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Second order chromaticity correction of LHC V6.0 at collision

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Abstract

The low-β triplets of the LHC induce strong chromatic aberrations in collision. The most spectacular one is a large second order chromaticity Q'' which can reach 60'000 units in the ultimate configuration (for protons) where the betatron functions are squeezed to β^* = 0.25 m in IP1 and IP5. This effect is accompanied by an off-momentum β-beating $\beta'(s) \equiv \left(\partial\beta/\partial\delta\right)(s)$ all around the ring which has to be carefully controlled at the level of the LHC inner triplets for mechanical aperture reasons. Two different correction strategies are discussed. The first one consists in constraining the betatron phase advances between the different collision points of the machine. It is found to be very efficient in the configuration where only IR1 and IR5 are tuned and is made possible by the relatively large tunability of IR4 and IR6. In the case of three IP's (i.e. β^* = 0.5 m in IP1, IP2 & IP5 in the nominal configuration for ions), the conditions required are difficult to obtain due to the lack of tunability of the insertion IR2. The second option is based on the use of several sextupole families, at least two per transverse plane, in order to correct the linear chromaticity Q', cancel Q'' and, at the same time, reduce to an acceptable level the off-momentum β-beating in the LHC inner triplets. As shown by the results obtained for the LHC Version 6.0 for different tune splits (ΔQ = 4 or 5) and different configurations (two or three IP's), this option seems to be less constraining than the previous one. Nevertheless, the use of sextupole families remains well-suited to LHC as long as the arc cells have phase advances close to 90° and is found to be inefficient for too large tune splits as ΔQ = 7.

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Introduction and motivation

The compensation of the chromatic aberrations arising from low-β insertions of a large storage ring is an old problem for which two different types of solution can be proposed:

- Option I: the careful adjustment of the betatron phase advances between IP's.
- Option II: the use of several sextupole families, more than one per transverse plane, which was the choice of LEP [1].

The first option was tried out for the LHC Version 5 [2] in the case where only two insertions, IR1 and IR5, were tuned in collision mode. The formula for Q'' given in this note is incomplete but however leads to the well-known condition which is to space the two IP's by $\pi/2 + k\pi$ (k integer) in betatron phase. As shown in [2], this condition can be fulfilled by a careful re-matching of IR4 and IR6. On the other hand, the phase conditions required in the case of three low-β insertions (for instance $\beta^* = 0.5$ m in IP1, IP2 and IP5) have not been discussed.
The use of two sextupole families per plane (Option II) was successfully implemented in LHC Version 4.1 [3] in order to correct the second order chromaticity induced at collision. Knowing that the efficiency of this correction depends strongly on the phase advance in the arc cells and then on the tune split (which was zero in Version 4), this study has to be repeated with the present lattice of the LHC and different optics.
The purpose of this report is to review the feasibility of these two correction methods for the LHC Version 6.0.
We will begin by specifying some acceptability criteria concerning the chromatic aberrations of the LHC at collision (Chapter 1). The latter will be defined, on the one hand, by imposing some limits on the variations of the optical functions $Q(\delta)$ and $\beta^*(\delta)$ over the natural momentum spread of the beam; on the other hand, they will be chosen according to the ability to accelerate and decelerate safely the beam in a relevant momentum range for linear chromaticity measurements at collision (Section 1.1). These criteria will then be expressed in terms of tolerances on Q'', Q''', $\beta^{*'} \equiv \partial\beta^*/\partial\delta$ and $\beta^{*''} \equiv \partial^2\beta^*/\partial\delta^2$ (Section 1.2). After compensation of the linear chromaticity Q', simulation results performed with MAD [4] on different collision optics of LHC Version 6.0 will show that, among these four quantities, only Q'' and $\beta^{*'}$ require a correction (Section 1.3).
In order to clarify the problem, the latter will be computed analytically in Chapter 2 by a simple use of the R-matrix formalism.
Starting from these analytic expressions, the strategy consisting in correcting Q'' by phasing the IP's will be reviewed for LHC Version 6.0 (Chapter 3). In the case of three low-β insertions (IP1, IP2 and IP5), this option will be found too constraining.
By using the $4 \times 8 = 32$ sextupole families per ring foreseen in the LHC lattice Version 6.0, the second option will be tested out on some collision optics available in the data base[1] (Chapter 4). The obtained results will show that, even after correction of Q', Q''

[1] Tune split of 4, 5 or 7; IP1, IP2 and IP5 with $\beta^* = 0.5$ m; IP1 and IP5 with $\beta^* = 0.25$ m.

and $\beta^{*'}$, the available safety margin on the sextupole gradients remains quite reasonable (close to 20%). Nevertheless, this method will be found inefficient for a tune split of 7 because the phase advances per arc cell are then too far from 90°.

1 Chromatic aberrations due to the low-β triplets at collision: acceptability criteria and tolerances

1.1 Momentum window at collision and acceptability criteria for chromatic aberrations

We begin by defining some acceptability criteria concerning the chromatic aberrations of the LHC at top energy. For that purpose, we will consider requirements of different nature, all related to the chromatic behaviour of the optical functions $Q(\delta)$ and $\beta^*(\delta)$ over a given energy range. In order to define this range, we must consider both the natural momentum spread of the beam at collision, and a minimal momentum window which must be accessible under **nominal** conditions for the measurement of the linear chromaticity after the β-squeeze:

- the relative energy spread (RMS) of the beam at collision is about $\sigma_\delta \simeq \mathbf{0.11 \times 10^{-3}}$ (0.111×10^{-3} for the protons and 0.114×10^{-3} for the ions $^{208}Pb^{82+}$ [5]). Over the range $\pm 2\sigma_\delta$, the average chromatic detuning of the beam (due to Q''), as well as the tune ripple (due to Q'' and Q''') sampling by the non-synchronous particles has to be minimised. On the other hand, in order to preserve the performance of the machine, the chromatic variation of the β function at the IP's has to be carefully controlled. As shown in the next section, this does only constrain the value of the second derivative of β^* with respect to the energy noted $\beta^{*''}$. The criterion used to fix the tolerance on $\beta^{*'}$ is given hereafter.

- it is estimated that a good accuracy on the linear chromaticity measurement is reached by fitting the points obtained in a window defined by $\delta = \pm\mathbf{5 \times 10^{-4}}$. Indeed, at both extremities of this window, the tune shift induced by a Q' of 2 units (which is the nominal value chosen for LHC) is equal to $\pm 10^{-3}$, which is sufficient assuming an absolute resolution of 10^{-4} for the tune-metre. In order to avoid the head-tail instability of the off-momentum beam, the slope of the function $Q(\delta)$ must remain positive over the whole range $[-5 \times 10^{-4}\,,\, 5 \times 10^{-4}]$, which, obviously, imposes a second constraint on the non-linear chromaticities Q'' and Q'''. The changes with momentum of the β-functions must also been controlled, especially at the level of the inner triplets, in order not to introduce additional aperture limitations. As we will see in Chapter 2, the relative off-momentum β-beating, i.e. $1/\beta\,(\partial\beta/\partial\delta)$, oscillates with twice the betatron phase all along the machine; since the inner triplets are spaced from the interaction point by $\pm\pi/2$ in phase, this last requirement fixes automatically an upper bounds on the value of the chromatic functions $\beta^{*'}/\beta^*$ at the IP's.

1.2 Tolerances

Chromatic aberrations	Tolerances at collision
First derivative of β^* with respect to the energy	$-200 < \beta^{*'}/\beta^* < 200$
Second derivative of β^* with respect to the energy	$-5 \times 10^5 < \beta^{*''}/\beta^* < 5 \times 10^5$
Second order chromaticity	$-2000 < Q'' < 2000$
Third order chromaticity	$-0.8 \times 10^7 < Q''' < 5 \times 10^7$

Table 1: Tolerances on the chromatic aberrations of the LHC at collision. The lower limit given for Q''' can be multiplied by two if Q'' is fully corrected.

We start from the following simple equations

$$Q(\delta) = Q_0 + Q'\delta + \frac{1}{2}\, Q''\delta^2 + \frac{1}{6}\, Q'''\delta^3 + \cdots , \tag{1}$$

$$\frac{\Delta\beta^*(\delta)}{\beta^*} = \frac{\beta^{*'}}{\beta^*}\,\delta + \frac{1}{2}\,\frac{\beta^{*''}}{\beta^*}\,\delta^2 + \cdots , \tag{2}$$

$$\frac{\Delta\sigma^*}{\sigma^*} = \sqrt{1 + \frac{1}{2}\,\frac{\beta^{*''}}{\beta^*}\,\sigma_\delta^2 + \cdots}\; - \;1 = \frac{1}{4}\,\frac{\beta^{*''}}{\beta^*}\,\sigma_\delta^2 + \cdots , \tag{3}$$

where σ^* denotes the beam spot-size in collision (on-momentum beam) at a given IP. We can now deduce from the previous criteria the tolerances on Q'', Q''', $(\beta^{*'}/\beta^*)$ and $(\beta^{*''}/\beta^*)$ (see also Table 1).

- *Ability to measure the linear chromaticity.* In order to avoid a head-tail instability of the beam during the measurement of Q', the function $Q'(\delta) = 2 + Q''\delta + Q'''\delta^2/2 + \cdots$ must remain positive in the energy range defined by $-5 \times 10^{-4} \le \delta \le 5 \times 10^{-4}$. This requirement can be replaced by the more restrictive condition:

  $$Q''\delta > -1 \;\text{ and }\; Q'''\delta^2/2 > -1 \;\text{ for }\; \delta = \pm 5 \times 10^{-4} \;\Rightarrow\; |Q''| < 2000 \text{ and } Q''' > -8 \times 10^6 \ . \tag{4}$$

  Nevertheless, if Q'' is fully corrected, the tolerance on Q''' can be doubled, i.e.

  $$Q''' > -1.6 \times 10^7 \ . \tag{5}$$

  In the inner triplets, on both sides of a given IP, the relative change with momentum of the β function, $(\Delta\beta/\beta)_{Triplet} \simeq \pm\beta^{*'}\delta/\beta^*$ ($-5 \times 10^{-4} \le \delta \le 5 \times 10^{-4}$), must stay within its uncertainty due to linear optics errors, say lower than 10%, leading to

  $$\left|\frac{\beta^{*'}}{\beta^*}\right| < 200 \ . \tag{6}$$

- *Tune ripple and performance.* Due to the non-linear chromaticities Q'' and Q''', the non-synchronous particles sample a tune ripple at one, two or three times the synchrotron frequency Ω_s, i.e. below 100 Hz. In this low frequency range, the ripple amplitude has to be carefully controlled at the level of a few 10^{-5} in order to preserve the beam life-time [6]. For a particle belonging to the core of the (on-momentum) beam, say $\delta = \sigma_\delta$, this condition yields

$$|Q''|\,\sigma_\delta^2/2 \lesssim 10^{-5} \text{ and } |Q'''|\,\sigma_\delta^3/6 \lesssim 10^{-5} \;\Rightarrow\; |Q''| < 2000 \text{ and } |Q'''| < 5\times 10^7 \ . \tag{7}$$

  Finally, by imposing to the relative luminosity loss to be less than a few per mil, we obtain (see Eq. (3))

$$\left|\frac{\Delta\sigma^*}{\sigma^*}\right| \lesssim 10^{-3} \quad\Rightarrow\quad \left|\frac{\beta^{*''}}{\beta^*}\right| < 5\times 10^5\ . \tag{8}$$

1.3 MAD calculation on LHC lattice version 6.0 for different optics

In order to demonstrate that the correction of chromatic aberrations of the LHC is imperative after (or/and during) the β-squeeze, the values of Q'', Q''', $\beta^{*'}$ and $\beta^{*''}$ have been computed with MAD [4] for different collision optics of the LHC lattice Version 6.0 (Ring-1, Tune split of 4, 5 and 7) and for different tunings of the low-β insertions IR1, IR2, and IR5:

- Tuning I. $\beta^* = 0.5\,\text{m}$ in IP1 and IP5; $\beta^* = 10\,\text{m}$ in IP2 and IP8 (nominal configuration for protons).

