Simultaneous Estimation of Noise and Signal in Cosmic Microwave Background Experiments

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16 September 1999

ABSTRACT

To correctly analyse data sets from current microwave detection technology, one is forced to estimate the sky signal and experimental noise simultaneously. Given a time-ordered data set we propose a formalism and method for estimating the signal and associated errors without prior knowledge of the noise power spectrum. We derive the method using a Bayesian formalism and relate it to the standard methods; in particular we show how this leads to a change in the estimate of the noise covariance matrix of the sky signal. We study the convergence and accuracy of the method on two mock observational strategies and discuss its application to a currently-favoured calibration procedure.

Key words: cosmic microwave background

1 INTRO

Observations of temperature fluctuations in the Cosmic Microwave Background (CMB) are poised to become perhaps the most precise tools to probe cosmological models (*e.g.*, Bond & Jaffe 1998). Characteristics of microwave detection technology—in particular, long-time-scale noise correlations—make these observations especially challenging (Delabrouille 1998; De Bernardis & Masi 1998; Górski, Hivon & Wandelt 1999; Tegmark 1997). Such difficult observations—and such high scientific stakes—require equally careful analyses of the data. A map of the CMB sky is not enough; detector characteristics will ensure that the noise contribution to such a map is highly correlated from pixel to pixel, in a pattern depending in detail upon the detector noise spectrum and the observing strategy. But correct estimation of the map and the noise is especially crucial for cosmological analyses which are themselves trying to measure a variance, the intrinsic power spectrum, C_ℓ of the CMB—mis-estimation of the noise can directly add or subtract (*i.e.*, bias) the estimated C_ℓ (Górski 1994a; Górski 1994b; Lineweaver 1994).

Unfortunately, we usually do not have an *independent* determination of the detector noise power spectrum, at least not one that has been made under the same operating conditions as the actual experiment. Moreover, there are noise contributions such as atmospheric emission which may have similar characteristics but are completely independent of the instrument. For this reason, we prefer to estimate the noise directly from the experimental data, perhaps using prior estimates of the noise under different conditions as a guide. In this paper, we

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discuss how to jointly estimate the microwave sky map and the noise correlation matrix using a maximum-likelihood formalism. This brings together two strands of CMB data analysis that had previously been seen as separate steps. We also discuss the further use of the map and correlation matrix for power spectrum estimation, and the pitfalls that this may incur relative to the ideal (but impossible to implement!) full analysis. Although CMB experiments have had to address this problem in the past, only cursory descriptions have been made of the methods used to estimate the noise and the validity of the approximations (Lineweaver 1994; Devlin 1998).

In Section 2 we begin with a discussion of CMB observations and a review of the Bayesian formalism for CMB mapmaking and parameter estimation. We extend this formalism to new maximum likelihood estimates for both the map and noise correlations. In Section 3 we discuss an iterative method for determining this maximum likelihood and apply it to a two model observational strategies as a test. In Section 4 we present the problem of estimating the noise for a data stream with large signal-to-noise, namely calibration with the dipole. In Section 5 we conclude.

2   BAYESIAN FORMALISM

As has become customary, we start our analysis with Bayes' theorem

$$P(\theta|DI) \propto P(\theta|I)P(D|\theta I) \tag{1}$$

where θ are the parameters we are trying to determine, D is the data, and I is the "background information" describing the problem (Lupton 1991), which we will often omit from our probability expressions. The quantity $P(D|\theta I)$ is thus the likelihood, the probability of the data given a specific set of parameters, $P(\theta|I)$ is the prior probability for the parameters, and the left-hand side of the equation is the posterior probability for the parameters given the data.

Here, we will take the data, d_i, as given by a time series of CMB measurements,

$$d_i = s_i + n_i = \sum_p A_{ip}T_p + n_i, \tag{2}$$

where i labels the time, $t = i\ \delta t$, s_i and n_i are the experimental noise and sky signal contributions at that time. The signal is in turn given by the operation of a "pointing matrix," A_{ip} on the sky signal at pixel p, T_p (*i.e.*, the "map"); we take the latter to be already pixelized and smeared by the experimental beam, so A is a very sparse matrix with a single "1" entry for each time corresponding to the observed pixel. In what follows we will often rely on the summation convention and write $\sum_p A_{ip}T_p = A_{ip}T_p$, or occasionally use matrix notation, as in AT, etc.

We will assume that the observed noise n_i is a realization of a stationary Gaussian process with power spectrum $\widetilde{N}(\omega)$. This means that the correlation matrix of the noise is given by

$$\langle n_i n_{i'} \rangle = N_{ii'} = \int \frac{d\omega}{2\pi}\ \widetilde{N}(\omega)e^{-i\omega(t_i - t_{i'})}. \tag{3}$$

The stationarity of the process requires (or is defined by) $N_{ii'} = N(t_i - t_{i'})$.

Most generally, we will take the parameters to be

- The observed CMB signal on the sky, T_p;
- The power spectrum of the noise, $\widetilde{N}(\omega)$
- (Possibly) any cosmological parameters which describe the distribution of the T_p (*i.e.*,

the CMB power spectrum, C_ℓ, although we could also directly use the cosmological parameters such as H_0 and Ω).

Sometimes, we will assign a prior distribution for the CMB signal on the sky itself (*i.e.*, uniform in the sky temperature) such that the cosmological parameters (or C_ℓ) will be irrelevant—we will see that this doing this as an intermediate step retains all of the information in the dataset. At other times we will marginalize over the CMB signal itself and determine those parameters.

With these parameters and the data, d_i, Bayes' theorem becomes

$$P[T_p, \widetilde{N}(\omega), C_\ell | d_i, I] \;\; \propto \;\; P[\widetilde{N}(\omega)|I] P(T_p, C_\ell | I)$$
$$\times P[d_i | \widetilde{N}(\omega), T_p, I]. \tag{4}$$

Here, we have used two pieces of information to simplify slightly. First, the noise power spectrum, $\widetilde{N}(\omega)$ does not depend at all on the signal, so we can separate out its prior distribution.[‡] Second, given the noise power spectrum and the sky signal, the likelihood does not depend upon the cosmological parameters. For the Gaussian noise we assume, the likelihood is simply

$$-2\ln\mathcal{L} \;\; \equiv \;\; -2\ln P[d_i | \widetilde{N}(\omega), T_p]$$
$$= \;\; \ln|N_{ii'}| + \sum_{ii'}(d_i - s_i) N_{ii'}^{-1}(d_{i'} - s_{i'})$$
$$= \;\; \sum_k \left[\ln\widetilde{N}_k + |\tilde{d}_k - \tilde{s}_k|^2 / \widetilde{N}_k \right] \tag{5}$$

(ignoring an additive constant); recall that $s_i = \sum_p A_{ip} T_p$. The second equality uses tildes to denote the discrete Fourier transform at angular frequency ω_k.

We will now apply these general formulae to various cases.

2.1 Known noise power spectrum

We will start with the simplest case, where we have complete prior knowledge of the noise power spectrum. This is the case that has been previously discussed in the literature, but we emphasize that it is very unrealistic.

