PROJECTION QUASICRYSTALS I: TORAL ROTATIONS

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Abstract. We present a systematic treatment of the commutative and non-commutative topology of quasicrystal point patterns and tilings produced in finite dimensional Euclidean space by the projection or strip method. With no conditions on the projection plane and with a general acceptance domain only weakly constrained, we examine two sets of points constructed by the projection method, one a finite decoration of the other, and their corresponding dynamical systems. We define a projection method pattern or tiling as one whose dynamical system is intermediate to these two systems, concluding that for fixed projection data this allows only a finite number of possible patterns up to topological conjugacy. In all cases the dynamical systems associated to the pattern are almost 1-1 extensions of minimal rotation actions on a torus and we compute these factors explicitly. We establish equivalence between the tiling groupoid and the transformation groupoid of these dynamical systems. In this way, we generalize results of Robinson and of Le and place them in a wider context. The results here provide the necessary groundwork for our second paper in this series, which describes qualitatively the cohomology of projection quasicrystals.

Key words. Quasicrystal, projection method, tiling dynamical system, tiling groupoid.

§**1 Introduction** Of the many examples of aperiodic tilings of the plane and higher dimensional Euclidean space found in recent years, two classes stand out as particularly interesting and æsthetically pleasing: the *substitution*, or self-similar, tilings [**GS**] [**AP**] and the *projection*, or *strip*, method tilings [**KN**] [**dB1**] [**KD**]. The overlap of these two classes includes some of the better studied examples of aperiodic tiling, for example the Penrose tiling [**Pe**] and the octagonal tiling (see [**Soc**]). In this paper we consider the second class in full generality.

The projection method was developed originally as a model for physical quasicrystals and for this it has proved quite acceptable [**1**] [**2**]. But it also has great mathematical appeal. It is elementary and geometric and, once the acceptance domain and the dimensions of the spaces used in the construction are chosen, has a finite number of degrees of freedom. The projection method is also a natural generalisation of low dimensional examples such as Sturmian sequences [**HM**] which have strong links with classical number theory.

In this paper our broad goal is to study the commutative and non-commutative topology of projection method patterns with few restrictions on the freedom of the construction. It is the first of a short series of papers describing a calculus for computing the topological invariants associated to projection method patterns and tilings and provides the necessary theory to allow a qualitative description in [**FHK1**] of the cohomology groups of a general tiling. A procedure for quantitative calculation will be described in [**FHK2**]. These invariants are studied as part of a wider programme of classification of quasicrystals.

In common with the papers [**Ru**] [**Ra**] [**R1**] [**K1**] [**K3**] on the topology of tilings (but, as we shall explain, unlike [**BCL**]), we are interested in an individual tiling or pattern in Euclidean space and in the topological spaces formed by its Euclidean translations. This allows us to apply to the individual pattern whatever deductions we can make from our construction, for example the labelling of gaps in the spectrum of the discrete Schrödinger operator describing the physical systems connected with the pattern [**B1**] [**K1**].

For an introduction to tilings and quasicrystals we recommend the well-illustrated monographs of Grünbaum and Sheppard and of Senechal [**GS**] [**S**]. We refer also to the original papers of de Bruijn [**dB1**], Katz and Duneau [**KD**] and its sequel [**OKD**] for the definitions (which we repeat below) and physical motivation. Our work on groupoids in this paper rests heavily on the theory and ideas found in [**Ren**] [**C**] and [**K1**].

The projection method starts with the choice of a d dimensional subspace, E, of $\mathbb{R}^N$ $(0 < d < N)$, with orthocomplement $E^\perp$, together with an acceptance domain, $K \subset E^\perp$ (see 2.1).

Classically the acceptance domain is an appropriate projection of the unit cube [**dB1**] [**OKD**], the *canonical* case, but greater freedom is allowed in more modern studies [**H**] [**S**] [**BKS**]. We impose only the weakest reasonable topological conditions on K in this paper (see 2.1 again).

It is commonly assumed that $E \cap \mathbb{Z}^N = 0$, which requires that the patterns produced have no periodic directions. For many purposes, periodic directions can be "quotiented out" and our general and natural approach takes this case in its stride (see Remark 5.5).

The similar condition, $E^\perp \cap \mathbb{Z}^N = 0$, is often assumed for convenience or technical reasons, but it excludes, for example, the Penrose tiling and is therefore a significant restriction to an inclusive study of projection method tilings. Many of the constructions and results for the $E^\perp \cap \mathbb{Z}^N = 0$ case pass to the general case without much adaption, but there are significant points where this is not so, especially in the comparison of projection point patterns and projection tilings (see section 8).

In this paper, we avoid making the assumption $E^\perp \cap \mathbb{Z}^N = 0$ completely and only adopt the assumption $E \cap \mathbb{Z}^N = 0$ occasionally in later sections. This allows us to develop a coherent theory of projection method patterns and tilings and makes the general case more approachable.

The second paper in this series, [**FHK1**], starts with the observation that the cohomology of a projection method pattern or tiling (*i.e.* of its groupoid) can be equated with the Čech cohomology of the space, $M\mathcal{T}$, of the pattern dynamical system, $(M\mathcal{T}, \mathbb{R}^d)$, defined by Rudolf [**Ru**] (see 4.2). This fact compels us to look more closely at the pattern dynamical system and provides the focus for the present paper.

A precursor to our description of the pattern dynamical system can be found in the work of Robinson [**R2**], who examined the dynamical system of the Penrose tiling and showed that it is an almost 1-1 extension of a minimal $\mathbb{R}^2$ action by rotation (2.15) on a 4-torus. Although Robinson used quite special properties of the tiling, Hof [**H**] has noted that the techniques are generalizable without being specific about the extent of the generalization. Our approach is quite different from that of Robinson and, by constructing a larger topological space from which the pattern dynamical system is formed by a quotient, we follow most closely the approach pioneered by Le [**Le**].

Our results are summarized precisely in section 12. We give a brief overview here, but we emphasise that the definitions and discussions of sections 2 and 4 are essential stepping stones to the more technical results of later sections.

Given a subspace, E, acceptance domain, K, and a positioning parameter u, we can distinguish *two* $\mathbb{R}^d$ dynamical systems constructed by the projection method, $(MP_u, \mathbb{R}^d)$ and $(M\widetilde{P}_u, \mathbb{R}^d)$, the first automatically the factor of the second. This allows us to define a *projection method pattern* (with data (E, K, u)) as a pattern, $\mathcal{T}$, whose dynamical system, $(M\mathcal{T}, \mathbb{R}^d)$, is intermediate to these two extreme systems. Sections 3 to 9 of this paper provide a complete description of the spaces and the extension $M\widetilde{P}_u \longrightarrow MP_u$, showing, under further weak assumptions on the acceptance domain, that it is a finite isometric extension. In section 7 we conclude that this restricts $(M\mathcal{T}, \mathbb{R}^d)$ to one of a finite number of possibilities, and that any projection method pattern is a finite decoration of its corresponding point pattern P_u (2.1).

In section 10 we describe yet another dynamical system connected with a projection method pattern, this time a $\mathbb{Z}^d$ action on a Cantor set X, whose mapping torus is the space of the pattern dynamical system. For the canonical case with $E^\perp \cap \mathbb{Z}^N = 0$ this is the same system as that constructed in [**BCL**] (see below).

All the dynamical systems produced in this paper are almost 1-1 extensions (2.15) of an action of $\mathbb{R}^d$ or $\mathbb{Z}^d$ by rotations (2.15) on a torus (or torus $\oplus$ finite abelian group). In each case the dimension of the torus and the generators of the action can be computed explicitly. This gives a clear picture of the orbits of non-singular points in the pattern dynamical system.

Comparing this analysis with the available descriptions of the tiling dynamical system in the canonical case (*e.g.* [**Le**] [**BCL**]) we see that our constructions both avoid and justify their intuitive approaches, satisfactory till now, which "Cantorize" Euclidean space by corners or cuts. We believe that the care we take in this paper is necessary for further progress and to allow general acceptance domains. Even in the canonical case, Corollary

7.2 and Proposition 8.4 of this paper, for example, require this precision despite being direct generalizations of Theorem 3.8 in [**Le**].

In section 11 we complete the circle of ideas by showing that the transformation group ([**Ren**]), $\mathcal{G}(X, \mathbb{Z}^d)$, of the system found in section 10, and the pattern groupoid, $\mathcal{GT}$ (section 11), are both regular reductions of the same groupoid. Therefore the two C^* algebras, $C(X) \rtimes \mathbb{Z}^d$ (the dynamical crossed product) and $C^*(\mathcal{GT})$ (the groupoid C*-algebra) are strongly Morita equivalent and their ordered K-theories agree. In the canonical cases such a correspondence is known from [**K2**] although with a different description of the dynamical system.

We remark that the C^*-algebras, $C^*(\mathcal{GT})$ and $C(X) \rtimes \mathbb{Z}^d$, can be thought of equally as non-commutative versions of the space $M\mathcal{T}$. In general they are not $*$-isomorphic.

The results of section 11 can be contrasted with the recently announced results of Bellissard *et al.* [**BCL**]. Recall that in this paper we have been examining a single projection method pattern, $\mathcal{T}$, and constructing the spaces, $M\mathcal{T}$ and $\mathcal{GT}$, produced by its Euclidean translations. This is not the approach in [**BCL**] where a groupoid is formed directly from the lattice inside the strip, and, in effect, the non-commutative version of the space $M\widetilde{P}$ (4.2) is analysed. These two constructions are the same if and only if $E^\perp \cap \mathbb{Z}^N = 0$.

This difference becomes clear when we compare Robinson's computations [**R2**] for the Penrose tiling and the conclusion to be derived from [**BCL**]. In this case, the latter paper would produce a dynamical system which is an almost 1-1 extension of a rotation on a 3-torus, whereas from [**R2**] (and Theorem 10.3 below) we produce $(X_\mathcal{T}, \mathbb{Z}^d)$ (in our notation), an almost 1-1 extension of a rotation on a 2-torus. This disagreement is significant at the non-commutative level as well since the K-theory of (the C^*-algebra associated with) the first system is the tensor product of the K-theory of the second system with $\mathbb{Z}^\infty$; and these are different.

This example underlines some of the difficulties we must address in accommodating the most general cases of projection method pattern.

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§**2 The projection method and associated geometric constructions** We use the construction of point patterns and tilings given in Chapters 2 and 5 of Senechal's monograph [**S**] throughout this paper, adding some assumptions on the acceptance domain in the following definitions.

Definitions 2.1 Suppose that E is a d dimensional subspace of $\mathbb{R}^N$ and $E^\perp$ its orthocomplement. For the time being we shall make no assumptions about the position of either of these planes.

Let π be the projection onto E and $\pi^\perp$ the projection onto $E^\perp$.

Let $Q = \overline{E + \mathbb{Z}^N}$ (Euclidean closure).

Let K be a compact subset of $E^\perp$ which is the closure if its interior (which we write $IntK$) in $E^\perp$. Thus the boundary of K in $E^\perp$ is compact and nowhere dense. Let $\Sigma = K + E$, a subset of $\mathbb{R}^N$ sometimes refered to as *the strip with acceptance domain* K.

A point $v \in \mathbb{R}^N$ is said to be *non-singular* if the boundary, $\partial\Sigma$, of Σ does not intersect $\mathbb{Z}^N + v$. We write NS for the set of non-singular points in $\mathbb{R}^N$. These points are also called *regular* in the literature.

Let $\widetilde{P}_v = \Sigma \cap (\mathbb{Z}^N + v)$, the *strip point pattern.*

Define $P_v = \pi(\widetilde{P}_v)$, a subset of E called the *projection point pattern.*

In what follows we assume E and K are fixed and suppress mention of them as a subscript or argument.

Lemma 2.2 *With the notation above,*

i/ NS is a dense G_δ subset of $\mathbb{R}^N$ invariant under translation by E.

ii/ If $u \in NS$, then $NS \cap (Q + u)$ is dense in $Q + u$.

iii/ If $u \in NS$ and F is a vector subspace of $\mathbb{R}^N$ complementary to E, then $NS \cap (Q + u) \cap F$ is dense in $(Q + u) \cap F$.

Proof i/ Note that $\mathbb{R}^N \setminus NS$ is a translate of the set $\cup_{v \in \mathbb{Z}^N}(\partial K + E + v)$ (where the boundary is taken in $E^\perp$) and our conditions on K complete the proof.

ii/ $NS \cap (Q + u) \supset E + \mathbb{Z}^N + u$.

iii/ $\overline{(E + \mathbb{Z}^N + u) \cap F} = (Q + u) \cap F$. •

Remark 2.3 The condition on the acceptance domain K is a topological version of the condition of [**H**]. We note that our conditions include the examples of acceptance domains with fractal boundaries which have recently interested quasicrystalographers [**BKS**].

In the original construction [**dB1**] [**KD**] $K = \pi^\perp([0,1]^N)$. We call this the *canonical* acceptance domain. The *canonical tiling*, defined by [**OKD**] with this choice of acceptance domain, is formed by picking $u \in NS$ and projecting onto E those d-dimensional faces of the lattice $\mathbb{Z}^N + u$ which are contained entirely in Σ. We write this $\mathcal{T}_u$.

The following notation and technical lemma makes easier some calculations in future sections.

Definition 2.4 If X is a subspace of Y, both topological spaces, and $A \subset X$, then we write $Int_X A$ to mean the interior of A in the subspace topology of X.

Likewise we write $\partial_X A$ for the boundary of A taken in the subspace topology of X.

Lemma 2.5 *a/ If $u \in NS$, then $(Q+u) \cap IntK = Int_{(Q+u)\cap E^\perp}((Q+u)\cap K)$ and $(Q+u) \cap \partial_{E^\perp} K = \partial_{(Q+u)\cap E^\perp}((Q+u)\cap K)$.*

b/ If $u \in NS$, then $((Q+u) \cap E^\perp) \setminus NS = \partial_{(Q+u)\cap E^\perp}((Q+u)\cap K) + \pi^\perp(\mathbb{Z}^N)$.

Proof a/ To show both facts, it is enough to show that $(\partial_{E^\perp} K) \cap (Q+u)$ has no interior as a subspace of $(Q+u) \cap E^\perp$.

Suppose otherwise and that U is an open subset of $\partial K \cap (Q+u)$ in $(Q+u) \cap E^\perp$. By the density of $\pi^\perp(\mathbb{Z}^N)$ in $Q \cap E^\perp$, we find $v \in \mathbb{Z}^N$ such that $u \in U + \pi^\perp(v)$. But this implies that $u \in \partial K + \pi^\perp(v)$ and so $u \notin NS$ - a contradiction.

b/ By defintion the left-hand side of the equation to be proved is equal to $(\partial_{E^\perp} K + \pi^\perp(\mathbb{Z}^N)) \cap (Q+u)$ which equals $(\partial_{E^\perp} K \cap (Q+u)) + \pi^\perp(\mathbb{Z}^N)$ since $\pi^\perp(\mathbb{Z}^N)$ is dense in $Q \cap E^\perp$. By part a/ therefore we obtain the right-hand side of the equation. •

Condition 2.6 We exclude immediately the case $(Q+u) \cap IntK = \emptyset$ since when $u \in NS$, Lemma 2.5 shows this is equivalent to $P_u = \emptyset$.

Examples 2.7 We note the parameters of two well-studied examples, both with canonical acceptance domain.

The *octagonal tiling* [**Soc**] [**B2**] has $N = 4$ and $d = 2$, where E is a vector subspace of $\mathbb{R}^4$ invariant under the action of the linear map which maps orthonormal basis vectors $e_1 \mapsto e_2$, $e_2 \mapsto e_3$, $e_3 \mapsto e_4$, $e_4 \mapsto -e_1$. Its orthocomplement, $E^\perp$, is the other invariant subspace. Here $Q = \mathbb{R}^4$ and so many of the distinctions made in subsequent sections are irrelevant to this example.

The *Penrose tiling* [**Soc**] [**S**] has $N = 5$ and $d = 2$ (although we note that there is a non-projection method construction using the root lattice of A_4 in $\mathbb{R}^4$ [**BJKS**]). The linear map which maps $e_i \mapsto e_{i+1}$ (indexed modulo 5) has two 2 dimensional and one 1 dimensional invariant subspaces. Of the first two subspaces, one is chosen as E and the other we name V. Then in fact $Q = E \oplus V \oplus \widetilde{\Delta}$, where $\widetilde{\Delta} = \frac{1}{5}(e_1 + e_2 + e_3 + e_4 + e_5)\mathbb{Z}$, and Q is therefore a proper subset of $\mathbb{R}^5$, a fact which allows the construction of *generalised Penrose tilings* using a parameter $u \in NS \setminus Q$.

Note that we speak of tilings and yet only consider point patterns. In both examples, the projection tiling [**OKD**] is conjugate to both the corresponding strip point pattern and projection point pattern, a fact proved in greater generality in section 8.

We develop these geometric ideas in the following lemmas. The next is Theorem 2.3 from [**S**].

Theorem 2.8 *Suppose that $\mathbb{Z}^N$ is in standard position in $\mathbb{R}^N$ and suppose that $\phi\colon \mathbb{R}^N \longrightarrow \mathbb{R}^n$ is a surjective linear map. Then there is a direct sum decompostion $\mathbb{R}^n = V \oplus W$ into real vector subspaces such that $\phi(\mathbb{Z}^N) \cap V$ is dense in V, $\phi(\mathbb{Z}^N) \cap W$ is discrete and $\phi(\mathbb{Z}^N) = (V \cap \phi(\mathbb{Z}^N)) + (W \cap \phi(\mathbb{Z}^N))$.* •

We proceed with the following refinement of Proposition 2.15 of [**S**].