- Tuning II. $\beta^* = 0.5\,\text{m}$ in IP1, IP2 and IP5; $\beta^* = 10\,\text{m}$ in IP8 (nominal configuration for ions).

- Tuning III. $\beta^* = 0.25\,\text{m}$ in IP1 and IP5; $\beta^* = 10\,\text{m}$ in IP2 and IP8 (ultimate configuration for protons).

In each case, the gradients of the arc sextupoles (two families per ring, SF and SD) was chosen to correct the linear chromaticity down to 2 units.
As shown in Table 2 and in view of the tolerances given previously, only Q'' and $\beta^{*'}$ require a special correction. More precisely, in the ultimate configuration where $\beta^* = 0.25$ m in IP1 and IP5, the second order chromaticity has to be corrected by more than one order of magnitude while $\beta^{*'}$ has to be reduced by a factor 2. Note that the values of $\beta^{*'}$ and $\beta^{*''}$ which are put in brackets in Table 2 are not constrained by the specifications given in Table 1: indeed, the latter concern the off-momentum β-beating at IP2 in the case where IR2 is not tuned in collision mode.
In the case of two IP's (Insertion 2 detuned) and for a given tuning of IR1 and IR5 (i.e. $\beta^* = 0.5\,\text{m}$ or $\beta^* = 0.25\,\text{m}$), it is worth noting that the chromatic aberrations induced (i.e. Q'', Q''', and $\beta^{*'}$ and $\beta^{*''}$ in IP1 and IP5) does (almost) not depend on the choice of

Optics	Optics 63.31/59.32			Optics 64.31/59.32			Optics 65.31/58.32	
Insertion tuning	Tuning I	Tuning II	Tuning III	Tuning I	Tuning II	Tuning III	Tuning I	Tuning II
$\mu_x[2\pi]$ at IP1	0	0	0	0	0	0	0	0
$\mu_x[2\pi]$ at IP2	8.322	8.247	8.404	8.461	8.381	8.523	8.604	8.514
$\mu_x[2\pi]$ at IP5	31.840	31.840	31.840	32.340	32.340	32.340	32.840	32.840
$\beta_x^{*'}/\beta_x^*$ at IP1	**223**	138	**469**	190	**368**	**405**	199	**245**
$\beta_x^{*'}/\beta_x^*$ at IP2	(150)	-154	(794)	(323)	**417**	(185)	(-232)	**212**
$\beta_x^{*'}/\beta_x^*$ at IP5	197	51	**428**	**210**	**352**	**440**	**230**	**345**
$\beta_x^{*''}/\beta_x^*$ $[10^5]$ at IP1	1.4	0.9	**5.2**	1.2	1.7	5.0	1.1	1.5
$\beta_x^{*''}/\beta_x^*$ $[10^5]$ at IP2	(1.1)	1.2	(5.9)	(1.9)	2.3	(9.1)	(1.7)	2.5
$\beta_x^{*''}/\beta_x^*$ $[10^5]$ at IP5	0.8	-0.3	3.5	0.8	1.0	3.5	1.0	1.9
Q_x'' $[10^3]$	**12.1**	0.3	**55.8**	**11.1**	**32.0**	**50.1**	**12.8**	**23.8**
Q_x''' $[10^7]$	0.4	-0.8	3.1	0.4	-0.1	3.3	0.4	1.2
$\mu_y[2\pi]$ at IP1	0	0	0	0	0	0	0	0
$\mu_y[2\pi]$ at IP2	7.467	7.486	7.389	7.446	7.500	7.322	7.309	7.388
$\mu_y[2\pi]$ at IP5	29.771	29.771	29.771	29.771	29.771	29.770	29.271	29.271
$\beta_y^{*'}/\beta_y^*$ at IP1	77	192	141	75	155	119	56	**257**
$\beta_y^{*'}/\beta_y^*$ at IP2	(229)	**237**	(809)	(306)	166	(562)	(232)	**497**
$\beta_y^{*'}/\beta_y^*$ at IP5	25	14	94	38	-12	120	62	**257**
$\beta_y^{*''}/\beta_y^*$ $[10^5]$ at IP1	0.6	0.9	2.7	0.7	0.8	2.7	0.6	0.4
$\beta_y^{*''}/\beta_y^*$ $[10^5]$ at IP2	(2.2)	3.6	(6.5)	(2.2)	3.8	(3.9)	(0.9)	2.7
$\beta_y^{*''}/\beta_y^*$ $[10^5]$ at IP5	1.1	2.2	4.5	1.2	2.1	4.3	1.1	1.6
Q_y'' $[10^3]$	**2.7**	**13.0**	**15.0**	**3.3**	**9.2**	**15.5**	**3.6**	**30.5**
Q_y''' $[10^7]$	0.5	0.9	3.5	0.5	1.0	3.4	0.5	-0.2

Table 2: Chromatic aberrations in Ring-1 (after Q' correction) and phase between IP's for different collision optics and configurations of IR1, IR2 and IR5 (LHC lattice Version 6.0).

the tune split. In fact, it can be observed that the phase differences between IP1 and IP5 are rigorously identical modulo $[\pi]$ for the three optics, 63-59, 64-59 and 65-58, considered here. This reveals the fact that only the betatron phase advances between IP's play some role and then could be used as correction knobs. In Chapter 3, we will show to what extent this assertion is right.

2 Analytical approach by the map formalism

In order to clarify the problem and before studying any strategy of correction for $\beta^{*'}$ and Q'', let us begin by deriving analytically the expressions of these two quantities.

2.1 Matricial Hill equation and perturbative series

We start from the Hill equation and assume some quadrupolar field errors along the machine:

$$\frac{d^2 z}{ds^2} + \Big(K_z(s) + k_z(s)\Big)\, z = 0 \ , \tag{9}$$

where the used notations have the following meaning:

- z stands for either the horizontal or vertical coordinate with respect to a given closed orbit with curvilinear abscissa s.
- $K_x(s) \stackrel{\mathrm{def}}{=} K_1(s) + h^2(s)$ and $K_y(s) \stackrel{\mathrm{def}}{=} -K_1(s)$; $K_1(s) = eB_1(s)/p_0 c$ is the quadrupolar strength seen by a particle of nominal energy p_0 and $h(s) = eB_0(s)/p_0 c$ denotes the local curvature of the reference orbit.
- the function $k_z(s)$ represents the distribution of quadrupolar field errors around the ring. In the particular case where the focusing errors are only related to chromatic effects, this function is given by

$$\left\{\begin{array}{rcl} k_x & = & (K_1 + h^2)\left(\dfrac{1}{1+\delta} - 1\right) + \dfrac{K_2}{1+\delta}\left(D_x\,\delta + D_x^{(2)}\,\delta^2 + \cdots\right) \\ & = & \left(K_2\,D_x - (K_1 + h^2)\right)\delta + \Big((K_1 + h^2) - K_2\,D_x + K_2\,D_x^{(2)}\Big)\,\delta^2 + \mathrm{O}(\delta^3) \\ k_y & = & -K_1\left(\dfrac{1}{1+\delta} - 1\right) - \dfrac{K_2}{1+\delta}\left(D_x\,\delta + D_x^{(2)}\,\delta^2 + \cdots\right) \\ & = & (K_1 - K_2\,D_x)\,\delta + \Big(K_2\,D_x - K_2\,D_x^{(2)} - K_1\Big)\,\delta^2 + \mathrm{O}(\delta^3) \ , \end{array}\right. \tag{10}$$

  where $\delta \stackrel{\mathrm{def}}{=} (p - p_0)/p_0$ is the relative momentum deviation, $K_2(s) = eB_2(s)/p_0 c$ is the sextupolar strength seen by a particle of nominal energy p_0 and where D_x and $D_x^{(2)}$ denote the horizontal dispersion function and its first derivative with respect to the energy[2].

Note that in writing the Hill equation, the focusing terms related to the pole face rotation and to the fringing fields of the dipoles have been omitted. These terms proportional to h and h' are found to have no influence on the chromatic properties of the LHC.
The solution of Eq. (9) can be written in a vectorial form as

$$Z(s) = R(s_0; s)\, Z(s_0) \quad \text{with} \quad Z(s) \stackrel{\mathrm{def}}{=} \begin{pmatrix} z(s) \\ z'(s) \end{pmatrix} \ ,$$

[2]By writing Eq. (10), it has been implicitly assumed that the magnetic field seen by the beam does not contain any octupolar components.

where the matrix R satisfies the following equation:

$$\left\{\begin{array}{l} R(s_0;s_0) = \mathbf{1} \\ \dfrac{dR}{ds} = \left(A^{(0)} + A^{(1)}\right) R \text{ with } A^{(0)} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 1 \\ -K_z & 0 \end{pmatrix} \text{ and } A^{(1)} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 \\ -k_z & 0 \end{pmatrix} . \end{array}\right. \tag{11}$$

In order to solve this equation, the matrix R is searched as a perturbative series with respect to the errors $k_z(s)$:

$$R(s_0;s) = R^{(0)}(s_0;s) + \sum_{n=1}^{\infty} R^{(n)}(s_0;s) \ , \tag{12}$$

where, for a given n, $R^{(n)}$ is of order n in the perturbation and satisfies (see Eq. (11))

$$\begin{array}{l} \left\{\begin{array}{lcl} R^{(0)}(s_0;s_0) & = & \mathbf{1} \\ \dfrac{dR^{(0)}}{ds} & = & A^{(0)}\, R^{(0)} \end{array}\right. \\ \left\{\begin{array}{lcl} R^{(n)}(s_0;s_0) & = & \mathbf{0} \\ \dfrac{dR^{(n)}}{ds} & = & A^{(0)}\, R^{(n)} + A^{(1)}\, R^{(n-1)} \end{array}\right. , \ n \geq 1 \ . \end{array} \tag{13}$$

