Nonperturbative Calculation of the Shear Viscosity in Hot ϕ^4 Theory in Real Time

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Starting from the Kubo formula we calculate the shear viscosity in hot ϕ^4 theory nonperturbatively by resumming ladders with a real-time version of the Bethe-Salpeter equation at finite temperature. In the weak coupling limit, the generalized Fluctuation-Dissipation Theorem is shown to decouple the Bethe-Salpeter equations for the different real-time components of the 4-point function. The resulting scalar integral equation is identical with the one obtained by Jeon using diagrammatic "cutting rules" in the Imaginary Time Formalism.

A systematic field theoretical calculation of the viscosity from the Kubo formula involving the stress tensor correlation function [1] has been the subject of intense theoretical interest [2–5]. First results from a one-loop calculation [2–4] of the viscosity in hot ϕ^4 theory were incomplete because an infinite number of higher order diagrams in the loop expansion contribute to the lowest order in powers of the coupling constant [5]. Their resummation requires a nonperturbative calculation. Jeon [5] identified the dominant class of diagrams contributing to the viscosity at leading order in the coupling constant and resummed them by using so-called diagrammatic "cutting" rules in the Imaginary Time Formalism (ITF). In this paper we reproduce his results using an alternative and, we believe, more economic approach based on a solution of the Bethe-Salpeter equation in real time using the Closed Time Path (CTP) formalism [6–8]. Our calculation draws heavily on recently derived [9] general relations among the different thermal components of real-time n-point Green functions.

The CTP formalism has the advantage over the ITF that it is easily generalized to non-equilibrium situations and that it does not require analytic continuation of the n-point Green functions to real time at the end of the calculation. It's popularity is, however, decreased by the doubling of degrees of freedom and the resulting complicated matrix structure of the Green functions. We begin by reviewing some important relations among the components of the n-point Green functions in the CTP formalism which help to reduce the complexity of the real-time calculation before evaluating any Feynman diagrams. In the so-called single-time representation of the CTP formalism the n-point Green function is defined as

$$G_{a_1\dots a_n}(1,\dots,n) \equiv (-i)^{n-1}\langle T_p[\hat\phi_{a_1}(1)\cdots\hat\phi_{a_n}(n)]\rangle\,. \quad (1)$$

Here $\langle\dots\rangle$ denotes the thermal expectation value, the numbers $1,\dots,n$ stand for Minkowski space coordinates $x_1,\dots,x_n$, T_p represents the time ordering operator along the closed time path and corresponds to normal (antichronological) time ordering of operators with time arguments on its upper (lower) branch, and $a_1,a_2,\dots,a_n = 1,2$ indicate on which of the two branches the fields are located. (For details see, e.g., [8,9].) Using the KMS condition [10] one derives in momentum space [8]

$$G^*_{a_1a_2\dots a_n}(k_1,k_2,\dots,k_n) = (-1)^{n-1}G_{\bar a_1\bar a_2\dots\bar a_n}(k_1,k_2,\dots,k_n)\prod_{\{i|a_i=2\}} e^{\beta k_i^0}\,, \quad (2)$$

where $k_1{+}k_2{+}\dots{+}k_n = 0$, the star denotes complex conjugation, β is the inverse temperature, and $\bar a_i = 2,1$ for $a_i = 1,2$, respectively. The "physical" or r/a representation is defined by setting [8]

$$\hat\phi_a(x) = \hat\phi_1(x){-}\hat\phi_2(x), \quad \hat\phi_r(x) = \frac{\hat\phi_1(x){+}\hat\phi_2(x)}{2}, \quad (3)$$

and writing the n-point Green function as

$$G_{\alpha_1\dots\alpha_n}(1,\dots,n) \equiv \frac{2^{n_r-1}}{i^{n-1}}\langle T_p[\hat\phi_{\alpha_1}(1)\cdots\hat\phi_{\alpha_n}(n)]\rangle\,. \quad (4)$$