Lemma 2.9 *Suppose that $\mathbb{Z}^N$ is in standard position in $\mathbb{R}^N$ and suppose that $\phi\colon \mathbb{R}^N \longrightarrow F$ is an orthogonal projection onto F a subspace of $\mathbb{R}^N$. With the decompostion of F implied by Theorem 2.8, $(F \cap \mathbb{Z}^N) + (V \cap \phi(\mathbb{Z}^N)) \subset \phi(\mathbb{Z}^N)$ as a finite index subgroup.*

Also, the lattice dimension of $F \cap \mathbb{Z}^N$ equals $\dim F - \dim V$ and the real vector subspace generated by $F \cap \mathbb{Z}^N$ is orthogonal to V.

Proof Suppose that U is the real linear span of $\Delta = F \cap \mathbb{Z}^N$. Note that, since Δ is discrete, the lattice dimension of Δ equals the real space dimension of U.

The argument of the proof of Proposition 2.15 in [**S**] shows that each element of $F \cap \mathbb{Z}^N$ is orthogonal to V. Therefore we have $\dim_{\mathbb{R}}(U) \leq \dim_{\mathbb{R}}(F) - \dim_{\mathbb{R}}(V)$ immediately.

Consider the rational vector space $\mathbb{Q}^N$, contained in $\mathbb{R}^N$ and containing $\mathbb{Z}^N$, both in canonical position. Let U' be the rational span of Δ and note that $U' = U \cap \mathbb{Q}^N$ and that $\dim_{\mathbb{Q}}(U') = \dim_{\mathbb{R}}(U)$. Let $U'^{\perp}$ be the orthocomplement of U' with respect to the standard inner product in $\mathbb{Q}^N$ so that, by simple rational vector space arguments, $\mathbb{Q}^N = U' \oplus U'^{\perp}$. Thus $(U' \cap \mathbb{Z}^N) + (U'^{\perp} \cap \mathbb{Z}^N)$ forms a discrete lattice of dimension N.

Extending to the real span, we deduce that $(U \cap \mathbb{Z}^N) + (U^{\perp} \cap \mathbb{Z}^N)$ is a discrete sublattice of $\mathbb{Z}^N$ of dimension N, hence a subgroup of finite index. Also the lattice dimension of $U \cap \mathbb{Z}^N$ and $U^{\perp} \cap \mathbb{Z}^N$ are equal to $\dim_{\mathbb{R}}(U)$ and $\dim_{\mathbb{R}}(U^{\perp})$ respectively.

Let $L = (U^{\perp} \cap \mathbb{Z}^N)$ be considered as a sublattice of $U^{\perp}$. It is integral (with respect to the restriction of the inner product on $\mathbb{R}^N$) and of full dimension. The projection ϕ restricts to a orthogonal projection $U^{\perp} \longrightarrow U^{\perp} \cap F$ and, by construction, $U^{\perp} \cap F \cap L = 0$. Therefore Proposition 2.15 of [**S**] applies to show that $\phi(L)$ is dense in $U^{\perp} \cap F$ and that ϕ is 1-1 on L.

However $\phi(L) \subset \phi(\mathbb{Z}^N)$ and so, by the characterisation of Theorem 2.2, we deduce that $U^{\perp} \cap F \subset V$. However, since $U^{\perp} \supset V$, we have $U^{\perp} \cap F = V$.

We have $U \cap \mathbb{Z}^N = F \cap \mathbb{Z}^N$ and $\phi(U^\perp \cap \mathbb{Z}^N) = \phi(\mathbb{Z}^N) \cap V$ automatically. Therefore $(\phi(\mathbb{Z}^N) \cap V) + (F \cap \mathbb{Z}^N) = \phi((U^\perp \cap \mathbb{Z}^N) + (U \cap \mathbb{Z}^N))$. As proved above, this latter set is the image of a finte index subgroup of the domain, $\mathbb{Z}^N$, and therefore it is a finite index subgroup of the image $\phi(\mathbb{Z}^N)$ as required.

The remaining properties follow quickly from the details above. •

Definition 2.10 Let $\Delta = E^\perp \cap \mathbb{Z}^N$ and $\widetilde{\Delta} = U \cap \overline{\pi^\perp(\mathbb{Z}^N)}$ where U is the real vector space generated by Δ.

Note that the discrete group Δ defined here is not the real vector space $\Delta(E)$ defined in [**Le**], but it is a cocompact sublattice and so the dimensions are equal.

Corollary 2.11 *With the notation before* $\overline{\pi^\perp(\mathbb{Z}^N)} = V \oplus \widetilde{\Delta}$ *and* $Q = E \oplus V \oplus \tilde{\Delta}$ *are orthogonal direct sums. Moreover,* $\Delta = E^\perp \cap \mathbb{Z}^N$ *is a subgroup of* $\widetilde{\Delta}$ *with finite index.*

Example 2.12 For example the octagonal tiling has $\Delta = 0$ and the Penrose tiling has $\Delta = (e_1 + e_2 + e_3 + e_4 + e_5)\mathbb{Z}$, a subgroup of index 5 in $\widetilde{\Delta}$.

And finally a general result about isometric extensions of dynamical systems.

Definition 2.13 Suppose that $\rho\colon (X, G) \longrightarrow (Y, G)$ is a factor map of topological dynamical systems with group, G, action. If every fibre $\rho^{-1}(y)$ has the same finite cardinality, n, then we say that (X, G) is an n-to-1 extension.

The structure of such extensions, a special case of isometric extensions, is well-known [**F**].

Lemma 2.14 *Suppose that* $\rho\colon (X, G) \longrightarrow (Y, G)$ *is an* n*-to-1 extension and that* (X, G) *is minimal. Suppose further that there is an abelian group* H *which acts continuously on* X*, commutes with the* G *action, preserves* ρ *fibres and acts transitively on each fibre. If* $(X, G) \xrightarrow{\rho'} (Z, G) \xrightarrow{\rho''} (Y, G)$ *is an intermediate factor, then* (Z, G) *is an* m*-to-1 extension where* m *divides* n*, and we can find a subgroup,* H' *of* H*, so that*

i/ H/H' *acts continuously on* Z*, commutes with the* G *action, preserves* ρ'' *fibres and acts transitively on each fibre and*

ii/ H' *acts on* X *as a subaction of* H*, preserving* ρ' *fibres and acting transitively on each fibre.*

Proof Given $h \in H$, consider $X_h = \{x \mid \rho'(x) = \rho'(hx)\}$ which is a closed G-invariant subset of X. Therefore, by minimality, $X_h = \emptyset$ or X. Let $H' = \{h \in H \mid X_h = X\}$ which can be checked is a subgroup of H. The properties claimed follow quickly. •

Definitions 2.15 We will call an extension which obeys the conditions of Lemma 2.14 a *finite isometric extension.*

An *almost 1-1 extension* of topological dynamical systems $\rho\colon (X,G) \longrightarrow (Y,G)$ is one in which the set $\rho^{-1}(y)$ is a singleton for a dense G_δ of $y \in Y$. In the case of minimal actions, it is sufficient to find just one point $y \in Y$ for which $\rho^{-1}(y)$ is a singleton.

We say that an abelian topological group, G, acting on a compact abelian topological group, Z say, acts *by rotation* if there is a group homomorphism, $\psi : G \longrightarrow Z$ such that $gz = z + \psi(g)$ for all $z \in Z$ and $g \in G$.

§**3 Topological spaces for point patterns** When v is non-singular, P_v forms an almost periodic pattern of points in the sense that each spherical window, whose position is shifted over the infinite pattern, reveals the same configuration at a syndetic (relatively dense) set of positions [**S**]. A precise formulation of this fact is well-known and we note the following relevant constructions and lemmas.

Definition 3.1 Let $B(r)$ be the closed ball in E, centre 0 and of radius r with boundary $\partial B(r)$. Given a closed subset, A, of $\mathbb{R}^N$, define $A[r] = (A \cap B(r)) \cup \partial B(r)$, a closed subset of $B(r)$. Consider the Hausdorff metric d_r defined among closed subsets of $B(r)$ and define a metric (after [**R1**], [**Sol**]) on closed subsets of the plane by

$$D(A, A') = \inf\{1/(r+1) \mid d_r(A[r], A'[r]) < 1/r\}.$$

The following is proved in much greater generality in [**Ru**] (see also [**R1**] and [**Ra**]).

Proposition 3.2 *If $u \in NS$, then the sets $\{P_v \mid v \in NS\}$ and $\{P_v \mid v \in u+E\}$ are precompact with respect to D.* •

Definition 3.3 The compact sets obtained from the closure of the sets of the lemma are written respectively MP and MP_u.

Remark 3.4 Note that $\Delta = 0$ if and only if $MP = MP_u$ for all $u \in NS$, which happens if and only if $MP = MP_u$ for some $u \in NS$.

Also P_v forms a Delone set (see [**Sol**]), so we deduce that, for $w \in E$ and $\|w\|$ small enough, $D(P_v, P_{v+w}) = \|w\|/(1+\|w\|)$.

Proposition 3.5 *Suppose that $w \in E$, then the map $P_v \mapsto P_{v+w}$, defined for $v \in NS$, may be extended to a homeomorphism of MP, and the family of homeomorphisms defined by taking all choices of $w \in E$ defines a group action of $\mathbb{R}^d \equiv E$ on NS.*

Also for each $u \in NS$, MP_u is invariant under the action above and E acts minimally on MP_u. $\bullet$

The dynamical system MP_u with the action by $E \equiv \mathbb{R}^d$ is the dynamical system, analogous to that constructed by Rudolf [**Ru**] for tilings, associated with the point pattern P_u. We modify this to an action by E on a non-compact cover of MP_u as follows.

Definition 3.6 For $v, v' \in \mathbb{R}^N$, write $\overline{D}(v, v') = D(P_v, P_{v'}) + \|v - v'\|$; this is clearly a metric. Let Π be the completion of NS with respect to this metric.

The following lemma starts the basic topological description of these spaces.

Lemma 3.7 *a/ The canonical injection $NS \longrightarrow \mathbb{R}^N$ extends to a continuous surjection $\mu: \Pi \longrightarrow \mathbb{R}^N$. Moreover, if $v \in NS$, then $\mu^{-1}(v)$ is a single point.*

b/ The map $v \mapsto P_v$, $v \in NS$, extends to a continuous E-equivariant surjection, $\eta: \Pi \longrightarrow MP$, which is an open map.

c/ The action by translation by elements of E on NS extends to a continuous action of $\mathbb{R}^d \equiv E$ on Π.

d/ Similarly the translation by elements of $\mathbb{Z}^N$ is $\overline{D}$-isometric and extends to a continuous action of $\mathbb{Z}^N$ on Π. This action commutes with the action of E found in part c/.

e/ If $a \in MP$ and $b \in \mathbb{R}^N$, then $|\eta^{-1}(a) \cap \mu^{-1}(b)| \leq 1$.

Proof a/ The only non-elementary step of this part is the latter sentence.

We must show that if $v \in NS$ then for all $\epsilon > 0$ there is a $\delta > 0$ such that $\|w - v\| < \delta$ and $w \in NS$ implies that $D(P_w, P_v) < \epsilon$. However, we know that if B is a ball in $\mathbb{R}^N$ of radius much bigger than $1/(2\epsilon)$, then $(\mathbb{Z}^N + v) \cap B$ is of strictly positive distance, say at least 2δ with $\delta > 0$ chosen $< \epsilon/2$, from $\partial\Sigma$. Therefore, whenever $\pi(v - w) = 0$ and $\|v - w\| < \delta$, we have $P_v \cap B = P_w \cap B$ and hence $D(P_v, P_w) < \delta$. On the other hand, if $\pi(v - w) \neq 0$ but $\|v - w\| < \delta$ then we may replace w by $w' = w + \pi(v - w)$, a displacement by less than δ. By the remark (3.4), we deduce that $D(P_w, P_{w'}) < \delta$ and so we have $D(P_w, P_v) < 2\delta < \epsilon$ in general, as required.

b/ The extension to η, and the equivariance and surjectivity, are immediate. The open map condition is quickly confirmed using remark (3.4).

c/ follows from the uniform action of E noted in Remark 3.4. d/ follows similarly where uniform continuity is immediate from the isometry.

Note that e/ is a direct consequence of the definition of the metric $\overline{D}$. $\bullet$

Definition 3.8 For $u \in NS$, let Π_u be the completion of $E + \mathbb{Z}^N + u$ with respect to the $\overline{D}$ metric.

Lemma 3.9 *For $u \in NS$, Π_u is a closed $E+\mathbb{Z}^N$-invariant subspace of Π. If $x \in \Pi_u$, then $(E+\mathbb{Z}^N)x$, the orbit of x under the E and $\mathbb{Z}^N$ actions, is dense in Π_u. Consequently*

a/ The injection, $E+\mathbb{Z}^N+u \longrightarrow \mathbb{R}^N$ extends to a continuous map, equal to the restriction of μ to Π_u, $\mu_u\colon \Pi_u \longrightarrow \mathbb{R}^N$, whose image is $Q+u$.

b/ By extending the action by translation by elements of $E+\mathbb{Z}^N$ on $E+\mathbb{Z}^N+u$, $E+\mathbb{Z}^N$ acts continuously and minimally on Π_u. This is the restriction of the action of Lemma 3.7 c/ and d/.

c/ The map $v \mapsto P_v$, $v \in E+\mathbb{Z}^N+u$, extends to an open continuous E-equivariant surjection, $\eta_u\colon \Pi_u \longrightarrow MP_u$, which is the restriction of η.

d/ If $x \in \Pi_u$ and $v \in E+\mathbb{Z}^N$ acts on Π_u fixing x, then in fact $v=0$.

Proof The first sentence is immediate since, by definition, Π_u is the closure of an $E+\mathbb{Z}^N$ orbit in Π.

Suppose that $x \in \Pi_u$ and that $y \in E+\mathbb{Z}^N+u$ which we consider as a subset of Π_u. Then there are $x_n \in E+\mathbb{Z}^N+u$ such that $x_n \to x$ in the $\overline{D}$ metric. Write $\beta_n\colon \Pi_u \longrightarrow \Pi_u$ for the translation action by $-x_n$ and write α for the translation action by y. Then we have $\mu(\beta_n(x)) \to 0$ and so $\mu(\alpha\beta_n(x)) = y + \mu(\beta_n(x)) \to y$.

But, since μ is 1-1 at $y \in NS$ by Lemma 3.7 a/, we deduce that $\overline{D}(\alpha\beta_n(x), y) \to 0$ and so y is in the closure of the $E+\mathbb{Z}^N$ orbit of x. However the orbit of y is dense and so we have the density of the x orbit as well.

The lettered parts follow quickly from this. •

By the results of parts b/ and c/ of Lemma 3.9, we may drop the suffix u from the maps μ_u and η_u without confusion, and this is what we do unless it is important to note the domain explicitly.

The aim of the next few sections is to fill in the fourth corner of the commuting square

$$\begin{array}{ccc} \Pi_u & \xrightarrow{\eta} & MP_u \\ \Big\downarrow{\scriptstyle \mu} & & \Big\downarrow{\scriptstyle ?} \\ (Q+u) & \xrightarrow{?} & ? \end{array}$$

in a way which illuminates the underlying structure.

§**4 Tilings and Point Patterns** We now connect the original construction of projection tilings due to Katz and Duneau [**KD**] with the point patterns that we have been considering until now. We refer to [**OKD**] and [**S**] for precise descriptions of the construction; we extract the points essential for our argument below.

We note two developements of the D metric (3.1) which will be used ahead. The first development is also E.A.Robinson's original application of D [**R1**].

Definition 4.1 Suppose that $\mathcal{T}$ is a *pattern* in E considered as a locally finite arrangement of uniformly bounded compact subsets of E (the *units* of the pattern). For example we could take a tiling of E and let the pattern consist of the boundaries of the tiles with superimposed decorations, *i.e.* small compact sets, in their interior giving further asymmetries or other distinguishing features. Or we could take a point pattern, perhaps replacing each point with one of a finite number of decorations. See [**GS**] for a thorough discussion of this process in general.

By taking the union of all the elements of the pattern, we obtain a locally compact subset $P(\mathcal{T})$ of E which can be shifted by elements of E, $P(\mathcal{T})+v$ and these various subsets of E can be compared using D literally as defined above (the addition of further decorations can also solve the problem of ambiguous overlap of adjacent elements of the pattern, a complication which we ignore therefore without loss of generality). Under natural conditions (see [**Ru**] [**Sol**]), which are always satisfied in our examples, the space $\{P(\mathcal{T})+v \mid v \in E\}$ is precompact with respect to the D metric and its closure, written $M\mathcal{T}$ here, supports a natural continuous E action. The *pattern dynamical system* of $\mathcal{T}$ is this dynamical system $(M\mathcal{T}, E)$.

Definition 4.2 The second development adapts D to compare subsets of Σ. Let $C(r) = \pi^{-1}(B(r)) \cap \Sigma$ and let $dC(r) = \pi^{-1}(\partial B(r)) \cap \Sigma$.