1. The matrix $R^{(0)}$ is known a priori:

$$R^{(0)}(s_0;s_1) = \begin{pmatrix} \sqrt{\dfrac{\beta_1^{(0)}}{\beta_0^{(0)}}}\left(\cos(\mu^{(0)}) + \alpha_0^{(0)} \sin(\mu^{(0)})\right) & \sqrt{\beta_0^{(0)}\beta_1^{(0)}} \sin(\mu^{(0)}) \\ \dfrac{1}{\sqrt{\beta_0^{(0)}\beta_1^{(0)}}}\left(\cos(\mu^{(0)})(\alpha_0^{(0)} - \alpha_1^{(0)}) - \sin(\mu^{(0)})(1 + \alpha_0^{(0)}\alpha_1^{(0)})\right) & \sqrt{\dfrac{\beta_1^{(0)}}{\beta_0^{(0)}}}\left(\cos(\mu^{(0)}) - \alpha_0^{(0)} \sin(\mu^{(0)})\right) \end{pmatrix}, \tag{14}$$

where $\alpha_0^{(0)} = \alpha^{(0)}(s_0)$, $\alpha_1^{(0)} = \alpha^{(0)}(s_1)$, $\beta_0^{(0)} = \beta^{(0)}(s_0)$ and $\beta_1^{(0)} = \beta^{(0)}(s_1)$ are the unperturbed Twiss parameters in s_0 and s_1 and where $\mu^{(0)} = \displaystyle\int_{s_0}^{s_1} \frac{ds}{\beta^{(0)}(s)}$ is the phase advance from s_0 to s_1 (without quadrupole field error).
2. Finally, by using Eq. (13) for $n \geq 1$, it can be easily checked that the matrices $R^{(n)}$ and $R^{(n-1)}$ are linked by the relation

$$R^{(n)}(s_0;s) = R^{(0)}(s_0;s) \int_{s_0}^{s} ds'\, R^{(0)}(s';s_0)\, A^{(1)}(s')\, R^{(n-1)}(s_0;s') \ , \text{ leading to}$$

$$R^{(n)}(s_0;s) = \int_{s_0<s_1<\ldots<s_n<s} ds_1 \ldots ds_n \ R^{(0)}(s_n;s) \times A^{(1)}(s_n) \times \ldots \times R^{(0)}(s_1;s_2) \times A^{(1)}(s_1) \times R^{(0)}(s_0;s_1) \, . \tag{15}$$

2.2 Integral formula for Q'' and $\dfrac{\partial\beta}{\partial\delta}$

We have now all the ingredients required to calculate the tunes $Q_{x,y}$ as well as the β functions $\beta_{x,y}$ at least at the second order in the perturbation k_z. Then, by using Eq. (10), we will directly obtain the expressions of the second order chromaticity Q'' and of the off-momentum β-beating $\beta' \equiv \partial\beta/\partial\delta$.
For that purpose, at a given abscissa s within the ring, the one-turn R-matrix $R(s; s+C)$ is parametrised in the following usual way:

$$\begin{aligned} R(s;s+C) &\equiv \begin{pmatrix} \cos(2\pi Q) + \alpha(s)\,\sin(2\pi Q) & \beta(s)\,\sin(2\pi Q) \\ -\dfrac{1+\alpha(s)^2}{\beta(s)}\,\sin(2\pi Q) & \cos(2\pi Q) - \alpha(s)\,\sin(2\pi Q) \end{pmatrix} \\ &= R^{(0)}(s;s+C) + \sum_{n=1}^{\infty} R^{(n)}(s;s+C)\ , \end{aligned} \tag{16}$$

where C is the machine circumference and where the quantities

$$\left\{ \begin{aligned} Q &= Q^{(0)} + Q^{(1)} + Q^{(2)} + \ldots = \arccos\left(\frac{R_{11}+R_{22}}{4\pi}\right) \\ \beta(s) &= \beta^{(0)}(s) + \beta^{(1)}(s) + \ldots = \frac{R_{12}}{\sin(2\pi\, Q)} \\ \alpha(s) &= \alpha^{(0)}(s) + \alpha^{(1)}(s) + \ldots \end{aligned} \right. \tag{17}$$

denote the (horizontal or vertical) tune of the ring and the Twiss parameters at abscissa s in the presence of linear imperfections $k \equiv k_z$. Then, by using the relations (15) and (17) and after some algebra, we obtain

$$\begin{aligned} Q^{(1)} &= -\frac{R_{11}^{(1)}(0;C) + R_{22}^{(1)}(0;C)}{4\pi \sin\left(2\pi Q^{(0)}\right)} = \frac{1}{4\pi}\int_0^C ds'\ k(s')\,\beta^{(0)}(s') \\ \frac{\beta^{(1)}(s)}{\beta^{(0)}(s)} &= \frac{R_{12}^{(1)}(s;s+C)/\beta^{(0)}(s) - 2\pi\,\cos\left(2\pi Q^{(0)}\right) Q^{(1)}}{\sin\left(2\pi Q^{(0)}\right)} \\ &= -\frac{1}{2\,\sin\left(2\pi Q^{(0)}\right)}\int_0^C ds'\ k(s')\,\beta^{(0)}(s')\,\cos\left(2\left|\mu^{(0)}(s') - \mu^{(0)}(s)\right| - 2\pi Q^{(0)}\right) \\ Q^{(2)} &= -\frac{R_{11}^{(2)}(0;C) + R_{22}^{(2)}(0;C) + 4\pi^2\,\cos\left(2\pi Q^{(0)}\right)\left(Q^{(1)}\right)^2}{4\pi \sin\left(2\pi Q^{(0)}\right)} \\ &= \frac{-1}{16\pi \sin\left(2\pi Q^{(0)}\right)}\int_0^C\! ds_1\int_0^C\! ds_2\,\left(k\beta^{(0)}\right)\!(s_1)\,\left(k\beta^{(0)}\right)\!(s_2)\cos\left(2\left|\mu^{(0)}(s_1)-\mu^{(0)}(s_2)\right|-2\pi Q^{(0)}\right). \end{aligned} \tag{18}$$

Finally, the second order chromaticity Q'' and the off-momentum β-beating β' can be deduced from the previous relations by replacing the chromatic focusing error k by its expression given in Eq. (10):

$$
\begin{cases}
Q'_{x,y} = -\dfrac{1}{4\pi} \displaystyle\int_0^C ds_1\, K^{(1)}_{x,y}(s_1)\ \beta_{x,y}(s_1) \\
Q''_{x,y} = \dfrac{1}{2\pi} \displaystyle\int_0^C ds_1\, K^{(2)}_{x,y}(s_1)\ \beta_{x,y}(s_1)\ - \\
\qquad \dfrac{1}{8\pi \sin(2\pi Q_{x,y})} \displaystyle\int_0^C ds_1 \int_0^C ds_2\, K^{(1)}_{x,y}(s_1)\beta_{x,y}(s_1)\, K^{(1)}_{x,y}(s_2)\beta_{x,y}(s_2)\ \cos(2\,|\mu_{x,y}(s_1)-\mu_{x,y}(s_2)| - 2\pi Q_{x,y}) \\
\dfrac{\beta'_{x,y}}{\beta_{x,y}} = \dfrac{1}{2\sin(2\pi Q_{x,y})} \displaystyle\int_0^C ds_1\, K^{(1)}_{x,y}(s_1)\ \beta_{x,y}(s_1)\ \cos(2\,|\mu_{x,y}(s_1)-\mu_{x,y}(s)| - 2\pi Q_{x,y})\ ,
\end{cases}
\tag{19}
$$

where we have defined the following functions:

$$
\begin{cases}
K^{(1)}_x(s) &=& (K_1(s) + h^2(s)) - K_2(s)\, D_x(s) \\
K^{(1)}_y(s) &=& -K_1(s) + K_2(s)\, D_x(s) \\
K^{(2)}_x(s) &=& K^{(1)}_x(s) + K_2(s)\, D^{(2)}_x(s) \\
K^{(2)}_y(s) &=& K^{(1)}_y(s) - K_2(s)\, D^{(2)}_x(s)\ .
\end{cases}
\tag{20}
$$

For any ring, the optical functions D_x and $D^{(2)}_x$ are generally of the same order of magnitude since they satisfy two similar differential equations (see e.g. [7, p. 66]). Therefore, if the lattice sextupoles are tuned to compensate the linear chromaticity of the ring, the first term occurring in the expression of Q'' becomes of the order of Q' before its correction, that is a few hundred for the LHC, and then can be neglected by comparing with the values obtained in Table 2. Under this approximation, note that second order chromaticity and off-momentum β-beating satisfy the following relation:

$$
Q''_{x,y} = -\frac{1}{4\pi} \int_0^C ds\, K^{(1)}_{x,y}(s)\, \beta'_{x,y}(s)\ . \tag{21}
$$

3 Phasing the IP's to correct Q''

The purpose of this chapter is to evaluate the possibility to correct the chromatic aberrations of the LHC induced at collision by adjusting the betatron phases between its different IP's. For any ring containing N low β-insertions, we will start by expressing Q'' and $\beta^{*'}$ as a function of the phase advances between IP's (Section 3.1). Then, we will study the particular cases where the ring contains 2 or 3 identical low-β insertions and will derive in each case the phase conditions required for a self-compensation of the induced Q'' (Sections 3.2 & 3.3). Numerical applications will be given for different collision optics and different configurations of the LHC Version 6.0 showing that, in the case of three IP's (IR1, IR2 and IR5 tuned in collision mode), this solution cannot be easily implemented in practice due to the lack of tunability of the insertion 2.

3.1 Q'' and $\dfrac{\partial\beta^*}{\partial\delta}$ as a function of the phase advances between IP's

We begin with some notations and definitions. Let us consider a ring containing N low-β insertions IR$_i$, $i = 1 \ldots N$, separated by a large number of **identical** FODO cells (arc cells). For each insertion IR$_i$, the horizontal and vertical β-functions $\beta^*_{x_i,y_i}$ at IP$_i$ (interaction point) are assumed to be made small (in a sense defined hereafter) by means of two sets of quadrupoles, say two triplets, without loss of generality, symmetrically placed with respect to IP$_i$: the triplets TR$_i$ (on the right of IP$_i$) and TL$_i$ (on the left). If L^*_i denotes the distance between one of these two triplets and the collision point IP$_i$, the ratio β^*_i/L^*_i is assumed to be much lower than unity so that the triplets TR$_i$ and TL$_i$ are spaced from IP$_i$ by $\pm\pi/2$ in phase:

$$\mu^{R_i} = \mu^i + \pi/2 \ \text{ and } \ \mu^{L_i} = \mu^i - \pi/2 \ , \tag{22}$$

where μ^{R_i}, μ^{L_i} and μ^i are the (horizontal or vertical) phase advances at TR$_i$, TL$_i$ and IP$_i$ respectively.

Now, by coming back to the relations (19), let us show that the contribution of the arc magnets can be neglected in the calculation of Q'' and $\beta'(s)$. Basically, this is due to the fact that the integrand functions occurring in the expressions giving $\beta'(s)$ and Q'' oscillate with twice the betatron phase. As a result, integrating these functions over a large number of **identical** cells or over a single cell gives quantitatively similar results. Finally, since the β-functions are of the order of ${L^*}^2/\beta^* \gg L^*$ in the triplets, that is several kilometres for the LHC to be compared to one or two hundred meters within the arcs, the contribution of a single cell is clearly negligible. However, note that all this would not be true anymore if, for instance, the sextupole strength would change sign from cell to cell in a 90° FODO lattice. This last remark reveals precisely that an other solution to correct Q'' or β' consists in increasing the number of sextupole families per arc, which will form the subject of Chapter 4.