Here $\alpha_1,\dots,\alpha_n = a,r$, and n_r is the number of r indices among $(\alpha_1,\alpha_2,\dots,\alpha_n)$. One can show [8] that

$$G_{aa\dots a}(1,\dots,n) = 0\,, \quad (5)$$

that the functions with only one r index, $G_{ra\dots a}$, $G_{ara\dots a}$, $\dots$, $G_{a\dots ar}$, are the fully retarded n-point Green functions defined in [11], and that (except for $G_{rr\dots r}$) all other components $G_{\alpha_1\dots\alpha_n}(1,\dots,n)$ involve both retarded and advanced relations among their n time arguments. The simplest example is the 2-point function:

$$\Delta_{aa}(k) = 0,\ \Delta_{ra}(k) = \Delta^{\rm ret}(k),\ \Delta_{ar}(k) = \Delta^{\rm adv}(k), \quad (6)$$

$$\Delta_{rr}(k) = \big(1{+}2n(k^0)\big)\big(\Delta_{ra}(k) - \Delta_{ar}(k)\big)\,. \quad (7)$$

Eq. (7) is the fluctuation-dissipation theorem [12], with $n(k^0) = 1/(e^{\beta k^0}{-}1)$. The retarded and advanced 2-point functions satisfy

$$\Delta^*_{ra}(k) = \Delta_{ar}(k), \quad (8)$$

$$\Delta_{ra}(-k) = \Delta_{ar}(k), \quad \Delta_{rr}(-k) = \Delta_{rr}(k). \quad (9)$$

The the single-time and r/a representations of the n-point Green functions are related by [8]

$$G_{a_1\dots a_n}(1,\dots,n) = 2^{1-\frac{n}{2}} G_{\alpha_1\dots\alpha_n}(1,\dots,n) \times Q_{\alpha_1 a_1}\cdots Q_{\alpha_n a_n}\,, \tag{10}$$

where repeated indices are summed over and

$$Q_{a1} = -Q_{a2} = Q_{r1} = Q_{r2} = \frac{1}{\sqrt{2}} \tag{11}$$

are the four elements of the orthogonal Keldysh transformation for 2-point functions [10]. n-point Green functions involving only a single field ϕ are symmetric under particle exchange:

$$G_{\dots\alpha_i\dots\alpha_j\dots}(\dots,k_i,\dots,k_j,\dots) = G_{\dots\alpha_j\dots\alpha_i\dots}(\dots,k_j,\dots,k_i,\dots) \tag{12}$$

In the following we study massless ϕ^4 theory in the weak coupling limit:

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{\lambda}{4!}\phi^4, \qquad \lambda \ll 1. \tag{13}$$

The Kubo formula for the shear viscosity [2,3]

$$\eta = \frac{\beta}{20}\lim_{p^0,\boldsymbol{p}\to 0}\int d^4x\, e^{ip\cdot x}\langle \pi_{lm}(x)\pi^{lm}(0)\rangle \tag{14}$$

involves the correlation function of the traceless pressure tensor π_{lm} which in the thermal rest frame reads

$$\pi_{lm}(x) = \left(\delta_{li}\delta_{mj} - \tfrac{1}{3}\delta_{lm}\delta_{ij}\right)\partial_i\phi\,\partial_j\phi \tag{15}$$

with $i,j,l,m = 1,2,3$. From the definitions (1,14) one sees that η is related to the 12-component of the 2-point Green function $\Delta_{\pi\pi}$ of the composite field π_{lm} in the single-time representation:

$$\eta = \frac{i\beta}{20}\lim_{p^0,\boldsymbol{p}\to 0}\Delta^{12}_{\pi\pi}(p) = -\frac{\beta}{20}\lim_{p^0,\boldsymbol{p}\to 0}\mathrm{Im}\,\Delta^{12}_{\pi\pi}(p)\,. \tag{16}$$

In the second equation we used that the (12)-component of any 2-point function is purely imaginary as follows from Eqs. (7)-(10). In (15) each composite field π_{lm} is composed of two single particle fields ϕ. Substituting (15) into (14) one can rewrite (16) as

$$\eta = \frac{\beta}{5}\lim_{p^0,\boldsymbol{p}\to 0}\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4q}{(2\pi)^4}J_{lm}(-k,p{+}k) \times \mathrm{Im}\,G_{1122}(-k,p{+}k,q,-p{-}q)J^{lm}(q,-p{-}q) \tag{17}$$

where G_{1122} denotes the (1122)-component of the 4-point Green function for the ϕ field, $J_{lm}(p,q)$ joins two ϕ propagators to a point (see Fig. 1):

$$J_{lm}(p,q) = p_l q_m - \tfrac{1}{3}\delta_{lm}\,\boldsymbol{p}\cdot\boldsymbol{q}\,, \tag{18}$$

and a symmetry factor 4 accounts for the different possibilities of doing so.

