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(Generalized) Conformal Quantum Mechanics of 0-Branes and Two-Dimensional Dilaton Gravity

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Abstract

We study the relation between the (generalized) conformal quantum mechanics of 0-branes and the two-dimensional dilaton gravity. The two-dimensional actions obtained from the supergravity effective actions for the (dilatonic) 0-branes through the compactification on a sphere are related to known two-dimensional dilaton gravity models. The $SL(2, \mathbf{R})$ symmetry of the (generalized) conformal quantum mechanics is realized within such two-dimensional models. The two-dimensional dilatonic gravity model derived from the non-dilatonic 0-brane action is related to the Liouville theory and therefore is conformal, whereas the two-dimensional model derived from the dilatonic 0-brane action does not have the conformal symmetry.

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1   Introduction

According to the AdS/CFT duality conjecture [1] and its generalization [2], the bulk (gravity) theory on the near-horizon manifold of the supergravity brane solution is equivalent to the boundary (field) theory, which is the worldvolume theory of the brane in the decoupling limit. This conjecture relates closed string theory and the open string theory in the appropriate limit. Namely, in taking the decoupling limit of the brane worldvolume theory, which is the open string theory on the brane, to obtain the gauge field theory (an extensive review on the gauge field theory as the decoupling limit of the brane worldvolume theory can be found in Ref. [3]), one ends up with the near horizon region [4, 5] of the brane supergravity solution, which is the low energy limit of the closed string theory. This AdS/CFT duality conjecture can be regarded as a concrete string theory realization of the holographic principle [6, 7, 8] and the previously conjectured equivalence [9, 4] between the bulk theory on the AdS space and the supersingleton field theory, which is the effective worldvolume theory of the brane solutions, on the boundary of the AdS space (for a review on this subject, see Ref. [10]).

The isometry symmetry of the near-horizon manifold manifests as a symmetry of the boundary field theory and the boundary field theory is conformal when the near-horizon geometry of the supergravity brane solution (in the string-frame) contains the AdS space [4, 5]. On the other hand, according to Refs. [11, 4, 5, 12], any p-brane supergravity solutions in the near-horizon limit take the AdS$\times S^n$ form in the so-called dual-frame. So, the isometry symmetry of the AdS space in the near-horizon region of the dilatonic brane solutions in the dual-frame is also expected to be present in the corresponding boundary field theories but the boundary theories are not genuinely conformal due to non-trivial dilaton field. Nevertheless, one can still define generalized conformal field theory [13, 14], where string coupling is now regarded as a part of background fields that transform under the symmetry.

In the 0-brane case, the corresponding boundary theory can be thought of as the (generalized) conformal quantum mechanics [15, 16, 17, 18, 19, 20, 21] of the probe 0-brane in the near-horizon background of the source 0-brane supergravity solution. The (generalized) conformal quantum mechanics of the 0-brane is shown [18, 20, 21] to have the $SL(2, \mathbf{R})$ symmetry, as expected from the fact that the near-horizon limit of the 0-brane solution (in the dual-frame) is of the form AdS$_2 \times S^n$, where the isometry of the AdS$_2$ part is $SO(1, 2) \cong SL(2, \mathbf{R})$. The $SL(2, \mathbf{R})$ symmetry of the probe 0-brane is conformal for the non-dilatonic 0-branes [18], and is not genuinely conformal but can be extended to the generalized conformal symmetry for the dilatonic 0-branes [20, 21].

Since the supergravity 0-brane solution (in the dual-frame) in the near-horizon region takes AdS$_2 \times S^n$ form, one can perform the Freund-Rubin compactification [22] of the

0-brane supergravity action on S^n. The resulting theory is the two-dimensional dilaton gravity with the (dilaton dependent) cosmological constant term. Therefore, one would expect the relevance of the (generalized) conformal quantum mechanics of 0-branes to the two-dimensional dilaton gravity. It is the purpose of this paper to elaborate on such relation. We observe that the two-dimensional dilaton gravity compactified from the 0-brane effective action has the $SL(2, \mathbf{R})$ affine symmetry, in accordance with the fact that the (generalized) conformal quantum mechanics of 0-brane has the $SL(2, \mathbf{R})$ symmetry. Also, consistently with the case of the (generalized) conformal quantum mechanics of 0-branes, the two-dimensional gravity model obtained from the non-dilatonic 0-brane action is conformal, whereas the one obtained from the dilatonic 0-brane action is not conformal.

The paper is organized as follows. In section 2, we summarize the general supergravity 0-brane solutions in D dimensions. In section 3, we summarize the (generalized) conformal quantum mechanics of 0-branes. In section 4, we compactify the supergravity actions for the 0-branes on S^{D-2} and relate the resulting two-dimensional actions to various two-dimensional dilaton gravity models for the purpose of studying the relevance of the (generalized) conformal quantum mechanics of 0-branes to the two-dimensional dilaton gravity theories.

2 Supergravity 0-Brane Solutions

In this section, we summarize the general supergravity 0-brane solution with an arbitrary dilaton coupling parameter a in arbitrary spacetime dimensions D. The supergravity action in the Einstein-frame for such dilatonic 0-brane is given by

$$S_E = \frac{1}{2\kappa_D^2} \int d^D x \sqrt{-G^E} \left[\mathcal{R}_{G^E} - \frac{4}{D-2} (\partial \phi)^2 - \frac{1}{2 \cdot 2!} e^{2a\phi} F_2^2 \right], \tag{1}$$

where κ_D is the D-dimensional Planck constant, ϕ is the D-dimensional dilaton, and $F_2 = dA_1$ is the field strength of the 1-form potential $A_1 = A_M dx^M$ ($M = 0, 1, ..., D-1$). The extreme 0-brane solution to the equations of motion of this action is given by

$$\begin{aligned}
ds_E^2 &= -H^{-\frac{4(D-3)}{(D-2)\Delta}} dt^2 + H^{\frac{4}{(D-2)\Delta}} (dr^2 + r^2 d\Omega_{D-2}^2), \\
e^\phi &= H^{\frac{(D-2)a}{2\Delta}}, \quad A_t = -H^{-1},
\end{aligned} \tag{2}$$

where $H = 1 + \left(\frac{\mu}{r} \right)^{D-3}$ and $\Delta = \frac{(D-2)a^2}{2} + \frac{2(D-3)}{D-2}$. The examples on the values of Δ and a for some interesting cases are

- D0-brane in $D = 10$: $(a, \Delta) = (\frac{3}{4}, 4)$

- d0-brane in $D = 6$: $(a, \Delta) = (\frac{1}{2}, 2)$

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- dilatonic black hole in $D = 4$: $(a,\Delta)=(\sqrt{3},4)$, $(1,2)$, $(\frac{1}{\sqrt{3}},\frac{4}{3})$, $(0,1)$

One can perform the Weyl rescaling transformation $G^E_{MN} = e^{-\frac{4}{D-2}\phi}G^s_{MN}$ to obtain the following "string-frame" effective action for the 0-brane in D dimensions:

$$S_s = \frac{1}{2\kappa_D^2}\int d^Dx\sqrt{-G^s}\left[e^{-2\phi}\left\{\mathcal{R}_{G^s} + 4(\partial\phi)^2\right\} - \frac{1}{2\cdot 2!}e^{2(a-\frac{D-4}{D-2})\phi}F_2^2\right]. \qquad (3)$$