This being said, for each insertion IR$_i$, we introduce the following quantities:

$$\left\{ \begin{array}{lllll} I^{R_i}_x & \stackrel{\text{def}}{=} \displaystyle\int_{\text{TR}_i} ds\, K_1(s)\,\beta_x(s) & \text{and} & I^{L_i}_x & \stackrel{\text{def}}{=} \displaystyle\int_{\text{TL}_i} ds\, K_1(s)\,\beta_x(s) \\ I^{R_i}_y & \stackrel{\text{def}}{=} -\displaystyle\int_{\text{TR}_i} ds\, K_1(s)\,\beta_y(s) & \text{and} & I^{L_i}_y & \stackrel{\text{def}}{=} -\displaystyle\int_{\text{TL}_i} ds\, K_1(s)\,\beta_y(s)\,. \end{array} \right. \tag{23}$$

By using Eq. (19) and Eq. (22) and by considering only the contribution coming from the triplets, the off-momentum β-beating $\beta'(s)$ can be expressed in the following way:

$$\begin{aligned} \frac{\beta'_{x,y}(s)}{\beta_{x,y}(s)} &= \frac{1}{2\sin(2\pi Q_{x,y})} \sum_{i=1}^{N} \left[I^{R_i}_{x,y} \cos\left(2\left|\mu^{R_i}_{x,y} - \mu_{x,y}(s)\right| - 2\pi Q_{x,y}\right) + I^{L_i}_{x,y} \cos\left(2\left|\mu^{L_i}_{x,y} - \mu_{x,y}(s)\right| - 2\pi Q_{x,y}\right) \right] \\ &= -\frac{1}{2\sin(2\pi Q_{x,y})} \sum_{i=1}^{N} \left[I^{R_i}_{x,y} + I^{L_i}_{x,y} \right] \cos\left(2\left|\mu^{i}_{x,y} - \mu_{x,y}(s)\right| - 2\pi Q_{x,y}\right) \ . \end{aligned} \tag{24}$$

Insertions	I_x^L	I_x^R	I_y^L	I_y^R	$I^R + I^L$
IR_1 for $\beta^* = 0.5$ m	188.52	158.72	158.72	188.52	$\simeq 350$
IR_1 for $\beta^* = 0.25$ m	375.04	316.92	316.92	375.04	$\simeq 700$
IR_5 for $\beta^* = 0.5$ m	188.52	158.72	158.72	188.52	$\simeq 350$
IR_5 for $\beta^* = 0.25$ m	375.04	316.92	316.92	375.04	$\simeq 700$
IR_2 for $\beta^* = 0.5$ m	162.02	201.73	201.73	162.02	$\simeq 360$

Table 3: Coefficients I^R and I^L for the insertions 1, 2 and 3 of the LHC Version 6.0 with β^* =0.5 m or β^* =0.25 m.

Strictly speaking, the last equality is inexact in the drift located between two consecutive triplets IL_i and IR_i but remains valid at the collision point IP_i for which

$$\frac{\beta_i^{*\prime}}{\beta_i^*} = -\frac{1}{2\sin(2\pi Q)} \sum_{j=1}^{N} \left[I^{R_j} + I^{L_j}\right] \cos\left(2\left|\mu^{ij}\right| - 2\pi Q\right) = -\left(\frac{\beta'}{\beta}\right)_{\mathrm{TR}_i} = -\left(\frac{\beta'}{\beta}\right)_{\mathrm{TL}_i} \quad (25)$$

where, in order to simplify the notations, the subscripts x or y have been omitted, $\mu^{ij} \equiv \mu^i - \mu^j$ is the phase difference (horizontal or vertical) between IP_i and IP_j, and where $(\beta'/\beta)_{\mathrm{TR}_i}$ and $(\beta'/\beta)_{\mathrm{TL}_i}$ denotes the relative off-momentum β-beating in the triplets TR_i and TL_j (considered here as thin lenses).
The second order chromaticity is then deduced from Eq. (21):

$$\begin{aligned} Q'' &= -\frac{1}{4\pi} \sum_{i=1}^{N} \left[I^{R_i} \left(\frac{\beta'}{\beta}\right)_{\mathrm{TR}_i} + I^{L_i} \left(\frac{\beta'}{\beta}\right)_{\mathrm{TL}_i} \right] \\ &= -\frac{1}{8\pi \sin(2\pi Q)} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[I^{R_i} I^{R_j} + I^{L_i} I^{L_j} + 2\, I^{L_i} I^{R_j}\right] \cos\left(2\left|\mu^{ij}\right| - 2\pi Q\right) . \end{aligned} \quad (26)$$

The coefficients $I_{x,y}^R$ and $I_{x,y}^L$ are listed in Table 3 for the insertions 1, 2 and 5 of the LHC in the configuration where $\beta^* = 0.5$ m and for the insertions 1 and 5 with $\beta^* = 0.25$ m. IR1 and IR5 are rigorously identical and, as expected, the coefficients I^R and I^L scale with $1/\beta^*$. Moreover, for a given insertion, it is worth noting that $I_x^R = I_y^L$ and $I_y^R = I_x^L$, which is due to the fact that the left and right inner triplets are identical but with inverse polarities (antisymmetric design). Finally, note that the three insertions are similar in the sense that

$$I_{x,y}^{R_1} = I_{x,y}^{R_5} \simeq I_{x,y}^{R_2} \quad \text{and} \quad I_{x,y}^{L_1} = I_{x,y}^{L_5} \simeq I_{x,y}^{L_2} \ ,$$

leading to

$$I_x^{R_i} + I_x^{L_i} = I_y^{R_i} + I_y^{L_i} \equiv I^R + I^L \text{ for } i\text{=1,2 or 5.} \quad (27)$$

Under these conditions, the relations (25) and (26) reduce to

$$\left\{ \begin{array}{lcl} \dfrac{\beta^{*\prime}_{i_{x,y}}}{\beta^{*}_{i_{x,y}}} & = & -\dfrac{I^R + I^L}{2\sin\left(2\pi Q_{x,y}\right)} \displaystyle\sum_{j=1}^{N} \cos\left(2\left|\mu^{ij}_{x,y}\right| - 2\pi Q_{x,y}\right) \\ Q''_{x,y} & = & -\dfrac{\left(I^R + I^L\right)^2}{8\pi\sin\left(2\pi Q_{x,y}\right)} \displaystyle\sum_{i=1}^{N}\sum_{j=1}^{N} \cos\left(2\left|\mu^{ij}_{x,y}\right| - 2\pi Q_{x,y}\right) \ . \end{array} \right. \tag{28}$$

3.2 Case of two IP's

In this section, we deal with the case of a ring containing only two low-β insertions. This case corresponds to the proton operation of the LHC where the β-functions are made extremely small in IP1 and IP5. By using Eq. (28) and after some trigonometric manipulations, the non-linear chromaticity Q'' and the off-momentum β-beating at IP1 and IP5 can be expressed as

$$\left\{ \begin{array}{l} \left(\dfrac{\beta^{*\prime}_{x,y}}{\beta^{*}_{x,y}}\right)_{\mathrm{IP1}} = \left(\dfrac{\beta^{*\prime}_{x,y}}{\beta^{*}_{x,y}}\right)_{\mathrm{IP5}} = -\dfrac{I^R + I^L}{\sin\left(2\pi Q_{x,y}\right)} \cos\left(\mu^{15}_{x,y}\right) \cos\left(\mu^{15}_{x,y} - 2\pi\, Q_{x,y}\right) \\ Q''_{x,y} = -\dfrac{\left(I^R + I^L\right)^2}{2\pi\sin\left(2\pi Q_{x,y}\right)} \cos\left(\mu^{15}_{x,y}\right) \cos\left(\mu^{15}_{x,y} - 2\pi\, Q_{x,y}\right) \ , \end{array} \right. \tag{29}$$

where $\mu^{15}_{x,y} > 0$ is the (horizontal or vertical) phase advance from IP1 to IP5 for Ring-1 (or from IP5 to IP1 for Ring-2). Therefore, Q'' and $\beta^{*\prime}$ are simultaneously vanishing if

$$\mu^{15}_{x,y} = \frac{\pi}{2} \ \mathrm{mod}[\pi] \quad \text{or} \quad \mu^{15}_{x,y} = 2\pi\, Q_{x,y} + \frac{\pi}{2} \ \mathrm{mod}[\pi] \ . \tag{30}$$

It is also interesting to study the behaviour of the function $\beta'(s)$ all around the ring. The latter can be easily deduced from Eq. (24):

$$\left\{ \begin{array}{lcll} \dfrac{\beta'(s)}{\beta(s)} & = & -\dfrac{I^R + I^L}{\sin(2\pi Q)} \cos\left(\mu^{15} - 2\mu(s)\right) \cos\left(\mu^{15} - 2\pi Q\right) & \text{between IP1 and IP5} \\ \dfrac{\beta'(s)}{\beta(s)} & = & -\dfrac{I^R + I^L}{\sin(2\pi Q)} \cos\left(\mu^{15} - 2\mu(s) + 2\pi Q\right) \cos\left(\mu^{15}\right) & \text{between IP5 and IP1,} \end{array} \right. \tag{31}$$

where, in order to simplify the notations, the subscripts x and y have been omitted. Thus, if $\mu^{15}_{x,y} = 0.25 \times 2\pi \ \mathrm{mod}[\pi]$, the function $\beta'(s)$ vanishes everywhere between IP5 and IP1 (closed "β-bump" between IP1 and IP5) and vice versa if $\mu^{15}_{x,y} = 2\pi\, Q_{x,y} + 0.25 \times 2\pi \ \mathrm{mod}[\pi]$.
The first condition seems well-suited to the present collision optics of the LHC Version 6.0. Indeed, for the three optics considered in Table 2, this constraint can be satisfied by reducing the horizontal and vertical phase advances of IR4 by $\Delta Q_x = -0.090 \sim -32.5°$ and $\Delta Q_y = -0.021 \sim -7.5°$, and vice versa for IR6 in order to keep the tunes constant:

according to References [8] and [9], this is well within the tuning range of IR4 and accessible to IR6. In this case, the measurement of the relative off-momentum β-beating between IP5 and IP1 is certainly the simplest way to appreciate the exactness of this correction.

In order to guaranty the validity of these results, the quantities Q'' and $\beta^{*'}_{\mathrm{IP}_{1,5}}$ have been computed with MAD [4] as a function of the betatron phase between IP1 and IP5. For that purpose, without re-matching explicitly the insertions IR4 and IR6, their phase advances have been artificially modulated within a range of $\pm 180°$ around their nominal value by inserting in IP4 and IP6 a thin linear element represented by the following 6×6 symplectic matrix:

$$R(\Delta\mu_x, \Delta\mu_y) = \begin{pmatrix} R_{11} & R_{12} & 0 & 0 & 0 & R_{16} \\ R_{21} & R_{22} & 0 & 0 & 0 & R_{26} \\ 0 & 0 & R_{33} & R_{34} & 0 & 0 \\ 0 & 0 & R_{43} & R_{44} & 0 & 0 \\ R_{51} & R_{52} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \text{ with}$$

$$\begin{cases} R_{11}{=}\cos(\Delta\mu_x){+}\alpha_x\sin(\Delta\mu_x) \\ R_{12}{=}\beta_x\sin(\Delta\mu_x) \\ R_{21}{=}{-}\gamma_x\sin(\Delta\mu_x) \\ R_{22}{=}\cos(\Delta\mu_x){-}\alpha_x\sin(\Delta\mu_x) \end{cases} \begin{cases} R_{33}{=}\cos(\Delta\mu_y){+}\alpha_y\sin(\Delta\mu_y) \\ R_{34}{=}\beta_y\sin(\Delta\mu_y) \\ R_{43}{=}{-}\gamma_y\sin(\Delta\mu_y) \\ R_{44}{=}\cos(\Delta\mu_y){-}\alpha_y\sin(\Delta\mu_y) \end{cases} \begin{cases} R_{16}{=}D_x\,(1{-}R_{11}){-}D'_x\,R_{12} \\ R_{26}{=}D'_x\,(1{-}R_{22}){-}D'_x\,R_{21} \\ R_{51}{=}D'_x\,(1{-}R_{11}){+}D_x\,R_{21} \\ R_{52}{=}D_x\,(R_{22}{-}1){-}D'_x\,R_{12} \end{cases},$$

where $\beta_{x,y}$, $\alpha_{x,y}$, $\gamma_{x,y}$, D_x and D'_x denote the Twiss parameters, the dispersion and the angular dispersion at IP4 or IP6.