We assign a delta-function prior distribution to N, transforming it in effect from a parameter to part of the prior knowledge. First, we assume no cosmological information about the distribution of temperatures on the sky: $P(T_p, C_\ell | I) = P(T_p | I) P(C_\ell | I)$; with this separation the posterior for C_ℓ is simply the prior—the experiment gives us no new information. We will also assign a uniform prior to T_p, lacking further information. Now, the posterior distribution for the sky temperature is simply proportional to the likelihood, which can be rewritten by completing the square in the exponential as

$$P(T_p | \widetilde{N}, d, I) \;\; \propto \;\; P(d | T_p, \widetilde{N}, I)$$
$$\propto \;\; \frac{1}{|2\pi C_{Npp'}|^{1/2}}$$
$$\times \exp\left[-\frac{1}{2} \sum_{pp'} \left(T_p - \bar{T}_p \right) C_{N,pp'}^{-1} \left(T_{p'} - \bar{T}_{p'} \right) \right] \tag{6}$$

with the mean (also, the likelihood maximum) given by

[‡] In principle, the noise may depend on the signal such as in the case of a nonlinear response of the system to large-amplitude signals such as planets or the galaxy; in practice data thus contaminated is discarded.

$$\bar{T} = \left(A^T N^{-1} A\right)^{-1} A^T N^{-1} d \tag{7}$$

(in matrix notation), and the noise correlation matrix by

$$C_N = \left(A^T N^{-1} A\right)^{-1}. \tag{8}$$

Occasionally, the inverse of this correlation matrix is referred to as the *weight matrix*. As is usual for linear Gaussian models, the mean is just the multidimensional least-squares solution to $d = AT$ with noise correlation N. This is just the standard mapmaking procedure (Lupton 1991; Tegmark 1996; Wright 1996b), cast into the form of a Bayesian parameter-estimation problem.

For the case of known noise, however, this map is more than a just a visual representation of the data. Even if we wish to determine the cosmological parameters, it is an essential quantity: we can write the prior for both the map and the spectrum as

$$P(C_\ell, T_p|I) = P(T_p|C_\ell, I)P(C_\ell|I) \tag{9}$$

using the laws of probability, and so we can see that our rewriting of the likelihood in the form of Eq. 6 remains useful. That is, the full distribution is only a function of the data through the maximum-likelihood map, $\bar{T}$—in statistical parlance, $\bar{T}$ is a *sufficient statistic*. Thus for known noise, we can *always* start by making a map (and calculating its noise matrix, C_N).

2.2 Cosmological CMB priors

Here, we briefly examine the specific form of the signal prior, $P(T_p|C_\ell, I)$, motivated by simple Gaussian models. That is, we take the sky temperature, T_p, to be an actual realization of a Gaussian CMB sky, with covariance specified by the power spectrum, C_ℓ,

$$\langle T_p T_{p'} \rangle = C_{T,pp'} = \sum_\ell \frac{2\ell + 1}{4\pi} C_\ell B_\ell^2 P_\ell \left(\hat{x}_p \cdot \hat{x}_{p'}\right) \tag{10}$$

(note that we include beam-smearing by a symmetric beam with spherical harmonic transform B_ℓ in this definition); $\hat{x}_p \cdot \hat{x}_{p'}$ gives the cosine of the angle between the pixels. With this covariance, the prior becomes

$$P(T_p|C_\ell, I) = \frac{1}{|2\pi C_{Tpp'}|^{1/2}} \exp\left[-\frac{1}{2} \sum_{pp'} T_p C_{T,pp'}^{-1} T_{p'}\right]. \tag{11}$$

We thus have a posterior distribution for T_p and C_ℓ which is the product of two Gaussians, $P(C_\ell, T_p|d, I) \propto P(T_p|C_\ell, I)P(d|T_p, I)$, given by Eqns. 6 and 11. In the usual cosmological likelihood problem, we don't care about the actual sky temperature *per se*, but are concerned with the C_ℓ (or the parameters upon which the power spectrum depends). Thus, we can marginalize over the T_p,

$$
\begin{aligned}
P(C_\ell|d, I) &= \int dT_p \, P(C_\ell, T_p|d, I) = P(C_\ell|I) \int dT_p \, P(T_p|C_\ell, I)P(d|T_p, I) \\
&= P(C_\ell|I)P(\bar{T}(d)|C_\ell I), \\
&= P(C_\ell|I) \int dT_p \, \frac{1}{|2\pi C_T|^{1/2}} \frac{1}{|2\pi C_N|^{1/2}} \\
&\quad \times \exp\left[-\frac{1}{2} \sum_{pp'} \left(T_p C_{T,pp'}^{-1} T_{p'} + \left(T_p - \bar{T}_p\right) C_{N,pp'}^{-1} \left(T_{p'} - \bar{T}_{p'}\right)\right)\right], \tag{12}
\end{aligned}
$$

where we have included a prior for the power spectrum itself, so we can write the Gaussian factor as the likelihood for the map given C_ℓ, $P(C_\ell|d, I) \propto P(C_\ell|I)P(\bar{T}|C_\ell)$. Equation 12 defines the effective likelihood for the map ($\bar{T}$, now considered as the data rather than the timestream itself, d), which is easily computed again by completing the square, giving

$$P(\bar{T}_p|C_\ell I) = \frac{1}{|2\pi (C_{Tpp'} + C_{Npp'})|^{1/2}} \exp\left[-\frac{1}{2}\sum_{pp'} \bar{T}_p (C_T + C_N)^{-1}_{pp'} \bar{T}_{p'}\right]. \tag{13}$$

This is just the usual CMB likelihood formula: the "observed map," $\bar{T}_p$, is just the sum of two quantities (noise and signal) distributed as independent Gaussians. Note again that the data only enter through the maximum likelihood map, $\bar{T}$, although that calculation is only implicit in this formula. Further, the power spectrum C_ℓ only enters through the signal correlation matrix, C_T, and in a very nonlinear way.

We can also play a slightly different game with the likelihood. If we retain the Gaussian prior for the CMB temperature but *fix* the CMB power spectrum, we can estimate the map with this additional prior knowledge. We will again be able to complete the square in the exponential and see that T_p is distributed as a Gaussian:

$$P(T_p|C_\ell, d, I) = \frac{1}{|2\pi C_W|^{1/2}} \exp\left[-\frac{1}{2}\chi^2(T_p|C_\ell, d, I)\right] \tag{14}$$

with

$$\chi^2(T_p|C_\ell, d, I) \equiv \sum_{pp'} \left(\bar{T}_p - (W\bar{T})_p\right) C^{-1}_{W,pp'} \left(\bar{T}_{p'} - (W\bar{T})_{p'}\right). \tag{15}$$

Now, the mean is given by

$$(W\bar{T}) = C_T(C_T + C_N)^{-1}\bar{T} \tag{16}$$

which is just the *Wiener Filter*. It has correlation matrix given by

$$C_W = C_T(C_T + C_N)^{-1}C_T. \tag{17}$$

Note that the maximum-likelihood map, $\bar{T}$ still appears in these formulae, but it is no longer the maximum of the *posterior* distribution, now given by the Wiener filter, $W\bar{T}$.