In the special cases pointed out in the previous paragraph, i.e. the cases with the dilaton coupling $a = (D-4)/(D-2)$, this action takes the form of the string-frame effective actions of the string theory. The spacetime metric for the dilatonic 0-brane in this frame is given by

$$ds_s^2 = -H^{-\frac{4(D-3)-2a(D-2)}{(D-2)\Delta}}dt^2 + H^{\frac{4+2a(D-2)}{(D-2)\Delta}}(dr^2 + r^2d\Omega^2_{D-2}). \qquad (4)$$

Note, in the case of $D = 10$ and $(a,\Delta)=(\frac{3}{4},4)$, one recovers the following familiar string-frame D0-brane solution:

$$
\begin{aligned}
ds_s^2 &= -H^{-\frac{1}{2}}dt^2 + H^{\frac{1}{2}}(dr^2 + r^2d\Omega^2_8), \\
e^\phi &= H^{\frac{3}{4}}, \quad A_t = -H^{-1}; \qquad H = 1 + \left(\frac{\mu}{r}\right)^7.
\end{aligned}
\qquad (5)
$$

Also, in the case of $D = 6$ and $(a,\Delta)=(\frac{1}{2},2)$, the solution (4) reduces to the following string-frame solution for d0-brane in $D = 6$:

$$
\begin{aligned}
ds_s^2 &= -H^{-1}dt^2 + H(dr^2 + r^2d\Omega^2_4), \\
e^\phi &= H^{\frac{1}{2}}, \quad A_t = -H^{-1}; \qquad H = 1 + \left(\frac{\mu}{r}\right)^3.
\end{aligned}
\qquad (6)
$$

One can also think of the dilatonic 0-brane in D dimensions as being magnetically charged under the $(D-2)$-form field strength F_{D-2}. The corresponding Einstein-frame effective action is

$$\tilde{S}_E = \frac{1}{2\kappa_D^2}\int d^Dx\sqrt{-G^E}\left[\mathcal{R}_{G^E} - \frac{4}{D-2}(\partial\phi)^2 - \frac{1}{2\cdot(D-2)!}e^{-2a\phi}F_{D-2}^2\right], \qquad (7)$$

where this action is obtained from the action (1) through the electric-magnetic duality transformation. So, the dilatonic 0-brane solution (2) takes the following form in terms of the "dual" field parametrization:

$$
\begin{aligned}
ds_E^2 &= -H^{-\frac{4(D-3)}{(D-2)\Delta}}dt^2 + H^{\frac{4}{(D-2)\Delta}}(dr^2 + r^2d\Omega^2_{D-2}), \\
e^\phi &= H^{\frac{(D-2)a}{2\Delta}}, \quad F_{D-2} = \star(dH\wedge dt).
\end{aligned}
\qquad (8)
$$

3

The following effective action in the dual-frame [21], in which all the 0-brane solutions in the near-horizon region take the $\text{AdS}_s \times S^{D-2}$ form, is obtained through the Weyl rescaling transformation $G^E_{MN} = e^{-\frac{2a}{D-3}\phi}G^d_{MN}$:

$$\tilde{S}_d = \frac{1}{2\kappa_D^2}\int d^D x \sqrt{-G^d}e^{\delta\phi}\left[\mathcal{R}_{G^d} + \gamma(\partial\phi)^2 - \frac{1}{2\cdot(D-2)!}F_{D-2}^2\right], \tag{9}$$

where

$$\delta \equiv -\frac{D-2}{D-3}a, \qquad \gamma \equiv \frac{D-1}{D-2}\delta^2 - \frac{4}{D-2}. \tag{10}$$

In the dual-frame, the dilatonic 0-brane solution takes the following form:

$$ds_d^2 = -H^{\frac{2\Delta-4(D-3)}{(D-3)\Delta}}dt^2 + H^{\frac{2}{D-3}}(dr^2 + r^2d\Omega_{D-2}^2), \tag{11}$$

where the dilaton and the $(D-2)$-form field strength take the same form as in Eq. (8). In the near horizon region ($r \ll \mu$), which corresponds to the decoupling limit of the corresponding boundary worldvolume theory, the metric (11) takes the following $\text{AdS}_2 \times S^{D-2}$ form [21]:

$$ds_d^2 \approx -\left(\frac{\mu}{r}\right)^{2-\frac{4(D-3)}{\Delta}}dt^2 + \left(\frac{\mu}{r}\right)^2 dr^2 + \mu^2 d\Omega_{D-2}^2, \tag{12}$$

and when the 0-brane is regarded as being electrically charged under the 1-form potential, which is relevant for the (generalized) conformal quantum mechanics of the probe 0-brane, the metric takes the same form as the above and the dilaton and the non-zero component of the $U(1)$ gauge field are

$$e^\phi \approx \left(\frac{\mu}{r}\right)^{\frac{(D-2)(D-3)a}{2\Delta}}, \qquad A_t \approx -\left(\frac{r}{\mu}\right)^{D-3}. \tag{13}$$

One can bring this $\text{AdS}_2 \times S^{D-2}$ metric to the following standard form in the horospherical coordinates by introducing new radial coordinate $\bar{r} = \frac{\Delta}{2(D-3)-\Delta}\mu^{\frac{2\Delta-2(D-3)}{\Delta}}r^{\frac{2(D-3)-\Delta}{\Delta}}$:

$$ds_d^2 \approx -\left(\frac{\bar{r}}{\bar{\mu}}\right)^2 dt^2 + \left(\frac{\bar{\mu}}{\bar{r}}\right)^2 d\bar{r}^2 + \mu^2\Omega_{D-2}^2, \tag{14}$$

and when the 0-brane is regarded as being electrically charged under the 1-form potential the metric takes the same form as the above and the dilaton and the non-zero component of the $U(1)$ gauge field are

$$e^\phi \approx \left(\frac{\bar{\mu}}{\bar{r}}\right)^{\frac{(D-2)(D-3)a}{4(D-3)-2\Delta}}, \qquad A_t = -\left(\frac{\bar{r}}{\bar{\mu}}\right)^{\frac{(D-3)\Delta}{2(D-3)-\Delta}}, \tag{15}$$

where $\bar{\mu} \equiv \frac{\Delta\mu}{2(D-3)-\Delta}$.

4

3 (Generalized) Conformal Quantum Mechanics of 0-Branes

The boundary theory counterpart to the bulk gravity theory in the near-horizon background of (dilatonic) 0-branes can be thought of as the (generalized) conformal quantum mechanics. The (generalized) conformal mechanics is described by the dynamics of the probe (dilatonic) 0-brane moving in the near-horizon field background (12) or (14) of the stack of the large number of source (dilatonic) 0-branes. The action for the probe (dilatonic) 0-brane with the mass m and the charge q is

$$S_{\text{probe}} = \int d\tau \mathcal{L} = \int d\tau \left(m e^{-\frac{D-4}{D-3}\phi} \sqrt{-G^d_{MN} \dot{x}^M \dot{x}^N} - q \dot{x}^M A_M \right), \tag{16}$$

where G^d_{MN}, A_M and ϕ are the dual-frame metric, the $U(1)$ gauge field and the dilaton field produced by the source (dilatonic) 0-brane.