The results obtained are shown in Fig. 1. This computation concerns the collision optics 64-59 (tune split of 5) of the LHC lattice Version 6.0 with $\beta^* = 0.25$ m in IP1 and IP5 (ultimate configuration for protons). For this optics, the nominal situation corresponds to $\mu_x^{15} = 32.340 \times 2\pi$ and $\mu_y^{15} = 29.771 \times 2\pi$ (see Table 2), which is indicated by the small circles on Fig. 1. As expected, the second order chromaticity Q'' and the off-momentum β-beating $\beta^{*'}$ at IP1 and IP5 are simultaneously vanishing when the condition (30) is satisfied.

The same exercise has been done for the chromatic aberrations Q''' and $\beta^{*''}$, showing that, regardless of the phase difference between IP1 and IP5, the latter remain within the specifications given in Table 1 (see Fig. 2).

3.3 Case of three IP's

We consider here a ring containing three low β-insertions, say the LHC in ion operation with $\beta^* = 0.5$ m in IP1, IP2 and IP5. By using Eq. (28) and after some trigonometric manipulations, the second order chromaticity induced by the triplets can be expressed as

$$Q''(\mu^{12}\,, \mu^{25}) = -\frac{\left(I^R + I^L\right)^2}{8\pi\sin(2\pi Q)}\left[\cos(2\pi Q) + 8\,\cos\left(\mu^{12}\right)\cos\left(\mu^{25}\right)\cos\left(\mu^{51}\right)\right]\,, \qquad (32)$$

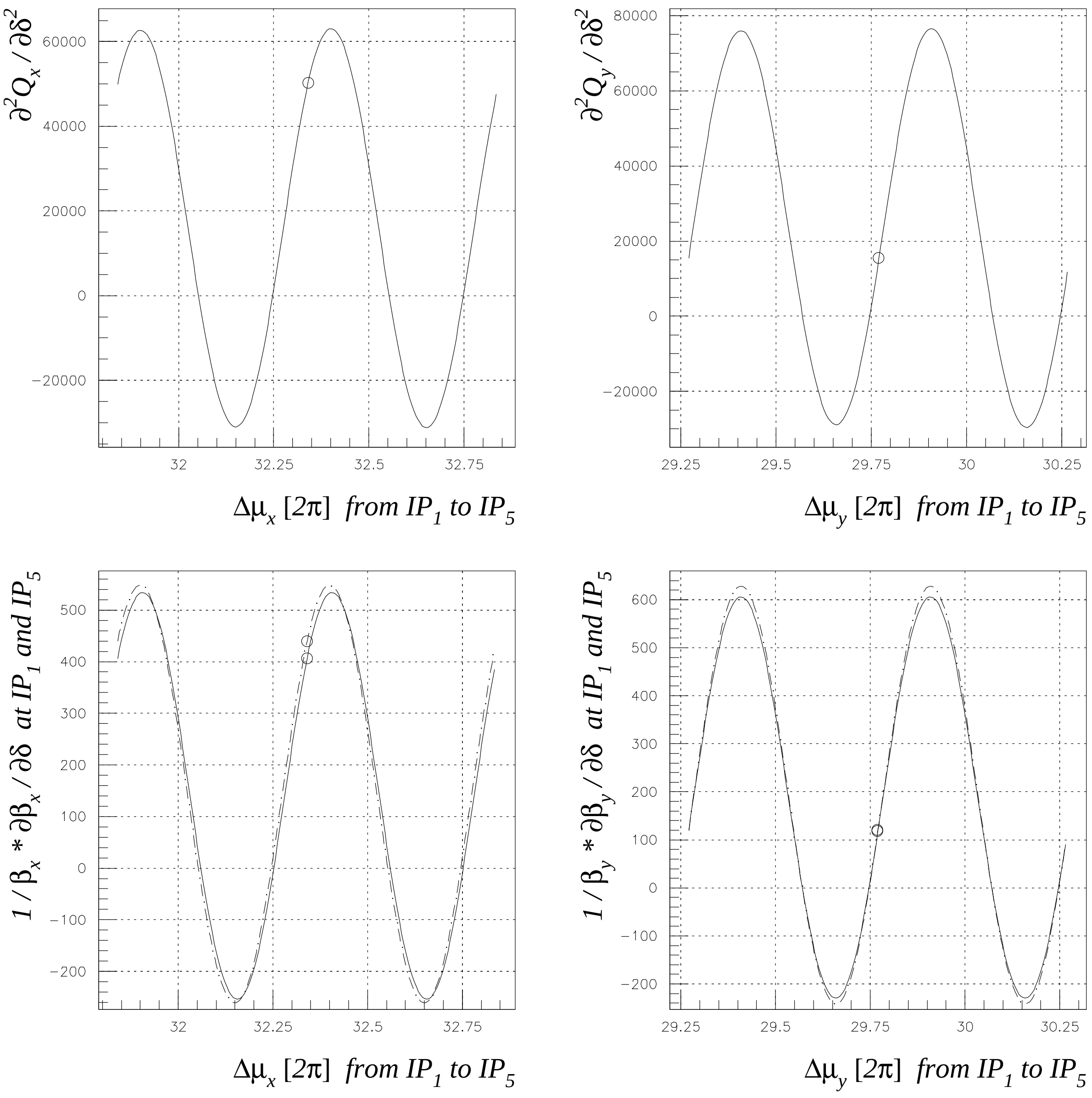

Figure 1: Chromatic aberrations Q'' and $\beta^{*\prime}$ at IP1 (solid lines) and IP5 (dashed lines) versus the phase advance from IP1 to IP5 (LHC Version 6.0, collision optics 64-59 with $\beta^* = 0.25$ m in IP1 and IP5).

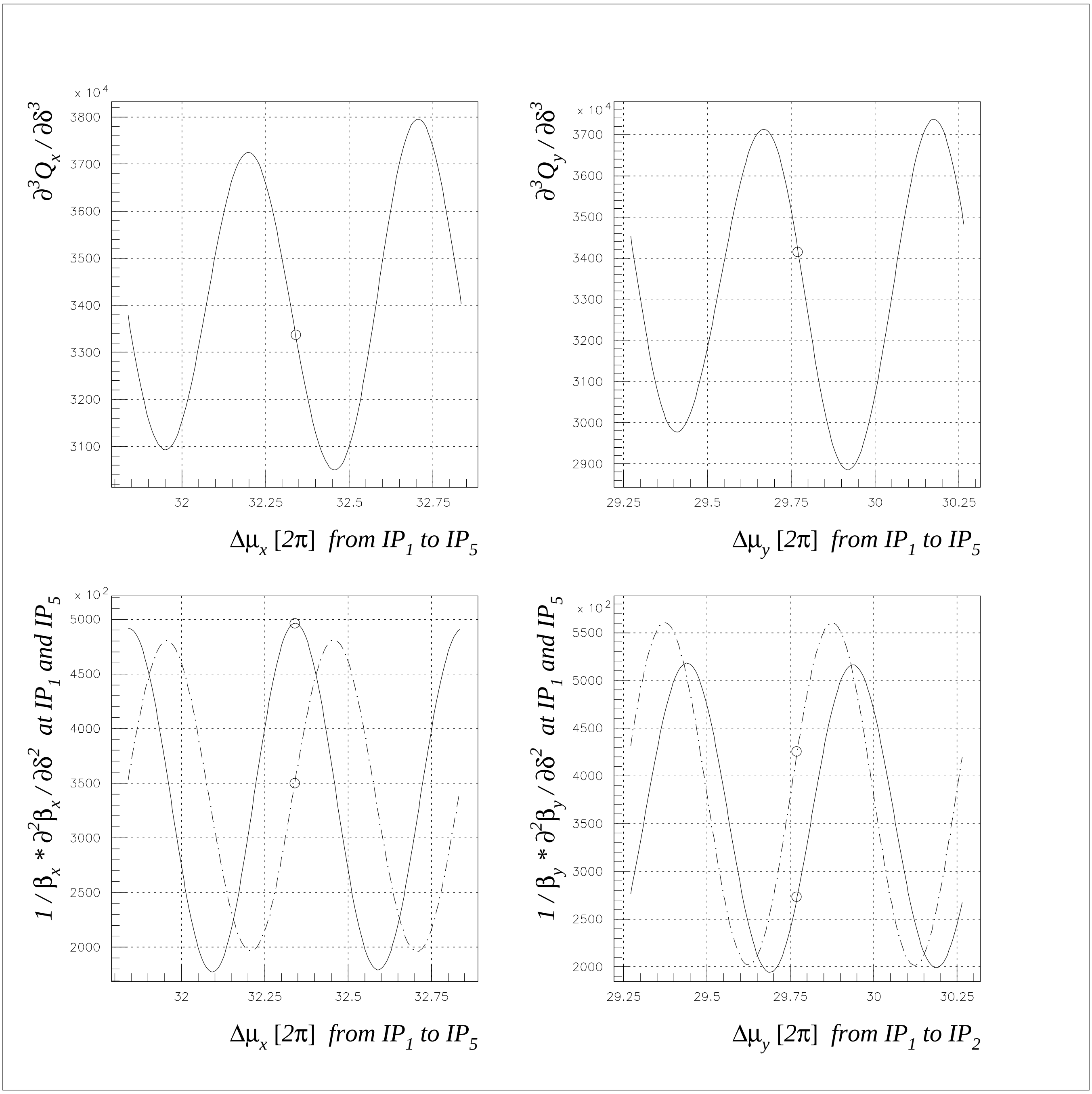

Figure 2: Chromatic aberrations Q''' and $\beta^{*''}$ at IP1 (solid lines) and IP5 (dashed lines) versus the phase advance from IP1 to IP5 (LHC Version 6.0, collision optics 64-59 with $\beta^* = 0.25$ m in IP1 and IP5).

with obviously $\mu^{51} = 2\pi Q - \mu^{15} = 2\pi Q - \mu^{12} - \mu^{25}$.
In view of Eq. (32), it is clear that for some value of the phase μ^{12}, for instance $\mu^{12} = 0.25 \times 2\pi \bmod[\pi]$ (corresponding to a closed "β-bump" between IP1 and IP2), the equation $Q''(\mu^{12}\,,\,\mu^{25}) = 0$ does not solve for μ^{25}. In fact, it is easy to see that the condition $Q'' = 0$ can also be written as

$$Q'' = 0 \Leftrightarrow \cos\left(2\pi Q - 2\mu^{25} - \mu^{12}\right) = -\frac{\cos(2\pi Q)}{4\,\cos\left(\mu^{12}\right)} - \cos\left(2\pi Q - \mu^{12}\right) \equiv \mathcal{G}\left(\mu^{12}\right) \ , \qquad (33)$$

which has no solution if $|\mathcal{G}| > 1$. For the present working point of the LHC at collision, $Q_x = .31$ and $Q_y = .32$, the condition $|\mathcal{G}| \leq 1$ is equivalent to

$$\begin{aligned} \mu_x^{12}/(2\pi) &\in [0\,,\,.242] \bigcup [.389\,,\,.742] \bigcup [.889\,,\,1] \mod[1] \\ \mu_y^{12}/(2\pi) &\in [0\,,\,.241] \bigcup [.402\,,\,.741] \bigcup [.902\,,\,1] \mod[1] \ , \end{aligned}$$

to be compared with the values of μ^{12} given in Table 2 for different LHC tune splits.
The solutions of the equation $Q''_{x,y}\left(\mu^{12}_{x,y}\,,\,\mu^{25}_{x,y}\right) \equiv 0$ are plotted in Fig. 3. The symbols $\circ$, $\times$ and $\bullet$ refer to the collision optics 63-59, 64-59 and 65-58, respectively, of the LHC lattice Version 6.0. Inside (resp. outside) the islands, the second order chromaticity is negative (resp. positive). On the frontier, Q" vanishes as it is the case (by chance) in the horizontal plane for the optics 63-59 ($Q''_x \sim 300$ in Table 2). On the other hand, the situation is much more delicate, for instance, in the horizontal plane for the optics 64-59 (symbol $\times$ on the top picture of Fig. 3). Different solutions are *a priori* possible to cancel Q''. The simplest, apparently, are the following:

- keep μ_x^{12} quasi-constant, change μ_x^{25} by $\sim \pm 0.25 \times 2\pi$ (see Fig. 3) and then μ_x^{51} by $\sim \mp 0.25 \times 2\pi$ in order to keep the horizontal tune constant. This is beyond the tuning range of IR4 and IR6 and then would require to re-match (almost) all the LHC insertions. This possibility is then too constraining.

