JOINT INSTITUTE FOR NUCLEAR RESEARCH, DUBNA

Report No. P9/3690, 1968

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Translated at CERN by B. Hodge
and revised by N. Mouravieff
(Original : Russian)

(CERN Trans. 68-6)

Geneva
May, 1968

Joint Institute for Nuclear Research
Dubna, 1968

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In paper[1] an examination was made, using the quasi-linear theory, of the pattern of development of aperiodic beam instability[2,3] right up to its disruption. No answer was given, however, to the question of the effect of the non-linear interaction of harmonics on the relaxation process which was considered. It is this question that we shall be investigating in this paper.

1. In order to evaluate the influence of the following terms of expansion over the field in Maxwell's equations, it will be sufficient in the present instance to calculate the non-linear flow of the second order[x)] (see also [4] for example).

$$j^{(2)}_{\vec{k}i} = \sum_a \int S^a_{iqr}(k,k_1,k_2)\, E_{\vec{k}_1 q}\, E_{\vec{k}_2 r}\, \delta(\vec{k} - \vec{k}_1 - \vec{k}_2)\, d\vec{k}_1\, d\vec{k}_2 \qquad (1.1)$$

x) This condition is related to the aperiodic character of the instability which is being investigated.

in which a = 1,2; index 1 refers to the beam and index 2 to the plasma. When obtaining equation (1.1) we used the following representation for the fields xx)

$$\vec{E}\,(\vec{r},\ t) = \int \vec{E}'_{\vec{k}}\, e^{i\,\vec{k}\,\vec{r} + \gamma_k t}\, d\vec{k} = \int \vec{E}_{\vec{k}}\, e^{i\,\vec{k}\,\vec{r}}\, d\vec{k} \quad . \tag{1.2}$$

Taking into account the non-linear flow (1.1) Maxwell's equations can be written in the following form :

$$\left(k^2\,\delta_{ij} - k_i k_j + \frac{\gamma^2}{c^2}\,\varepsilon^{\pi}_{ij}\right) E_{\vec{k}j} = -\,\frac{4\pi\gamma_{\vec{k}}}{c^2}\, j^{(2)}_{\vec{k}i} \tag{1.3}$$

where

$$\varepsilon^{\pi}_{ij} = \delta_{ij} - \sum_a \left[\delta_{ij} + \frac{v^{(a)}_{oi}\, v^{(a)}_{oj}}{\omega^2}\, k^2 + \frac{k_i\, v^{(a)}_{oj} + v^{(a)}_{oj} k_j}{\omega}\right]\left(\frac{\omega^{(a)}_L}{\omega}\right)^2$$

ω_L = Langmuir frequency

$$\vec{k} = (k_x,\ 0,\ 0)\ ,\ \vec{v}_o = (0,\ 0,\ v_{oz}) \quad .$$

Here the z axis is the direction of the relative movement of the beam and plasma.

In order to simplify the solution of equation (1.3) we shall change over to the moving system of coordinates in which (see ref. 3)

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xx) A representation of the type naturally narrows the range of perturbations under study.

$$\sum_a (\omega_L^{(a)})^2 \vec{v}^{(a)} = 0 \qquad (1.4)$$

In this system all the non-diagonal elements ε_{ij}^{π} are zero. In this manner we obtain

$$\left(k^2 - k_x^2 + \frac{\gamma^2}{c^2} + \sum_a \frac{(\omega_L^{(a)})^2}{c^2}\right) E_{\vec{k}x} = -\frac{4\pi\gamma_{\vec{k}}}{c^2} j_{\vec{k}x}^{(2)}$$

$$\left(k^2 + \frac{\gamma_{\vec{k}}^2}{c^2} + \sum_a \frac{(\omega_L^{(a)})^2}{c^2}\right) E_{\vec{k}y} = -\frac{4\pi\gamma_{\vec{k}}}{c^2} j_{\vec{k}y}^{(2)} \quad .$$

$$\left(k_x^2 + \frac{\gamma_{\vec{k}}^2}{c^2} - \sum_a \frac{(\omega_L^{(a)})^2}{c^2} \left(k_x^2 \frac{(v_{oz}^{(a)})^2}{\gamma^2} - 1\right)\right) E_{kz} = -\frac{4\pi\gamma_{\vec{k}}}{c^2} j_{kz}^{(2)}$$

$$(1.5)$$

Assuming the non-linearity to be small, we shall solve system (1.5) by the method of consecutive approximations. It follows from the solution of the linear problem that only those waves with an electrical vector directed along the beam[5] can be unstable. We shall therefore confine ourselves to investigating the oscillations in which $\vec{E} = (0, 0, E_z)$. In the non-linear approximation we shall consider only the interaction of transverse oscillations E^t (assuming the amplitude of the longitudinal linear oscillations E^l to be small, which may be the case, for example in a system which is limited along z, with open ends).