From the following mass-shell constraint for the probe (dilatonic) 0-brane:

$$G^{d\,MN}(P_M - qA_M)(P_N - qA_N) + m^2 e^{-2\frac{D-4}{D-3}\phi} = 0, \tag{17}$$

where $P_M = -\delta\mathcal{L}/\delta\dot{x}^M$ is the canonical momentum, one obtains the expression for the Hamiltonian $\mathcal{H} = -P_t$ describing the mechanics of the probe 0-brane. With the following general spherically symmetric Ansatz for the spacetime metric (in the dual-frame):

$$G^d_{MN} dx^M dx^N = -A(r)dt^2 + B(r)dr^2 + C(r)d\Omega^2_{D-2}, \tag{18}$$

one obtains the following expression for the Hamiltonian [18, 20, 21]:

$$\mathcal{H} = \frac{P_r^2}{2f} + \frac{g}{2f}, \tag{19}$$

where f and g are given by

$$
\begin{aligned}
f &\equiv \frac{1}{2} A^{-\frac{1}{2}} B e^{-\frac{D-4}{D-3}\phi} \left[\sqrt{m^2 + e^{2\frac{D-4}{D-3}\phi}(P_r^2 + BC^{-1}\vec{L}^2)/B} + qA^{-\frac{1}{2}} A_t e^{\frac{D-4}{D-3}\phi} \right], \\
g &\equiv B e^{-2\frac{D-4}{D-3}\phi} \left[(m^2 - q^2 A^{-1} A_t^2 e^{2\frac{D-4}{D-3}\phi}) + C^{-1} e^{2\frac{D-4}{D-3}\phi} \vec{L}^2 \right],
\end{aligned} \tag{20}
$$

where $\vec{L}^2$ is the angular momentum operator of the probe 0-brane, and the expressions for A, B, C, ϕ and A_t that should enter in these expressions for f and g can be read off from the near horizon 0-brane solution (12) with (13) or (14) with (15). For particular values of the dilaton coupling parameter given by $a = (D-4)/(D-2)$ [2],

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[2]For this particular value of the dilatonic coupling parameter a, the kinetic term for the $U(1)$ gauge field A_1 in the "string-frame" (Cf. see the corresponding action (3) in the "string-frame") does not have the dilaton ϕ dependence, which is also the characteristic of the kinetic terms for the RR form fields of string theories.

which include the cases of D0-brane, d0-brane and $D = 4$ Reissner-Nordström black hole, the expressions for f and g get simplified significantly [20, 21] as follows:

$$
\begin{aligned}
f &= \frac{1}{2} A^{-\frac{1}{2}} B e^{-\frac{D-4}{D-3}\phi} \left[\sqrt{m^2 + e^{2\frac{D-4}{D-3}\phi}(P_r^2 + BC^{-1}\vec{L}^2)/B} + q \right], \\
g &= B e^{-2\frac{D-4}{D-3}\phi} \left[(m^2 - q^2) + C^{-1} e^{2\frac{D-4}{D-3}\phi}\vec{L}^2 \right].
\end{aligned}
\tag{21}
$$

In this case, in the extreme limit $(m = q)$ of the probe 0-brane the first term in g drops out and when the (extreme) probe's motion is restricted along the radial direction (i.e. $\vec{L}^2 = 0$), $g = 0$.

The mechanics of the probe 0-brane has the $SL(2, \mathbf{R})$ symmetry with the following generators:

$$
\mathcal{H} = \frac{P_r^2}{2f} + \frac{g}{2f^2}, \quad \mathcal{K} = -\frac{1}{2}fr^2, \quad \mathcal{D} = \frac{1}{2}rPr,
\tag{22}
$$

where the Hamiltonian $\mathcal{H}$ generates the time translation, $\mathcal{K}$ generates the special conformal transformation and $\mathcal{D}$ generates the scale transformation or the dilatation. These generators satisfy the following $SL(2, \mathbf{R})$ algebra [18, 20, 21]:

$$
[\mathcal{D}, \mathcal{H}] = \mathcal{H}, \quad [\mathcal{D}, \mathcal{K}] = -\mathcal{K}, \quad [\mathcal{H}, \mathcal{K}] = 2\mathcal{D}.
\tag{23}
$$

In the non-dilatonic case $(a = 0)$, i.e. the case of the Reissner-Nordström solution, this $SL(2, \mathbf{R})$ symmetry is the genuine conformal symmetry [18]. But when the dilaton field is non-trivial $(a \neq 0)$, the above $SL(2, \mathbf{R})$ symmetry is no longer conformal, because the string coupling $g_s = e^{\phi_\infty}$ changes under the dilatation and the special conformal transformation. But one can still think of the "generalized" $SL(2, \mathbf{R})$ conformal symmetry of the quantum mechanics of the dilatonic 0-brane [20, 21], in which the string coupling is now regarded as a part of background fields that transform under this symmetry [13, 14].

4 Relations to Two-Dimensional Dilaton Gravity

In this section, we elaborate on the relation of the (generalized) conformal quantum mechanics of the (probe) 0-branes to the two-dimensional dilaton gravity theories. We begin by discussing the symmetries of the general class of two-dimensional dilaton gravity theories, to which we shall relate the S^{D-2}-compactified effective supergravity action for the (source) 0-branes.

The most general coordinate invariant action functional of the metric $g_{\mu\nu}$ and the dilaton ϕ, which depends at most on two derivatives of the fields, in two spacetime dimensions has the following form [23]:

$$
S[g, \phi] = \frac{1}{2\kappa_2^2} \int d^2x \sqrt{-g} \left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + V(\phi) + D(\phi)\mathcal{R} \right],
\tag{24}
$$

where $V(\phi)$ is an arbitrary function of ϕ, $D(\phi)$ is a differentiable function of ϕ such that $D(\phi) \neq 0$ and $dD(\phi)/d\phi \neq 0$, and $\mathcal{R}$ is the Ricci scalar of the metric $g_{\mu\nu}$. As pointed out in Ref. [23], although the action (24) is expressed in terms of two functions $V(\phi)$ and $D(\phi)$ of ϕ, the physics of this model does not depend on two arbitrary functions of ϕ. In fact, it is later explicitly shown (following a comment in Ref. [23]) in Ref. [24] that a ϕ-dependent Weyl rescaling transformation $\bar{g}_{\mu\nu} = \Omega^2(\phi)g_{\mu\nu}$, where $\Omega(\phi)$ is the solution to the following differential equation:

$$\frac{1}{2} - 2\frac{dD}{d\phi}\frac{d\ln\Omega}{d\phi} = 0, \tag{25}$$

followed by redefinition of the dilaton field $\bar{\phi} \equiv D(\phi)$ leads to the following action expressed in terms of only one function of the dilaton:

$$S = \frac{1}{2\kappa_2^2} \int d^2x \sqrt{-\bar{g}} \left[\bar{\phi}\mathcal{R}_{\bar{g}} + \bar{V}(\bar{\phi}) \right], \tag{26}$$

where the potential $\bar{V}$ is given by

$$\bar{V}(\bar{\phi}) = \frac{V(\phi(\bar{\phi}))}{\Omega^2(\phi(\bar{\phi}))}. \tag{27}$$

From now on, we suppress the bars in the fields.