Given a subset, A, of Σ define $A[r] = (A \cap C(r)) \cup dC(r)$. Let d_r' be the Hausdorff metric defined among closed subsets of $C(r)$ and define a metric on subsets of Σ by

$$D'(A, A') = \inf\{1/(r+1) \mid d_r'(A[r], A'[r]) < 1/r\}$$

Let $\overline{D}'(v,w) = D'(\widetilde{P}_v, \widetilde{P}_w) + \|v-w\|$, where we recall that $\widetilde{P}_v = \Sigma \cap (\mathbb{Z}^N + v)$.

Let $M\widetilde{P}_u$ be the D'-closure of the space $\{\widetilde{P}_v \mid v \in E+u\}$, and let $\widetilde{\Pi}_u$ be the $\overline{D}'$ completion of $NS \cap (Q+u)$. Let $M\widetilde{P}$ be the D' closure of the space $\{\widetilde{P}_v \mid v \in NS\}$.

The analogues of Proposition 3.5 and Lemma 3.9 with respect to $\widetilde{P}$, $M\widetilde{P}$, $\widetilde{\Pi}_u$, $M\widetilde{P}_u$ and $Q+u$, continue to hold and so we define maps $\widetilde{\mu}\colon \widetilde{\Pi}_u \longrightarrow Q+u$ and $\widetilde{\eta}\colon \widetilde{\Pi}_u \longrightarrow M\widetilde{P}_u$.

We use the projection π to compare the strip pattern with the projection pattern. It will turn out that $\widetilde{\Pi}_u$ will be more convenient than Π_u and we use this comparison to work with both spaces.

Theorem 4.3 *There are E-equivariant maps π_* induced by the projection π which complete* *the commuting square*

$$\begin{array}{ccc} \widetilde{\Pi}_u & \xrightarrow{\pi_*} & \Pi_u \\ \Big\downarrow \widetilde{\eta} & & \Big\downarrow \eta \\ M\widetilde{P}_u & \xrightarrow{\pi_*} & MP_u \end{array}$$

Furthermore we have the following commuting square

$$\begin{array}{ccc} \widetilde{\Pi}_u & \xrightarrow{\pi_*} & \Pi_u \\ \Big\downarrow \widetilde{\mu} & & \Big\downarrow \mu \\ Q+u & = & Q+u \end{array}$$

in which all the labelled maps are 1-1 on NS. •

Consider the example of the canonical tiling, $\mathcal{T}_u$ (2.3). If we know $\widetilde{P}_u$ then we have all the information needed to reconstruct $\mathcal{T}_u$ by its definition. Conversely, the usual assumption that the projected faces are non-degenerate (see [**Le**] (3.1)) allows us to distinguish the orientation of the lattice face (in $\mathbb{Z}^N$) from which a given tile came. Piecing together all the faces defined this way obtains $\widetilde{P}_u$. So the canonical tiling is conjugate (in the sense defined ahead 4.5) to $\widetilde{P}_u$.

On the other hand, the well-known Voronoi or Dirichlet tiling [**GS**] obtained from a point pattern in E is a tiling conjugate to the original point pattern provided we decorate each tile with the point which generates it.

With these two examples of tiling in mind, we consider the pattern $\widetilde{P}_u$ to represent the most elaborate tiling or pattern that can be produced by the projection method, without imposing further decorations not directly connected with the geometry of the construction, and at the other extreme, the point pattern, P_u, represents the least decorated tiling or pattern which can be produced by the projection method.

Definition 4.4 For a given E and K as in (2.1), we include in the *class of projection method patterns* all those patterns, $\mathcal{T}$, of $\mathbb{R}^d$ such that there is a $u \in NS$ and two E-equivariant surjections

$$M\widetilde{P}_u \longrightarrow M\mathcal{T} \longrightarrow MP_u$$

whose composition is π_*.

We call (E, K, u) the *data* of the projection method and by presenting these data we require tacitly that K has the properties of Definition 2.1, that $u \in NS$ and that $(Q + u) \cap IntK \neq \emptyset$ (2.6).

It is clear from the discussion above that the tilings of [**OKD**] and the Voronoi tilings defined above are examples from this class when $K = \pi^{\perp}([0,1]^N)$. In order to compare these two constructions, or to consider projection method patterns in the general sense defined above, we aim to describe $\pi_*: M\widetilde{P}_u \longrightarrow MP_u$.

First we adopt the following definitions which possibly duplicate notions already existing in the literature.

Definition 4.5 Adapting a definition of Le [**Le**], we say that two patterns, $\mathcal{T}, \mathcal{T}'$, in E are *topologically conjugate* if there is an E-equivariant homeomorphism, $M\mathcal{T} \leftrightarrow M\mathcal{T}'$.

The two patterns, $\mathcal{T}, \mathcal{T}'$, are *pointed conjugate* if there is an E-equivariant homeomorphism, $M\mathcal{T} \leftrightarrow M\mathcal{T}'$ which maps $\mathcal{T}$ to $\mathcal{T}'$.

A pattern $\mathcal{T}'$ is a *finite decoration* of a pattern $\mathcal{T}$ if there is a radius r and a rule which forms $\mathcal{T}'$ by choosing and superimposing one of a finite number of decorations on each unit, T, of the pattern (*c.f.* Definition 4.1); and the rule depends only on $\mathcal{T}$ within distance r of T.

We note that topological conjugacy is strictly weaker than local isomorphism (as in [**Le**] for example) and strictly stronger than equal quasicrystal type [**R1**]. Pointed conjugacy is strictly stronger than mutual local derivability [**BSJ**] and topological equivalence [**K3**], but has no strong relation with local isomorphism and quasicrystal type. Finite decoration is strictly stronger than local derivability [**BSJ**].

However, we have the following, an immediate application of the defintions to the fact that an n-to-1 factor map (see 2.13) is an open map [**F**].

Lemma 4.6 *Suppose we have two patterns, $\mathcal{T}, \mathcal{T}'$, in E and a continuous E-equivariant surjection $M\mathcal{T}' \longrightarrow M\mathcal{T}$ which is n-to-1, sending $\mathcal{T}'$ to $\mathcal{T}$. Then $\mathcal{T}'$ is pointed conjugate to a finite decoration of $\mathcal{T}$.* •

§**5 Comparing** Π_u **and** $\widetilde{\Pi}_u$ We start by examining $\pi_*: \widetilde{\Pi}_u \longrightarrow \Pi_u$ from (4.3) and seek conditions under which it is a homeomorphism.

Recall the space V, one of the orthocomponents of the decomposition of Q in Corollary 2.11.

Lemma 5.1 *Suppose that $u \in NS$ and that, for all $v \in Q + u$ such that $v \in \partial((V + v) \cap IntK)$ (the boundary taken in $V + v$), we have $(\Delta + v) \cap K = \{v\}$; then $\pi_*: \widetilde{\Pi}_u \longrightarrow \Pi_u$ is an E-equivariant homeomorphism.*

Proof We ask under what circumstances could we find $x \in \Pi_u$ with two preimages under π_* in $\widetilde{\Pi}_u$? We would need two sequences $v_n, w_n \in (Q + u) \cap NS$ both converging to x in the $\overline{D}$ metric such that $\widetilde{P}_{v_n}$ and $\widetilde{P}_{w_n}$ have different $\overline{D}'$ limits, say A and B respectively. From this we see that $A\Delta B \subset \partial\Sigma$ (symmetric difference) and yet $\pi(A) = \pi(B)$.

Let $p \in \pi(A\Delta B)$ and consider the set $(A\Delta B) \cap \pi^{-1}(p)$. As noted above, this set is a subset of the boundary of $\Sigma \cap \pi^{-1}(p) \equiv K$ and each pair of elements is separated by some element of Δ.

Suppose that $a \in (A \setminus B) \cap \pi^{-1}(p)$. By construction, there are $a_n \in (Q+u) \cap NS \cap \widetilde{P}_{v_n}$ converging to a implying that $a \in \partial((Q+u) \cap IntK)$. But by hypothesis, we deduce $B \cap \pi^{-1}(p) = \emptyset$ - a contradiction to the fact that $p \in \pi(A) = \pi(B)$.

A symmetric argument produces a contradiction from $b \in (B \setminus A) \cap \pi^{-1}(p)$. •

Note that if $\Delta = 0$ or, more generally, if $K \cap (K + \delta) = \emptyset$ whenever $\delta \in \Delta$, $\delta \neq 0$, then the hypothesis of the Lemma is satisfied trivially.

Corollary 5.2 *If $\Delta = 0$, then $\pi_*: \widetilde{\Pi}_u \longrightarrow \Pi_u$ is an E-equivariant homeomorphism.* •

In special cases the hypothesis is satisfied less trivially. We give a slightly more special condition here.

Proposition 5.3 *Suppose that J is the closure of a fundamental domain for Δ in $E^\perp$. If K is contained in some translate of J, then $\pi_*: \widetilde{\Pi}_u \longrightarrow \Pi_u$ is a homeomorphism. In particular, if $K = \pi^\perp([0,1]^N)$, then $\pi_*: \widetilde{\Pi}_u \longrightarrow \Pi_u$ is a homeomorphism.*

Proof For the first part, suppose that $a, b \in K$ and $0 \neq a - b = \delta \in \Delta$, then, by construction, a and b sit one in each of two hyperplanes orthogonal to δ between which K lies. Note that then these hyperplanes are therefore both parallel to V and each intersects K only in a subset of ∂K. Therefore, $a, b \in \partial K$ and further, since $V + a$ and $V + b$ are contained one in each of the hyperplanes, we have $a \notin \partial((V + a) \cap IntK)$ (boundary in $V + a$) and $b \notin \partial((V + b) \cap IntK)$ (*similis*). Therefore the conditions of Lemma 5.1 are fulfilled vacuously.

In the second part, suppose that $K = \pi^\perp([0,1]^N)$ and that $\delta = (\delta_1, \delta_2, ..., \delta_N) \in \Delta$, $\delta \neq 0$ (the case $\Delta = 0$ is easy). Consider the set $I = \{\langle \delta, t\rangle \mid t \in K\}$, where $\langle ., .\rangle$ is the inner product on $\mathbb{R}^N$. This is a closed interval. Also, since δ is fixed by the orthonormal projection $\pi^\perp$, $I = \{\langle \delta, s\rangle \mid s \in [0,1]^N\}$, from which we deduce that the length of I is $\sum_j |\delta_j|$. But since $|\delta_j| < 1$ implies that $\delta_j = 0$, we have $\langle \delta, \delta\rangle = \sum |\delta_j|^2 \geq \sum |\delta_j|$ and so K can be fitted between two hyperplanes orthogonal to δ and separated by δ.

Therefore K is contained in a translate of $\cap_{\delta\in\Delta,\delta\neq 0}\{v \in E^\perp \mid |\langle v, \delta\rangle| \leq (1/2)\langle \delta, \delta\rangle \}$, which in turn is contained in the closure of a fundamental domain for Δ. So we have confirmed the conditions of the first part. •

Remark 5.4 Using Lemma 2.5, the condition of Proposition 5.3 is equivalent to the following condition: $((IntK)-(IntK))\cap\Delta=\{0\}$, where we write $A-A=\{a-b\mid a,b\in A\}$ for the arithmetic (self-)difference of A, a subset of an abelian group. Compare with 8.2 ahead.

All of these results say that if K is small enough relative to Δ then π_* is a homeomorphism. The following construction gives a procedure to reduce the size of a general acceptance domain appropriately.

Suppose that J is the closure of a fundamental domain for Δ in $E^\perp$ and suppose, as we always can, that $\partial J\cap(Q+u)=\emptyset$. Let $K'=(K+\Delta)\cap J$, then K' is a subset of $E^\perp$ which obeys the conditions required in the original definition of (2.1). Also the placement of J ensures that the points in $Q+u$, in particular u itself, which are non-singular with respect to K are also non-singular with respect to K'.

Moreover, if we define $\Sigma'=K'+E$, then, by construction, $\pi(\Sigma'\cap(v+\mathbb{Z}^N))=\pi(\Sigma\cap(v+\mathbb{Z}^N))$ for all $v\in\mathbb{R}^N$. Therefore, working with Σ' instead of Σ we can retrieve the projection point pattern and have, by Proposition 5.3 and the fact that $K'\subset J$, an equation between the spaces $\widetilde{\Pi}_u(\Sigma')$ and Π_u.

Remark 5.5 We note a second process of reduction without loss of generality. Until now we have assumed nothing about the rational position of E, but it is convenient to assume and is often required in the literature that $E\cap\mathbb{Z}^N=0$.

But if $E\cap\mathbb{Z}^N\neq 0$, then by Theorem 2.8 applied to the map π, we have a decomposition of E into complementary spaces, $E=V\oplus W$, where $\pi(\mathbb{Z}^N)=(\pi(\mathbb{Z}^N)\cap W)+(\pi(\mathbb{Z}^N)\cap V)$ is a decompostion into a discrete subset of W and a dense subset of V; the dimension of W is equal to the rank of $E\cap\mathbb{Z}^N$. We may form a complemented subgroup $\Gamma=\{v\in\mathbb{Z}^N\mid\pi(v)\in V\}$ of $\mathbb{Z}^N$ and consider the projection method construction with $\mathbb{R}^N$ replaced by $E^\perp\oplus V$ (with the restriction of the canonical inner product), E replaced by V, and $\mathbb{Z}^N$ replaced by Γ; K is unchanged and u is replaced by $u'=u \bmod W$.

We leave the details to the reader and simply state that if P_u is the projection point pattern with the original parameters, and $P'_{u'}$ is the projection point pattern with these adjusted parameters (which produce a projection plane containing no non-zero lattice point since $\Gamma\cap V=0$), then there is a natural E-equivariant homeomorphism $MP'_{u'}\times\mathbb{T}^{\dim W}\equiv MP_u$. A similar correspondence exists for the strip point pattern.

§**6 Calculating** $M\widetilde{P}_u$ **and** MP_u We now describe $M\widetilde{P}_u$ and MP_u as quotients of $\widetilde{\Pi}_u$ and Π_u respectively. First we examine $M\widetilde{P}_u$ and prove a generalisation of (3.8) of [**Le**].

Proposition 6.1 *Suppose that $u\in NS$, then there is an isometric action of $\mathbb{Z}^N$ on $\widetilde{\Pi}_u$, which factors by $\widetilde{\mu}$ to the translation action by $\mathbb{Z}^N$ on $Q+u$, and $M\widetilde{P}_u=\widetilde{\Pi}_u/\mathbb{Z}^N$. Thus* *we obtain a commutative square of E equivariant maps*

$$\begin{array}{ccc} \widetilde{\Pi}_u & \xrightarrow{\widetilde{\eta}} & M\widetilde{P}_u \\ \Big\downarrow \widetilde{\mu} & & \Big\downarrow \widetilde{\mu} \\ Q+u & \longrightarrow & (Q+u)/\mathbb{Z}^N. \end{array}$$

The left vertical map is 1-1 precisely at the points in $NS \cap (Q+u)$. The right vertical map is 1-1 precisely on the same set, modulo the action of $\mathbb{Z}^N$.

Proof The action of $\mathbb{Z}^N$ on $\widetilde{\Pi}_u$, as an extension of the action on $Q+u$ by translation, is easy to define since the maps are $\overline{D}'$-isometries.

If $v, w \in NS$ then it is clear that $\widetilde{P}_v = \widetilde{P}_w$ if and only if $v - w \in \mathbb{Z}^N$. Moreover, there is $\delta > 0$ so that $\|v - w\| < \delta$ implies that $D'(\widetilde{P}_v, \widetilde{P}_w) \geq \|v - w\|/2$.

From this we see that, if $\widetilde{P}_v = \widetilde{P}_w$ and $\widetilde{P}_{v'} = \widetilde{P}_{w'}$ and $\|v - v'\| < \delta/2$ and $\|w - w'\| < \delta/2$, then $v - w = v' - w'$. The uniformity of δ irrespective of the choice of v, w, v' and w' shows that the statement $\widetilde{\eta}(v) = \widetilde{\eta}(w)$ implies $\widetilde{\mu}(v) - \widetilde{\mu}(w) \in \mathbb{Z}^N$, which is true for $v, w \in NS \cap (Q+u)$, is in fact true for all pairs in $\widetilde{\Pi}_u$, the $\overline{D}'$ closure.

To show the 1-1 properties for the map on the left, suppose that $v \in Q+u$ and that $p \in \partial\Sigma \cap (\mathbb{Z}^N + v)$, i.e. $v \notin NS$. Then since K is the closure of its interior and since NS is dense in $\mathbb{R}^N$ (Lemma 2.2), there are two sequences $v_n, v_n' \in NS$ both converging to v in Euclidean topology and such that $p + (v_n - v) \in \Sigma$ and $p + (v_n' - v) \notin \Sigma$. This implies that any D' limit point of $\widetilde{P}_{v_n}$ contains p and any D' limit point of $\widetilde{P}_{v_n'}$ does not contain p. But both such limit points (which exist by compactness of $M\widetilde{P}$) are in $\widetilde{\mu}^{-1}(v)$ which is a set of at least two elements therefore.

The 1-1 property for the map on the right follows directly from this and the commuting diagram. •

The space $(Q+u)/\mathbb{Z}^N$ and its E action, which is being compared with $M\widetilde{P}_u$, also have a simple description.