Jeon showed [5] that in ϕ^4 theory at leading order of the coupling constant ($\sim \frac{1}{\lambda^2}$) all planar ladder diagrams (see Fig. 1) contribute to the shear viscosity. On the other hand, contributions from crossed diagrams can be ignored. In the CTP formalism the infinite number of ladder diagrams can be resummed by using the Bethe-Salpeter (BS) equation. Fig. 2 illustrates the corresponding BS integral equation for the 4-point Green function, with ladders consisting of rungs formed by a one-loop diagram connecting two propagators. In general this BS integral equation will couple the different components of the 4-point Green function in the CPT formalism with each other. The key technical issue is therefore whether and how these equations can be decoupled. In the following we will show that in the weak coupling limit this is indeed possible, and that the most convenient basis for doing so is the r/a representation of the CTP formalism. The previously derived generalized fluctuation-dissipation theorem (FDT) [9] plays an important role in the decoupling procedure.

From Eq. (2) we deduce

$$G_{1122}(-k,p{+}k,q,-p{-}q) = -n(p^0) \times[G_{1122}(-k,p{+}k,q,-p{-}q) + G^*_{2211}(-k,p{+}k,q,-p{-}q)] \tag{19}$$

so that Eq. (17) can be rewritten as

$$\eta = -\frac{\beta}{5}\lim_{p^0,\boldsymbol{p}\to 0}n(p^0)\int_{k,q}J_{lm}(-k,p{+}k)J^{lm}(q,-p{-}q) \times\mathrm{Im}\,[G_{1122}{+}G^*_{2211}](-k,p{+}k,q,-p{-}q)]. \tag{20}$$

In the physical representation we have (cf. Eq. (10))

$$\mathrm{Im}\,[G_{1122}+G^*_{2211}] = -\tfrac{1}{4}\mathrm{Im}\,[G_{rrra}{+}G_{aaar} {+}G_{rrar}{+}G_{aara}{-}G_{rarr}{-}G_{araa}{-}G_{arrr}{-}G_{raaa}]\,. \tag{21}$$

From the generalized FDT [9] we know that only 7 of the 16 components of $G_{\alpha_1\alpha_2\alpha_3\alpha_4}$ are independent; choosing them as the 4 fully retarded functions plus G_{arra}, G_{arar} and G_{aarr} and using the relations derived in Sec. III C of Ref. [9] we find

$$\mathrm{Im}\left(G_{1122}{+}G^*_{2211}\right) = \mathrm{Im}\left(aG_{raaa} + bG_{araa} + cG_{aara} +dG_{aaar} + eG_{arra} + fG_{arar} + gG_{aarr}\right), \tag{22}$$

where the coefficients $a,b,\dots,g$ are sums and products of thermal distribution functions. In the limit $p^0{\to}0$ the latter are given by

$$a = \beta p^0\,N_k(N_q^2{-}1) + \mathcal{O}\big((\beta p^0)^2\big)\,, \tag{23a}$$

$$b = \frac{(\beta p^0)^2}{4}(N_k^2{-}1)(N_q^2{-}1) + \mathcal{O}\big((\beta p^0)^3\big)\,, \tag{23b}$$

$$c = \frac{\beta p^0}{2}\left[N_k(N_q^2{-}1) + N_q(N_k^2{-}1)\right] + \mathcal{O}\big((\beta p^0)^2\big)\,, \tag{23c}$$

$$d = \frac{\beta p^0}{2}\left[N_k(N_q^2{-}1) - N_q(N_k^2{-}1)\right] + \mathcal{O}\big((\beta p^0)^2\big)\,, \tag{23d}$$

$$e = \frac{\beta p^0}{2}\left[(N_k^2{-}1) + (N_q^2{-}1)\right] + \mathcal{O}\big((\beta p^0)^2\big)\,, \tag{23e}$$

$$f = \frac{\beta p^0}{2}\left[(N_k^2{-}1) + (N_q^2{-}1)\right] + \mathcal{O}\big((\beta p^0)^2\big)\,, \tag{23f}$$

$$g = -\beta p^0(N_k^2{-}1) + \mathcal{O}\big((\beta p^0)^2\big)\,, \tag{23g}$$

where $N_k = 1 + 2n(k^0)$. With $\lim_{p^0\to 0} \beta p^0\, n(p^0) = 1$ and using the symmetry of the integrations over k and q in Eq. (20) as well as the symmetry relation (12), the shear viscosity (20) is found as