In the conformal coordinates, in which the metric takes the form $g_{\mu\nu} = e^{2\rho}\eta_{\mu\nu}$, the general action (26) can be put into the following form of the non-linear sigma model:

$$S = \frac{1}{2\kappa_2^2} \int d^2x \left[G_{ij}(X)\partial_\mu X^i \partial^\mu X^i + \Lambda e^{W(X)} \right], \tag{28}$$

where for the case of the two-dimensional model with the action (26)

$$(X^1, X^2) = (\phi, \rho), \quad W(X) = 2\rho + \ln V(\phi), \quad \Lambda = 1, \quad G_{ij} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}. \tag{29}$$

The conditions for the non-linear sigma model action (28) (with arbitrary G_{ij} and W, not necessarily of the forms in Eq. (29)) to be invariant under a general variation δX^i of the target space coordinates $X^i(x)$ are [25]

$$\delta X^i = \frac{\epsilon^{ij}}{\sqrt{-|G|}}W_{,j}, \qquad \nabla_i\nabla_j W = 0, \tag{30}$$

where $|G|$ is the determinant of G_{ij}, $W_{,i} = \nabla_i W$ and ∇_i is the covariant derivative with respect to the target space metric $G_{ij}(X)$. From the second condition in Eq. (30), one can see that the two-dimensional target space (with the metric G_{ij}) of the general non-linear sigma model of the form (28) has to be flat: $0 = [\nabla_i, \nabla_j]W_{,k} = \mathcal{R}_{\ell kij}W^{,\ell}$.

Substituting the expression for $W(X)$ in Eq. (29) into the second condition in Eq. (30), one obtains the following condition [26] for the classical conformal invariance of the general two-dimensional dilaton gravity model (26):

$$\frac{d^2 \ln V(\phi)}{d\phi^2} = 0, \tag{31}$$

from which one obtains the following general expression for the potential $V(\phi)$ for the conformally invariant two-dimensional dilaton gravity models:

$$V(\phi) = 4\lambda^2 e^{\beta\phi}, \tag{32}$$

where λ and β are arbitrary constants.

Furthermore, the invariance condition (30) implies the existence of a free field $F(X)$ defined by $F_{,k} = -\sqrt{-|G|}\epsilon_{k\ell}W^{,\ell}$ [25]. This free field F is directly related to the Noether current j_μ associated with the symmetry under the general variation δX^i as follows:

$$j_\mu = \partial_\mu X^i G_{ij}\delta X^j = \partial_\mu X^i \sqrt{-|G|}\epsilon_{ij}W^{,j} = -\partial_\mu F. \tag{33}$$

Making use of the symmetry invariance, one can put the general non-linear sigma model action (28) (with arbitrary G_{ij} and W) into the following form:

$$S = \frac{1}{2\kappa_2^2}\int d^2x \frac{1}{\Upsilon}\left[-\partial_\mu F\partial^\mu F + \partial_\mu W\partial^\mu W + \Lambda\Upsilon e^W\right], \tag{34}$$

where $\Upsilon \equiv W^{,i}W_{,i}$ is a constant. So, when it is invariant under the symmetry, the non-linear sigma model (28) is described [25] by a free field $F(X)$ and a field $W(X)$ that satisfies the Liouville field equation in the flat spacetime:

$$\Box W = \frac{1}{2}\Lambda\Upsilon e^W. \tag{35}$$

We shall come back to this point later in this section.

In general, with an arbitrary potential $V(\phi)$ the two-dimensional dilaton gravity model with the action (26) has the symmetry under the following transformations [26]:

$$\begin{aligned}
\delta_E\phi &= 0, \quad &\delta_E g_{\mu\nu} &= g_{\mu\nu}a_\rho\nabla^\rho\phi - \frac{1}{2}(a_\mu\nabla_\nu\phi + a_\nu\nabla_\mu\phi),\\
\delta_1\phi &= 0, \quad &\delta_1 g_{\mu\nu} &= \varepsilon_1\left[\frac{g_{\mu\nu}}{(\nabla\phi)^2} - 2\frac{\nabla_\mu\phi\nabla_\nu\phi}{(\nabla\phi)^4}\right],\\
\delta_2\phi &= \varepsilon_2, \quad &\delta_2 g_{\mu\nu} &= \varepsilon_2 V\left[\frac{g_{\mu\nu}}{(\nabla\phi)^2} - 2\frac{\nabla_\mu\phi\nabla_\nu\phi}{(\nabla\phi)^4}\right],\\
\delta_3\phi &= 0, \quad &\delta_3 g_{\mu\nu} &= -\frac{\varepsilon_3}{2}\left[g_{\mu\nu} + J\left\{\frac{g_{\mu\nu}}{(\nabla\phi)^2} - 2\frac{\nabla_\mu\phi\nabla_\nu\phi}{(\nabla\phi)^4}\right\}\right],
\end{aligned} \tag{36}$$

where a_μ is an arbitrary constant vector and $J'(\phi) = V(\phi)$. The transformations δ_1, δ_2 and δ_3 close down to a non-Abelian Lie algebra, in which δ_2 is a central generator and δ_1 and δ_3 generate the affine subalgebra $[\delta_1, \delta_3] = \frac{1}{2}\delta_1$. The Noether currents associated with the transformations in Eq. (36) are respectively

$$J^{\mu\nu} = Eg^{\mu\nu}, \quad j_1^\mu = \frac{\nabla^\mu\phi}{(\nabla\phi)^2}, \quad j_2^\mu = j_\mathcal{R}^\mu + V\frac{\nabla^\mu\phi}{(\nabla\phi)^2}, \quad j_3^\mu = Ej_1^\mu, \qquad (37)$$

where $E \equiv \frac{1}{2}[(\nabla\phi)^2 - J(\phi)]$ is interpreted as the local energy of the configuration and $\nabla_\mu j_\mathcal{R}^\mu = \mathcal{R}$. The conservation of $J^{\mu\nu}$ implies that E is a conserved scalar (independent of the spacetime coordinates), i.e. $\nabla_\mu E = 0$. When fields are on-shell, the symmetry δ_2 is identified as a diffeomorphism with the vector field $f^\mu = \nabla^\mu\phi/(\nabla\phi)^2$. In particular, due to the conservation $\nabla_\mu E = 0$ of the local energy E, the general action (26) is invariant under the following transformation [27]:

$$\begin{aligned}
\delta_f\phi &= 0, \\
\delta_f g_{\mu\nu} &= -\varepsilon f'(E)\left[g_{\mu\nu} - \frac{\nabla_\mu\phi\nabla_\nu\phi}{(\nabla\phi)^2}\right] + \varepsilon f(E)\left[\frac{g_{\mu\nu}}{(\nabla\phi)^2} - 2\frac{\nabla_\mu\phi\nabla_\nu\phi}{(\nabla\phi)^4}\right], \qquad (38)
\end{aligned}$$

associated with the conserved Noether current $j_f^\mu = f(E)j_1^\mu$. The particular cases of δ_f with $f(E) = 1, E$ respectively correspond to δ_1 and δ_3. The transformation (38) closes under the algebra $[\delta_f, \delta_g] = \frac{1}{2}\delta_{f'g-g'f}$ and in particular δ_f's with $f(E) = 1, E, E^2$ close an $SL(2, \mathbf{R})$ algebra. When the potential $V(\phi)$ takes the form (32), the two-dimensional model (26) is invariant under the conformal transformation, which is the linear combination $\delta_\beta = \delta_2 + 2\beta\delta_3$ of the symmetry transformations in Eq. (36) [26]:

$$\delta_\beta g_{\mu\nu} = -\beta\varepsilon g_{\mu\nu}, \qquad \delta_\beta\phi = \varepsilon, \qquad (39)$$

where keep in mind that the bars are suppressed in the above and we have let $\varepsilon := \varepsilon_2 = \varepsilon_3$.