- keep μ_x^{25} constant, reduce μ_x^{12} by $\sim -0.14 \times 2\pi$ (see Fig. 3) and then increase μ_x^{51} by $\sim 0.14 \times 2\pi$. According to Ref. [9], the change of the phase μ_x^{51} between IP5 and IP1 can be achieved by re-matching IR6. On the other hand, since the IR2 tunability is quasi null due to the special constraints imposed by the injection elements in Ring-1 [11], the condition required for μ_x^{12} can be fulfilled only by acting on the *right side* of the insertion IR1, i.e. by reducing its horizontal phase advance by $\Delta Q_x \sim -0.14$. In view of the study done in Ref. [12], this is beyond the tuning range of IR1.

Finally, it is worth noting that whether by acting on the main arc quadrupoles MQ or whether by acting on the tune correctors MQT, the arcs cannot be used in a simple way as tunable phase trombones between IP's. Indeed, on the one hand, in a given arc, the MQ's are powered in series in Ring-1 and Ring-2 (and the phase difference between two given IP's is not necessarily the same for Ring-1 and Ring-2). On the other hand, the

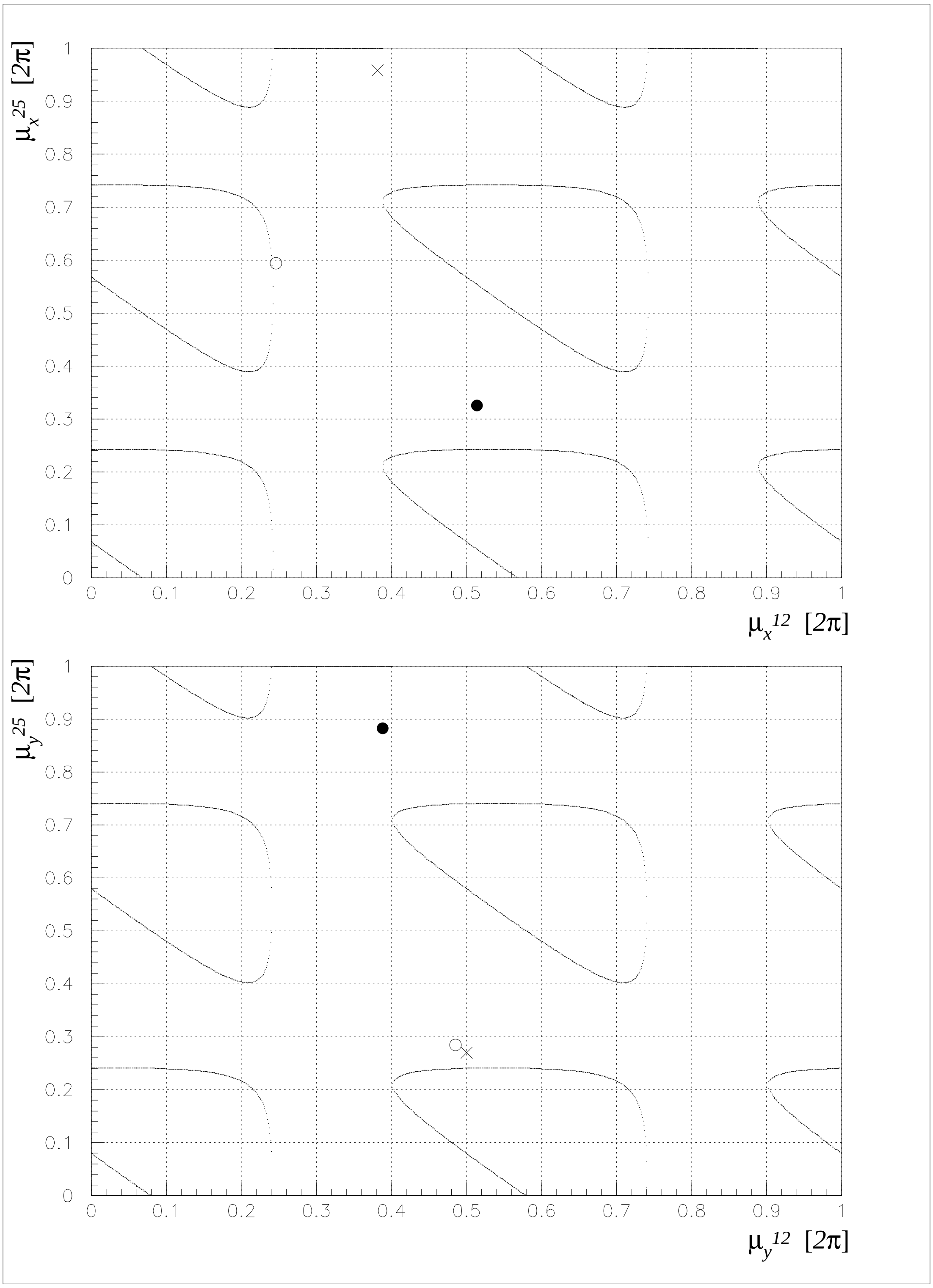

Figure 3: Solution of the equation $Q''(\mu^{12}, \mu^{25}) \equiv 0$ in the horizontal plane (top picture) and the vertical plane (bottom picture); the phases μ^{12} (from IP1 to IP2) and μ^{25} (from IP2 to IP5) are given modulo 1 in units of 2π. The symbols $\circ$, $\times$ and $\bullet$ refer to the nominal collision optics 63-59, 64-59 and 65-58, respectively, of the LHC lattice Version 6.0 (Ring-1).

MQT's (2 families of 8 MQT's per arc) are only foreseen for rapid and small changes of tune in operation. To that end, it is true that their use is limited by the optical aberrations (mismatch of the β-functions) that they induce at too high gradient so that the global change of tune that they can safely guaranty remains lower than ± 0.3 per ring [10] and corresponds to only 3/13 of their available strength at top energy. Therefore, they could also be used for phasing IP1, IP2 and IP5 (independently in Ring-1 and Ring-2) but provided that the β-functions are carefully re-matched at the entrances and exits of IR1, IR2 and IR5. By assuming that this re-matching does not change too much the tunability diagrams of IR1, IR2 and IR5 and by using say up to 10/13 of the MQT strength available at top energy, the variability of the phases μ^{12}, μ^{25} and μ^{51} could be increased by ± 0.125, $\pm 3 \times 0.125 = \pm 0.375$ and $\pm 4 \times 0.125 = \pm 0.5$, respectively, which is sufficient for our concern. This alternative would require a specific study of feasibility.
Due to the limited tunability of the LHC insertions, we conclude that the strategy consisting in phasing the IP's is not well-suited to the case of three low-β insertions. Moreover, note that if the LHC was constrained to run temporarily with only one low-β insertion, the correction of the second order chromaticity could evidently not be performed at collision. Therefore, as mentioned in the introduction, it is recommended to keep the 4×8 sextupole families presently foreseen in LHC Version 6.0. This will form the subject of the next chapter.

4 Using sextupole families to correct Q'' and $\dfrac{\partial \beta^*}{\partial \delta}$

In Section 4.1, we begin by describing the layout of the chromaticity sextupoles in the LHC lattice Version 6.0, i.e. four families per arc and per ring. Then, by using the analytical results derived in Chapter 2, we will obtain explicitly the dependence of the chromatic aberrations Q'' and $\beta^{*'}$ on the strengths of these different families (Section 4.2). Finally, in Section 4.3, we will develop a minimisation method suited to our problem and will test it on different collision optics and different configurations of the LHC Version 6.0.

4.1 Sextupole budget and available gradient

In the previous LHC versions, the sextupoles were only dedicated to the correction of the linear chromaticity Q'. In each arc, they were split in two distinct families, the SF family and the SD family, placed in the vicinity of the arc quadrupoles of type QF and QD respectively. Since the Version 6, the total number of sextupole families has been doubled, passing from 2×8 to $4 \times 8 = 32$ families per ring, in order to guaranty also the correction of Q'' at collision. In a given arc, the four families are named SFa, SFb, SDa and SDb respectively (see Fig. 4). The rings 1 and 2 are independent in terms of the sextupole supplying; each ring contains 188 sextupoles of type SD and 154 sextupoles of type SF [3].

[3]In each arc, it is foreseen to tilt by 90° four chromaticity sextupoles of type SF in order to use them as a_3 correctors [13].