$$\eta = \frac{\beta}{5}\int \frac{d^4k}{(2\pi)^4} n(k^0)\big(1{+}n(k^0)\big) I_{\pi,lm}(k) \times \int \frac{d^4q}{(2\pi)^4} \mathrm{Im}\,\bar{G}(-k,k,q,-q) I_\pi^{lm}(q)\,, \tag{24}$$

where $I_{\pi,lm}(k) \equiv -J_{lm}(-k,k)$ as in [5] and

$$\bar{G} = 2\,G_{arra} - G_{aarr}\,. \tag{25}$$

We now set up the BS integral equation for $\bar{G}$ according to Fig. 2. The Feynman rules of the r/a representation [9] give

$$\begin{aligned} &i^3 G_{\alpha_1\alpha_2\alpha_3\alpha_4}(-k,k,q,-q) = \\ &[i\Delta_{\alpha_1\alpha_3}(-k)]\,[i\Delta_{\alpha_2\alpha_4}(k)]\,(2\pi)^4\delta^4(k-q) + \\ &\frac{1}{2}\,[i\Delta_{\alpha_1\beta_1}(-k)]\,[i\Delta_{\alpha_2\gamma_1}(k)]\,(-i\lambda_{\beta_1\beta_2\beta_3\beta_4})(-i\lambda_{\gamma_1\gamma_2\gamma_3\gamma_4}) \\ &\times \int \frac{d^4s}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\,[i\Delta_{\gamma_2\beta_2}(s)]\,[i\Delta_{\beta_3\gamma_3}(s+l-k)] \\ &\quad\times \left[i^3 G_{\beta_4\gamma_4\alpha_3\alpha_4}(-l,l,q,-q)\right]. \end{aligned} \tag{26}$$

Here the factor $\frac{1}{2}$ on the r.h.s. is the symmetry factor associated with the bubble connecting the two lines, and in the r/a representation the bare 4-point vertex is given by [9]

$$\lambda_{\alpha_1\alpha_2\alpha_3\alpha_4} = \frac{\lambda}{4}\,[1-(-1)^{n_a}]\,. \tag{27}$$

n_a is the number of a indices among $(\alpha_1,\alpha_2,\alpha_3,\alpha_4)$.

In general, the full retarded 2-point Green function $\Delta_{ra}(k)$ can be expressed as

$$\Delta_{ra}(k) = \frac{1}{k^2 + \mathrm{Re}\,\Sigma(k) + i\,\mathrm{Im}\,\Sigma(k)}\,. \tag{28}$$

The advanced Green function $\Delta_{ar}(k)$ is given by the complex conjugate expression (see (8)). For massless ϕ^4 theory $\mathrm{Re}\,\Sigma(k) = -\lambda T^2/24 + \mathcal{O}(\lambda^2)$ whereas $\mathrm{Im}\,\Sigma(k) \sim \mathcal{O}(\lambda^2)$ [13,14]. For weak coupling the propagators thus have quasiparticle poles at $\mathrm{Re}\,E_k = \pm\sqrt{\boldsymbol{k}^2 + \lambda T^2/24}$ with small imaginary parts, and in the limit $\lambda \to 0$ the product $\Delta_{ra}(k)\Delta_{ar}(k)$ develops a pinch singularity at E_k due to the converging pair of poles in the upper and lower half of the complex energy plane. Since the products $\Delta_{ra}(k)\Delta_{ra}(k)$ and $\Delta_{ar}(k)\Delta_{ar}(k)$ have no pinch singularities, the corresponding contributions to the shear viscosity can be neglected for $\lambda \ll 1$.

We now evaluate the BS equation for $\bar{G}$ in this approximation. Using (6)-(9) in Eq. (26) one obtains for the first term on the r.h.s. of (25)

$$G_{arra}(-k,k,q,-q) = \frac{\lambda^2}{4} N(k^0)\Delta_{ra}(k)\Delta_{ar}(k) \times \int \frac{d^4s}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\Delta_{rr}(s)\Delta_{rr}(s{+}l{-}k)G_{aara}(-l,l,q,-q)\,. \tag{29}$$

Obviously this doesn't decouple: G_{arra} couples to G_{aara}. However, changing the integration variables $s \to -s$ and $l \to -l$ and using the relations (9) and (12) together with $N(-k^0) = -N(k^0)$ one finds that $G_{arra}(-k,k,q,-q)$ is an odd function of k. The corresponding contribution to the shear viscosity (24) thus vanishes by symmetric integration.