4.1 Dilatonic 0-brane case

From the near horizon metric (12) or (14) for the dual-frame dilatonic 0-brane solution, one can see that there is the Freund-Rubin compactification [22] on S^{D-2} of the action (9) down to the following two-dimensional effective gauged supergravity action [21]:

$$S = \frac{1}{2\kappa_2^2}\int d^2x\sqrt{-g}e^{\delta\phi}\left[\mathcal{R}_g + \gamma(\partial\phi)^2 + \Lambda\right], \qquad (40)$$

where the parameters in this action are defined as

$$\delta \equiv -\frac{(D-2)a}{D-3}, \qquad \gamma \equiv \frac{D-1}{D-2}\delta^2 - \frac{4}{D-2},$$

$$\Lambda \equiv \frac{(D-3)}{2\mu^2}\left[2(D-2) - \frac{4(D-3)}{\Delta}\right]. \tag{41}$$

To bring the action (40) to the standard form (24), one redefines the dilaton as $\Phi = e^{\delta\phi}$ and then applies the Weyl rescaling of the metric $g_{\mu\nu} = \Phi^{-\frac{\gamma}{\delta^2}}e^{\frac{\Phi}{2}}\tilde{g}_{\mu\nu}$. Then, the action (40) takes the following standard form:

$$S = \frac{1}{2\kappa_2^2}\int d^2x\sqrt{-\tilde{g}}\left[\Phi\mathcal{R}_{\tilde{g}} + \frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi + \Lambda\Phi^{1-\frac{\gamma}{\delta^2}}e^{\frac{\Phi}{2}}\right]. \tag{42}$$

To remove the kinetic term for Φ to bring the action (42) to the form (26), one applies one more Weyl rescaling transformation $\tilde{g}_{\mu\nu} = e^{-\frac{\Phi}{2}}\bar{g}_{\mu\nu}$. The resulting action has the following form:

$$S = \frac{1}{2\kappa_2^2}\int d^2x\sqrt{-\bar{g}}\left[\Phi\mathcal{R}_{\bar{g}} + \Phi^{1-\frac{\gamma}{\delta^2}}\Lambda\right]. \tag{43}$$

This resulting two-dimensional dilaton gravity model has the $SL(2,\mathbf{R})$ affine symmetry, as will be discussed in the next subsection. This $SL(2,\mathbf{R})$ symmetry is expected from the fact that the generalized conformal mechanics of the probe dilatonic 0-brane also has the $SL(2,\mathbf{R})$ symmetry. Since the potential term in the action (43) is not of the form (32), the two-dimensional gravity model (43), derived from the dilatonic 0-brane supergravity action in D dimensions, is not conformal in general. This is also consistent with the fact that the boundary theory counterpart to the bulk near-horizon dilatonic 0-brane theory, namely the generalized conformal quantum mechanics of the probe dilatonic 0-brane, is not genuinely conformal. From these facts, we are lead to the speculation that the generalized conformal quantum mechanics of the probe dilatonic 0-brane is dual to the two-dimensional dilaton gravity model with the classical action given by Eq. (42) or Eq. (43). In particular for the D0-brane case, which is relevant to the M-theory in the light-cone frame [28], the exponent in the potential is given by $1 - \gamma/\delta^2 = 5/9$.

The model with the action (43) is conformal when $\gamma = \delta^2$. From the expressions for γ and δ in Eq. (41), one can see that this happens when the dilaton coupling parameter a takes the following special value:

$$a = 2\frac{D-3}{D-2}. \tag{44}$$

A special case of interest is the $D = 4$ case. In this case, the dilaton coupling parameter becomes $a = 1$, which corresponds to the string theory inspired model of the $D = 4$ Einstein-Maxwell-dilaton theory. Generally, for any values of D with a taking the values specified by Eq. (44), one can bring the action (43) to the form of the action of the Callan, Giddings, Harvey and Strominger (CGHS) model [29]. This can be done by

first redefining the dilaton field as $\Phi = e^{-2\phi}$ and then by applying the Weyl rescaling transformation $g_{\mu\nu} = e^{2\phi}\bar{g}_{\mu\nu}$. The resulting action has the following form:

$$S = \frac{1}{2\kappa_2^2} \int d^2x \sqrt{-g}\, e^{-2\phi} \left[\mathcal{R}_g + 4g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \Lambda \right]. \tag{45}$$

Indeed, the CGHS model is known to be conformally invariant. Namely, the CGHS model is invariant under the following transformation:

$$\delta\phi = \varepsilon e^{2\phi}, \qquad \delta g_{\mu\nu} = 2\varepsilon e^{2\phi} g_{\mu\nu}. \tag{46}$$

This transformation, when expressed in terms of the fields of the action (43) with $\gamma = \delta^2$, corresponds to the linear combination [26] $\delta = \delta_2 - 2\lambda^2\delta_1$ of the symmetry transformations (36) of the generic action (26) with the potential given by Eq. (32) with $\beta = 0$:

$$\delta\Phi = \varepsilon, \qquad \delta\bar{g}_{\mu\nu} = 0. \tag{47}$$

Another interesting case is when $\gamma = 0$, in which case the action (43) describes the Jackiw-Teitelboim [30, 31] model with the following action:

$$S = \frac{1}{2\kappa_2^2} \int d^2x \sqrt{-\bar{g}} \left[\Phi\mathcal{R}_{\bar{g}} + \Phi\Lambda \right]. \tag{48}$$

This happens when the dilaton coupling a takes the following form:

$$a = \frac{2}{\sqrt{D-1}} \frac{D-3}{D-2}. \tag{49}$$

An example is the case of $D = 4$ dilatonic 0-brane with the dilaton coupling $a = 1/\sqrt{3}$.

4.2   Non-dilatonic 0-brane case

The non-dilatonic case (i.e. $a = 0$ and $\phi = 0$) requires special treatment, because the general two-dimensional action (40) (as well as the two-dimensional actions (42) and (43) in standard forms) is not well-defined for this case, behaving singularly. For example, the general formula for the cosmological constant Λ in Eq. (41) becomes zero when $a = 0$, although the compactified $D = 2$ solution in the near-horizon region is the AdS$_2$ space, which requires non-zero cosmological constant term in the action. So, in the following we first derive the compactified two-dimensional action for the non-dilatonic 0-brane case separately.

The effective action for the non-dilatonic 0-brane in D spacetime dimensions is

$$S = \frac{1}{2\kappa_D^2} \int d^Dx \sqrt{-G} \left[\mathcal{R}_G - \frac{1}{2\cdot 2!} F_{MN}F^{MN} \right], \tag{50}$$