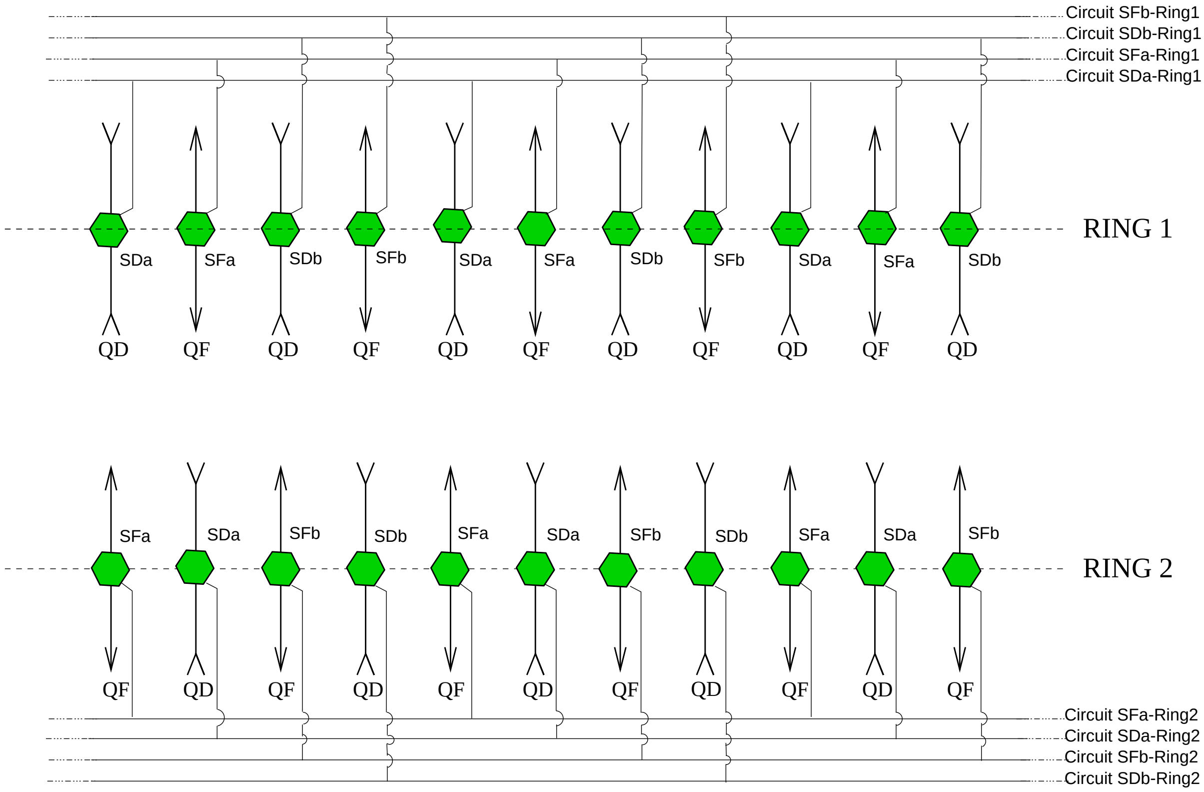

Figure 4: Sextupole families in LHC lattice Version 6.0.

The maximum strength available per sextupole is $0.129\,\mathrm{m}^{-2}$ at 7 TeV (for a nominal field of 1500 $\mathrm{T/m}^2$ [5]).

4.2 Q'' and $\dfrac{\partial\beta}{\partial\delta}$ as a function of the gradient of each sextupole family

We now come back to the results obtained in Chapter 2, more precisely to the relations (19) which give the quantities Q', Q'' and $\beta'(s)$ in integral form. The purpose of this section is to find explicitly the dependence of these chromatic aberrations on the strengths of the different sextupole families present in the ring.

Let us begin with some notations. We assume that the LHC runs with N low β-insertions and that each ring possesses 32 sextupole families. For a given ring, M_i and K_{2_i}, $i = 1 \ldots 32$, denote the number and the strength of the sextupoles belonging to the i^{th} family. When the sextupole gradients are all set to zero, the linear and second order chromaticities, Q' and Q'', the off-momentum β-beating $\beta'(s)$ and the second order dispersion $D_x^{(2)}(s)$ along the machine will be labelled by the superscript $^{(0)}$. Note that neither the β functions nor the dispersion D_x depend on the sextupolar strength $K_2(s)$ but the second order dispersion does. The latter varies linearly with K_2 and is given by (see e.g. [7, p. 66]):

$$D_x^{(2)}(s)=D_x^{(2)^{(0)}}(s) - \frac{1}{4\sin(\pi Q_x)}\int_0^C ds'\, K_2(s')D_x(s')^2\sqrt{\beta_x(s)\beta_x(s')}\,\cos\Big(\big|\mu_x(s')-\mu_x(s)\big|-\pi Q_x\Big)\,. \tag{34}$$

Finally, $\mathbf{K}$ will represent the 32-dimensional vector containing the strengths K_{2_i} of the 32 sextupole families (column vector) and its transpose will be noted $\mathbf{K}^{\mathrm{t}}$ (row vector). In view of Eq. (19), the linear chromaticity Q' and the off-momentum β-beating $\beta'(s)$ vary linearly with $\mathbf{K}$:

$$\begin{cases} Q'_{x,y} = Q'^{(0)}_{x,y} + \mathbf{K}^{\mathrm{t}}\cdot\mathbf{Q^1_{x,y}} \quad \text{with} \\ \left(\mathbf{Q^1_{x,y}}\right)_i \stackrel{\text{def}}{=} \pm\dfrac{L}{4\pi}\displaystyle\sum_{j=1}^{M_i}\beta_{x_j,y_j}D_{x_j}\,,\ i=1\ldots 32 \\ \dfrac{\beta'_{x,y}(s)}{\beta_{x,y}(s)} = \dfrac{\beta'^{(0)}_{x,y}(s)}{\beta_{x,y}(s)} + \mathbf{K}^{\mathrm{t}}\cdot\mathbf{B_{x,y}}(s) \quad \text{with} \\ \left(\mathbf{B_{x,y}}(s)\right)_i \stackrel{\text{def}}{=} \mp\dfrac{L}{2\sin\left(2\pi Q_{x,y}\right)}\displaystyle\sum_{j=1}^{M_i}\beta_{x_j,y_j}D_{x_j}\cos\left(2\left|\mu_{x_j,y_j}-\mu_{x,y}(s)\right|-2\pi Q_{x,y}\right)\,,\ i=1\ldots 32\,, \end{cases} \tag{35}$$

where L denotes the sextupole length and where the subscript j, $1\leq j\leq M_i$, refers to the location of the j^{th} sextupole of the family number i.
On the other hand, the second order chromaticity Q'' contains both a linear and a quadratic term in $\mathbf{K}$ and can be expressed in the following way (see Eqs. (19) and (34)):

$$Q''_{x,y} = Q''^{(0)}_{x,y} + \mathbf{K}^{\mathrm{t}}\cdot\mathbf{Q^2_{x,y}} + \frac{1}{2}\,\mathbf{K}^{\mathrm{t}}\cdot\mathbf{A_{x,y}}\cdot\mathbf{K}\,, \tag{36}$$

where the 32-dimensional vector $\mathbf{Q^2_{x,y}}$ and the 32×32 symmetric matrix $\mathbf{A_{x,y}}$ are defined by

$$\begin{cases} \left(\mathbf{Q^2_{x,y}}\right)_i \stackrel{\text{def}}{=} \underbrace{\mp\dfrac{L}{2\pi}\displaystyle\sum_{j=1}^{M_i}\beta_{x_j,y_j}\left(D_{x_j}-D_{x_j}^{(2)^{(0)}}\right)}_{\text{Term I}} \underbrace{\pm\,\mathbf{2}\times\dfrac{1}{4\pi}\displaystyle\sum_{j=1}^{M_i}\beta'^{(0)}_{x_j,y_j}D_{x_j}}_{\text{Term II}}\,,\ i=1\ldots 32 \\ \left(\mathbf{A_{x,y}}\right)_{ij} \stackrel{\text{def}}{=} \underbrace{\mp\dfrac{L^2}{8\pi\sin(\pi Q_x)}\displaystyle\sum_{r=1}^{M_i}\sum_{s=1}^{M_j}\left[\beta_{x_r,y_r}(D_{x_s})^2+\beta_{x_s,y_s}(D_{x_r})^2\right]\sqrt{\beta_{x_r}\beta_{x_s}}\cos\left(|\mu_{x_r}-\mu_{x_s}|-\pi Q_x\right)}_{\text{Terms linked to the second order dispersion}} - \\ \qquad \underbrace{\dfrac{L^2}{4\pi\sin\left(2\pi Q_{x,y}\right)}\displaystyle\sum_{r=1}^{M_i}\sum_{s=1}^{M_j}\beta_{x_r,y_r}\beta_{x_s,y_s}D_{x_r}\,D_{x_s}\cos\left(2\left|\mu_{x_r,y_r}-\mu_{x_s,y_s}\right|-2\pi Q_{x,y}\right)}_{\text{Terms linked to the off-momentum }\beta\text{-beating induced by the sextupoles}}\,, \\ \qquad i=1\ldots 32\,,\ j=1\ldots 32\,. \end{cases} \tag{37}$$

Concerning the quadratic variation of Q'', we can make the following two remarks by examining the dependence of the coefficients $(\mathbf{A_{x,y}})_{ij}$ on the phase advances between the different sextupoles of the ring:

- it is worth noting that two consecutive sextupoles of a same family are spaced roughly by π in betatron phase. As a result, the first term (terms in "$\cos(\mu)$" linked to the second order dispersion) is expected to be small.
- the second term is related to the off-momentum β-beating generated at the level of a given family of sextupoles and contributing to Q'' by means of an other family. This β-beating varies with "$\cos(2\mu)$" so that its influence on Q'' will be minimised if the gradient differences are not too high between the two families SFa and SFb (resp. SDa and SDb) of a same arc.

Let us now examine the linear dependence of Q'' on the sextupolar gradients of the 32 families:

- if the first constraint imposed on the sextupole strengths is the correction of the natural linear chromaticity of the ring (including that of the triplets), the first term occurring in the definition of $\mathbf{Q^2_{x,y}}$ (Term I) will be of the order of $Q^{'(0)}$, i.e. a few hundred for the LHC, and then can be neglected.
- the second term (Term II) is much more interesting. In fact, the latter contains two contributions which are equal (note the factor 2 in bold-face character in Eq. (37)) but of different origin. The first one (Contribution I) comes from the off-momentum β-beating $\beta^{'(0)}$ induced by the triplets and contributing to Q'' by means of the sextupole families. The second one (Contribution II) is related to the off-momentum β-beating generated by the sextupoles and contributing to Q'' at the level of the inner triplets. After some algebra, it is easy to see that these two contributions are equal. As a result, if, by means of the sextupoles, the off-momentum β-beating $\beta'(s_i^*)\,,1 \leq i \leq N$, is cancelled at each IP (and then in the inner triplets), the contributions I and II become both equal to $-Q^{''(0)}$; in other words, the second order chromaticity is over-compensated in this case, roughly by a factor 2:

$$\beta'(s_i^*) \equiv 0\,,1 \leq i \leq N \Rightarrow Q'' \simeq -Q^{''(0)} + \mathrm{O}(\mathbf{K}^2)\ . \tag{38}$$

  On the other hand, if the off-momentum β-beating is not set to zero at the IP's but only reduced by a given factor λ (close to one-half), the second order chromaticity can be easily cancelled:

$$\beta'(s_i^*) \equiv \lambda\,\beta^{'(0)}(s_i^*)\,,1 \leq i \leq N \Rightarrow Q'' \simeq (2\lambda-1)\,Q^{''(0)} + \mathrm{O}(\mathbf{K}^2) = 0 \text{ for } \lambda \sim 0.5\,. \tag{39}$$

  In view of the different optics studied in Chapter 1, note that the situation is particularly favourable for the LHC since the values obtained for $\beta^{*'}$ do never exceed their specification by a factor larger than 2, whereas the second order chromaticity shall be strongly reduced.

In conclusion, it seems difficult (or even impossible) to satisfy simultaneously the conditions $Q'' \equiv 0$ and $\beta'(s_i^*) \equiv 0$, $1 \le i \le N$, and to **minimise** at the same time the sextupole strengths in order to keep under control the anharmonicities that the latter induce.
The method that we will use for the LHC is based on Eq. (39) and is described in the next section.