Lemma 6.2 *With the data above, $(Q+u)/\mathbb{Z}^N$ is a coset of the closure of E mod $\mathbb{Z}^N$ in $\mathbb{R}^N/\mathbb{Z}^N \equiv \mathbb{T}^N$. Therefore $(Q+u)/\mathbb{Z}^N$ with its E action is isometrically conjugate to a minimal action of $\mathbb{R}^d$ by translation on a torus of dimension $N - \dim \Delta$.*

Proof The space Q mod $\mathbb{Z}^N$ is equal to the closure of E mod $\mathbb{Z}^N$ and its translate by u mod $\mathbb{Z}^N$ is an isometry which is E equivariant. The action of E on its closure is isometric and transitive, hence minimal, and is by translations. E is a connected subgroup of $\mathbb{T}^N$ and so also is the closure of E, which is therefore equal to a torus of possibly smaller dimension. The codimension of this space agrees with the codimension of $V+E$ (the continuous component of Q) in $\mathbb{R}^N$ which, by Lemma 2.9 and Corollary 2.11, equals $\dim\Delta$ as required. •

Now we turn to a description of MP_u which is similar in form to that of $M\widetilde{P}_u$, but as to be shown in examples 8.7 and 8.8, need not be equal.

Lemma 6.3 *Suppose that $u\in NS$. If $v,w\in NS\cap(Q+u)$ and $P_v=P_w$ then there are $v^*\in v+\mathbb{Z}^N$ and $w^*\in w+\mathbb{Z}^N$ such that $v^*,w^*\in\Sigma$ and $\pi(v^*)=\pi(w^*)$, and with this choice $\widetilde{P}_v+\Delta-\pi^\perp(v^*)=\widetilde{P}_w+\Delta-\pi^\perp(w^*)$.*

Proof Fix $p_o\in P_v=P_w$ and let $v^*\in\widetilde{P}_v$ be chosen so that $\pi(v^*)=p_o$ and similarly, let $w^*\in\widetilde{P}_w$ be chosen so that $\pi(w^*)=p_o$. Clearly v^* and w^* obey the conditions required. Also $\widetilde{P}_w-\pi^\perp(w^*)$ and $\widetilde{P}_v-\pi^\perp(v^*)$ are both contained in $p_o+\mathbb{Z}^N$ and project under π to the same set P_v. Thus the difference of two points, one in $\widetilde{P}_w-\pi^\perp(w^*)$ and the other in $\widetilde{P}_v-\pi^\perp(v^*)$, and each with the same image under π, is an element of Δ as required. •

Proposition 6.4 *Suppose that $x,y\in\Pi_u$ and that $\eta(x)=\eta(y)$, then there is a $v\in Q$ and a $\overline{D}$ isometry $\phi\colon\Pi_u\longrightarrow\Pi_u$ so that the following diagram commutes*

$$\begin{array}{ccc}\Pi_u & \xrightarrow{\phi} & \Pi_u\\ \Big\downarrow{\scriptstyle\mu} & & \Big\downarrow{\scriptstyle\mu}\\ Q+u & \xrightarrow{w\mapsto w+v} & Q+u\end{array}$$

and $\eta_u\phi=\eta_u$ (here the restriction to Π_u is important to note). In this case we deduce $v+u\in NS$.

Conversely, if we have such an isometry in such a diagram and if $P_{u+v}=P_u$, then $v+u\in NS$ and $\eta_u\phi=\eta_u$ automatically.

Proof Suppose $w\in E+\mathbb{Z}^N$ and that $\alpha_w\colon\Pi_u\longrightarrow\Pi_u$ is the map completed from the map $z\mapsto z+w$ defined first for $z\in NS\cap(Q+u)$ (see Proposition 3.5). Then, since $\eta(x)=\eta(y)$ and η is $(E+\mathbb{Z}^N)$-equivariant, we have $\eta(\alpha_w(x))=\eta(\alpha_w(y))$ for all $w\in E+\mathbb{Z}^N$. So, by definition, the map $\alpha_w(x)\mapsto\alpha_w(y)$ defined point-by-point for $w\in E+\mathbb{Z}^N$ is a $\overline{D}$ isometry from the $(E+\mathbb{Z}^N)$-orbit of x onto the $(E+\mathbb{Z}^N)$-orbit of y (By Lemma 3.9 d/ the mapping is well-defined). These two orbits being dense (Lemma 3.9) in Π_u, this map extends as an isometry onto, $\phi\colon\Pi_u\longrightarrow\Pi_u$.

Since μ is $(E+\mathbb{Z}^N)$-equivariant, we deduce the intertwining with translation by $v=\mu(y)-\mu(x)$. Also since $\eta\phi=\eta$ on the $E+\mathbb{Z}^N$ orbit of x, the E-equivariance of η extends this equality over all of Π_u.

Conversely suppose we have an isometry which intertwines the translation by v on $Q+u$. Then for general topological reasons the cardinality of the μ preimage of a point in $Q+u$ is preserved by translation by v and we deduce that $NS\cap(Q+u)$ is invariant under the translation by v. In particular $u+v\in NS$. The equation follows since it applies, by hypothesis and Lemma 6.3, at u and therefore, by equivariance, at all points in $E+\mathbb{Z}^N+u$, a dense subset. •

Definition 6.5 For $u\in NS$, let $R_u=\{v\in Q\mid v+u\in NS, P_{u+v}=P_u\}$.

Corollary 6.6 *Suppose that $u\in NS$ and $w\in NS\cap(Q+u)$, then $R_w=R_u$. Therefore, if $v\in R_u$, then $v+w\in NS\cap(Q+u)$ for all $w\in NS\cap(Q+u)$.*

Proof By Lemma 3.9, we know that $\Pi_w \xrightarrow{\eta_w} Q+w$ equals $\Pi_u \xrightarrow{\eta_u} Q+u$ and so any isometry of Π_u which factors by η through to a translation by v also does the same for Π_w. Proposition 6.4 completes the equivalence.

The second sentence follows directly from the definition of R_w. •

Remarks 6.7 It would be natural to hope that the condition $u+v\in NS$ could be removed from the definition of R_u. We have been unable to do this in general. But since NS is a dense G_δ set (2.2) and, anticipating Theorem 7.1, R_u is countable, we see that for a dense G_δ set of $u\in NS$ (*generically*) we can indeed equate $R_u=\{v\in Q: P_{u+v}=P_u\}$.

This is bourne out in Corollary 6.6 where we see that R_u is defined independently of the choice of u generically, and R_u can be thought of as an invariant of Π_u. This result also shows that R_u is a subset of the translations of $\mathbb{R}^N$ which leave $NS\cap(Q+u)$ invariant.

Note that, since μ is 1-1 only on NS, R_u could as well have been defined as $\{v\in Q\mid \eta\mu^{-1}(u+v)=\{P_u\}\ \}$.

It is clear that $\mathbb{Z}^N\subset R_u$.

Theorem 6.8 *If $u\in NS$, then R_u is a closed subgroup of Q. Also R_u acts by ϕ isometrically on Π_u and defines a homeomorphism $\Pi_u/R_u\equiv MP_u$. Moreover the R_u action commutes with the E-action, so the homeomorphism is E-equivariant.*

Proof The main point to observe is that R_u consists precisely of those elements v such that there is an isometry ϕ_v as in Proposition 6.4 with $\eta_u\phi_v=\eta_u$. Since the inverse of such an isometry is another such, and the composition of two such isometries produces a third, we deduce the group property for R_u immediately. The isometric action is given to us and Proposition 6.4 shows directly that $\Pi_u/R_u\equiv MP_u$.

Closure of R_u is more involved. Suppose that $v_n\in R_u$ and that $v_n\to v$ in the Euclidean topology. Then ϕ_{v_n} is uniformly Cauchy and so converges uniformly to a bijective isometry, ψ, of Π_u which intertwines the translation by v on $Q+u$. Therefore, if $\mu^{-1}(u+v)$ has at least two elements, then so also does $\psi^{-1}\mu^{-1}(u+v)$, but this set is contained in $\mu^{-1}(u)$, a contradiction since $\mu^{-1}(u)$ is a singleton. Therefore $u+v \in NS$ and $\mu^{-1}(u+v) = \psi\mu^{-1}(u) = \lim \phi_{v_n}\mu^{-1}(u) = \lim \mu^{-1}(u+v_n) = \mu^{-1}(u)$. Thus $v \in R_u$ and so R_u is closed.

The commutation with the E action on Π_u is immediate from the corresponding commutation on $Q+u$. •

§7 Comparing MP_u with $M\widetilde{P}_u$ The discussion of the previous section has defined projection method patterns as those whose dynamical system sits intermediate to $M\widetilde{P}_u$ and MP_u. We discover in this section how closely these two spaces lie and circumstances under which they are equal.

To compare MP_u with $M\widetilde{P}_u$ we start with the assumption $\widetilde{\Pi}_u = \Pi_u$, which loses some generality but which was also justified by the discussion of (5.4). By Proposition 6.1 and Theorem 6.8, therefore, the problem becomes the comparison of R_u with $\mathbb{Z}^N$. Perhaps surprisingly, under general conditions we find that R_u is not much larger than $\mathbb{Z}^N$ and under special conditions the two groups are equal.

Theorem 7.1 *For all $u \in NS$, $\mathbb{Z}^N \subset R_u$ as a finite index subgroup. In fact, with the notation of (2.10), $R_u \subset \mathbb{Z}^N + \widetilde{\Delta}$.*

Proof Suppose that $v \in R_u$. Then in particular, by (6.7), $P_{v+u} = P_u$. Therefore there is an $a \in \mathbb{Z}^N$ such that $\pi(v+u+a) = \pi(u)$ and so by translating if necessary, we may assume without loss of generality that $v \in E^\perp$; and this defines v uniquely $\mod \Delta$.

With this assumption we deduce from Lemma 6.3 that $\widetilde{P}_{u+v} + \Delta = \widetilde{P}_u + \Delta + v$ In particular, $\pi^\perp(\widetilde{P}_{u+v}) + \Delta = \pi^\perp(\widetilde{P}_u) + \Delta + v$.

Now each of $\pi^\perp(\widetilde{P}_{u+v})$ and $\pi^\perp(\widetilde{P}_u)$ is contained in K a compact set. Suppose that $\alpha \in \Delta^\perp$ and that $\langle v, \alpha\rangle \neq 0$, then there is $t \in \mathbb{Z}$ such that $|\langle tv, \alpha\rangle| > 2\|\alpha\|\ diamK$. However, since $tv \in R_u$ by Theorem 6.3, we have $\pi^\perp(\widetilde{P}_{tv+u}) + \Delta = \pi^\perp(\widetilde{P}_u) + \Delta + tv$. Applying the function $\langle ., \alpha\rangle$ to both sets produces a contradiction by construction. Thus we have $v \in U$, the space generated by Δ (see 2.9).

But if $v \in R_u$ then we restrict attention to Q and so we have $v \in \widetilde{\Delta}$ a group which, by Corollary 2.11, contains Δ with finite index. •

We note, for use in section 10, that therefore R_u is free abelian on N generators.

Corollary 7.2 *If $\Delta = 0$, then $R_u = \mathbb{Z}^N$ and $\pi_*\colon M\widetilde{P}_u \leftrightarrow MP_u$.* •

The following combines Propositions 6.1 and 6.8 and fits the present circumstances to the conditions of Lemma 2.14. Recall the definitions of n-to-1 extension (2.13) and finite isometric extension (2.15).

Proposition 7.3 *Suppose that* $u \in NS$ *and that* $\widetilde{\Pi}_u = \Pi_u$. *The map* $\pi_*: M\widetilde{P}_u \longrightarrow MP_u$ *is p-to-1 where p is the index of* $\mathbb{Z}^N$ *in* R_u. *The group* R_u *acts isometrically on* $M\widetilde{P}_u$, *commuting with the* E *action, preserving* π_* *fibres and acting transitively on each fibre. This action is mapped almost 1-1 by* $\widetilde{\mu}$ *to an action by* R_u *on* $(Q+u)/\mathbb{Z}^N$ *by rotation, and so we complete a commuting square*

$$\begin{array}{ccc} M\widetilde{P}_u & \xrightarrow{\pi_*} & MP_u \\ \Big\downarrow \widetilde{\mu} & & \Big\downarrow \mu \\ (Q+u)/\mathbb{Z}^N & \xrightarrow{modR_u} & (Q+u)/R_u. \end{array}$$

•

From this and the construction of Lemma 2.14 applied to the case $G = E$ and $H = R_u$, we deduce the main Theorem of the section.

Theorem 7.4 *Suppose that,* E, K *and* $u \in NS$ *are given, that* $\widetilde{\Pi}_u = \Pi_u$, *and that* $\mathcal{T}$ *is a projection method pattern. Then there is a group* $H_\mathcal{T}$, *intermediate to* $\mathbb{Z}^N < R_u$, *which fits into a commutative diagram of* E *equivariant maps*

$$\begin{array}{ccccc} M\widetilde{P}_u & \longrightarrow & M\mathcal{T} & \longrightarrow & MP_u \\ \Big\downarrow \widetilde{\mu} & & \Big\downarrow & & \Big\downarrow \mu \\ (Q+u)/\mathbb{Z}^N & \xrightarrow{modH_\mathcal{T}} & (Q+u)/H_\mathcal{T} & \xrightarrow{modR_u} & (Q+u)/R_u \end{array}$$

where the top row maps are finite isometric extensions and the bottom row maps are group quotients.

Conversely, every choice of group H' *intermediate to* $\mathbb{Z}^N < R_u$ *admits a projection method pattern,* $\mathcal{T}$, *fitting into the diagram above with* $H' = H_\mathcal{T}$. •

With the considerations of section 4 (in particular using 4.6) we can count the projection method patterns up to topological conjugacy or pointed conjugacy in the following corollary.

Corollary 7.5 *With fixed projection data and the conditions of Theorem 7.4, the set of topological conjugacy classes of projection method patterns is in bijection with the lattice of subgroups of* $R_u/\mathbb{Z}^N$. *Moreover, each projection method pattern,* $\mathcal{T}$, *is pointed conjugate to a finite decoration of* P_u, *and* $\widetilde{P}_u$ *is pointed conjugate to a finite decoration of* $\mathcal{T}$. •

Also we deduce a generalisation of the result of Robinson [**R2**] and the topological version of the result of Hof [**H**].

Corollary 7.6 *With the conditions assumed in Theorem 7.4, the pattern dynamical system* $M\mathcal{T}$ *is an almost 1-1 extension (2.15) of a minimal* $\mathbb{R}^d$ *action by rotation on a* $(N - \dim\Delta)-$*torus.*

Proof It suffices to show that the central vertical arrow in the diagram of Theorem 7.4 is 1-1 at some point. But this is immediate since each of the end arrows is 1-1 at u say. •

§**8 Examples and Counter-examples** In this section we give sufficient conditions, similar to and stronger than 5.3, under which P_u or $\mathcal{T}$ is pointed conjugate to $\widetilde{P}_u$, and show why these conjugacies are not true in general.

Definition 8.1 For data (E, K, u), define $B_u = \overline{(Q+u)\cap IntK}$ (Euclidean closure in $E^\perp$).

Proposition 8.2 *Suppose that* E*,* K *and* $u \in NS$ *are chosen so that* $E\cap\mathbb{Z}^N = 0$*,* $\Pi_u = \widetilde{\Pi}_u$ *and* $\Delta\cap[(B_u - B_u) - (B_u - B_u)] = \{0\}$*, then* $R_u = \mathbb{Z}^N$*. In this case, therefore,* P_u *is pointed conjugate to* $\widetilde{P}_u$*.*

Proof This follows from the fact, deduced directly from the condition given, that if $v \in NS\cap(Q+u)$ and $a, b \in P_v$, then we can determine $w - w'$ whenever $w, w' \in \widetilde{P}_v$ are such that $a = \pi(w)$ and $b = \pi(w')$. Knowing the differences of elements of $\widetilde{P}_v$ forces the position of $\widetilde{P}_v$ in Σ by the density of $\pi^\perp(\widetilde{P}_v)$ in B_u. So we can reconstruct $\widetilde{P}_v$ uniquely from P_v and we have $R_u = \mathbb{Z}^N$. •

Corollary 8.3 *In the canonical case, the condition that the points* $\pi(w) \mid w \in \{-1, 0, 1\}^N$ *are all distinct is sufficient to show that* $R_u = \mathbb{Z}^N$ *for all* $u \in NS$*. In this case, therefore,* P_u *is pointed conjugate with* $\widetilde{P}_u$*.*

Proof The condition implies that $\Delta\cap[(K-K)-(K-K)] = \{0\}$ and this gives the condition in the proposition since $B_u \subset K$. •

If we are interested merely in the equation between $\mathcal{T}$ and $\widetilde{P}_u$, then the canonical case also allows simple sufficient conditions weaker than 8.3.

We observe first that the construction of [**OKD**] can be extended to admit non-generic parameters, provided that we are comfortable with "tiles" which, although convex polytopes, have no interior in E and are unions of faces of the true tiles. But we retain these degenerate tiles as components of our "tiling", *i.e.* really as units of a pattern, giving essential information about the pattern dynamics. We call such patterns *degenerate* canonical tilings.

We write e_j with $1 \leq j \leq N$ for the canonical unit basis of $\mathbb{Z}^N$.

Proposition 8.4 *In the canonical case, the condition that no two points from* $\{\pi(e_j) \mid 1 \leq j \leq N\}$ *are collinear, is sufficient to show that,* $\mathcal{T}_u$*, the canonical (but possibly degenerate) tiling, is pointed conjugate to* $\widetilde{P}_u$ *for all* $u \in NS$.