11

where $\mathcal{R}_G$ is the Ricci scalar of the D-dimensional spacetime metric G_{MN} ($M, N = 0, 1, ..., D-1$) and F_{MN} is the field strength of the $U(1)$ gauge field A_M. We take the following Ansatz for the D-dimensional spacetime metric:

$$G_{MN}dx^M dx^N = g_{\mu\nu}(x^\mu)dx^\mu dx^\nu + \exp\left[-\frac{4}{D-2}\sigma(x^\mu)\right]d\Omega_{D-2}^2, \qquad (51)$$

where $\mu, \nu = 0, 1$. This general metric Ansatz includes the metric for the 0-brane solution as a special case. We assume that the $U(1)$ gauge field A_M is only electric, i.e. its only non-zero component is A_t. Then, by solving the Maxwell's equation $\nabla_M F^{MN} = 0$ with the above metric Ansatz (51), one obtains the following expression for the electric field $E = F_{tr}$:

$$F_{tr} = Qe^{2\sigma}\sqrt{-g}, \qquad (52)$$

where Q is a constant related to the electric charge and $g \equiv \det(g_{\mu\nu})$. Then, the D-dimensional action (50), upon dimensional reduction on S^{D-2}, becomes of the following form:

$$
\begin{aligned}
S &= \frac{1}{2\kappa_2^2}\int d^2x\sqrt{-g}e^{-2\sigma}\left[\mathcal{R}_g - 4\frac{D-3}{D-2}\partial_\mu\sigma\partial^\mu\sigma - (D-2)(D-3)e^{\frac{4}{D-2}\sigma} + \frac{Q^2}{2}e^{4\sigma}\right], \\
&= \frac{1}{2\kappa_2^2}\int d^2x\sqrt{-g}\left[\bar{\sigma}\mathcal{R}_g - \frac{D-3}{D-2}\frac{\partial_\mu\bar{\sigma}\partial^\mu\bar{\sigma}}{\bar{\sigma}} - (D-2)(D-3)\bar{\sigma}^{\frac{D-4}{D-2}} + \frac{Q^2}{2}\bar{\sigma}^{-1}\right], (53)
\end{aligned}
$$

where $\mathcal{R}_g$ is the Ricci scalar for the two-dimensional metric $g_{\mu\nu}$ defined in Eq. (51) and $\bar{\sigma} \equiv e^{-2\sigma}$. The field equations that follow from this action are

$$\mathcal{R}_g + \frac{D-3}{D-2}\left[\frac{\partial_\mu\bar{\sigma}\partial^\mu\bar{\sigma}}{\bar{\sigma}^2} - 2\nabla_\mu\left(\frac{\partial^\mu\bar{\sigma}}{\bar{\sigma}}\right)\right] - (D-3)(D-4)\bar{\sigma}^{-\frac{2}{D-2}} - \frac{Q^2}{2}\bar{\sigma}^{-2} = 0,$$

$$\nabla_\mu\nabla_\nu\bar{\sigma} + \frac{D-3}{D-2}\left[\frac{1}{2}g_{\mu\nu}\frac{\partial_\rho\bar{\sigma}\partial^\rho\bar{\sigma}}{\bar{\sigma}} - \frac{\partial_\mu\bar{\sigma}\partial_\nu\bar{\sigma}}{\bar{\sigma}}\right]$$

$$+\frac{1}{2}g_{\mu\nu}\left[(D-2)(D-3)\bar{\sigma}^{\frac{D-4}{D-2}} - \frac{Q^2}{2}\bar{\sigma}^{-1}\right] = 0, \quad (54)$$

where ∇_μ is the covariant derivative with respect to the metric $g_{\mu\nu}$. The solution to these field equations, through the relations (51) and (52) between the 2-dimensional fields and the D-dimensional fields, reproduces non-dilatonic 0-brane solution in D dimensions.

Note, in this paper we are particularly interested in the two-dimensional theory associated with the near-horizon approximation of the 0-brane solutions. The solution for the non-dilatonic 0-brane in D dimensions is

$$ds^2 = -H^{-2}dt^2 + H^{\frac{2}{D-3}}(dr^2 + r^2d\Omega_{D-2}^2), \qquad A_t = -H^{-1}, \qquad (55)$$

where $H = 1 + \left(\frac{\mu}{r}\right)^{D-3}$. In the near-horizon region ($r \ll \mu$), the solution takes the following form:

$$ds^2 = -\left(\frac{r}{\mu}\right)^{2(D-3)} dt^2 + \left(\frac{\mu}{r}\right)^2 dr^2 + \mu^2 d\Omega_{D-2}^2, \qquad A_t = -\left(\frac{r}{\mu}\right)^{D-3}. \tag{56}$$

As expected, the spacetime in the near-horizon region is $AdS_2 \times S^{D-2}$. So, the two-dimensional scalar field $\sigma(x^\mu)$ (or $\bar{\sigma}(x^\mu)$) defined in Eq. (51) becomes constant in the near-horizon region of the 0-brane solution. The field equations (54) therefore reduce to the following:

$$\mathcal{R}_g - \Lambda = 0, \tag{57}$$

where $\Lambda = 2\left(\frac{D-3}{\mu}\right)^2$. This constant Λ is exactly the curvature $\mathcal{R}_g$ of the (t,r) part $g_{\mu\nu}dx^\mu dx^\nu = -\left(\frac{r}{\mu}\right)^{2(D-3)} dt^2 + \left(\frac{\mu}{r}\right)^2 dr^2$ of the metric G_{MN} in Eq. (56). Note, the curvature $\mathcal{R}_g$ of this AdS_2 metric has the positive sign due to the choice of the signature $(-+)$ for the metric.

We realize that the field equation (57) describes the Liouville theory (for a review on the Liouville theory, see Ref. [32]). This can be seen by going to the coordinate system where the spacetime becomes conformally flat, which is always possible for the two-dimensional spacetime. For the two-dimensional subset of the spacetime described by the metric in Eq. (56), this is achieved by defining new radial coordinate as $\rho \equiv \frac{\mu}{D-3}\left(\frac{\mu}{r}\right)^{D-3}$. In this new coordinates, the solution (56) takes the following form:

$$ds^2 = \left(\frac{\mu}{D-3}\right)^2 \frac{1}{\rho^2}\left[-dt^2 + d\rho^2\right] + \mu^2 d\Omega_{D-2}^2, \qquad A_t = -\frac{\mu}{D-3}\frac{1}{\rho}, \tag{58}$$

where the (t,ρ) part of the metric is conformally flat. If we denote the two-dimensional (subspace) metric $g_{\mu\nu}$ in the conformally flat coordinate system as $g_{\mu\nu} = e^\varphi \eta_{\mu\nu}$, then Eq. (57) reduces to the following:

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\varphi - \Lambda e^\varphi = 0, \tag{59}$$

which is the Liouville field equation in flat spacetime.

In general, the Liouville theory in a curved spacetime with the metric $\hat{g}_{\mu\nu}$ is defined by the following action [33]:

$$S = \frac{1}{8\pi} \int d^2x \sqrt{-\hat{g}} \left[\hat{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + Q\phi\mathcal{R}_{\hat{g}} + \frac{2\Lambda}{\gamma^2}e^{\gamma\phi}\right]. \tag{60}$$

The requirement of conformal invariance of the classical Liouville action determines the classical background charge coefficient Q to be $Q = 2/\gamma$. With this choice of Q, the Liouville action (60) is invariant under the following Weyl rescaling transformations:

$$\hat{g}_{\mu\nu} \to e^{2\sigma}\hat{g}_{\mu\nu}, \qquad \gamma\phi \to \gamma\phi - 2\sigma, \tag{61}$$

which is equivalent to demanding the metric $g_{\mu\nu} = e^{\gamma\phi}\hat{g}_{\mu\nu}$ to be invariant.