For the second term in (25) we obtain in the same approximation from Eq. (26)

$$\begin{aligned} &G_{aarr}(-k,k,q,-q) = -\Delta_{ra}(k)\Delta_{ar}(k)\Big\{ i(2\pi)^4\delta^4(k{-}q) \\ &+\frac{\lambda^2}{8}\int\frac{d^4s}{(2\pi)^4}\frac{d^4l}{(2\pi)^4} G_{aarr}(-l,l,q,-q) \\ &\times\Big[\Delta_{ra}(s)\Delta_{ar}(s{+}l{-}k) + \Delta_{ar}(s)\Delta_{ra}(s{+}l{-}k) \\ &\quad +\Delta_{rr}(s)\Delta_{rr}(s{+}l{-}k) \\ &\quad +N(l^0)\big([\Delta_{ra}(s)-\Delta_{ar}(s)]\,\Delta_{rr}(s{+}l{-}k) \\ &\quad +\Delta_{rr}(s)\,[\Delta_{ar}(s{+}l{-}k)-\Delta_{ra}(s{+}l{-}k)]\big)\Big]\Big\} \end{aligned} \tag{30}$$

Already here one sees that the BS equation decouples: the integral equation involves only the single component G_{aarr} of the 4-point function. To make further progress it is convenient to introduce a function M which is obtained by truncating two of the external legs of the 4-point Green function G:

$$\begin{aligned} &G_{\alpha_1\alpha_2\alpha_3\alpha_4}(-k,k,q,-q) = \\ &[i\Delta_{\alpha_1\beta_1}(-k)]\,[i\Delta_{\alpha_2\beta_2}(k)]\,M_{\beta_1\beta_2\alpha_3\alpha_4}(-k,k,q,-q)\,. \end{aligned} \tag{31}$$

We also introduce the 2-point spectral density

$$\rho(k) = i\,[\Delta_{ra}(k) - \Delta_{ar}(k)] \tag{32}$$

and derive from Eq. (8)

$$\Delta_{ra}(k)\Delta_{ar}(k) = \frac{\rho(k)}{2\mathrm{Im}\,\Sigma(k)}. \tag{33}$$

In the pinching pole approximation the first two terms in the square brackets in Eq. (30) can be approximated as

$$\begin{aligned} &\Delta_{ra}(s)\Delta_{ar}(s{+}l{-}k) + \Delta_{ar}(s)\Delta_{ra}(s{+}l{-}k) \\ &\approx [\Delta_{ra}(s)-\Delta_{ar}(s)][\Delta_{ar}(s{+}l{-}k)-\Delta_{ra}(s{+}l{-}k)] \\ &= \rho(s)\rho(s{+}k{-}l)\,. \end{aligned} \tag{34}$$

With these ingredients, and using Eq. (7), Eq. (30) can then be simplified as

$$\begin{aligned} &\mathrm{Im}\,M_{rrrr}(-k,k,q,-q) = (2\pi)^4\delta^4(k{-}q) + \\ &\frac{\lambda^2}{4}\int\frac{d^4s}{(2\pi)^4}\frac{d^4l}{(2\pi)^4}\frac{\rho(l)\,\rho(s)\,\rho(s{+}k{-}l)}{\mathrm{Im}\,\Sigma(l)}\times \\ &\frac{[1{+}n(l^0)][1{+}n(s^0{+}k^0{-}l^0)]n(s^0)}{1+n(k^0)}\mathrm{Im}\,M_{rrrr}(-l,l,q,-q)\,. \end{aligned} \tag{35}$$

Note that the integral equation remains decoupled when expressed through the truncated 4-point function M.

For the shear viscosity one thus gets

$$\eta = \frac{\beta}{10} \int \frac{d^4k}{(2\pi)^4}\, n(k^0)[1+n(k^0)] I_\pi(k) \frac{\rho(k)}{\mathrm{Im}\,\Sigma(k)} \times \int \frac{d^4q}{(2\pi)^4} I_\pi(q) \mathrm{Im}\, M_{rrrr}(-k,k,q,-q)\,. \qquad (36)$$

This can be brought into the form given by Jeon [5] by defining

$$D_\pi(k) = \int \frac{d^4q}{(2\pi)^4}\, I_\pi(q)\, \mathrm{Im}\, M_{rrrr}(-k,k,q,-q)\,. \qquad (37)$$

Inserting Eq. (35) into this definition, the BS equation derived here is found to coincide with the results obtained by Jeon in Eqs. (4.16)–(4.21) of Ref. [5]).