Proof We show that the conditions given imply that the shape of a tile (even in degenerate cases) determines from which face of the lattice cube it is projected. In fact we shall show that if $I \subset \{1, 2, ..., N\}$ then knowing $\pi(\gamma^I)$ and the cardinality of I determines I (we write $\gamma^I = \{\sum_{i \in I} \lambda_i e_i \mid 0 \leq \lambda_i \leq 1, \ \forall\ i \in I\}$).

Suppose that $\pi(\gamma^I) = \pi(\gamma^J)$ and $I, J \subset \{1, 2, .., N\}$ are of the same cardinality. It is possible always to distinguish an edge on the polyhedron $\pi(\gamma^I)$ which is parallel to a vector $\pi(e_i)$ for some $i \in I$; and i is determined from this edge by hypothesis. The same is true of this same edge with respect to J and so $i \in J$ also.

Writing $I' = I \setminus \{i\}$ and $J' = J \setminus \{i\}$ we deduce that $\pi(\gamma^{I'}) = \pi(\gamma^I) \cap (\pi(\gamma^I) - \pi(e_i)) = \pi(\gamma^J) \cap (\pi(\gamma^J) - \pi(e_i)) = \pi(\gamma^{J'})$. Now we can apply induction on the cardinality of I, and deduce that $I' = J'$ and so $I = J$. Induction starts at cardinality 1 by hypothesis.

Now, given this correspondence between shape of tile and its preimage under π, we reconstruct $\widetilde{P}_u$ from $\mathcal{T}_u$ much as we did in Proposition 8.2 above. To complete the argument we must check that no other element of $M\widetilde{P}_u$ maps onto $\mathcal{T}_u$ in $M\mathcal{T}_u$. But if there were such an element, then the argument above shows that it cannot be of the form $\widetilde{P}_{u'}$ for $u' \in NS$. Also, by Theorem 7.4, we deduce that the map $M\widetilde{P}_u \longrightarrow M\mathcal{T}_u$ is p-to-1 with $p \geq 2$, and so, using Lemma 2.2ii/, we find $\mathcal{T}_v$ with two preimages of the form $\widetilde{P}_v$ and $\widetilde{P}_{v'}$. But this contradicts the principle of the previous sentence. •

The conditions of this proposition include all the non-degenerate cases of the canonical tiling usually treated in the literature (including the Penrose tiling), so from the equation $M\mathcal{T}_u = \widetilde{\Pi}_u/\mathbb{Z}^N$, deduced from Proposition 8.4 as a consequence, we retrieve many of the results stated (but not proved) in section 3 of [**Le**].

Now we turn to conditions under which R_u differs from $\mathbb{Z}^N$. We can extend the argument of 7.1 to give a geometric condition for elements of R_u, of considerable use in computing examples.

Lemma 8.5 *Suppose that* $E \cap \mathbb{Z}^N = 0$, $u \in NS$ *and* $v \in E^\perp$*. Then* $v \in R_u \cap E^\perp$ *if and only if* $v + u \in NS$ *and* $v + (B_u + \Delta) = B_u + \Delta$.

Proof We suppose that $u, v+u \in NS$. Then $\overline{\pi^\perp(\widetilde{P}_u)} = \overline{(Q+u)\cap IntK}$ and $\overline{\pi^\perp(\widetilde{P}_{u+v})} = \overline{(Q+u+v)\cap IntK}$.

If $v \in R_u \cap E^\perp$, then, by Lemma 6.3, we have $\widetilde{P}_u + \Delta + v = \widetilde{P}_{u+v} + \Delta$. Also, since by 7.1, v is in $\widetilde{\Delta}$, we have $Q+u = Q+u+v$ and $\pi^\perp(\widetilde{P}_u) + \Delta + v = \pi^\perp(\widetilde{P}_{u+v}) + \Delta$. Putting all these together gives the required equality $v + (B_u + \Delta) = B_u + \Delta$.

Conversely, if $v + (B_u + \Delta) = B_u + \Delta$, then, by the argument of 7.1, $v \in \widetilde{\Delta}$ and so, as above, $Q+u = Q+u+v$ and $\pi^\perp(\widetilde{P}_u) + \Delta + v = \pi^\perp(\widetilde{P}_{u+v}) + \Delta$. So, if $a \in \widetilde{P}_u$, then there is a $b \in \widetilde{P}_{u+v}$ such that $\pi^\perp(a) - \pi^\perp(b) \in \Delta - v$. However, since $\pi^\perp$ is 1-1 on $\mathbb{Z}^N$, we can retrieve the set $\widetilde{P}_u$ as the inverse $\pi^\perp$ image of $\pi^\perp(\widetilde{P}_u) \cap IntK$ and similarly for $\widetilde{P}_{u+v}$. This forces $a - b \in \Delta - v$ therefore, and so $\pi(a) = \pi(b)$. Thus we see that $P_u = P_{u+v}$, as is required to show that $v \in R_u$ •

Example 8.6 By Corollary 7.2 and the fact that $\Delta = 0$, the octagonal tiling is pointed conjugate to both its projection and strip point patterns.

Also it is from Lemma 8.5 (and not 8.3) that we can deduce that $R_u = \mathbb{Z}^5$ for the (generalised) Penrose tiling for all choices of $u \in NS$, and so the generalised Penrose tiling is pointed conjugate to both its projection and strip point patterns.

Now it is quite easy to construct counter-examples to the possibility that $R_u = \mathbb{Z}^N$ always, even under the conditions $E \cap \mathbb{Z}^N = 0$ and $\Pi_u = \widetilde{\Pi}_u$.

Example 8.7 We start with a choice of E for which $\Delta \neq 0$ and $\widetilde{\Delta}$ contains Δ properly. The E used to construct the Penrose tiling is such an example. As in the proof of Lemma 2.6, let U be the real span of Δ and V the orthocomplement of U in $E^\perp$. Choose a closed unit disc, I, in V and let J be the closure of a rectangular fundamental domain for Δ in U. Let $K = I + J \subset E^\perp$ and note that K has all the propeties required of an acceptance domain in this paper and that, by Lemma 5.1, we have $\Pi_u = \widetilde{\Pi}_u$.

With $u \notin V + \widetilde{\Delta}$ (equivalently $u \in NS$), the rectilinear construction of K ensures that $((Q+u)\cap IntK) + \Delta$ is invariant under the translation by any element, v, of $\widetilde{\Delta}$. Also $v + u \in NS$ since the boundary of $((Q+u)\cap IntK) + \Delta$ in $Q+u$ is invariant under translations by $\widetilde{\Delta}$. So, with the characterisation of Lemma 8.5, this shows that $R_u = \widetilde{\Delta} + \mathbb{Z}^N$ which is strictly larger than $\mathbb{Z}^N$.

By varying the shape of J in this example, we can get $R_u/\mathbb{Z}^N$ equal to any subgroup of $(\widetilde{\Delta} + \mathbb{Z}^N)/\mathbb{Z}^N$, and we can make it a non-constant function of $u \in NS$ as well.

Example 8.8 Take $N = 3$ with unit vectors e_1, e_2 and e_3. Let L be the plane orthonormal to $e_1 - e_2$ and let E be a line in L placed so that $E \cap \mathbb{Z}^3 = 0$. Then $E^\perp$ is a plane which contains $e_1 - e_2$ and we have $\Delta = \{n(e_1 - e_2) \mid n \in \mathbb{Z}\}$.

Write $e_1^\perp$, $e_2^\perp$, and $e_3^\perp$ for the image under $\pi^\perp$ of e_1, e_2 and e_3 respectively. Then $e_1^\perp + e_2^\perp$ and $e_3^\perp$ are collinear in $E^\perp$ and they are both contained in V (the continuous subspace of $V + \widetilde{\Delta} = \overline{\pi^\perp(\mathbb{Z}^3)}$). Δ is orthogonal to V and $e_1^\perp - e_2^\perp = e_1 - e_2$. However $\widetilde{\Delta} = \{n(e_1 - e_2)/2 \mid n \in \mathbb{Z}\}$ which contains Δ as an index 2 subgroup.

The set $K = \pi^\perp([0,1]^3)$ is a hexagon in $E^\perp$ with a centre of symmetry. It is contained in the closed strip defined by two lines, $V + a$ and $V + b$, where $b - a = e_1 - e_2$, and it is in fact reflectively symmetric around an intermediate line, $V + c$ where $c - a = (e_1 - e_2)/2$. The boundary of the hexagon on each of $V + a$ and $V + b$ is an interval congruent to $e_3^\perp$ (*i.e.* a translate of $\{te_3^\perp \mid 0 \leq t \leq 1\}$). The four other sides are intervals congruent to $e_1^\perp$ or $e_2^\perp$, two of each. The vertices of the hexagon are on $V + a$, $V + b$ or $V + c$, two on each.

The point of all this is that there is a choice of non-singular u (in $E^\perp$ without loss of generality) such that $B_u = \overline{(Q + u) \cap IntK}$ consists of the two intervals $K \cap (V + a')$ and $K \cap (V + b')$, where $2a' = a + c$ and $2b' = b + c$ (we can choose $u \in (V + a') \cap NS$ for example), and these intervals are a translate by $\pm(e_1 - e_2)/2$ of each other. Thus we deduce that $B_u + \Delta = B_u + \Delta + v$ for all $v \in \widetilde{\Delta}$.

Upon confirming that $v + u \in NS$ for all $v \in \widetilde{\Delta}$ as well, we use 8.5 to show that $R_u = \mathbb{Z}^3 + \widetilde{\Delta}$, which contains $\mathbb{Z}^3$ with index 2.

Remark 8.9 We note that Example 8.8 is degenerate and Proposition 8.4 shows why this must be the case. However, under any circumstances, there exist projection method tilings, in the sense of 4.4, pointed conjugate to P_u or to $\widetilde{P}_u$. The point here is that these tilings will not necessarily be constructed by the special method of Katz and Duneau.

Also, leaving the details to the reader, we mention that Example 8.8 and its analogues in higher dimensions are the only counter-examples to the assertion $R_u = \mathbb{Z}^N$ in the canonical case when Δ is singly generated (and here we find always that $R_u/\mathbb{Z}^N$ is a cyclic group of 2 elements). When Δ is higher dimensional we have no concise description of the exceptions allowed.

§**9 The topology of** $\widetilde{\Pi}_u$ Section 5 justifies the assumption, which we continue to make, that $\Pi_u = \widetilde{\Pi}_u$ for all $u \in NS$. Apart from the results of the previous section, the main advantage of this equality is that $\widetilde{\Pi}_u$ is more easily described than Π_u *a priori*.

Definition 9.1 Let F be a plane complementary, but not necessarily orthonormal, to E and let π' be the skew projection (idempotent map) onto F parallel to E. Let $K' = \pi'(K)$.

Set $F_u^o = NS \cap (Q + u) \cap F$ and let F_u be the $\overline{D}'$-closure of F_u^o in $\widetilde{\Pi}_u$.

Note that, since $R_u \subset Q$, F_u^o is invariant under translation by $\pi'(r)$, $r \in R_u$ and by extension F_u supports a continuous R_u action. R_u acts freely on F_u when $E \cap \mathbb{Z}^N = 0$ (i.e. with any fixed $x \in F_u$, the equation $gx = x$ implies $g = 0$).

Similarly, R_u acts on E by translation by $\pi(r)$, $r \in R_u$.

Lemma 9.2 *With the data above, $F_u = \widetilde{\mu}^{-1}(F \cap (Q+u))$ and there is a natural equivalence $\widetilde{\Pi}_u \equiv F_u \times E$ and a surjection $\nu: F_u \longrightarrow ((Q+u) \cap F)$ which fits into the following commutative square*

$$\begin{array}{ccc} \widetilde{\Pi}_u & \longleftrightarrow & F_u \times E \\ \Big\downarrow{\scriptstyle \widetilde{\mu}} & & \Big\downarrow{\scriptstyle \nu \times id} \\ Q+u & \longleftrightarrow & ((Q+u) \cap F) \times E. \end{array}$$

Moreover these maps are E-equivariant where we require that E acts trivially on F_u. The set $\nu^{-1}(v)$ is a singleton whenever $v \in NS \cap F \cap (Q+u)$.

The canonical action of R_u on $\widetilde{\Pi}_u$ is represented in this equivalence as the direct sum (i.e. diagonal) of the action of R_u on F_u and E described in (9.1).

Proof This follows quickly from the observation that there is a natural, $\overline{D}'$-uniformly continuous and E equivariant equivalence $NS \cap (Q+u) = F_u^o + E \equiv F_u^o \times E$, which can be completed. •

Definitions 9.3 Let $\mathcal{A}_u$ be the algebra of subsets (*i.e.* closed under finite union, finite intersection and symmetric difference) of F_u^o generated by the sets $(NS \cap (Q+u) \cap K') + \pi'(v)$ as v runs over $\mathbb{Z}^N$. It is clear that this algebra is countable and invariant under the action of R_u.

Write $C^*(\mathcal{A}_u)$ for the smallest C^* algebra which contains the indicator functions of the elements of $\mathcal{A}_u$.

Let $\mathbb{Z}\mathcal{A}_u$ be the ring (pointwise addition and multiplication) generated by this same collection of indicator functions.

Let $CC(F_u; \mathbb{Z})$ be the group of continuous integer valued functions compactly supported on F_u.

These three algebraic objects support a canonical R_u action induced by the action of R_u on F_u described in (8.1) and so we define three $\mathbb{Z}[R_u]$ modules. As $\mathbb{Z}^N$ sits inside R_u, this action can be restricted to a canonical subaction by $\mathbb{Z}^N$ thereby defining three $\mathbb{Z}[\mathbb{Z}^N]$ modules.

Let $\mathcal{B}_u = \{\overline{A} \mid A \in \mathcal{A}_u\}$ where the bar refers to $\overline{D}'$ closure in F_u.

And finally, we give the main theorem of this section which will be of much use in [**FHK1**].

Theorem 9.4 *With data (E, K, u) and $\widetilde{\Pi}_u = \Pi_u$,*

a/ The collection $\mathcal{B}_u$ is a base of clopen neighbourhoods which generates the topology of F_u.

b/ We have the $*$-isomorphisms of C^* algebras $C_o(F_u) \equiv C^*(\mathcal{A}_u)$ and $C_o(\widetilde{\Pi}_u) \equiv C^*(\mathcal{A}_u) \otimes C_o(E)$ which respect the maps defined in Lemma 8.2.

c/ $CC(F_u;\mathbb{Z}) \equiv \mathbb{Z}\mathcal{A}_u$ as a $\mathbb{Z}[R_u]$ module (and by pull-back as a $\mathbb{Z}[\mathbb{Z}^N]$ module).

d/ F_u is locally a Cantor Set.

First we have a lemma also of independent interest in the next section.

Definition 9.5 Write $\overline{K}$ for the $\overline{D}'$-closure of the set $K' \cap NS \cap (Q+u)$.

Lemma 9.6 *$\overline{K}$ is a compact open subset of F_u.*

Proof Closure is by definition so compactness follows immediately on observing that $K' \cap (Q+u) \cap NS$ is embedded $\overline{D}'$-isometrically in the space $M\widetilde{P}_u \times K'$ with metric $D' + \|.\|$ as the closed subset $\{(\widetilde{P}_v, v) \mid v \in K' \cap (Q+u) \cap NS\}$. But $M\widetilde{P}_u \times K'$ is compact.

For openness, we appeal to an argument similar to that of (6.1). Suppose, for a contradiction, that v_n is a $\overline{D}'$-convergent sequence in $(F \cap NS) \cap K'$ and that v'_n is a $\overline{D}'$ convergent sequence in $(F \cap NS) \setminus K'$ and that both sequences have the same limit $x \in \widetilde{\Pi}_u$. Therefore $v = \widetilde{\mu}(x)$ is the Euclidean limit of the v_n and v'_n and so $v \in \partial K$. But by construction $\widetilde{P}_{v_n}$ and $\widetilde{P}_{v'_n}$ have a different D' limit – a contradiction since the limit in each case must be $\widetilde{\eta}(x)$. •

Proof of Theorem 9.4 a/ The sets in $\mathcal{B}_u$ are clopen by Lemma 9.6 above. It remains to check that the collection $\mathcal{B}_u$ contains a decreasing set of neighbourhoods around any point in F_u.

Certainly, if $a \neq b$ with $a, b \in F \cap (Q+u)$, then the assumption that $Int(K) \cap (Q+u) \neq \emptyset$ (2.6) (hence $Int(K') \cap (Q+u) \neq \emptyset$, interior taken in F) and the facts that K' is bounded and that $\pi'(\mathbb{Z}^N)$ is dense in $Q \cap F$, imply that there is some $v \in \mathbb{Z}^N$ such that $a \in (Int(K') \cap (Q+u)) + \pi'(v)$ and $b \notin \overline{(K' \cap (Q+u))} + \pi'(v)$ (Euclidean closure in $F \cap (Q+u)$). *i.e.* a and b are separated by the topology induced by $\widetilde{\mu}(\mathcal{B}_u)$.

In particular, if $x, y \in F_u$ and $y \in \cap\{B \in \mathcal{B}_u \mid x \in B\}$ then $\widetilde{\mu}(x) = \widetilde{\mu}(y)$.