This subsection has shown how many of the usual CMB data calculations can be seen as different uses of the Bayesian formalism:

- the least-squares map (seeing it as a "sufficient statistic"),
- the CMB cosmological-parameter likelihood function, and
- the Wiener filter map of the CMB signal.

The differences depend on what quantity is estimated (the map or the power spectrum) and what prior information is included.

2.3 Unknown noise

The previous subsection briefly outlined the Bayesian approach to CMB statistics, assuming a known noise power spectrum. Now, we will relax this assumption and approach the more realistic case when we must estimate both the experimental noise and the anisotropy of the CMB. We will take as our model a noise power spectrum of amplitude $\bar{N}_\alpha$ in bands numbered α, with a shape in each band given by a fixed function P_k with a width of n_α; $n_\alpha = \frac{\Delta_\omega}{\omega_0}$ is the number of discrete modes in the band α where Δ_ω is the width of the band and ω_0 is the minimum frequency of the data-stream. We will usually take $P_k = \text{const}$ so $\widetilde{N}$ is piecewise constant. That is,

$$\widetilde{N}(\omega_k) = \bar{N}_\alpha P_k. \tag{18}$$

We again assign a constant prior to the sky map, T_p. As the prior for the noise we will take $P(\bar{N}_\alpha) \propto 1/\bar{N}_\alpha^\nu$. With $\nu = 1$ and a single band, this is the usual Jeffereys prior advocated for "scale parameters" and the units on $\bar{N}_\alpha$ are irrelevant (Kendall & Stuart 1977).

With this model and priors, the joint likelihood for the noise and the map becomes

$$
\begin{aligned}
P(T_p, \bar{N}_\alpha | d, I) &\propto \prod_\alpha \frac{1}{\bar{N}_\alpha^{\nu + n_\alpha/2}} \exp\left[-\frac{1}{2\bar{N}_\alpha} \sum_{k \in \alpha} \frac{1}{P_k} \left| \tilde{d}_k - \widetilde{A}_{kp} T_p \right|^2 \right] \\
&= \prod_\alpha \frac{1}{\bar{N}_\alpha^{\nu + n_\alpha/2}} \exp\left[-\frac{1}{2\bar{N}_\alpha} \sum_{k \in \alpha} \frac{1}{P_k} \left| \tilde{\varepsilon}_k \right|^2 \right]
\end{aligned}
\tag{19}
$$

where $k \in \alpha$ refers to a sum over modes in band α. We also define the estimate of the noise as $\varepsilon = d - AT$ for future use.

We can simultaneously solve for the maximum-probability noise and signal. Carrying out the necessary derivatives gives

$$
\begin{aligned}
\frac{\partial \ln \mathcal{L}}{\partial T_p} &= \sum_\alpha \left[\frac{1}{\bar{N}_\alpha} \sum_{k \in \alpha} \frac{1}{P_k} \left(\tilde{d}_k - \widetilde{A}_{kp'} T_{p'} \right) \widetilde{A}_{kp} \right] \\
&= (d - AT)^T N^{-1} A = \varepsilon^T N^{-1} A
\end{aligned}
\tag{20}
$$

(switching between indices in Fourier space and matrix notation) and

$$\frac{\partial \ln \mathcal{L}}{\partial \bar{N}_\alpha} = -\frac{1}{2} \frac{1}{\bar{N}_\alpha} \left[(n_\alpha + 2\nu) - \frac{1}{\bar{N}_\alpha} \sum_{k \in \alpha} \frac{1}{P_k} \left| \tilde{d}_k - \widetilde{A}_{kp'} T_{p'} \right|^2 \right]. \tag{21}$$

Setting these to zero and solving, we then find we must simultaneously satisfy

$$T = (A^T N^{-1} A)^{-1} A^T N^{-1} d \tag{22}$$

(using matrix notation for simplicity) and

$$\bar{N}_\alpha = \frac{1}{n_\alpha + 2\nu} \sum_{k \in \alpha} \frac{1}{P_k} \left| \tilde{d}_k - \widetilde{A}_{kp'} T_{p'} \right|^2 = \frac{1}{n_\alpha + 2\nu} \sum_{k \in \alpha} \frac{1}{P_k} \left| \tilde{\varepsilon}_k \right|^2. \tag{23}$$

Equation 22 is just the usual maximum-likelihood map solution; Equation 23 is the average "periodogram" (*i.e.*, the naive power spectrum calculated from the Fourier transform) of the noise over the band α, with a slight modification for the prior probability, parameterized by ν; for wide bands this modification is irrelevant. As we will see below, iteration is actually a very efficient way to solve Equations 22 and 23 simultaneously. For future reference, we also write down the derivative with respect to the map at the joint maximum (*i.e.*, substituting Eq. 23 into Eq. 20),

$$\frac{\partial \ln \mathcal{L}}{\partial T_p} = \sum_\alpha \left[(n_\alpha + 2\nu) \frac{\sum_{k \in \alpha} \tilde{\varepsilon}_k \widetilde{A}_{kp}/P_k}{\sum_{k' \in \alpha} \left| \tilde{\varepsilon}_{k'} \right|^2 / P_{k'}} \right]. \tag{24}$$

If we fix the noise at the joint maximum, then the problem reduces to that of the previous section, a Gaussian likelihood in T_p for which the usual tools can be applied. This is not a rigorous approach to the problem, however due to correlation between noise and signal estimation. To get a handle on this, we calculate the curvature of the distribution around this joint maximum, which we define as

$$\mathcal{F} = \begin{pmatrix} \mathcal{G}_{pp'} & \mathcal{G}_{p\alpha} \\ \mathcal{G}_{p\alpha} & \mathcal{G}_{\alpha\alpha'} \end{pmatrix} \tag{25}$$

where a subscript α or p refers to a derivative with respect to $\bar{N}_\alpha$ or T_p, respectively. $\mathcal{F}$ refers to the full matrix; $\mathcal{G}$ to the sub-blocks. Explicitly, the parameter derivatives are

$$\frac{\partial^2 \ln \mathcal{L}}{\partial T_p \partial T_{p'}} = \mathcal{G}_{pp'} = -\sum_\alpha \frac{1}{\bar{N}_\alpha} \sum_{k \in \alpha} \frac{\widetilde{A}_{kp} \widetilde{A}_{kp'}}{P_k} = -A^T N^{-1} A \tag{26}$$

and

$$\frac{\partial^2 \ln \mathcal{L}}{\partial \bar{N}_\alpha \partial \bar{N}_{\alpha'}} = \mathcal{G}_{\alpha\alpha'} = -\frac{n_\alpha + 2\nu}{2\bar{N}_\alpha^2} \delta_{\alpha\alpha'} \tag{27}$$

with the cross-curvature

$$\frac{\partial^2 \ln \mathcal{L}}{\partial T_p \partial \bar{N}_\alpha} = \mathcal{G}_{p\alpha} = -\frac{1}{\bar{N}_\alpha^2} \sum_{k \in \alpha} \frac{\tilde{\varepsilon}_k \widetilde{A}_{kp}}{P_k}. \tag{28}$$