In accordance with this we have

$$\frac{\partial E^{t}_{\vec{k}z}}{\partial t} = \gamma_{\vec{k}} E^{t}_{\vec{k}z} - \frac{4\pi\gamma_{\vec{k}} \sum_a \int S^{(a)}_{zzz}(k,k_1,k_2)\, E^{t}_{\vec{k}_1 z}\, E_{\vec{k}_2 z}\,\delta(\vec{k} - \vec{k}_1 - \vec{k}_2)\,d\vec{k}_1 d\vec{k}_2}{2\gamma_{\vec{k}} + \dfrac{\sum_a 2(\omega_L^{(a)})^2 (v_{oi}^{(a)})^2 k^2}{\begin{matrix}3\\ \gamma_{\vec{k}}\end{matrix}}} \tag{1.6}$$

2. Let us now turn to the calculation of the non-linear polarizability $S_{iqr}(k, k_1, k_2)$. By expanding the kinetic equation for the distribution functions of the beam and plasma, with an accuracy up to the terms of the second order in the field, we obtain (see [6] for example) :

$$S^{(a)}_{iqr} = -e^{3} \int \frac{v_i}{\gamma_{\vec{k}} + i\vec{k}\,\vec{v}} \left[\left(1 + \frac{i}{\gamma_1}(\vec{k}_1\,\vec{v})\right) \frac{\partial}{\partial p_q} - \frac{i v_q}{\gamma_1} \left(\vec{k}_1 \frac{\partial}{\partial \vec{p}}\right) \right] \times \tag{2.1}$$

$$\times \frac{1}{\gamma_2 + i\vec{k}_2\vec{v}} \left[\left(1 + \frac{1}{\gamma_2}(\vec{k}_2\vec{v})\right) \frac{\partial}{\partial p_r} - \frac{i v_r}{\gamma_2} \left(\vec{k}_2 \frac{\partial}{\partial \vec{p}}\right) \right] f_o^{(a)} \frac{\partial \vec{p}}{(2\pi)^3}$$

or, in the hydrodynamic range which is at present of interest to us :

$$S^{(a)}_{iqr} = -\frac{(\omega_L^{(a)})^2 e}{8\pi\, m\omega_1\omega_2} \left[\left(\frac{k^2}{\omega^2} v^{(a)}_{oi} \delta_{qr} + \frac{k_1^2}{\omega_1^2} v^{(a)}_{oq}\, \delta_{ir}\right) + \frac{k_2^2}{\omega_2^2} v^{(a)}_{or}\, \delta_{iq}\right) + \right.$$

$$\left. + \left(\frac{k^2}{\omega^2} \frac{(\vec{k}_1\vec{k}_2)}{\omega_1\omega_2} + \frac{k_1^2}{\omega_1^2} \frac{(\vec{k}\,\vec{k}_2)}{\omega\,\omega_2} + \frac{k_2^2}{\omega_2^2} \frac{(\vec{k}\,\vec{k}_1)}{\omega\,\omega_1}\right) v^{(a)}_{oi}\, v^{(a)}_{oq}\, v^{(a)}_{or} \right] . \tag{2.2}$$

Considering that the condition $(v_o^{(2)})^2 \ll \frac{n_1}{n_2}(v_o^{(1)})^2$ (which follows from formula (1.4)) is fulfilled, the expression for non-linear polarisation of the plasma can easily be simplified x)

$$S_{iqr}^{(2)} = -\frac{(\omega_L^{(2)})^2 e}{8\pi\, m\omega_1\omega_2}\left(\frac{k^2}{\omega^2}\, v_{oi}^{(2)}\,\delta_{qr} + \frac{k_1^2}{\omega_1^2}\, v_{oq}^{(2)}\,\delta_{ir} + \frac{k_2^2}{\omega_2^2}\, v_{or}^{(2)}\delta_{iq}\right) . \tag{2.3}$$

As shown in paper[1] the linear increment of the aperiodic beam instability can be written in the following manner :

$$\gamma_{\vec{k}} = \omega_L^{(2)}\left(\frac{n_1}{n_2}\right)^{1/2}\frac{v_{oz}^{(1)}}{c} - \frac{k_x}{\sqrt{k_x^2 + (\omega_L^{(2)})^2}} . \tag{2.4}$$