But, if $x \neq y$ and $\widetilde{\mu}(x) = \widetilde{\mu}(y) = v$, then, by (3.7)e/, $\widetilde{\eta}(x) \neq \widetilde{\eta}(y)$ and we may suppose that there is a point $p \in \widetilde{\eta}(x) \setminus \widetilde{\eta}(y)$. We use the argument of Lemma 8.6 to show that then there are two sequences $v_n, v'_n \in \pi'(\mathbb{Z}^N)$ both converging to v in Euclidean topology and such that $p + (v_n - v) \in \Sigma$ and $p + (v'_n - v) \notin \Sigma$. But then $y \notin \overline{A}$ (closure in $\overline{D}'$ metric) where $A = (NS \cap (Q+u) \cap K') + \pi'(p + v_n)$, and $x \in \overline{A}$, a contradiction to the construction of y.

Therefore $x = y$ and so, by the local compactness (Lemma 9.6) of F_u we have the required basic property of the collection $\mathcal{B}_u$.

b/ This will follow from a/ and the equivalences in Lemma 9.2 if we can show that $\mathcal{A}_u$ is isomorphic to $\mathcal{B}_u$ as a Boolean algebra. To show this, it is enough to show that

$A \mapsto \overline{A}$ (closure in $\overline{D}'$ metric) is 1-1 on $\mathcal{A}_u$; and for this it suffices to prove that if $A \in \mathcal{A}_u$ is non-empty, then its Euclidean closure has interior (in $(Q+u) \cap F$).

Note that $NS \cap K' = NS \cap Int(K')$, so that if $A \in \mathcal{A}_u$ then A is formed of the union and intersection of sets of the form $(NS \cap (Q+u) \cap Int(K')) + v$ $(v \in \pi'(\mathbb{Z}^N))$, and the subtraction of unions and intersections of sets of the form $(NS \cap (Q+u) \cap K') + v$. With this description and Lemma 2.5, A is equal to $NS \cap Int(\overline{A})$ (Euclidean closure and interior in $(Q+u) \cap F$), and from this our conclusion follows.

c/ Elements of $CC(F_u; \mathbb{Z})$ are finite sums of integer multiples of indicator functions of compact open sets. Such sets are finite unions of basic clopen sets from the collection in part a/. The isomorphism in part b/ completes the equation.

d/ Given the results of a/ and Lemma 9.6 it is sufficient to show that F_u has no isolated points. However, by the argument of part b/ and Lemma 2.5 we see that every clopen subset of F_u has $\widetilde{\mu}$ image with Euclidean interior (in $(Q+u) \cap F$) and so cannot be a single point. •

§10 A Cantor $\mathbb{Z}^d$ Dynamical System In this section we describe a $\mathbb{Z}^d$ dynamical system whose mapping torus is equal to MP_u. First, assuming $E \cap \mathbb{Z}^N = 0$, we find a suitable F to which to apply the construction of the previous section.

Definition 10.1 Suppose that G is a group intermediate to $\mathbb{Z}^N$ and R_u. The example in all our applications ahead is the group $H_{\mathcal{T}}$ found in Theorem 7.4, and so a projection method pattern $\mathcal{T}$ (and its data) defines G.

Fix a free generating set, $r_1, r_2, ..., r_N$, for G and suppose that the first $\dim \Delta$ of these generate the subgroup $G \cap E^\perp$ (this can be required by Lemma 2.9).

Let F be the real vector space spanned by $r_1, r_2, ..., r_n$, where $n = N - d$.

Note that, since $E \cap G = 0$ (by Lemma 2.9 and the assumption $E \cap \mathbb{Z}^N = 0$), F is complementary to E and, since $n \geq \dim \Delta$, F contains Δ.

Let $\pi' : \mathbb{R}^N \longrightarrow F$ be the idempotent map with kernel E and image F. Define $r'_j = \pi'(r_j)$ for $1 \leq j \leq N$ and note that $r'_j = r_j \in F$ for $1 \leq j \leq n$.

Note that by Theorem 7.1, any two such groups, G and G' say, differ only by some elements in $R_u \cap E^\perp$, a complemented subgroup of R_u. Thus we may fix their generating sets to differ only among those elements which generate $G \cap E^\perp$ or $G' \cap E^\perp$ respectively. With this convention therefore, the construction of F is independent of G and hence only on the data of $\mathcal{T}$. Likewise, $r_{\dim \Delta + 1}, ..., r_N$ depend only on the data of $\mathcal{T}$.

We assume this convention holds in all that follows.

Definition 10.2 Suppose that G_0 is the subgroup of G generated by $r_1, r_2, ..., r_n$, and that G_1 is the complementary subgroup generated by the other d generators (thus G_1

is independent of the choice of G). Both groups act on F_u and E as subactions of R_u (Definition 9.1).

Let $X_{\mathcal{T}} = F_u/G_0$, a space, depending on G, on which G_1 acts continuously.

Theorem 10.3 *Suppose that $\mathcal{T}$ is a projection method pattern with data (E, K, u) such that $E \cap \mathbb{Z}^N = 0$ and $\widetilde{\Pi}_u = \Pi_u$. Then $X_{\mathcal{T}}$ is a Cantor set on which G_1 acts minimally and there is a commutative square of G_1 equivariant maps*

$$\begin{array}{ccc} F_u & \xrightarrow{q} & X_{\mathcal{T}} \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu'} \\ F \cap (Q+u) & \longrightarrow & (F \cap (Q+u))/G_0. \end{array}$$

The set $\nu'^{-1}(v)$ is a singleton whenever $v \in (NS \cap F \cap (Q+u))/G_0$.

The space $(F \cap (Q+u))/G_0$ is homeomorphic to a finite union of tori each of dimension $(N - d - \dim \Delta)$. Indeed, this space can be considered as a topological group, in which case it is the product of a subgroup of $\widetilde{\Delta}/\Delta$ with the $(N - d - \dim \Delta)$-torus. The action of G_1 on this space is by rotation and is minimal.

Proof Assuming we have proved the fact that $X_{\mathcal{T}}$ is compact then the commuting square and its properties follow quickly. Therefore we look at $X_{\mathcal{T}}$.

Since G_0 acts isometrically on F_u with uniformly discrete orbits (Theorem 6.8), q is open and locally a homeomorphism and so $X_{\mathcal{T}}$ inherits the metrisability of F_u, a base of clopen sets and the lack of isolated points (see Theorem 9.4 d/).

Now, let $Y_o = \{\sum_{1 \le j \le n} \lambda_j r'_j \mid 0 \le \lambda_j < 1\} \cap (Q+u)$, a subset of $F \cap (Q+u)$. Choose $J \subset \mathbb{Z}^N$ finite but large enough that $Y_1 = \cup_{v \in J}((K' \cap (Q+u)) + \pi'(v))$ contains $\overline{Y}_o$ (Euclidean closure). In particular $q(\cup_{v \in J}(\overline{K} + \pi'(v))) = X_{\mathcal{T}}$, the image of a compact set (Lemma 9.6) under a continuous map. So $X_{\mathcal{T}}$ is also compact.

Therefore, we have checked all conditions that show $X_{\mathcal{T}}$ is a Cantor set.

Minimality follows from the minimality of the G action on F_u which in turn follows from the minimality of the $\mathbb{Z}^N + E$ action on $\widetilde{\Pi}_u$, proved analogously to (3.9).

The structure of the rotational factor system follows quickly from the first part of this lemma, the structure of $F \cap (Q+u)$, and Lemma 9.2. •

Now we describe the quotient definition of $X_{\mathcal{T}}$ as a fundamental domain.

Definition 10.4 From the details of 10.3 we constuct a clopen fundamental domain for the action of G_0 on F_u. We let $G_0^+ = \{\sum_{1 \le j \le n} \alpha_j r_j \mid \alpha_j \in \mathbb{N}\}$ and set $Y^+ = \cup_{r \in G_0^+}(Y_1 + r)$ and define $Y = Y^+ \setminus \cup_{r \in G_0^+, r \ne 0}(Y^+ + r)$.

Define $Y_{\mathcal{T}} = \overline{\nu^{-1}(Y \cap NS)}$ (closure in the $\overline{D}'$ metric), a subset of F_u.

The following is immediate from this construction, using Lemma 9.6 and the equivariance of ν and ν' in Lemma 9.3 with respect to the R_u action.

Lemma 10.5 *With data (E, K, u), $E \cap \mathbb{Z}^N = 0$ and $\widetilde{\Pi}_u = \Pi_u$, and the definitions above, Y is a fundamental domain for the translation action by G_0 on $F \cap (Q+u)$. Moreover, $Y_{\mathcal{T}}$ is a compact open subset of F_u, and a fundamental domain for the action by G_0 on F_u.*

There is a natural homeomorphism $X_{\mathcal{T}} \leftrightarrow Y_{\mathcal{T}}$ which is G_1 equivariant. •

Definition 10.6 Define $C(X_{\mathcal{T}}; \mathbb{Z})$ and $C(F_u; \mathbb{Z})$ to be the rings of continuous integer valued functions defined on the respective spaces without restriction on support, uniformity or magnitude. As $\mathbb{Z}[G_0]$ modules, the first is trivial and the second is defined as usual using the subaction of the R_u action on F_u. Both are non-trivial $\mathbb{Z}[G_1]$ modules.

The following combines Lemmas 10.3, 10.5 and Proposition 9.4 and will be of much importance in **[FHK1]**.

Corollary 10.7 *With the data of Lemma 10.5,*

$$CC(F_u; \mathbb{Z}) \equiv C(X_{\mathcal{T}}; \mathbb{Z}) \otimes \mathbb{Z}[G_0]$$

and

$$C(F_u; \mathbb{Z}) \equiv Hom_{\mathbb{Z}}(\mathbb{Z}[G_0], C(X_{\mathcal{T}}; \mathbb{Z}))$$

as $\mathbb{Z}[G_0]$ modules. •

Definition 10.8 Let E' be the real span of $r_{n+1}, ..., r_N$ selected in Definition 10.1. This space contains G_1 as a subgroup.

Recall the definition of dynamical mapping torus **[PT]** which for the G_1 action on $X_{\mathcal{T}}$ is

$$MT(X_{\mathcal{T}}, G_1) = (X_{\mathcal{T}} \times E')/\langle (gx, v) \sim (x, v - g) \mid g \in G_1 \rangle.$$

Proposition 10.9 *Suppose that $\mathcal{T}$ is a projection method pattern with data (E, K, u) such that $E \cap \mathbb{Z}^N = 0$ and $\widetilde{\Pi}_u = \Pi_u$. With the definitions above, E' is a d-dimensional subspace of $\mathbb{R}^N$ complementary to F and $E^\perp$.*

Also $MT(X_{\mathcal{T}}, G_1) \equiv \widetilde{\Pi}_u / G$.

Proof The transformation $E' \longrightarrow E$ defined by $r_j \mapsto \pi(r_j)$, $n+1 \leq j \leq N$ is bijective since G_1 is complementary to G_0 and hence to the subset $G \cap E^\perp$ of G_0. From this we deduce the complementarity immediately.

From Lemma 10.3 we see that G_0 acts naturally on $MT(F_u, G_1)$ and that $MT(X_{\mathcal{T}}, G_1) \equiv MT(F_u, G_1)/G_0$.

To form $MT(F_u, G_1)$ we take $F_u \times E'$ and quotient by the relation $(ga, v) \equiv (a, v - g)$, $g \in G_1, a \in F_u, v \in \mathbb{R}^d$. Applying the inverse of the map of the first paragraph, we can re-express the mapping torus as $F_u \times E$ quotiented by the relation $(ga, w + \pi(g)) \sim (a, w)$, $g \in G_1, a \in F_u, w \in E$.

However, the action of G_1 on the F_u is that induced by translation on F_u^o by elements $\pi'(g) \mid g \in G_1$. So, working first on the space F_u^o, we have the equations

$$MT(F_u^o, G_1) = (F_u^o \times E)/\langle (a + \pi'(g), w + \pi(g)) \sim (a, w) \mid g \in G_1 \rangle$$

$$= NS/\langle v = v + g \mid g \in G_1 \rangle = NS/G_1$$

(recall that NS is G_1 invariant by Corollary 6.6). Then, by completing, we deduce the equation $MT(F_u, G_1) = \widetilde{\Pi}_u/G_1$ directly. A further quotient by G_0 completes the construction. •

Corollary 10.10 *Suppose that $\mathcal{T}$ is a projection method pattern with data (E, K, u) such that $E \cap \mathbb{Z}^N = 0$ and $\widetilde{\Pi}_u = \Pi_u$. Then $(X_{\mathcal{T}}, G_1)$ is a minimal Cantor $\mathbb{Z}^d$ dynamical system, whose mapping torus $MT(X_{\mathcal{T}}, G_1)$ is homeomorphic to $M\mathcal{T}$. The pattern dynamical system, $(M\mathcal{T}, E)$ is equal to the canonical $\mathbb{R}^d$ action on the mapping torus $(MT(X_{\mathcal{T}}, G_1), \mathbb{R}^d)$ up to a constant automorphic time change.*

Proof Choose $G = H_{\mathcal{T}}$ from Theorem 7.4 which gives $M\mathcal{T} \equiv \widetilde{\Pi}_u/G$. From this, all but the time change information follows quickly from 10.9 and 10.3, noting that $G_1 \equiv \mathbb{Z}^d$.

To compare the two $\mathbb{R}^d$ actions, we apply the constant time change which takes the canonical $\mathbb{R}^d (\equiv E')$ action on $MT(X_{\mathcal{T}}, \mathbb{Z}^d)$ to the canonical $\mathbb{R}^d (\equiv E)$ action on $M\mathcal{T}$ by the isomorphism $\pi|_{E'}: E' \longrightarrow E$, mapping generators of the G_1 action $r_j \mapsto \pi(r_j)$ for $n + 1 \leq j \leq N$. •

Examples 10.11 The dynamical system of 10.10 for the octagonal tiling is a $\mathbb{Z}^2$ action on a Cantor set, an almost 1-1 extension of a $\mathbb{Z}^2$ action by rotation on $\mathbb{T}^2$ (see [**BCL**]). For the Penrose tiling, it is also a Cantor almost 1-1 extension of a $\mathbb{Z}^2$ action by rotation on $\mathbb{T}^2$ (see [**R1**]), where we must check carefully that the torus factor has only one component (the only alternative of 5 components is excluded *ad hoc*).

The correspondence in 10.9 and 10.10 respects the structures found in Theorem 7.4 and so we deduce an analogue.

Corollary 10.12 *Suppose that we have data (E, K, u) such that $E \cap \mathbb{Z}^N = 0$ and $\widetilde{\Pi}_u = \Pi_u$. Then we can construct two Cantor dynamical systems, (X_u, G_1) and $(\widetilde{X}_u, G_1)$, the latter a* *finite isometric extension of the former, together with a compact abelian group, M, which is a finite union of $(N-d-\dim\Delta)$-dimensional tori (independent of u) on which G_1 acts minimally by rotation, and a finite subgroup, Z_u, of M.*

These have the property that, if $\mathcal{T}$ is a projection method pattern with data (E, K, u), then there is a finite subgroup $Z_{\mathcal{T}}$ of Z_u and a commuting diagram of G_1-equivariant surjections

$$\begin{array}{ccccc} \widetilde{X}_u & \longrightarrow & X_{\mathcal{T}} & \longrightarrow & X_u \\ \downarrow & & \downarrow & & \downarrow \\ M & \longrightarrow & M/Z_{\mathcal{T}} & \longrightarrow & M/Z_u \end{array}$$

where the top row consists of finite isometric extensions, the bottom row of group quotients and the vertical maps are almost 1-1.

Taking the mapping torus of this diagram produces the diagram of Theorem 7.4.

Proof Set $X_u = X_{P_u}$ and $\widetilde{X}_u = X_{\widetilde{P}_u}$ as in Definition 10.2. Write G_u for the group G_0 defined in 10.2 with respect to the choice $G = \mathbb{Z}^N$. Write $\widetilde{G}_u$ for the G_0 produced from the choice $G = R_u$ and write $G_{\mathcal{T}}$ for the G_0 produced from the choice $G = H_{\mathcal{T}}$ (7.4). By the remark after 10.1, we know that $G_u < G_{\mathcal{T}} < \widetilde{G}_u$. Using the notation of sections 7, 9 and 10, set $M = ((Q+u)\cap F)/G_u$, $Z_u = \widetilde{G}_u/G_u \equiv R_u/\mathbb{Z}^N$ and $Z_{\mathcal{T}} = G_{\mathcal{T}}/G_u \equiv H_{\mathcal{T}}/\mathbb{Z}^N$. Note that G_u is independent of the choice of non-singular u and hence so is M, to which we attach no subscript therefore. The description of the systems, X_u and $\widetilde{X}_u$, using Theorem 10.3 gives the result immediately. •

§11 Groupoids of Projection Method patterns We develop now the connections between the pattern dynamical systems described before and the pattern groupoid. As with the mapping torus, a pattern groupoid, which we write $\mathcal{GT}^*$, can be defined abstractly for any pattern, $\mathcal{T}$, of Euclidean space and we refer to [**K1**] [**K3**] for the most general definitions. We give a special form for projection method patterns below (11.12).

The (reduced) C^*-algebra, $C^*(\mathcal{GT}^*)$, of this groupoid is a non-commutative version of the mapping torus which is regarded as a more precise detector of physical properties of the quasicrystal. The discrete Schrödinger operators for the quasicrystal are naturally members of this algebra.