The Liouville theory is known to have the hidden $SL(2,\mathbf{R})$ symmetry [34, 35]. This hidden $SL(2,\mathbf{R})$ symmetry can be seen [3] easily in the light-cone (Polyakov) gauge, in which the metric takes the following form:

$$ds^2 = dx^+dx^- + h_{++}(x^+, x^-)(dx^+)^2. \tag{62}$$

The residual symmetry of this gauge choice contains the Virasoro (conformal) symmetry and the $SL(2,\mathbf{R})$ current symmetry of the Liouville theory. One can replace the metric component h_{++} by a new field f through the relation:

$$\partial_+ f = h_{++}\partial_- f. \tag{63}$$

By using the equation of motion for ϕ, one can also express the Liouville field ϕ in terms of f. The resulting Liouville action, expressed totally in terms of f, is invariant under the following variation of f, which corresponds to the reparametrization variation [4] $\delta h_{++} = \nabla_+\varepsilon^-$ of the metric component h_{++}:

$$\delta f = \varepsilon^-\partial_- f, \tag{64}$$

provided the infinitesimal parameter $\varepsilon^-(x)$ satisfies the condition $\partial_-^3\varepsilon^- = 0$. The most general infinitesimal variation parameter ε^- that satisfies this condition is

$$\varepsilon^-(x^+, x^-) = w_-(x^+) + x^- w_0(x^+) + (x^-)^2 w_+(x^+). \tag{65}$$

Under this reparametrization variation with the infinitesimal parameter given by (65), the metric component h_{++} transforms as

$$\delta h_{++} = \left[w_-j^- + w_0j^0 + w_+j^+\right]h_{++} + 2\left[\partial_+w_- + x^-\partial_+w_0 + (x^-)^2\partial_+w_+\right], \tag{66}$$

where $j^- \equiv \partial_-$, $j^0 \equiv x^-\partial_- - 1$ and $j^+ \equiv (x^-)^2\partial_- - 2x^-$. The generators $j^{-,0,+}$ of the above transformation satisfy the following $SL(2,\mathbf{R})$ algebra:

$$[j^0, j^-] = -j^-, \quad [j^0, j^+] = j^+, \quad [j^+, j^-] = -2j^0. \tag{67}$$

Another way of seeing the hidden $SL(2,\mathbf{R})$ is by considering the metric component constraint $\partial_-^3 h_{++} = 0$ that follows from the equations of motion. The general form of the metric component h_{++} that satisfies this constraint is

$$h_{++}(x^+, x^-) = J^+(x^+) - 2J^0(x^+)x^- + J^-(x^+)(x^-)^2. \tag{68}$$

---

[3]Before one applies the analysis described in the following, one has to first apply the Weyl rescaling transformation $\hat{g}_{\mu\nu} \to e^{\gamma\phi}\hat{g}_{\mu\nu}$ to remove the dependence on ϕ of the potential term in the action (60).

[4]When applying the reparametrization variation, one has to make sure that the light-cone gauge is maintained.

Under the reparametrization variation $\delta h_{++} = \nabla_+ \varepsilon^-$ with the infinitesimal parameter ϵ^- given by Eq. (65), $J^a(x^+)$ $(a = -, 0, +)$ transform as

$$\delta J^a = f^{abc} w_b g_{cd} J^d + 2g^{ab}\partial_+ w_b, \tag{69}$$

where f^{abc} and g_{ab} are respectively the structure constants and the Cartan's metric of the $SL(2,\mathbf{R})$ algebra. This $SL(2,\mathbf{R})$ affine Kac-Moody symmetry can also be seen by considering the Ward identity [34].

Similarly, one can show that the generic two-dimensional dilaton gravity action (42) obtained from the (source) dilatonic 0-brane action in D dimensions also has the $SL(2,\mathbf{R})$ current symmetry. So, the $SL(2,\mathbf{R})$ symmetry of the (generalized) conformal quantum mechanics of the (probe) dilatonic and non-dilatonic 0-branes can be realized within the two-dimensional dilaton gravity models obtained from the effective actions for the (source) 0-branes through the S^{D-2} compactification. Before one applies the analysis similar to the one described in the previous paragraph, one has to first apply the Weyl rescaling transformation of the metric to remove the dilaton dependence of the potential term of the action (42) and then redefine the dilaton to have the standard dilaton kinetic term. However, unlike the case of the two-dimensional dilaton gravity model associate with the (source) dilatonic 0-branes (discussed in the previous subsection), the Liouville theory, associated with the (source) non-dilatonic 0-branes, in addition has the conformal symmetry, which we discuss in the following.

The stress-energy tensor $T_{\mu\nu} = 2\pi \delta S/\delta \hat{g}^{\mu\nu}$ of the classical Liouville theory with the action (60) has the following form:

$$T_{z\bar{z}} = 0, \quad T_{zz} = -\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}Q\partial^2\phi \equiv T(z), \quad T_{\bar{z}\bar{z}} = \bar{T}(\bar{z}), \tag{70}$$

where $z = t + ix$. Since the trace of the stress-energy tensor $T^\mu_\mu = 4T_{z\bar{z}}$ is zero, the classical Liouville theory is conformally invariant, which is consistent with the fact that the conformal quantum mechanics of non-dilatonic 0-branes also has conformal symmetry. Under the conformal transformation $z \to w(z)$, the Liouville field and the stress-energy tensor transform as

$$\phi \to \phi - \frac{1}{\gamma}\ln\left|\frac{dw}{dz}\right|^2, \quad T(z) \to \left(\frac{dw}{dz}\right)^2 T_{ww} + \frac{1}{\gamma^2}S[w;z], \tag{71}$$

where $S[w;z] \equiv \frac{w'''}{w'} - \frac{3}{2}\left(\frac{w''}{w'}\right)^2$ is the Schwartzian derivative. Therefore, the coefficients (called Virasoro operators) of the following Laurent expansion of the stress-energy tensor

$$T(z) = \sum_{n\in\mathbf{Z}} z^{-n-2}L_n, \quad \bar{T}(\bar{z}) \sum_{n\in\mathbf{Z}} \bar{z}^{-n-2}\bar{L}_n, \tag{72}$$

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where $L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z)$ and $\bar{L}_n = \oint \frac{d\bar{z}}{2\pi i} \bar{z}^{n+1} \bar{T}(\bar{z})$, satisfy the following Virasoro algebra with the central charge $c = 12/\gamma^2$:

$$
\begin{aligned}
{[L_n, L_m]} &= (n - m)L_{n+m} + \frac{1}{\gamma^2}(n^3 - n)\delta_{n+m,0}, \\
{[\bar{L}_n, \bar{L}_m]} &= (n - m)\bar{L}_{n+m} + \frac{1}{\gamma^2}(n^3 - n)\delta_{n+m,0}, \\
{[L_n, \bar{L}_m]} &= 0.
\end{aligned}
\tag{73}
$$

Note, the conformal algebra of the Liouville theory already has the anomaly term (proportional to the central charge) at the classical level. The quantum effect due to the normal ordering of the Virasoro generators gives rise to the additional contribution to the central charge. This results in the renormalization of the parameter $1/\gamma^2$ in the Liouville action (60) [36].

In the flat background ($\hat{g}_{\mu\nu} = \eta_{\mu\nu}$), the Liouville action (60) takes the following form, after the field redefinition $\phi = \gamma^{-1}\varphi$:

$$
S = \frac{1}{4\pi\gamma^2} \int dx^2 \left[\frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi + \Lambda e^\varphi \right].
\tag{74}
$$

The field equation of this action yields the Liouville field equation (59) in flat spacetime, which is also the field equation of the non-dilatonic 0-brane in the near-horizon region. This flat spacetime Liouville action is invariant under the following conformal transformation:

$$
\begin{aligned}
x^\pm &\rightarrow f^\pm(x^\pm), \\
\varphi(x^+, x^-) &\rightarrow \varphi(f^+(x^+), f^-(x^-)) + \ln\left[f^{+\prime}(x^+) f^{-\prime}(x^-) \right],
\end{aligned}
\tag{75}
$$

where $x^\pm = t \pm x$ and $f^{\pm\prime} = df^\pm/dx^\pm$. Again, this is consistent with the fact that the boundary theory of the bulk theory on the AdS$_2$ space, i.e. the quantum mechanics of the (probe) non-dilatonic 0-brane, is also conformal.