The distribution about the joint maximum is *not* a Gaussian in the $\bar{N}_\alpha$ directions, so there is more information available than this. Nonetheless, if we treat the distribution as if it were Gaussian, we can calculate the covariance matrix, given by the inverse of this curvature matrix; if we assume that the matrix $I_{pp'} - \mathcal{G}_{pp''}^{-1} \mathcal{G}_{p''\alpha} \mathcal{G}_{\alpha\alpha'}^{-1} \mathcal{G}_{\alpha'p'}$ is invertible, then we find that the effective variance of the map is increased from $C_{Npp'} = \mathcal{G}_{pp'}^{-1}$ to

$$\begin{aligned} C_{Npp'}^{\mathrm{eff}} &= (C_{N\ pp'}^{-1} - \mathcal{G}_{p\alpha} \mathcal{G}_{\alpha\alpha'}^{-1} \mathcal{G}_{\alpha'p'})^{-1} \\ &= \left[\sum_\alpha \frac{1}{\bar{N}_\alpha} \sum_{k \in \alpha} \frac{\widetilde{A}_{kp} \widetilde{A}_{kp'}}{P_k} - 2 \sum_\alpha \frac{1}{(n_\alpha + 2\nu)\bar{N}_\alpha^2} \sum_{k \in \alpha} \frac{\tilde{\varepsilon}_k \widetilde{A}_{kp}}{P_k} \sum_{k' \in \alpha} \frac{\tilde{\varepsilon}_{k'} \widetilde{A}_{k'p'}}{P_{k'}} \right]^{-1} \end{aligned} \tag{29}$$

which we have evaluated at the simultaneous peak of the distribution to determine $\bar{N}_\alpha$.

To assess the importance of the correction term we can consider an approximate expression that relates $(C_N^{\mathrm{eff}})^{-1}$ to $(C_N)^{-1}$; we shall assume that $\tilde{N}(\omega)$ is approximately white, with a constant number of frequency samples per bin, *i.e.*, $n_\alpha = n$, $P_k = 1$ for all α, and $\nu = 1$. The key parameter is then

$$r_p = \frac{4\pi}{\Delta_\omega \tau_p} \tag{30}$$

where τ_p is the total time spent on a pixel p and Δ_ω is the dimensionful width of the bins. A simple way of obtaining r_p is by reordering the data stream in such a way that all time samples for a given pixel are grouped contiguously. Then the minumum frequency for a given pixel will correspond to $2\pi/\tau_p$ where τ_p is now the length of the data stream segment. We then have that the total number of discrete modes in bin of width Δ_ω is $2\pi/\Delta_\omega \tau_p$. The extra factor of two comes from equation (29).

If we take the ensemble average of Eq. 29 we obtain

$$\left\langle \left(C_{Npp'}^{\mathrm{eff}} \right)^{-1} \right\rangle = \left(1 - \sqrt{r_p r_{p'}} \right) \sum_\alpha \left\langle \frac{1}{\bar{N}_\alpha} \right\rangle \sum_{k \in \alpha} \widetilde{A}_{kp} \widetilde{A}_{kp'} = \left(1 - \sqrt{r_p r_{p'}} \right) \left\langle (C_{Npp'})^{-1} \right\rangle. \tag{31}$$

For an equal amount of time spent on each pixel we have that $n = \Delta_\omega T_{\mathrm{tot}}/2\pi$ where $T_{tot} \equiv N_{\mathrm{pix}} \tau_p$ so that Eq. 31 simplifies to

$$\left\langle \left(C_{Npp'}^{\mathrm{eff}} \right)^{-1} \right\rangle = \left(1 - \frac{2N_{\mathrm{pix}}}{n} \right) \left\langle (C_{Npp'})^{-1} \right\rangle. \tag{32}$$

This immediately shows us two limits. In the case of one frequency per band, the inverse variance is negative and the problem is ill-defined; in some sense one is trying to estimate too much from the data set. In the limit of just one band (*i.e.*, $n \to \infty$) the correction for the effective variance is negligible. For practical cases one needs enough bins to properly capture

the shape of the noise periodogram, so one has to consider some intermediate regime. For the case of one pixel and white noise, to get a correction of less than 1% to the $C_{\mathrm{N}pp'}$, one needs $n > 10^2 N_{pix}$. For more realistic cases of many pixels and non-white noise, much larger bins are necessary, and must be evaluated on a case by case basis.

We note a caveat to this procedure: consider an experiment that simply chops with a given frequency of rotation in a circle on the sky (as in a simple model for Planck's observing strategy). Any source of noise synchronous with this motion will necessarily be very difficult to distinguish from sky signal. That is, there will necessarily be strong covariance between chop- or spin- synchronous noise and the signal. This will remain somewhat the case even if the pattern moves slowly on the sky. In this case, other information, such as the smoothness of the noise power spectrum, must be used.

2.4 Noise Marginalization

Instead of this joint solution, formally, at least, we know the appropriate procedure: marginalize over the quantity we don't care about (the noise power spectrum) to obtain the distribution for the quantity we wish to know (the map, T_p). We can actually carry out the integral in this case:

$$
\begin{aligned}
P(T_p|d_i, I) &= \int d\widetilde{N}_k\, P(T_p, N_k|d_i, I) \\
&\propto \prod_\alpha \left[\sum_{k\in\alpha} \frac{1}{P_k} \left| \tilde{d}_k - \widetilde{A}_{kp} T_p \right|^2 \right]^{-(n_\alpha/2 + \nu - 1)}
\end{aligned}
\tag{33}
$$

(this is just Student's t distribution in a slightly different form than is usually seen). If we stay in Fourier space, the maximum of this distribution can be calculated to be the solution of

$$
\frac{\partial \ln P}{\partial T_p} = \sum_\alpha \left[(n_\alpha + 2\nu - 2) \frac{\sum_{k\in\alpha} \frac{1}{P_k} \left(\tilde{d}_k - \widetilde{A}_{kp'} T_{p'} \right) \widetilde{A}_{kp}}{\sum_{k'\in\alpha} \frac{1}{P_{k'}} \left| \tilde{d}_{k'} - \widetilde{A}_{k'p'} T_{p'} \right|^2} \right].
\tag{34}
$$

Note that this is *exactly the same form* as Eq. 24, the equation for the maximum probability map in the joint estimation case, with the prefactor $(n_\alpha + 2\nu)$ replaced by $(n_\alpha + 2\nu - 2)$. This is equivalent to changing the exponent of the prior probability from ν to $\nu - 1$: the marginalized maximum for $\nu = 1$ (Jefferys prior) is the same as the joint maximum for $\nu = 0$ (constant prior). We have also seen that for $n_\alpha \gg 1$, the value of ν is irrelevant, so that these maxima should be nearly equal. (Moreover, the numerical tools to calculate the joint solution, outlined below, can be used in this case, too.) Note also that if $n_\alpha + 2\nu - 2 \leq 0$, the equation isn't solved for any map. In this case, either our prior information is so unrestrictive or the bands are so narrow that it is impossible to distinguish between noise and signal, and the probability distribution has no maximum when the noise is marginalized.