Let us summarize: We have shown that a nonperturbative calculation of the shear viscosity in hot ϕ^4 theory can be performed in the real-time (CTP) formalism by solving a BS equation. In the weak coupling limit the BS integral equation decouples and involves only a single component of the real-time thermal 4-point function. Our results agree with those of Jeon [5], but our derivation is much more compact. This was made possible by using the generalized fluctuation-dissipation theorem which relates the different components of real-time thermal n-point functions.

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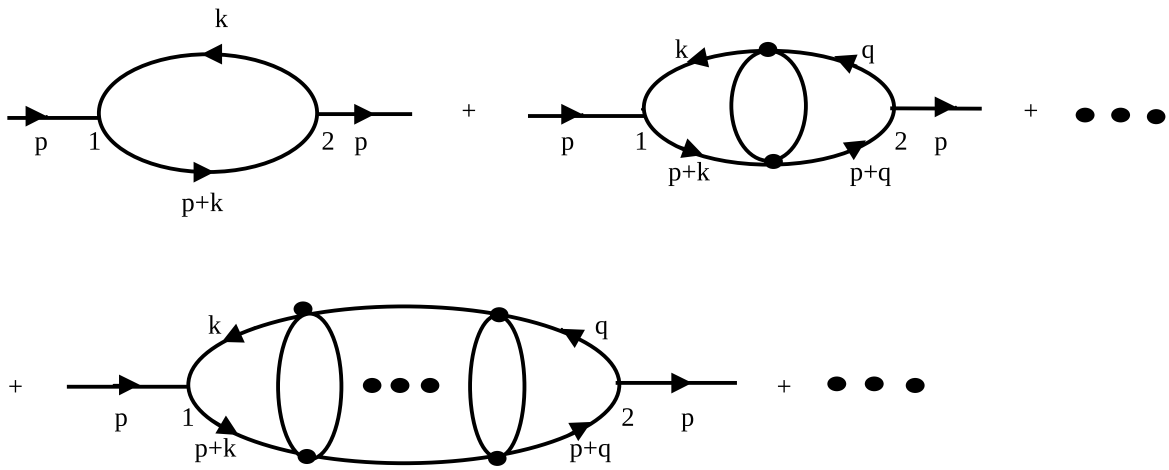

Fig. 1. The planar ladder diagrams contributing at the same order for shear viscosity.

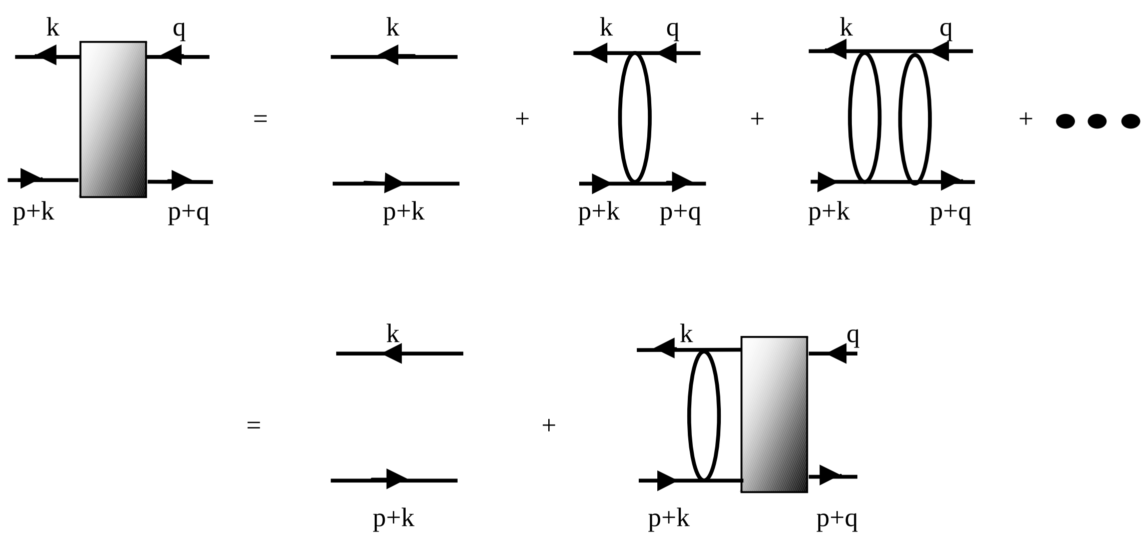

Fig. 2. Bethe-Salpeter equation for 4-point Green function.