The purpose of this section is to compare the non-commutative structure, *i.e.* the groupoid, of a pattern with the dynamical systems constructed before. In this regard, we cover similar ground to the work of Bellissard *etal.* [**BCL**] but, as noted in the introduction, applied to a groupoid sometimes different.

First we develop some general results about topological groupoids, appealing to the definitions in [**Ren**].

Definition 11.1 We write the unit space of a groupoid $\mathcal{G}$ as $\mathcal{G}^o$, and write the range and source maps, $r, s: \mathcal{G} \longrightarrow \mathcal{G}^o$ respectively. Both these maps are continuous and, due the existence of a Haar System in all our examples, we note that they are open maps as well.

Recall the *reduction* of a groupoid. Given a groupoid $\mathcal{G}$ with unit space, $\mathcal{G}^o$, and a subset, L, of $\mathcal{G}^o$, define the reduction of $\mathcal{G}$ to L as the subgroupoid ${}_L\mathcal{G}_L = \{g \in \mathcal{G} \mid r(g), s(g) \in L\}$ of $\mathcal{G}$, with unit space, L.

If L is closed then ${}_L\mathcal{G}_L$ is a closed subgroupoid of $\mathcal{G}$.

We also define $\mathcal{G}_L = \{g \in \mathcal{G} \mid s(g) \in L\}$ and note the maps $\rho: \mathcal{G}_L \longrightarrow \mathcal{G}^o$ and $\sigma: \mathcal{G}_L \longrightarrow L$ defined by r and s respectively.

We say $L \subset \mathcal{G}^o$ is *range-open* if, for all open $U \subset \mathcal{G}$, we have $r(\{x \in U : s(x) \in L\})$ open in $\mathcal{G}^o$.

Suppose a topological abelian group, H, acts by homeomorphisms on a topological space X, then we define a groupoid called the *transformation groupoid*, $\mathcal{G}(X, H)$, as the topological direct product, $X \times H$, with multiplication $(x, g)(y, h) = (x, g + h)$ whenever $y = gx$, and undefined otherwise. The unit space is $X \times \{0\}$.

This last construction is sometimes called the transformation group [**Ren**] or even the transformation group groupoid, but we prefer the usage to be found in [**Pa**].

We note that if H is locally compact, then naturally $C^*(\mathcal{G}(X, H)) = C_o(X) \rtimes H$, the crossed product [**Ren**].

Lemma 11.2 *Suppose that H is an abelian metric topological group acting homeomorphically on X. Let $\mathcal{G} = \mathcal{G}(X, H)$ be the transformation groupoid and suppose that L is a closed subset of $X \equiv \mathcal{G}^o$.*

a/ If H is discrete and countable, then $\mathcal{G}^o$ is a clopen subset of $\mathcal{G}$, and L is range-open if and only if it is clopen in X.

b/ If there is an $\epsilon > 0$ such that for all neighbourhoods, $B \subset B(0, \epsilon)$, of 0 in H and all A open in L, we have BA open in X, then L is range-open.

Proof Only part b/ presents complications. Suppose that U is open in $X \times H$. We want to show that $r((L \times H) \cap U)$ is open. Pick $x = r(y, h) \in r(L \times H \cap U)$ and let $C \times (B + h)$ be a neighbourhood of (y, h) inside U, with B sufficiently small. Then $A = s(C) \cap L$ is open in L and $x \in (B + h)A = h(BA)$ an open subset of X by hypothesis. However, $(B + h)A \subset r((L \times H) \cap U)$ by construction, and so we have found an open neighbourhood of x in $r((L \times H) \cap U)$ as required. •

We continue to use the constructions from [**MRW**] [**Rie**] without comment. In particular, we do not repeat the definition of (strong Morita) *equivalence* of groupoids or of C^* algebras, which is quite complicated. For separable C^*-algebras strong Morita equivalence implies stable equivalence and equates the ordered K-theory (without attention to the scale). All our examples are separable.

Lemma 11.3 *Suppose that $\mathcal{G}$ is a locally compact groupoid and that $L \subset \mathcal{G}^o$ is a closed, range-open subset which intersects every orbit of $\mathcal{G}$. Then ${}_L\mathcal{G}_L$ is equivalent to $\mathcal{G}$ (in the sense of* [**MRW**]*) and the two C^* algebras, $C^*({}_L\mathcal{G}_L)$ and $C^*(\mathcal{G})$ are strong Morita equivalent.*

Proof It is sufficient to show that $\mathcal{G}_L \xrightarrow{\rho} \mathcal{G}^o$ is a left $(\mathcal{G} \xrightarrow{r,s} \mathcal{G}^o)$-module whose $\mathcal{G} \xrightarrow{r,s} \mathcal{G}^o$ action is free and proper, and that $\mathcal{G}_L \xrightarrow{\sigma} L$ is a right $({}_L\mathcal{G}_L \xrightarrow{r,s} L)$-module whose ${}_L\mathcal{G}_L \xrightarrow{r,s} L$ action is free and proper. In short, ${}_L\mathcal{G}_L$ is an abstract transversal of $\mathcal{G}$ and $\mathcal{G}_L$ a $(\mathcal{G}, {}_L\mathcal{G}_L)$-equivalence bimodule from which we can construct the $(C^*(\mathcal{G}), C^*({}_L\mathcal{G}_L))$-bimodule which shows strong Morita equivalence of the two algebras directly, *c.f.* [**MRW**] Thm 2.8.

The definition of these actions is canonical and the freedom and properness of the actions is automatic from the fact that L intersects every orbit and from the properness and openness of the maps r, s. Indeed all the conditions follow quickly from these considerations except for the fact that $\mathcal{G}_L \xrightarrow{\rho} \mathcal{G}^o$ is a left $(\mathcal{G} \xrightarrow{r,s} \mathcal{G}^o)$-module; and the only trouble here is in showing that ρ is an open map. However, this is precisely the problem that range-openness is defined to solve. •

Together with Lemma 11.2 above, this result gives a convenient corollary which unifies the r-discrete and non-r-discrete cases treated seperately in [**AP**].

Corollary 11.4 *Suppose that (X, H) and $L \subset X$ obey either of the conditions of Lemma 11.2, then, writing $\mathcal{G} = \mathcal{G}(X, H)$, $C^*({}_L\mathcal{G}_L)$ and $C^*(\mathcal{G})$ are strong Morita equivalent.* •

Before passing to more special examples, we remark that there is no obstruction to the generalisation of results 11.2, 11.3 and 11.4 to the case of non-abelian locally compact group actions, noting only that, for notational consistency with the definition of transformation groupoid, the group action on a space should then be written on the right.

Now we define a selection of groupoids associated with projection method patterns, all of them transformation groupoids.

Definition 11.5 Now, given a projection method pattern, $\mathcal{T}$, with data (E, K, u), fix $G = H_{\mathcal{T}}$ as the group obtained from $\mathcal{T}$ Theorem 7.4, so that $M\mathcal{T} = \widetilde{\Pi}_u / G$.

We define in turn: $\mathcal{G}X_{\mathcal{T}} = \mathcal{G}(X_{\mathcal{T}}, G_1)$, from the G_1 action on $X_{\mathcal{T}}$, and $\mathcal{G}F_{\mathcal{T}} = \mathcal{G}(F_u, G)$, using the action of G on F_u, both defined in 8.1.

Also define $\mathcal{G}\widetilde{\Pi}_{\mathcal{T}} = \mathcal{G}(\widetilde{\Pi}_u, E+G)$.

All but the last of these groupoids are r-discrete (see [**Ren**]).

Lemma 11.6 *Suppose that $\mathcal{T}$ is a projection method pattern with data (E, K, u) such that $E \cap \mathbb{Z}^N = 0$ and $\widetilde{\Pi}_u = \Pi_u$. The groupoids $\mathcal{G}F_{\mathcal{T}}$ and $\mathcal{G}X_{\mathcal{T}}$ are each a reduction of $\mathcal{G}\widetilde{\Pi}_{\mathcal{T}}$ to the closed range-open sets, F_u and $Y_{\mathcal{T}}$.*

Proof It is clear that $\mathcal{G}F_{\mathcal{T}}$ is a reduction of $\mathcal{G}\widetilde{\Pi}_{\mathcal{T}}$ to F_u. To prove that this a range-open set using Lemma 11.2, we take an open subset of F_u and examine the action of small elements of $E+G$ on it. Only the E action enters our consideration and then it is clear from 9.2 that if B is an open subset of E and A is an open subset of F_u, then as topological spaces, $BA \equiv A \times B$, which is clearly open in $\widetilde{\Pi}_u$.

Recall the homeomorphism $X_{\mathcal{T}} \leftrightarrow Y_{\mathcal{T}}$ found in Lemma 10.5 which equates $X_{\mathcal{T}}$ with a fundamental domain of the G_0 action on F_u. This homeomorphism is G_1-equivariant if we equate the G_1 action on $Y_{\mathcal{T}}$ with the induced action of G/G_0 on $F_u/G_0 = X_{\mathcal{T}} \equiv Y_{\mathcal{T}}$. But this is precisely the correspondence needed to equate $\mathcal{G}(X_{\mathcal{T}}, G_1)$ with the reduction of $\mathcal{G}F_{\mathcal{T}}$ to $Y_{\mathcal{T}}$ considered as a subset of the unit space of $\mathcal{G}F_{\mathcal{T}}$. Thus $\mathcal{G}X_{\mathcal{T}}$ is the reduction of $\mathcal{G}\widetilde{\Pi}_{\mathcal{T}}$ to $Y_{\mathcal{T}}$, and since $Y_{\mathcal{T}}$ is clopen in F_u the same argument as above shows that $Y_{\mathcal{T}}$ is closed and range-open in $\widetilde{\Pi}_u$. •

Now we define a groupoid connected more directly with the pattern, $\mathcal{T}$.

Definition 11.7 Recall the two maps $M\widetilde{P}_u \longrightarrow M\mathcal{T} \longrightarrow^* MP_u$ whose composition is π_*. Without confusion we name the second (starred) map π_* as well.

We also define a map $\eta_{\mathcal{T}}$ which is the composite $\widetilde{\Pi}_u \xrightarrow{\widetilde{\eta}} M\widetilde{P}_u \longrightarrow M\mathcal{T}$.

Note that $\eta(x) = \pi_*(\eta_{\mathcal{T}}(x))$ for all $x \in \widetilde{\Pi}_u$ and that, being a composition of open maps (3.9), $\eta_{\mathcal{T}}$ is an open map.

Define the *hull* of $\mathcal{T}$ as $\Omega_{\mathcal{T}} = \{S \in M\mathcal{T} \mid 0 \in \pi_*(S)\}$.

The *pattern groupoid*, $\mathcal{G}\mathcal{T}$, is the space $\{(S, v) \in \Omega_{\mathcal{T}} \times E \mid v \in \pi_*(S)\}$ inheriting the subspace topology of $\Omega_{\mathcal{T}} \times E$. The restricted multiplication operation is $(S', v')(S, v) = (S', v+v')$, if $S = v'S'$, undefined otherwise. The unit space is $\mathcal{G}\mathcal{T}^o = \{(S, 0) \mid S \in \Omega_{\mathcal{T}}\}$, homeomorphic to $\Omega_{\mathcal{T}}$.

Also define $E_u^{\perp} = \widetilde{\mu}^{-1}(E^{\perp})$, a space which is naturally homeomorphic to F_u; a correspondence made by extending the application of $\pi^{\perp}$, inverted by the extension of π'.

Lemma 11.8 *Suppose that $\mathcal{T}$ is a projection method pattern with data (E, K, u) such that $E \cap \mathbb{Z}^N = 0$ and $\widetilde{\Pi}_u = \Pi_u$. The groupoid $\mathcal{G}\mathcal{T}$ is isomorphic to a reduction of $\mathcal{G}\widetilde{\Pi}_{\mathcal{T}}$ to a closed range-open set.*

Proof Let L be a compact open subset of $E_u^\perp$ so that $\eta_{\mathcal{T}}(L) = \Omega_{\mathcal{T}}$ and $\eta_{\mathcal{T}}$ is 1-1 on L. This can be constructed as follows. Define $L_o = \overline{NS \cap K \cap (Q+u)}$ where the closure is taken with respect to the $\overline{D}'$ metric - a clopen subset of $E_u^\perp$ by 9.6. Let $L = L_o \setminus \cup\{gL_o \mid g \in G \cap E^\perp, g \neq 0\}$ (using the G action on $\widetilde{\Pi}_u$).

We claim that the reduction of $\mathcal{G}\widetilde{\Pi}_{\mathcal{T}}$ to L is isomorphic to the pattern groupoid defined above.

Suppose that $(x; g, v) \in \mathcal{G}\widetilde{\Pi}_{\mathcal{T}}$ and $x \in L$ and $(g+v)x \in L$, then $0 \in \eta(x) = \pi_*(\eta_{\mathcal{T}}(x))$ and $0 \in \eta((g+v)x)$. But note that the action by $v \in E$ on $x \in NS$ is $vx = x - v$ and so $\eta((g+v)x) = \eta(gx) - v = \pi_*(\eta_{\mathcal{T}}(gx)) - v = \pi_*(\eta_{\mathcal{T}}(x)) - v$. Thus $0, v \in \pi_*(\eta_{\mathcal{T}}(x))$ and the map $\psi\colon (x; g, v) \mapsto (\eta_{\mathcal{T}}(x), v)$ is well defined ${}_L\mathcal{G}\widetilde{\Pi}_{\mathcal{T}L} \longrightarrow \mathcal{G}\mathcal{T}$. The E and G equivariance of the maps used to define ψ show that the groupoid structure is preserved.

Conversely, if $0, -v \in \pi_*(\eta_{\mathcal{T}}(x))$, then there are, by construction of L, $g, g' \in G$ such that $gx, (g'+v)x \in L$. Thus $(gx; g'-g, v) \in {}_L\mathcal{G}\widetilde{\Pi}_{\mathcal{T}L}$ showing that ψ is onto. Also, the g, g' are unique by the construction of L above, and so ψ is 1-1. The continuity of ψ and its inverse is immediate, so we have a topological groupoid isomorphism, as required.

Thus we have shown that $\mathcal{G}\mathcal{T}$ is isomorphic to a reduction of $\mathcal{G}\widetilde{\Pi}_{\mathcal{T}}$ to the set L which is clearly closed.

Also, L is a subset of $E_u^\perp$, transverse to E, so that the same argument as 11.6 shows that L is range open.

It remains to show that L hits every orbit of $\mathcal{G}\widetilde{\Pi}_{\mathcal{T}}$ and for this it is sufficient to show that for any $x \in \widetilde{\Pi}_u$, $Gx \cap (L \times E) \neq 0$ (where we exploit the equivalence: $\widetilde{\Pi}_u \equiv E_u^\perp \times E$ (see 9.2). But this is immediate from the fact that $L \times E$ is a clopen subset of $\widetilde{\Pi}_u$ (9.2), and by minimality of the $G + E$ action on $\widetilde{\Pi}_u$ (as in 3.9). •

Combining the Lemmas above, we obtain the following.

Theorem 11.9 *Suppose that $\mathcal{T}$ is a projection method pattern with data (E, K, u) such that $E \cap \mathbb{Z}^N = 0$ and $\widetilde{\Pi}_u = \Pi_u$. The C^* algebras $C^*(\mathcal{G}\mathcal{T})$, $C_o(F_u) \rtimes G$ and $C(X_{\mathcal{T}}) \rtimes G_1$ are strong Morita equivalent and thus their ordered K-theory (without attention to scale) is identical.* •

Remark 11.10 We can compare the construction above with the "rope" dynamical system constructed by the third author [**K2**] exploiting the generalised grid method introduced by de Bruijn [**dB1**]. The rope construction actually shows that, in a wide class of tilings including the canonical projection method examples, there is a Cantor minimal system $(X, \mathbb{Z}^d)$ such that $\mathcal{G}(X, \mathbb{Z}^d)$ is a reduction of $\mathcal{G}\mathcal{T}$. By comparing the details of the proof above with [**K2**] it is possible to show directly that, in the case of non-degenerate canonical tilings, the rope dynamical system is conjugate to $(X_{\mathcal{T}}, G_1)$.

We note that the construction of Lemma 11.8 depends only on the data (E, K, u) and on G and from this we deduce the following.

Corollary 11.11 *We have data (E, K, u), such that $E \cap \mathbb{Z}^N = 0$ and $\widetilde{\Pi}_u = \Pi_u$. Then, among projection method patterns, $\mathcal{T}$, with this data, the dynamical system $(M\mathcal{T}, E)$ determines $\mathcal{GT}$ up to groupoid isomorphism.*

Thus among projection method patterns with fixed data, the dynamical invariants are at least as strong as the non-commutative invariants. •

Finally we reconnect the work of this section with the original construction of the tiling groupoid due to Kellendonk [**K1**].

Definition 11.12 Recall the notation $A[r] = (A \cap B(r)) \cup \partial B(r)$ *etc.* defined in 3.1 and 4.2, for $r \geq 0$ and $A \subset \mathbb{R}^N$ or E. Given two closed sets, A, A' define the distance $D_o(A, A') = \inf\{1/(r+1) \mid r > 0, A[r] = A'[r]\}$.

As a metric this can be used to compare point patterns in E or $\mathbb{R}^N$ (as in 3.1 and 4.2), or decorated tilings in E as described in 4.1.