In principle, any further analysis of the map would have to rely on the full distribution of Eq. 33. In practice, the t-distribution is quite close to a Gaussian although the tails are suppressed by a power-law rather than an exponential. It will thus be an excellent approximation to take the distribution to be a Gaussian

$$
\begin{aligned}
P(T_p|d_i, I) &\approx \frac{1}{\left| 2\pi C_{N,pp'}^{\mathrm{eff}} \right|^{1/2}} \\
&\times \exp\left[-\frac{1}{2} \sum_{pp'} \left(T_p - \bar{T}_p \right) \left(C_{N,pp'}^{\mathrm{eff}} \right)^{-1} \left(T_{p'} - \bar{T}_{p'} \right) \right],
\end{aligned}
\tag{35}
$$

with noise covariance given by

$$
\begin{aligned}
\left(C_{Npp'}^{\text{eff}}\right)^{-1} &= -\frac{\partial^2 \ln P(T_p|d_i, I)}{\partial T_p \partial T_{p'}} \\
&= \sum_\alpha \left[\frac{(n_\alpha + 2\nu - 2)}{\sum_{k\in\alpha} |\tilde{\varepsilon}_k|^2} \sum_{k\in\alpha} \frac{\widetilde{A}_{kp}\widetilde{A}_{kp'}}{P_k} - 2\frac{(n_\alpha + 2\nu - 2)}{(\sum_{k\in\alpha}|\tilde{\varepsilon}_k|^2)^2} \sum_{k\in\alpha} \frac{\tilde{\varepsilon}_k \widetilde{A}_{kp}}{P_k} \sum_{k'\in\alpha} \frac{\tilde{\varepsilon}_{k'} \widetilde{A}_{k'p'}}{P_{k'}} \right]
\end{aligned}
\tag{36}
$$

where $\bar{T}$ is the solution to Eq. 34 and the derivatives are taken with respect to the *full* distribution of Eq. 34. Note that the inverse of C_N^{eff} (occasionally referred to as the weight matrix) is given by the sum of two terms. The first is just the weight matrix that would be assigned given the maximum probability solution for the noise, Eq. 8 (with the numerically irrelevant change of the prior $\nu \to \nu - 1$ as noted above). Again we find a correction term due to the fact that we are only able to *estimate* this noise power spectrum. This correction term can be minimized by taking sufficiently wide bands, as in the simultaneous estimation case above (and subject to the same caveats).

3   AN ITERATIVE METHOD: CONVERGENCE AND ACCURACY

In this section we propose an algorithm for finding the Maximum Likelihood Estimate (MLE) of the noise and signal and we apply it to two models that contain all the essential features of current CMB measurement processes. We parameterize the map and noise as described above, in terms of a set of pixel temperatures, T_p, and a set of bandpowers, $\bar{N}_\alpha$. The noise is stationary and Gaussian with a periodogram of the form:

$$
\widetilde{N}(\omega) \propto 1 + \frac{\omega_{\text{knee}}}{\omega}.
\tag{37}
$$

Given some prior knowledge of the position of ω_{knee} we divide the noise periodogram, $\widetilde{N}(\omega)$, into logarithmically equally spaced bins below ω_{knee} and linearly equally spaced bins above ω_{knee}.

Equations 22 and 23 are nonlinear and it is therefore impractical to obtain an explicit solution for $\bar{N}_\alpha$ and T_p. However the structure of this system lends itself to an iterative approximation scheme; given the ith estimate the map, $T_p^{(i)}$, we can find the the ith estimate of the noise in a band α, $\bar{N}_\alpha$ from Equation 23. As we shall see, this system will converge to the MLE of T_p and $\bar{N}_\alpha$. We note that this algorithm is similar to "Expectation-Maximization" (EM) algorithms (Moon 1996) which alternate between filling in unknown data (as in our mapmaking step) and calculating a maximum-likelihood estimation of some quantity (the power-spectrum estimation step).

The algorithm starts with initial conditions:

$$
\begin{aligned}
T^{(0)} &= 0 \\
\bar{N}_\alpha^{(0)} &= \frac{1}{n_\alpha + 2} \sum_{\omega\in\alpha} |\tilde{d}(\omega)|^2.
\end{aligned}
\tag{38}
$$

The iterative step is

$$
\begin{aligned}
T^{(i)} &= \left(A^T N^{(i-1)-1} A\right)^{-1} A^T N^{(i-1)-1} d \\
\tilde{s}^{(i)}(\omega) &= \tilde{A}(\omega) T^{(i-1)} \\
\bar{N}_\alpha^{(i)} &= \frac{1}{n_\alpha + 2} \sum_{\omega\in\alpha} |\tilde{d}(\omega) - \tilde{s}^{(i)}(\omega)|^2.
\end{aligned}
\tag{39}
$$
$$
\tag{40}
$$

The first model CMB experiment we will analyse is an idealized small sky mapping

experiment for which we shall adopt the name *fence* scan (Tegmark 1997): for the first half of the experiment the beam is swept back and forward horizontally, shifting downwards until the entire patch has been covered; for the second half of the scan the beam is swept back and forth vertically, shifting horizontally to the left, again until the patch is completely covered. The end result is a set of perpendicularly crossed linked scans. The second model scan is a small sky version of the original Planck Surveyor scan strategy, where great circles intersect at the two poles (we shall call it a *poles* scan): The beam sweeps horizontally back and forth along a square patch of the sky, drifting downwards until the bottom of the patch when it reaches the end of the scan; at the edge of each horizontal sweep the beam returns to the same pixels (one on the left and one on the right). The crosslinking is then simply at the two end pixels (which play the role of the poles in the full-sky, great circle Planck scan). We consider a map with 10×10 pixels for the fence scan and $8 \times 10 + 2$ pixels for the poles scan. We generate 56000 observations per scan. The noise is generated by a stationary, Gaussian number generator with periodogram given by Eq. 37 and we choose $\omega_{\mathrm{knee}}T_s=0.5$ where T_s is the duration of one horizontal sweep; we have found this to be typical values in the BOOMERanG and MAXIMA experiments.

The first thing to check is if the algorithm converges. We quantify this by comparing the map and periodogram in successive iterations using

$$
\begin{aligned}
\eta_T^i &= \frac{1}{N_{pix}} \sum_p \frac{(T_p^{(i)} - T_p^{(i-1)})^2}{(T_p^{(i)})^2 + (T_p^{(i-1)})^2}; \\
\eta_N^i &= \frac{1}{N_b} \sum_\omega \frac{(\bar{N}_\alpha^{(i)} - \bar{N}_\alpha^{(i-1)})^2}{(\bar{N}_\alpha^{(i)})^2 + (\bar{N}_\alpha^{(i-1)})^2},
\end{aligned}
\tag{41}
$$

where N_b is the number of bins that characterize the periodogram and N_{pix} is the number of pixels in the map. The form of the denominator guarantees that spurious null values of these quantities won't bias the measure of convergence. In Figure 1 we plot the average of 200 simulated experiments for three levels of signal-to-noise.