Comparing $k_x\, v_{oz}^{(1)}$ and the increment $\gamma_{\vec{k}}$ which is determined by means of formula (2.3) the following inequality can easily be obtained

$$\frac{\gamma_{\vec{k}}}{kv_o^{(1)}} \lesssim \left(\frac{n_1}{n_2}\right)^{1/2} . \tag{2.5}$$

Using this condition the equation for polarizability can be simplified :

$$S_{iqr}^{(1)} = \frac{(\omega^{(1)})^2 e}{8\pi\, m\gamma_1\gamma_2}\left(\frac{k^2}{\gamma^2}\frac{(\vec{k}_1\vec{k}_2)}{\gamma_1\gamma_2} + \frac{k_1^2}{\gamma_1^2}\frac{(\vec{k}\,\vec{k}_2)}{\gamma\gamma_2} + \frac{k_2^2}{\gamma_2^2}\frac{(\vec{k}\,\vec{k}_1)}{\gamma\gamma_1}\right) v_{oi}^{(1)} v_{oq}^{(1)} v_{or}^{(1)} . \tag{2.6}$$

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x) We would point out that as $(S_{iqr}^{(2)} \approx v_o^{(2)}$, its non-linear flow, in the system of coordinates where the plasma is at rest, is not included in the equation for the field E_{kz}^{t}.

3. In [1] the magnetic energy W_H was evaluated for the moment of disruption of the instability to which it is due

$$W_H \lesssim n_1 m (v_o^{(1)})^2 \frac{n_1}{n_2} \frac{(\omega_L^{(2)})^2}{k^2c^2 + (\omega_L^{(2)})^2} \quad . \tag{3.1}$$

Let us evaluate the non-linear term in the equation for the field E^t_{kz}, the value of the field being determined by the relationship (3.1).

The equation is of the form

$$\frac{\partial E^t_{\vec{k}z}}{\partial t} = \gamma_{\vec{k}} E^t_{\vec{k}z} + \frac{4\pi\gamma_{\vec{k}} \int (S^{(1)}_{zzz} + S^{(2)}_{zzz}) E^t_{\vec{k}_1 z} E^t_{\vec{k}_2 z} \,\delta\,(\vec{k}-\vec{k}_1-\vec{k}_2)d\vec{k}_1\, d\vec{k}_2}{-\dfrac{2(\omega_L^{(1)})^2 (v_{oz}^{(1)})^2}{\gamma_k^3} k^2} \tag{3.2}$$

where $S^{(1)}_{zzz}$ is determined by equation (2.6) and $S^{(2)}_{zzz}$ by equation (2.3). Using the same assumptions it is easy to show that $(S^{(1)}/S^{(2)}) \approx (\frac{n_2}{n_1})^{1/2}$. Consequently for the evaluations we shall leave the function $S^{(1)}_{zzz}$ everywhere in the non-linear flow.

The maximum increment is that of the harmonics in which k is included in the interval

$$\frac{\omega_L}{c} < k < (\frac{n_1}{n_2})^{1/2} (\frac{v_o^{(1)}}{v_T(t)}) \frac{\omega_L}{c} \tag{3.3.}$$

but this interval is shortened with the passage of time

$$v_T - v_T^* = v_o^{(1)} (\frac{n_1}{n_2})^{1/2}$$

(the instability passes into a quasi-linear stage, examined in paper (7). Consequently, in the evaluations we shall use for $|H_{\vec{k}}|^2$ the formula $|H_{\vec{k}}|^2 = |H|^2 \delta(\bar{k} - \bar{k}_o)$ where $k_o \approx \frac{\omega_L}{c}$.

Taking into consideration what has been stated above, and considering formula (3.1), we are able to determine that towards the end of the quasi-linear relaxation, the relation of the non-linear number to the linear one is of the order a, where a is :

$$a = (W_H/n_1\ m\ (v_o^{(1)})^2)\ \frac{n_2}{n_1} < 1 \quad .$$

It follows, therefore, that taking into account non-linear effects cannot noticeably alter the results obtained in the first part of the work.

Finally, let us evaluate the amplitude of the longitudinal field E^{ℓ} $((\vec{E}^{\ell}\vec{k}) \neq 0)$, which occurs as a result of the non-linear interaction at the moment of disruption of the instability. Using the first equation in the system (1.5), it is easy to obtain, in the approximation of weak non-linearity,

$$\frac{E^{\ell}}{E^{t}} = \frac{\omega_L}{ck}\left(\frac{n_1}{n_2}\right) \ll 1 \ .$$

In conclusion, the authors have pleasure in expressing their gratitude to V.N. Tsytovich for the most valuable discussions.

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Manuscript received by publishing

department 5 February 1968.