We consider only tilings $\mathcal{T}$ which are *translationally finite*, i.e. each tile of $\mathcal{T}$ is one of a finite number of possibilities up to translation (see [**Sol**]).

Given such a tiling, $\mathcal{T}$, of E, the construction of the hull in [**K1**] starts by placing a single puncture generically in the interior of each tile according to local information (usually just the shape, decoration and orientation of the tile itself). So we form the collection of punctures, $\tau(\mathcal{T})$ of $\mathcal{T}$, a discrete subset of points in E.

We consider the set $\Omega^o_{\mathcal{T}} = \{\mathcal{T} + x \mid 0 \in \tau(\mathcal{T} + x) = \tau(\mathcal{T}) + x\}$, and define a modified hull, which we write $\Omega^*_{\mathcal{T}}$ in this section, as the D_o completion of this selected set of shifts of $\mathcal{T}$.

From this hull, we define the groupoid, $\mathcal{GT}^*$ exactly as for $\mathcal{GT}$: $\mathcal{GT}^* = \{(S, v) \in \Omega^*_{\mathcal{T}} \times E \mid v \in \tau(S)\}$ with the analogous rule for partial multiplication.

The assumption of local information dictates more precisely that the map τ is continuous, E-equivariant, and 1-1 from $\Omega^*_{\mathcal{T}}$ with D_o metric to the space of Delone subsets of E also with D_o metric.

Remark 11.13 Although phrased in terms of tilings, this definition can in fact be applied to patterns as well, where the idea of puncture becomes now the association of a point with each unit of the pattern (4.1). In this case the condition of translational finiteness is equated with the condition that $\tau(\mathcal{T})$ is Meyer (see [**La**]), and this is sufficient to prove the analogues of all the Lemmas below. However, we continue to use the language of tilings and, since every projection method pattern is pointed conjugate to a decorated tiling with translational finiteness (decorating the Voronoi tiles for P_u for example (7.5)), we lose no generality in doing so.

We note that when a projection method pattern $\mathcal{T}$ is in fact a tiling, the two definitions of hull (11.7 and above) given here seldom coincide nor do we obtain the same groupoids (but we note the important exception of the canonical tiling in 11.16). The remainder of this section shows that, never-the-less, the two groupoids, $\mathcal{GT}$ and $\mathcal{GT}^*$, are equivalent. We start by comparing D and D_o.

Lemma 11.14 *Suppose that $\mathcal{T}$ is a tiling as above, then $\Omega^o_{\mathcal{T}}$ is precompact with respect to D_o. Further D and D_o generate the same topology on $\Omega^o_{\mathcal{T}}$.*

Proof The precompactness of $\Omega^o_{\mathcal{T}}$ is proved in [**K1**].

For any two tilings, we have $D(\mathcal{T},\mathcal{T}') \le D_o(\mathcal{T},\mathcal{T}')$ by definition, and so the topology of D_o is always finer than that of D.

Conversely, as a consequence of the translational finiteness of $\mathcal{T}$ there is a number $\delta_o < 1$ such that if $0 < \epsilon < \delta_o$, then $\mathcal{T}+x, \mathcal{T}+x' \in \Omega^o_{\mathcal{T}}$ and $D(\mathcal{T}+x,\mathcal{T}+x') < \epsilon$ together imply that $\mathcal{T}+x$ and $\mathcal{T}+x'$ actually agree up to a large radius ($1/\epsilon - 1$ will do) and we conclude $D_o(\mathcal{T}+x,\mathcal{T}+x') < 2\epsilon$ as required. •

Consequently, $\Omega^*_{\mathcal{T}}$ is canonically a subspace of $M\mathcal{T}$ and we can consider its properties as such.

Lemma 11.15 *With respect to the E action on $M\mathcal{T}$, both $\Omega_{\mathcal{T}}$ and $\Omega^*_{\mathcal{T}}$ are range-open.*

Proof With the notation of the proof of Lemma 11.8, $\Omega_{\mathcal{T}} = \eta_{\mathcal{T}}(L)$, where L is a compact open subset of $E_u^{\perp}$. As in 11.8, L is range-open in $\widetilde{\Pi}_u$ and, since $\eta_{\mathcal{T}}$ is an open E-equivariant map, we deduce the same of $\eta_{\mathcal{T}}(L)$.

For $\Omega^*_{\mathcal{T}}$, we note that (as in 3.4 before) by the translational finiteness of $\mathcal{T}$, there is a number δ_o so that, if $x, x' \in E$, $\mathcal{T}' \in M\mathcal{T}$ and $0 < \|x - x'\| < \delta_o$, then $D(\mathcal{T}'-x, \mathcal{T}'-x') \ge \|x-x'\|/2$ and $D_o(\mathcal{T}'-x,\mathcal{T}'-x') = 1$. In particular, since $\Omega^*_{\mathcal{T}}$ is D_o-compact and hence a finite union of radius $1/2$ D_o-balls, we deduce that the map $\Omega^*_{\mathcal{T}} \times B(\delta_o/2) \longrightarrow M\mathcal{T}$, defined as $(\mathcal{T}',x) \mapsto \mathcal{T}'+x$, is locally injective and hence open. From here the range-openness of $\Omega^*_{\mathcal{T}}$ is immediate. •

Theorem 11.16 *If $\mathcal{T}$ is at once a tiling and a projection method pattern with data (E,K,u), such that $E \cap \mathbb{Z}^N = 0$ and $\widetilde{\Pi}_u = \Pi_u$, then the tiling groupoid $\mathcal{GT}^*$ as defined in* [**K1**] *(11.12) is equivalent to $\mathcal{GT}$ (11.7). Thus the respective C^*-algebras are strong Morita equivalent also.*

In the case of a (non-degenerate) canonical projection method tiling there is a puncturing procedure for which the two groupoids are in fact isomorphic.

Proof Recall the definition of transformation groupoid and the action of E on $M\mathcal{T}$ and consider $\mathcal{G}(M\mathcal{T}, E)$. Using Lemmas 11.3 and 11.15, it suffices to show that the groupoids $\mathcal{G}\mathcal{T}$ and $\mathcal{G}\mathcal{T}^*$ are each a reduction of $\mathcal{G}(M\mathcal{T}, E)$ to the sets $\Omega_{\mathcal{T}}$ and $\Omega^*_{\mathcal{T}}$ respectively. But this is immediate from their definition.

To treat the canonical case, we note that the point pattern may be translated by a small generic fixed vector to give a collection of punctures for the tiling. Thus the point pattern is pointed conjugate to the puncturing decoration, giving a pointed conjugacy between the tiling with point pattern and the punctured tiling. More importantly the first conjugacy is a geometric isometry preserving the scale of the patterns, and so the second conjugacy passes to an isomorphism of the respective groupoids. •

§**12 Summary of results** Here we present concisely the most useful conclusions of this paper. The numbers in brackets refer to points in this paper.

From data (E, K, u) (4.4), we define two discrete sets of points P_u and $\widetilde{P}_u$ (2.1) whose dynamical systems [**Ru**], $(M\widetilde{P}_u, E)$ (4.2) and (MP_u, E) (3.3), are related by an E-equivariant surjection: $M\widetilde{P}_u \xrightarrow{\pi_*} MP_u$ (4.3). A general *projection method pattern*, $\mathcal{T}$, with data (E, K, u) is a pattern in E (4.4) whose pattern dynamical system, $M\mathcal{T}$ (4.1), is the middle space of some factorisation of π_* into two E-equivariant maps $M\widetilde{P}_u \longrightarrow M\mathcal{T} \longrightarrow MP_u$.

Under certain sufficient conditions on the data (e.g. (5.1), specialised to (5.2, 5.3)) which are not restrictive (5.4), the fact that $\mathcal{T}$ is a projection method pattern with these data implies that there is a torus, $\mathbb{T}^m$, of dimension $m = N - \dim\Delta$ (2.10), on which E acts minimally by rotation, together with two finite subgroups $Z_{\mathcal{T}} \subset Z_u$ of $\mathbb{T}^m$ (with $Z_{\mathcal{T}} \equiv H_{\mathcal{T}}/\mathbb{Z}^N$ (7.4) and $Z_u \equiv R_u/\mathbb{Z}^N$ (6.5)), and a commutative diagram of E-equivariant maps (7.4)

$$\begin{array}{ccccc} M\widetilde{P}_u & \longrightarrow & M\mathcal{T} & \longrightarrow & MP_u \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{T}^m & \longrightarrow & \mathbb{T}^m/Z_{\mathcal{T}} & \longrightarrow & \mathbb{T}^m/Z_u \end{array}$$

where the top row is dictated by definition and the bottom row is the sequence of group quotients. The vertical maps are almost 1-1 and the top row maps are finite isometric extensions (2.15). As a corollary, every projection method pattern under these conditions is a finite decoration (4.5) of P_u and, in turn, can be decorated finitely to give a pattern pointed conjugate to $\widetilde{P}_u$. The data and the subgroup $Z_{\mathcal{T}}$ (or $H_{\mathcal{T}}$) determines $(M\mathcal{T}, E)$ up to topological conjugacy.

All the horizontal arrows are bijective (equivalently $R_u = \mathbb{Z}^N$) when $\Delta = 0$ (5.2, 8.2). The left-hand horizontal arrows become trivial in the case of the canonical projection method tiling of **[OKD]** with no degeneracy (5.3, 8.4).

An analogous construction exists (10.10, 10.12) of a $\mathbb{Z}^d$ action on a Cantor set, $X_{\mathcal{T}}$, whose mapping torus with $\mathbb{R}^d$ action is naturally conjugate, up to homomorphic time change, to the E action on $M\mathcal{T}$. This $(X_{\mathcal{T}}, \mathbb{Z}^d)$ is an almost 1-1 extension of a $\mathbb{Z}^d$ action by rotation on a finite union of $m-d$ dimensional tori.

At the level of C^*-algebras, the C^*-algebra of the pattern $\mathcal{T}$ is defined to be the C^*-algebra of the pattern groupoid, $C^*(\mathcal{G}\mathcal{T})$ (11.7). By showing that the transformation groupoids for $(X_{\mathcal{T}}, \mathbb{Z}^d)$ and $(M\mathcal{T}, E)$ are both equivalent **[MRW]** to $\mathcal{G}\mathcal{T}$, we deduce (11.9) the strong Morita equivalence of $C^*(\mathcal{G}\mathcal{T})$, $C(X_{\mathcal{T}}) \rtimes \mathbb{Z}^d$ and $C(M\mathcal{T}) \rtimes E$ and hence the equation of their ordered K-theories (without attention to the scale). The data and the subgroup $H_{\mathcal{T}}$ (7.4) determines $\mathcal{G}\mathcal{T}$ up to topological groupoid isomorphism.

When $\mathcal{T}$ happens to be a tiling, the tiling groupoid defined by **[K1]** is equivalent to $\mathcal{G}\mathcal{T}$ (11.16) and in canonical cases the two groupoids are isomorphic.

References

[AP] J.E.Anderson, I.F.Putnam. Topological invariants for substitution tilings and their associated C*-algebras. To appear, Ergodic Theory and Dynamical Systems.

[BKS] M. Baake, R. Klitzing and M. Schlottmann, Fractally shaped acceptance domains of quasiperiodic square-triangle tilings with dodecagonal symmetry, Physica A 191 (1992) 554-558.

[BJKS] M. Baake, D. Joseph, P. Kramer and M. Schlottmann, Root lattices and quasicrystals, J. Phys. A 23 (1990) L1037-L1041.

[BSJ] M.Baake, M.Schlottman and P.D.Jarvis. Quasiperiodic tilings with tenfold symmetry and equivalence with respect to local derivability. J.Phys.A 24, 4637-4654 (1991).

[B1] J.Bellissard. Gap labelling theorems for Schrödinger operators. from Number Theory and Physics. Luck, Moussa, Waldschmit, eds. Springer, 1992.

[B2] J.Bellissard. K-theory for C*algebras in solid state physics. Lect. Notes Phys. 257. Statistical Mechanics and Field Theory, Mathematical Aspects, 1986, pp. 99-256.

[BCL] J.Bellissard, E.Contensou, A.Legrand. K-théorie des quasicristaux, image par la trace, le cas du réseau octagonal. C.R.Acad.Sci. Paris, t.326, Série I, p.197-200, 1998.

[dB1] N.G.de Bruijn. Algebraic Theory of Penrose's nonperiodic tilings of the plane. Kon.Nederl.Akad.Wetensch. Proc. Ser.A 84 (Indagationes Math. 43) (1981), 38-66.

[dB2] N.G.de Bruijn. Quasicrystals and their Fourier transforms. Kon.Nederl.Akad.Wetensch.Proc. Ser.A 89 (Indagationes Math. 48) (1986), 123-152.

[C] A.Connes. Non-commutative Geometry. Academic Press, 1994.

[FH] A.H.Forrest, J.Hunton. Cohomology and K-theory of Commuting Homeomorphisms of the Cantor Set. To appear Ergodic Theory and Dynamical Systems.

[FHK1] A.H.Forrest, J.Hunton, J.Kellendonk. Projection Quasicrystals II: versus substitutions. In preparation.

[FHK2] A.H.Forrest, J.Hunton, J.Kellendonk. Projection Quasicrystals III: cohomology. In preparation.

[F] H.Furstenberg. The structure of distal flows. Amer.J.Math. 85 (1963), 477-515.

[GS] B.Grünbaum, G.Shephard. Tilings and Patterns, San Francisco. W.Freeman, 1987.

[HM] G.A.Hedlund, M.Morse. Symbolic Dynamics II. Sturmian trajectories. Am.J.Math.62 (1940), 1-42.

[H] A.Hof. Diffraction by Aperiodic Structures. in The Mathematics of Long Range Aperiodic Order, 239-268. Klewer 1997.

[KD] A.Katz, M.Duneau. Quasiperiodic patterns and icosahedral symmetry. Journal de Physique, Vol.47, 181-96. (1986).

[K1] J. Kellendonk. Non Commutative Geometry of Tilings and Gap Labelling. Rev. Math. Phys. 7 (1995) 1133-1180.

[K2] J. Kellendonk. The local structure of Tilings and their Integer group of coinvariants. Commun. Math. Phys. 187 (1997) 115-157.

[K3] J. Kellendonk. Topological Equivalence of Tilings. J.Math.Phys., Vol 38, No4, April 1997.

[KN] P. Kramer and R. Neri. On Periodic and Non-periodic Space Fillings of E^m Obtained by Projection. Acta Cryst. A 40 (1984) 580-7.

[La] J.C.Lagarias. Meyer's concept of quasicrystal and quasiregular sets. Comm.Math.Phys. 179, 365-376.

[Le] T.T.Q.Le. Local Rules for Quasiperiodic Tilings. in The Mathematics of Long Range Aperiodic Order, 331-366. Klewer 1997.

[M] A.Mackay. Crystallography and the Penrose pattern. Physics 114A (1982), 609-613.

[MRW] P.S.Muhly, J.N.Renault, D.P.Williams. Equivalence and isomorphism for groupoid C* algebras. J.Operator Theory 17 (1987) 3-22.

[OKD] C.Oguey, A.Katz, M.Duneau. A geometrical approach to quasiperiodic tilings. Commun. Math. Phys. 118 (1988) 99-118.

[PT] W.Parry, S.Tuncel. Classification Problems in Ergodic Theory. LMS Lecture Notes series. Vol 65. CUP 1982.

[Pa] A.Paterson. Groupoids, inverse semigroups and their operator algebras. Birkhäuser. to appear.

[Pe] R.Penrose. Pentaplexity. Mathematical Intelligencer. 2(1979), 32-37.

[Ra] C.Radin, M.Wolff. Space tilings and local isomorphisms. Geometricae Dedicata 42, 355-360.

[Rie] M.A.Rieffel Applications of strong morita equivalence to transformation group C*algebras. Proc.Symp. Pure Math 38 (1) (1982) 299-310.

[Ren] J.Renault. A Groupoid Approach to C*algebras. Lecture Notes in Mathematics 793. Springer-Verlag 1980.

[R1] E.A.Robinson Jr.. The dynamical theory of tilings and quasicrystallography. in Ergodic Theory of Zd actions (M.Pollicott and K.schmidt, eds.) 451-473, CUP Cambridge, 1996.

[R2] E.A.Robinson Jr.. The dynamical properties of Penrose tilings. Trans.Amer.Math.Soc. 348 (1994), 4447-4469.

[Ru] D.J.Rudolf. Markov tilings of Rn and representations of Rn actions. Contemporary Mathematics 94, (1989) 271-290.

[S] M.Senechal. Quasicrystals and Geometry. CUP 1995.

[ST] M.Senechal, J.Taylor. Quasicrystals: The view from Les Houches. from The Mathematical Intelligencer, Vol 12, No 2, 1990, pp.54-64.

[Soc] J.E.S.Socolar. Simple octagonal and dodecagonal quasicrystals. Phys.Rev. B, 39(15), 10519-10551.

[Sol] B.Solomyak. Spectrum of Dynamical Systems Arising from Delone Sets. To Appear in Fields Institute Communications.

[1] "The Physics of Quasicrystals", editors P.J. Steinhardt and S. Ostlund, World Scientific Publishing Co. 1987.

[2] Proceedings of the NATO Advanced Study Institute on "The Mathematics of Long-Range Aperiodic Order", editor R.V. Moody, Kluwer Academic Publishers 1997

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