There are essentially only two parameters which may effect the existence of, and convergence to, a fixed point: the number of samples per bin, n_α, and the signal-to-noise of the experiment. As we saw in the previous section, n_α must be sufficiently large for $C_{N\mathrm{eff}}$ to be positive definite; otherwise the problem is ill-posed. As one would expect from Eq. 31, this is tied to the convergence of the iterative scheme we propose: if the n_α are small the algorithm either does not converge or takes a prohibitively long time to converge (of order 10^2 iterations). For simplicity of analysis we consider all linear bins to have the same number of samples. We then find that the threshold for both of the experimental strategies is $n_\alpha \simeq 2 - 5 \times 10^2$; the number of logarithmic bins between the minimum frequency and w_{knee} are $N_< = (w_{\mathrm{knee}}/w_0)(\ln n_\alpha)/n_\alpha$. For values greater than this, the system has difficulties in converging, for smaller values there is very little increase in speed. A few comments are in order. Firstly in choosing such values of n_α we are in a regime in which both the joint estimate of $(T_p, \bar{N}_\alpha)$ and the sole estimate of T_p using the marginalized likelihood give essentially the same results. Secondly one may worry that one is smoothing out important features in the noise periodogram, in particular mis-estimating the shape of the periodogram below ω_{knee} and smoothing over spectral "spikes". We have found that in choosing $N_<$ logarithmic bins below ω_{knee} we adequately resolve the shape while at the same time having enough samples per bin to get rapid convergence. Fine structures in the noise periodogram are typically wide enough to be resolved with the resolution we propose here; very fine structures are usually due to systematic effects and should be taken care of as such. Thirdly, as we saw in Section 2.3, the wider the bins, the smaller the correction to the noise covariance matrix;
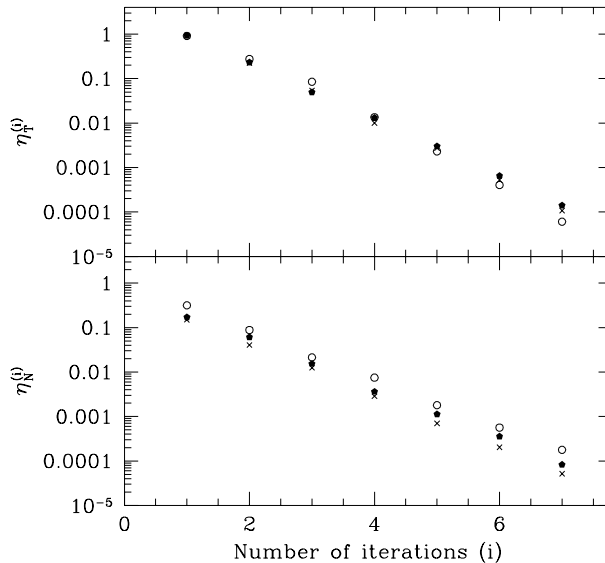
Figure 1. The convergence of the algorithm for the map, η_T^i and noise periodogram, η_N^i for experiments with signal-to-noise of 5 (open circles), 1 (closed pentagons) and 0.2 (stars).

this must be taken into account when subsequently using C_N for estimating C_ℓ. For the "borderline" values of n_α we use for Figure 1 we expect the correction terms to increase C_N by 50–100%. Larger values of n_α for which convergence is slightly more rapid should give smaller corrections.

The second factor we mentioned is the signal-to-noise of the experiment; the figure of merit to be used is the signal-to-noise per frequency band *in the bands where there is signal*. Let us clarify what we mean. For the ensemble average signal periodogram we know that there will only be a significant signal amplitude for frequencies corresponding to the scan frequency and its harmonics. Comparing to the ensemble average noise periodogram, there will be two limiting regimes: if the signal-to-noise is small, the noise will dominate and the "spikes" of the signal will be subdominant; if the signal-to-noise is large, the signal periodogram, *at the scan frequency and its harmonics* will stand out. The signal-to-noise in these bands is naturally related to the signal-to-noise per beam; for the case of white noise one has

$$\frac{S}{N}\bigg|_{\text{band}} \simeq \frac{S}{N}\bigg|_{\text{beam}} \tag{42}$$

where the constant of proportionality is dependent on the characteristics of the experiment (see Appendix A1). The larger the S/N of the experiment, the further away the initial estimate of the periodogram will be from the ML estimate. Prior expectation of the smoothness of $\widetilde{N}(\omega)$ can also be used to distinguish between noise and spikes from the signal, although, as mentioned above scan-synchronous noise (which produces similar spikes in the periodogram) is very difficult to disentangle from the signal. From Figure 1 we can read off the effect for different levels of signal-to-noise. For η_T^i, we can see that, regardless of the level of signal-to-noise, between $0.2 - 5$, the algorithm converges to better than one percent in the first 5 iterations. For signal-to-noise up to one, the speed of convergence is approximately constant, while for high signal-to-noise, after a transient of slow convergence, the map converges to the
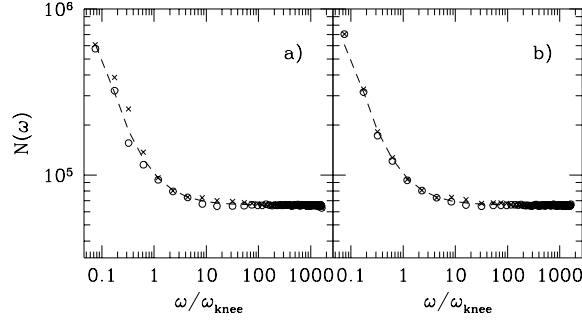
Figure 2. Results from 200 realization of the fence (a) and poles (b) strategy for signal-to-noise of 1. The dashed line is the ensemble average periodogram, open circles are the average of the MLE estimate after 5 iterations and crosses are the average of the initial conditions of the periodogram as defined in Equation 38
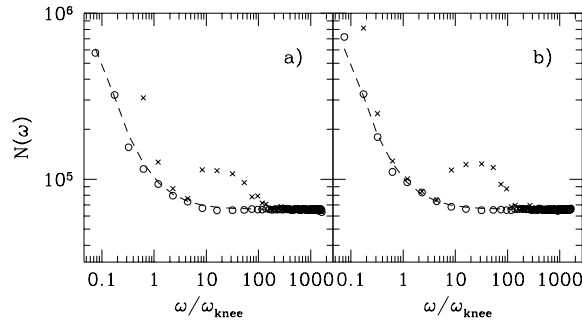

Figure 3. Results from 200 realization of the fence (a) and poles (b) strategy for signal-to-noise of 5. The dashed line is the ensemble average periodogram, open circles are the average of the MLE estimate after 5 iterations and crosses are the average of the initial conditions of the periodogram as defined in Equation 38

fixed point much faster. The evolution of η_N^i is much more sensitive to the signal-to-noise: the higher the signal-to-noise the slower the convergence at each iteration.