4.3 Minimisation method and application to LHC Version 6.0

The initial problem to solve is of quadratic programming type (QP problem). The objective functions to cancel are Q''_x and Q''_y which depends quadratically and linearly on the variable $\mathbf{K}$ (see Eq. (36)). The constraints vary linearly with $\mathbf{K}$ and are given by (see Eq. (35))

$$\left\{\begin{array}{lcl} Q'_{x,y} & = & Q^{'(0)}_{x,y} + \mathbf{K}^{\rm t} \cdot \mathbf{Q^1_{x,y}} = 2 \\ \dfrac{\beta'_{x,y}(s_i^*)}{\beta_{x,y}(s_i^*)} & = & \dfrac{\beta^{'(0)}_{x,y}(s_i^*)}{\beta_{x,y}(s_i^*)} + \mathbf{K}^{\rm t} \cdot \mathbf{B_{x,y}}(s_i^*) = 0 \, , \, 1 \le i \le N \\ |\mathbf{K}_j| & \le & K_{2_{max}} \, , \, 1 \le j \le 32 \; . \end{array}\right.$$

According to the different points discussed in the previous section and instead of seeking to solve this problem, we tackle successively the two following sub-problems.

- First, by using the method of the Lagrange multipliers, we compute the solution $\mathbf{K}(\lambda_x, \lambda_y)$ of the linear problem

$$\mathbf{Pb1} \left\{\begin{array}{l} Q'_x = 2 \text{ and } Q'_y = 2 \\ \beta'_x(s_i^*) = \lambda_x \, \beta^{'(0)}_x(s_i^*) \text{ and } \beta'_y(s_i^*) = \lambda_y \, \beta^{'(0)}_y(s_i^*) \, , \, 1 \le i \le N \\ \text{Minimise the penalty function } \displaystyle\sum_{j=1}^{16} w_j \left(\mathbf{K}_j - \text{KSF}\right)^2 + \sum_{j=17}^{32} w_j \left(\mathbf{K}_j - \text{KSD}\right)^2 , \end{array}\right.$$

  where the coefficients λ_x and λ_y are undetermined and where the sextupolar gradients KSF and KSD are obtained by a standard correction of Q' (using one single sextupole family per transverse plane). Note that we have implicitly assumed that the indices j in the range 1 to 16 refer to the families SFa and SFb and that those in the range 17 to 32 concern the families SDa and SDb. Finally, in most cases, the weights w_j will be taken equal to 1.
  Since the penalty function is quadratic in $\mathbf{K}$, the solution of **Pb1** is linear in $\lambda_{x,y}$:

$$\mathbf{K}(\lambda_x, \lambda_y) \stackrel{\text{def}}{=} \mathbf{K}_0 + \lambda_x \mathbf{K}_x + \lambda_y \mathbf{K}_y \; . \tag{40}$$

- After a numerical estimate of the vectors $\mathbf{K}_0$, $\mathbf{K}_x$ and $\mathbf{K}_y$, we complete our computation by finding the coefficients λ_x and λ_y satisfying

$$\mathbf{Pb2}\left\{\begin{array}{rcl} \mathbf{K} & = & \mathbf{K}_0 + \lambda_x \mathbf{K}_x + \lambda_y \mathbf{K}_y \\ Q''_x(\lambda_x,\lambda_y) & = & Q_x''^{(0)} + \mathbf{K}^{\mathrm{t}}\cdot\mathbf{Q^2_x} + \frac{1}{2}\,\mathbf{K}^{\mathrm{t}}\cdot\mathbf{A_x}\cdot\mathbf{K} \;\equiv\; 0 \\ Q''_y(\lambda_x,\lambda_y) & = & Q_y''^{(0)} + \mathbf{K}^{\mathrm{t}}\cdot\mathbf{Q^2_y} + \frac{1}{2}\,\mathbf{K}^{\mathrm{t}}\cdot\mathbf{A_y}\cdot\mathbf{K} \;\equiv\; 0\ . \end{array}\right.$$

The functions $Q''_{x,y}(\lambda_x,\lambda_y)$ are simple polynomials of degree 2 in λ_x and λ_y and the non-linear problem **Pb2** can be easily solved by using, for instance, the steepest descent method, starting from the approximate solution $\lambda_x = \lambda_y = 0.5$ (see Eq. (39)).

This method has been implemented in a FORTRAN program `minchroma.f` which is called from MAD by the command `system,"minchroma"`. The input file is produced in a first time by running the `OPTICS` command of MAD which computes the optical functions $\beta_{x,y}$, D_x, ... as well as the phase advances $\mu_{x,y}$ at each sextupole and at the N IP's. Then, the program generates the vectors $\mathbf{Q^1_{x,y}}$, $\mathbf{Q^2_{x,y}}$, $\mathbf{B_{x,y}}(\mathbf{s^*_i})$, $1\le i\le N$, and the matrices $\mathbf{A_{x,y}}$ (see Eqs. (35) and (36)). The problems **Pb1** and **Pb2** are solved successively and `minchroma` returns to MAD the strengths of the 32 sextupole families. Finally, the chromatic aberrations Q', Q'' and $\beta^{*'}$ (but also Q''' and $\beta^{*''}$) are re-computed with MAD in order to appreciate the quality of the correction.
Concerning the different collision optics and insertion tunings considered in Chapter 1, the results obtained after correction are reported in Table 4. The method works extremely well for a tune split of 4 and a tune split of 5. The second order chromaticity is fully corrected, the off-momentum β-beating β'^*/β^* is reduced in each case by more than a factor 2 and the other chromatic aberrations β''^* and Q''' are within their specification given in Table 1. In every case, the safety margin on the sextupole gradient remains greater than 20% and the anharmonicities induced are quite reasonable. Indeed, according to tracking results, satisfactory dynamic apertures are often correlated with low amplitude detuning, say lower than 10^{-3} at $8\,\sigma$'s. In terms of the anharmonicity coefficients given by the `STATIC` command of MAD, this criterion yields

$$\Delta Q_{x,y}(8\sigma) \lesssim 10^{-3} \;\Rightarrow\; \left|\frac{dQ_{x,y}}{dE_{x,y}}\right| \lesssim \frac{10^{-3}}{64\ \epsilon_{x,y}} \simeq 31000\,\mathrm{m}^{-1}\ ,$$

(where $\epsilon_{x,y} = 5.03\times 10^{-10}\,\mathrm{m}$ is the physical emittance r.m.s. of the LHC beam at top energy [5]), to be compared to the values obtained in Table 4.
The sextupole families are found to be inefficient for a tune split of 7. Indeed, according to [14], the regular sextupole families proposed for the LHC can work properly only if the phase advance per cell is close to $\pi/2$ within about $\pm 0.01\times 2\pi$. This is not the case for the optics 65-58 of the LHC Version 6.0 for which $\mu_x = 95.243°$ and $\mu_y = 83.758°$. However, it is worth noting that this is not a problem when only IR1 and IR5 are tuned in collision mode: in that case, IP1 and IP5 can always be separated by $\pi/2 + k\pi$ in phase (see Section 3.2). On the other hand, no solution has been found to correct Q'' and $\beta^{*'}$ for the optics 65-58 of the LHC when the β functions are squeezed to 0.5 m in IP1, IP2 and IP5.

Optics	Optics 63.31/59.32			Optics 64.31/59.32		
Insertion tuning	Tuning I	Tuning II	Tuning III	Tuning I	Tuning II	Tuning III
Coefficient λ_x	0.30	0.26	0.37	0.53	0.18	0.44
Coefficient λ_y	0.54	0.47	0.27	0.48	0.39	0.44
$\max_{1\le i\le 32} \lvert\mathbf{K}_i\rvert$ in [%] of the available strength	49.9	69.6	82.0	52.0	78.8	77.6
Chromatic Aberrations (horizontal plane)						
$\beta_x^{*'}/\beta_x^*$ at IP1	68	35	174	101	64	178
$\beta_x^{*'}/\beta_x^*$ at IP2	(203)	-40	(714)	(293)	73	(170)
$\beta_x^{*'}/\beta_x^*$ at IP5	60	13	158	111	62	194
$\beta_x^{*''}/\beta_x^*$ [10^5] at IP1	0.8	0.6	4.4	0.9	0.3	2.6
$\beta_x^{*''}/\beta_x^*$ [10^5] at IP2	(0.5)	0.2	(6.4)	(1.7)	-0.4	(5.7)
$\beta_x^{*''}/\beta_x^*$ [10^5] at IP5	0.3	0.0	2.1	0.7	0.2	2.3
Q'_x	2.0	2.0	2.0	2.0	2.0	2.0
Q''_x	0.0	0.0	0.1	0.0	0.2	0.2
Q'''_x [10^7]	0.2	-0.5	5.4	0.6	-0.2	2.9
Chromatic Aberrations (vertical plane)						
$\beta_y^{*'}/\beta_y^*$ at IP1	42	91	38	36	60	53
$\beta_y^{*'}/\beta_y^*$ at IP2	(201)	112	(692)	(272)	64	(555)
$\beta_y^{*'}/\beta_y^*$ at IP5	14	7	26	18	-4	54
$\beta_y^{*''}/\beta_y^*$ [10^5] at IP1	0.5	0.7	0.7	0.5	0.5	2.5
$\beta_y^{*''}/\beta_y^*$ [10^5] at IP2	(2.1)	2.9	(4.3)	(2.0)	2.9	(3.6)
$\beta_y^{*''}/\beta_y^*$ [10^5] at IP5	1.1	2.1	2.3	1.0	1.7	3.6
Q'_y	2.0	2.0	2.0	2.0	2.0	2.0
Q''_y	0.0	0.0	0.2	0.0	0.1	0.0
Q'''_y [10^7]	0.4	0.8	0.6	0.4	0.7	3.3
Anharmonicities						
dQ_x/dE_x [m^{-1}]	449	536	5004	943	7011	2896
dQ_x/dE_y [m^{-1}]	-5298	-7883	-12702	-5551	-10309	-10386
dQ_y/dE_y [m^{-1}]	985	1677	2688	1745	2927	5000

Table 4: Chromatic aberrations and anharmonicities in Ring-1 after Q'' correction for different collision optics and configurations of IR1, IR2 and IR5 (4×8 sextupole families per ring, LHC lattice Version 6.0).

Conclusions

In order to compensate the chromatic aberrations induced by the low-β triplets of the LHC in collision, mainly the second order chromaticity and the off-momentum β-beating $\beta' = \partial\beta/\partial\delta$, the following two options have been envisaged:

- to constrain the phase advances between the different IP's of the ring.
- to use more than one sextupole family per transverse plane.

In the case where only IR1 and IR5 are tuned with the same β^* (nominal or ultimate configurations for protons), it is easy to obtain the required phase advances between IP1 and IP5 by re-matching, for instance, the insertions IR4 and IR6. On the other hand, in the nominal configuration for ions (i.e $\beta^* = 0.5\,\mathrm{m}$ in IP1, IP2 and IP5), phasing the IP's becomes difficult due to the limited tunability of IR2. Nevertheless, this option remains possible if we allow using some tuning correctors MQT as phase trombones between IP's; this possibility is delicate to implement in practice since it imposes to re-match in collision the β functions at the entries and exits of several LHC insertions; this has not yet been studied.
It is worth noting that this correction strategy would become irrelevant if the LHC was constrained to run temporarily with only one low-β insertion, the others being partially or completely detuned in order to limit the background in their detector.
Therefore, it is recommended to keep the 4×8 sextupole families presently implemented in the LHC lattice Version 6.0. They have been successfully tested out on the optics 63-59 and 64-59 with different tunings of IR1, IR2 and IR5. Nevertheless, we have to keep in mind that the sextupole families become inefficient for too large tune splits, as, in particular, for the collision optics 65-58 of the LHC Version 6.0.

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