The general solution to the Liouville equation (59) in flat spacetime is given by [37]

$$
\varphi(x^+, x^-) = \ln \frac{8A'_+(x^+)A'_-(x^-)}{|\Lambda| \left[A_+(x^+) - \epsilon A_-(x^-) \right]^2},
\tag{76}
$$

where $A^\pm(x^\pm)$ is an arbitrary function such that $A'_\pm = dA_\pm/dx^\pm > 0$ and ϵ is the sign of Λ, i.e. $\epsilon = \Lambda/|\Lambda|$. This general solution is invariant under the following $SL(2,\mathbf{R})$ transformation of $A_\pm$:

$$
A_+(x^+) \rightarrow \frac{aA_+(x^+) + b}{cA_+(x^+) + d}, \qquad A_-(x^-) \rightarrow \frac{aA_-(x^-) + \epsilon b}{\epsilon cA_-(x^-) + d},
\tag{77}
$$

where real numbers a, b, c, d satisfy $ad - bc = 1$.

The underlying $SL(2, \mathbf{R})$ symmetry of the Liouville theory in flat spacetime becomes manifest in the Hamiltonian expressed in terms of free fields and their conjugate momenta. Such free fields are obtained from $A_+(x^+)$ through the 'inverse scattering method' followed by the 'bosonization'. As a side comment, the fields $\psi_{\pm 1/2}$ obtained from $A_+(x^+)$ through the inverse scattering method transform in the linear spin $1/2$ representation under $SL(2, \mathbf{R})$. The details on the hidden $SL(2, \mathbf{R})$ symmetry of the Liouville theory in flat spacetime can be found, for example, in Ref. [32].

The general solution (76) can be obtained from the following simple solution:

$$\varphi(x^+, x^-) = \ln \frac{8}{|\Lambda|(x^+ - \epsilon x^-)^2}, \tag{78}$$

which corresponds to the $A_\pm = x^\pm$ case of (76), by applying the conformal transformation (75). The resulting (conformal transformed) general solution is given by Eq. (76) with $A_\pm(x^\pm) = f^\pm(x^\pm)$. In fact, under the conformal transformation (75), the functions $A_\pm(x^\pm)$ in the general solution (76) transform as in Eq. (77), namely

$$A_+(f^+(x^+)) = \frac{aA_+(x^+) + b}{cA_+(x^+) + d}, \qquad A_-(f^-(x^-)) = \frac{aA_-(x^-) + \epsilon b}{\epsilon c A_-(x^-) + d}. \tag{79}$$

Therefore, all the solutions to the Liouville equation (59) in the flat spacetime is related to the simple solution (78) through the conformal transformation (75) or the $SL(2, \mathbf{R})$ transformation (77).

For the case under consideration in this section, namely the near horizon spacetime of the non-dilatonic 0-brane, the constant $\Lambda = 2\left(\frac{D-3}{\mu}\right)^2$ is always positive and therefore in the above $\epsilon = 1$. We notice that the simplest solution (78) to the Liouville equation (59) corresponds to the two-dimensional subspace of the near-horizon solution (58) of the non-dilatonic 0-brane in the conformal coordinates. Therefore, all the solutions of the Liouville field theory is locally related through the conformal transformation (75) or the $SL(2, \mathbf{R})$ transformation (77) to the near-horizon region solution of the non-dilatonic 0-brane. This provides one of the evidences for the equivalence of the conformal quantum mechanics of non-dilatonic 0-branes to the Liouville field theory.

We mentioned previously that the most general two-dimensional dilaton gravity model with conformal invariance has the following action:

$$S = \frac{1}{2\kappa_2^2} \int d^2 x \sqrt{-\bar{g}} \left[\bar{\phi} \mathcal{R}_{\bar{g}} + 4\lambda^2 e^{\beta \bar{\phi}} \right]. \tag{80}$$

For the $\beta \neq 0$ case, by redefining a new dilaton field as $\varphi \equiv 2\beta\bar{\phi}$ and then by applying the Weyl rescaling transformation $g_{\mu\nu} = e^{-\varphi/2} \bar{g}_{\mu\nu}$, one can transform the action (80) to the following form:

$$S = \frac{1}{4\beta\kappa_2^2} \int d^2 x \sqrt{-g} \left[\varphi \mathcal{R}_g + \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi + 8\beta\lambda^2 e^{\varphi} \right]. \tag{81}$$

We recognize that this is the Liouville action in curved spacetime. In the $\beta = 0$ case, one can also bring the action (80) to the Liouville-like form through the Weyl rescaling transformation $g_{\mu\nu} = e^{\bar{\phi}}\bar{g}_{\mu\nu}$. The resulting action has the following form:

$$S = \frac{1}{2\kappa_2^2} \int d^2x \sqrt{-g} \left[\bar{\phi}\mathcal{R}_g + g^{\mu\nu}\partial_\mu\bar{\phi}\partial_\nu\bar{\phi} + 4\lambda^2 e^{\bar{\phi}} \right], \tag{82}$$

with different numerical factor in front of the kinetic term of $\bar{\phi}$, and therefore has to be identified with the Liouville action (60) with non-critical value of Q different from $Q = 2/\gamma$. Anyhow, by going to the gauge and the coordinate frame in which the spacetime is flat, one can bring the actions (81) and (82) to the form of the action (74) for the Liouville theory in flat spacetime. This can also be seen directly from the following field equations for the action (80) in the conformal coordinates (in which the metric takes the form $g_{\mu\nu} = e^{2\rho}\eta_{\mu\nu}$):

$$\begin{aligned} \partial_\mu\partial^\mu(2\rho - \beta\bar{\phi}) &= 0 \\ \partial_\mu\partial^\mu(2\rho + \beta\bar{\phi}) &= 8\lambda^2\beta e^{\rho+\beta\bar{\phi}}. \end{aligned} \tag{83}$$

So, whereas $2\rho - \beta\bar{\phi}$ is a free field, $2\rho + \beta\bar{\phi}$ satisfies the Liouville field equation in flat spacetime. If one lets $2\rho = \beta\bar{\phi}$, which is equivalent to applying the Weyl rescaling transformation $g_{\mu\nu} = e^{-\beta\bar{\phi}}\bar{g}_{\mu\nu}$ and then going to the gauge where $g_{\mu\nu} = \eta_{\mu\nu}$, one is left only with a theory with the Liouville field $\varphi = 2\rho + \beta\bar{\phi} = 2\beta\bar{\phi}$ in flat spacetime, as pointed out in the above. Also, as it is pointed out in the paragraph that follows Eq. (32), the general non-linear sigma model with the action (28) with the conformal symmetry is described by a free scalar field and the (flat spacetime) Liouville field. Thus, one can see that all the conformally invariant two-dimensional dilaton gravity is at least locally equivalent to the Liouville theory in flat spacetime and therefore to the conformal quantum mechanics of the probe non-dilatonic 0-brane in the near-horizon background of the source non-dilatonic 0-brane.

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