Thus far we have shown that the algorithm converges to a fixed point which we know is the MLE estimate of T_p and $\bar{N}_\alpha$. It is now important to check how biased such an estimate is, *i.e.*, whether for an ensemble of realizations of the noise time series the average of the MLE $\bar{N}_\alpha$ is centred on the true ensemble average $\bar{N}_\alpha$. To do so we have generated 200 realizations of the mock experiments for both fence and poles scans. In Figure 2 we plot the results for an experiment with signal-to-noise per beam of order unity. The open circles are the MLE of the $\bar{N}_\alpha$ after 5 iterations of our algorithm, and as we see match up with the ensemble average noise periodogram (dashed lines) for both the poles and fence scans. Moreover we find that the initial estimate of the noise periodogram, $\bar{N}_\alpha^{(0)}$ (crosses), is also a good estimate of the $\bar{N}_\alpha$ at this level of signal-to-noise.
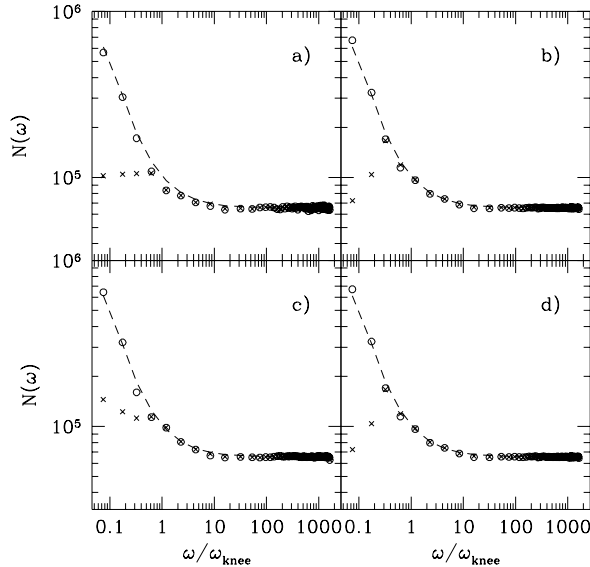
Figure 4. Results from 200 realizations of the fence (a) and poles (b) strategy for signal-to-noise of 1 and fence (c) and poles (d) strategy for signal-to-noise of 5. The dashed line is the ensemble average periodogram, open circles are the average of the MLE estimate after 5 iterations and crosses are the average of the initial conditions of the periodogram as defined in Equation 43

Naturally, for higher signal-to-noise per beam, the initial estimate is not that good. In Figure 3 we plot the ensemble average $\widetilde{N}(\omega)$ as a dashed line, $\bar{N}_\alpha^{(0)}$ as crosses and the MLE of $\bar{N}_\alpha$ as open circles for experiments with signal-to-noise of 5. In the high signal-to-noise range of the periodogram, which corresponds to $\omega > \omega_{\text{knee}}$, $\widetilde{N}_\alpha^{(0)}$ is a very bad estimate of $\bar{N}_\alpha$; one is misinterpreting signal as noise. We clearly see, however the the MLE estimate of $\bar{N}_\alpha$ is centred on the $\widetilde{N}(\omega)$, a result which is not surprising given what we found in our analysis of the convergence of the algorithm.

A possible alternative to initial conditions, $T_p^{(0)}$ and $\widetilde{N}_\alpha^{(0)}$, of Equation 38 is to start with the raw map:

$$T_{\text{raw}}^{(0)} = (A^T A)^{-1} A^T d. \tag{43}$$

Indeed in the absence of low frequency, $1/\omega$ noise, this is the MLE solution to the map. We have regenerated our monte carlo results using these initial conditions and present them in Figure 4. The first thing to note is that the MLE of $\bar{N}_\alpha$ is centred on the ensemble average $\widetilde{N}(\omega)$ just as in the original choice of $\widetilde{N}^{(0)}$. Again this is evidence that the MLE fixed point is a strong attractor of the iterative algorithm, and relatively insensitive to initial conditions. In addition we find that $\widetilde{N}_\alpha^{(0)}$ constructed from the raw map estimate of the signal is a bad estimator for $\bar{N}_\alpha$ in the low frequency range. This is true in both low and high signal-to-noise experiments given that at low frequencies the noise component of the periodogram always dominates: the raw map misrepresents the low frequency noise as signal and subtracts it away from the data stream as such.

4   DIPOLE CALIBRATION

In this section, we apply the formalism developed in the earlier sections to a somewhat different problem: the calibration of anisotropy experiments off the CMB dipole. Nowadays,

many anisotropy experiments have a "spinning mode" when they rotate in a circle of constant elevation on the sky (Lee *et al* 1999). (Over short periods of time, this scan is approximately a scan at constant "latitude" relative to some fixed pole on the sky.) The circle is chosen to have sufficiently large opening angle such that the measurement is sensitive to large-angle CMB anisotropy. The amplitude of the CMB dipole is sufficiently high that the data for these scans is dominated by the modulation of the dipole around the circle. In this case, the data are given by

$$d_i = n_i + A_{ip}(\alpha_1 x_p + T_p + \alpha_2 g_p) \tag{44}$$

where n_i and T_p are noise and anisotropy signal as before, and x_p is the dipole pattern at pixel p, which has a well-measured amplitude and well-understood pattern. The constant of proportionality for the dipole, α_1, is then the calibration of the detector. We also allow contamination from the Galaxy at that pixel, $\alpha_2 g_p$, where α_2 allows an extrapolation from the frequencies where the Galaxy is well-measured to those observed by the experiment under consideration. (Of course, if we take the signal, T_p, or the Galaxy, g_p, to be in real temperature units, they too must include the calibration parameter.)

We wish to basically apply the same rationale as before: determine α along with the other parameters, or ideally determine α marginalizing over the other parameters. For the purposes of dipole calibration, we can ignore the intrinsic CMB contribution, or, for a particular power spectrum, include it in the noise (because of the circular nature of the scan, the isotropy of the CMB translates to an effective stationarity for circular scans, so the CMB signal contribution could easily be included). Because the signal-to-noise of the dipole measurements is so high, it hardly matters. We can thus rewrite the above equation in a suggestive form:

$$d_i = n_i + B_{i0}\alpha_0 + B_{i1}\alpha_1 = \nu_i + B_{ij}\alpha_j \tag{45}$$

where $B_{i0} = A_{ip}x_p$ and $B_{i1} = A_{ip}g_p$. In this form, we immediately see that we can use the techniques developed above to estimate $\{\alpha_0, \alpha_1\}$. For known noise, the best-fit calibration would be

$$\alpha_i = \left(B^T N^{-1} B\right)_{ii'}^{-1} \left(B^T N^{-1} d\right)_{i'} \tag{46}$$

In fact, if our model of the Galaxy were perfect, we could just set $\alpha = \alpha_1 = \alpha_2$ and estimate that single parameter:

$$\alpha = \left[(B_{i0} + B_{i1})^T N^{-1}(B_{i0} + B_{i1})\right]^{-1} (B_{i0} + B_{i1})^T N^{-1} d \tag{47}$$

where now there are no matrix manipulations (except for N^{-1}). However, our extrapolation of galactic models and observations remains sufficiently uncertain that it is probably best to wholly ignore the data within the galactic plane, which amount to a small number of pixels in a dipole scan.

