitly the appearance of stable (infrared pseudo-fixed points) forms for U.

Appendix

We derive here the general form of the stable (infrared pseudo-fixed point) MNS matrix, U, in the presence of quasi-degenerate neutrino masses. The generic RGE of the neutrino mass matrix is given by ($t = \log\mu$)

$$\frac{d\mathcal{M}_\nu}{dt} = -(\kappa_U \mathcal{M}_\nu + \mathcal{M}_\nu P + P^T \mathcal{M}_\nu), \tag{120}$$

from which the RGEs for the mass eigenvalues m_i and the mixing matrix U are obtained as

$$\frac{dm_i}{dt} = -2m_i \hat{P}_{ii} - m_i Re(\kappa_U), \tag{121}$$

and

$$\frac{dU}{dt} = UT. \tag{122}$$

T is an anti-hermitian matrix given by

$$T_{ii} \equiv i\hat{Q}_{ii},$$

$$T_{ij} \equiv \nabla_{ij} Re(\hat{P}_{ij}) + i[\nabla_{ij}]^{-1} Im(\hat{P}_{ij}) + i\hat{Q}_{ij}, \qquad i \neq j, \tag{123}$$

with $\hat{P} \equiv \frac{1}{2}U^\dagger(P+P^\dagger)U$, $\hat{Q} \equiv -\frac{i}{2}U^\dagger(P-P^\dagger)U$ and $\nabla_{ij} \equiv (m_i+m_j)/(m_i-m_j)$.

Let us suppose that two mass eigenvalues, say m_1 and m_2, are almost degenerate at the initial high-energy scale Λ. Then, near the starting point of the running, $t \sim t_0$, T is dominated by the (real) terms proportional to ∇_{12}

$$T \sim \begin{pmatrix} 0 & -Re(T_{21}) & 0 \\ Re(T_{21}) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{124}$$

Then, the formal solution to eq. (122) is given by

$$U' \equiv U(t) \sim U\exp\left[\begin{pmatrix} 0 & \int_{t_0}^t Re(T_{21})dt & 0 \\ -\int_{t_0}^t Re(T_{21})dt & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] = UR_{12}(\Gamma), \tag{125}$$

where $R_{12}(\Gamma)$ is an ordinary rotation in the 1–2 plane by an angle Γ

$$R_{12}(\Gamma) = \begin{pmatrix} c_\Gamma & s_\Gamma & 0 \\ -s_\Gamma & c_\Gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{126}$$

The value of Γ will be such that dU/dt becomes non-singular, *i.e.* it renders $Re(T_{12}) \simeq 0$. Thus the stable matrix U' satisfies

$$Re(\hat{P}'_{12}) = \frac{1}{2} Re(U'^\dagger(P+P^\dagger)U')_{12} = 0\ , \tag{127}$$

which we can solve for Γ in terms of initial quantities obtaining

$$\tan 2\Gamma = \frac{2Re(\hat{P}_{12})}{\hat{P}_{22}-\hat{P}_{11}}. \tag{128}$$

Therefore, using (125), we get unambiguously the stable U' matrix in terms of the initial U.

One can further justify the condition (127) by showing that the RG evolution actually drives $Re(\hat{P}_{12})$ towards zero, as expected. The relevant RG when ∇_{12} dominates is

$$\frac{d}{dt} Re(\hat{P}_{12}) \simeq (\hat{P}_{11}-\hat{P}_{22})\nabla_{12} Re(\hat{P}_{12}). \tag{129}$$

For a sufficiently long running interval one has [using (121)]

$$\nabla_{12} \simeq \frac{1}{(\hat{P}_{11}-\hat{P}_{22})\log(\Lambda/\mu)}, \tag{130}$$

and inserting this in (129) we get $Re(\hat{P}_{12}) \to 0$ in the infrared.

Let us consider now the only remaining possibility, *i.e.* that all the mass eigenvalues m_1, m_2, m_3 are (almost) degenerate. Similarly to the previous case, for $t \sim t_0$, T is dominated by the (real) terms

$$T \sim \begin{pmatrix} 0 & -Re(T_{21}) & -Re(T_{31}) \\ Re(T_{21}) & 0 & -Re(T_{32}) \\ Re(T_{31}) & Re(T_{32}) & 0 \end{pmatrix} . \tag{131}$$

So, the solution to eq. (122) is formally given by

$$U' \equiv U(t) \sim U \exp\left[\begin{pmatrix} 0 & \int_{t_0}^t Re(T_{21})dt & \int_{t_0}^t Re(T_{31})dt \\ -\int_{t_0}^t Re(T_{21})dt & 0 & \int_{t_0}^t Re(T_{32})dt \\ -\int_{t_0}^t Re(T_{31})dt & -\int_{t_0}^t Re(T_{32})dt & 0 \end{pmatrix}\right] = UR, \tag{132}$$

where R is a general 3×3 rotation, depending on three angles, say Γ_1, Γ_2, Γ_3,

$$R = \begin{pmatrix} c_2 c_3 & c_2 s_3 & s_2 \\ -c_1 s_3 - s_1 s_2 c_3 & c_1 c_3 - s_1 s_2 s_3 & s_1 c_2 \\ s_1 s_3 - c_1 s_2 c_3 & -s_1 c_3 - c_1 s_2 s_3 & c_1 c_2 \end{pmatrix}, \tag{133}$$

with $s_i = \sin\Gamma_i$, $c_i = \cos\Gamma_i$ (the resemblance with the 'CKM' matrix is only formal). R is determined, as in the case of twofold degeneracy considered before, by the fact that dU/dt becomes non-singular, *i.e.* $Re(T_{ij}) \simeq 0$. Thus

$$Re(\hat{P}'_{12}) = \frac{1}{2} Re(U'^{\dagger}(P + P^{\dagger})U')_{12} = 0, \tag{134}$$

$$Re(\hat{P}'_{31}) = \frac{1}{2} Re(U'^{\dagger}(P + P^{\dagger})U')_{31} = 0, \tag{135}$$

$$Re(\hat{P}'_{32}) = \frac{1}{2} Re(U'^{\dagger}(P + P^{\dagger})U')_{32} = 0. \tag{136}$$

Substituting $U' = UR$, these equations determine completely the rotation R, and therefore U', in terms of U and the initial parameters. Another interesting remark is that if one has information on the mixing angles in the stable matrix U', the phases will be constrained, so as to satisfy the set of equations (134-136). And viceversa, if the phases are known, the mixing angles cannot take arbitrary values.

Again, one can show that in this case RG evolution drives $Re(\hat{P}_{ij}) \to 0$ for all i, j. The relevant RGE that shows this is now

$$\frac{1}{2}\frac{d}{dt}\left\{[Re(\hat{P}_{12})]^2 + [Re(\hat{P}_{23})]^2 + [Re(\hat{P}_{31})]^2\right\} = (\hat{P}_{11} - \hat{P}_{22})\nabla_{12}[Re(\hat{P}_{12})]^2 \\ +(\hat{P}_{22} - \hat{P}_{33})\nabla_{23}[Re(\hat{P}_{23})]^2 + (\hat{P}_{33} - \hat{P}_{11})\nabla_{31}[Re(\hat{P}_{31})]^2. \tag{137}$$

Using

$$\nabla_{ij} \simeq \frac{1}{(\hat{P}_{ii} - \hat{P}_{jj}) \log(\Lambda/\mu)}, \tag{138}$$

in the equation above, we get $Re(\hat{P}_{ij}) \to 0$ in the infrared for all i, j.

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