With unknown noise, we can again iterate to simultaneously determine the noise power spectrum and the calibration. However, there is one significant difference in this case: because the dipole scans describe nearly perfect circles on the sky, the dipole signal is nearly sinusoidal (and the Galaxy signal, although complicated, is nearly periodic with the same period). Thus, most of the signal power will be located in a very narrow frequency bin (or harmonics thereof) and will have much higher signal-to-noise than the CMB anisotropy. This has several practical effects. Outside of this bin, the noise power can be determined by simply taking the periodogram of the data. If we expect the noise power spectrum to be smooth compared to the width of the dipole spike, we may be able to estimate the noise outside of the spike and smoothly extrapolate into the region shared with the dipole. In this case, we expect

that the determination of the noise and calibration to be nearly independent, and so the marginalization exercise above should prove superfluous (as should any need for iteration).

The simplicity of this result is somewhat compromised by the presence of the galactic signal. To a very good approximation, we can assume that the contribution from the Galaxy is zero except for a small number of pixels where it overwhelms the rest of the data, and again that the dipole signal is much larger than the anisotropy signal. As our models of moderate-latitude galactic emission improve, this information can be incorporated. But because of the known angular frequency of the dipole signal, it is safe to remove any part of the data possibly contaminated by galactic emission.

In this case, the data are

$$d_i = n_i + \alpha A_{ip} x_p \tag{48}$$

where we also take

$$\langle n_i^2 \rangle \to \infty, \qquad \text{where } \theta_i \in \text{galaxy}. \tag{49}$$

The formally infinite noise in these pixels just amounts to ignoring those rows and columns in matrix manipulation.

How does this method compare with more traditional methods of dipole calibration? Rather then encompass all of the noise into a power spectrum, it is customary to split the noise into

$$n_i = w_i + (a + bi) \tag{50}$$

where a and b are constant over short segments of the data, and w_i is a white noise component. That is, the noise is described as an offset, a linear drift, and white noise, over sufficiently short timespans. The amplitude of the noise, $\sigma^2 = \langle w_i^2 \rangle$ is estimated and a least-squares fit to a, b and the calibration is done on the short segments and combined at the end of the process. This can be complicated somewhat by marginalizing over the (a, b) in each segment and finding a single, global calibration and errors. To the extent that this is a good model for the noise, this procedure should find correct results. Note, though, that Eq. 50 is not a completely general description of noise while our result simply assumes Gaussian and stationary noise.

Of course, this procedure can be extended to deal with any sort of "template," that is, signals of known shape but unknown amplitude. Such amplitudes can be estimated or even marginalized over in the usual way for Gaussian likelihoods. Here, though, we present it as a superior way to account for low-frequency noise when calibrating experiments.

5   DISCUSSION & CONCLUSIONS

In this paper we have addressed the task of correctly estimating noise in current CMB experiments. Our main points and conclusions are:

(i) Long term, correlated noise is ubiquitous in current CMB experiments. The dependence of the noise characteristics on the specifics of the actual observational conditions make it essential to devise an algorithm which can estimate the noise from the data stream itself.

(ii) Under the assumption of stationarity and Gaussianity of the noise we present the Maximum Likelihood estimates of both signal and noise from a data stream. These estimates are solutions to a coupled set of nonlinear equations which must be solved iteratively.

(iii) Given that we are now jointly estimating the signal and noise, there will be correlations between the map and the pixel noise covariance matrix. The standard expression for the

pixel noise covariance matrix is now modified by a correction term which is effectively proportional to the number of pixels and inversely proportional to the width of the bands one is using to characterize the noise periodogram.

(iv) If we forgo estimating the noise periodogram and marginalize over it, we obtain an estimate for the map and its covariance which is effectively equivalent to the joint estimate of map and noise in the limit of wide bands.

 (v) We show that, for wide enough bands, the iterative method converges quickly to the fixed point. For convergence better than a few percent, this can be achieved in 4-6 iterations, for $S/N \simeq 0.2 - 5$.

(vi) An analysis of two standard observational strategies shows that our estimate is unbiased. We show that the two conventional alternative methods are biased for high signal-to-noise.

(vii) It is conventional wisdom that an ideal observational strategy has signal-to-noise of unity (Tegmark 1997). We show that, in this regime, a good approximation to the estimator of the noise periodogram is the banded periodogram of the data stream. Thus one is able to avoid the time consuming process of performing multiple iterations.

A few additional comments should be made:

 (i) For Gaussian theories one can by-pass the map estimation step altogether, constructing an iterative algorithm that estimates $\tilde{N}(\omega)$ and C_ℓ directly. This sensibly puts the spatial power spectrum of the signal and the temporal power spectrum of the noise on the same footing. A priori, however, the computational expense of such an algorithm is prohibitive; however, it is possible to simplify the matrix manipulations exploiting the stationarity of the noise and the isotropy of the signal.

 (ii) We have not addressed the computationally feasibility of our method; it is well known that for data sets of current CMB experiments and the expected megapixel datasets there is a serious problem with actually performing all the matrix manipulations (Borrill 1998; Delabrouille 1998; Górski, Hivon & Wandelt 1999). In the case of our method these problems are of course worse: one must perform these matrix manipulations multiple times. We have limited ourselves to implementing our procedure assumimg the limitations of the standard map making procedure (we do not think that there is an alternative more efficient but accurate methods which can be applied for large data sets) and are therefore at the limit of computational capability with regards to data sets from experiments such as BOOMERanG and MAXIMA. A possible approach is to make local (in time) estimates of the noise periodogram by restricting oneself to subsections of the time stream which are long enough to show the long term correlations of the noise but small enough to be computationally tractable. This approach is indeed justified in some of the current experiments which have "AC coupled detectors" *i.e.*, whose signal is high-pass filtered by the hardware.

(iii) We have also not addressed the problem of "gaps" in the data *i.e.*, the fact that sections of the data may be contaminated with, for example, cosmic rays, high amplitude galactic emission, fast detector transients, etc. There exist a variety of techniques for estimating the Fourier transform of unevenly sampled data, from the more conventional ones like the Lomb periodogram (Press et al 1993) to advanced multitaper techniques (Komm *et al* 1999). For many data sets, only a small fraction of the time series is in gaps; a simple filling-in procedure followed by a standard FFT periodogram will be adequate.

(iv) In Section 4 we have only considered spatial templates (such as the Galaxy or the dipole). However, many CMB experiments suffer from the presence of signals which have a well defined structure in time (Kogut 1996). Examples of such signals are detector variations

due to the mechanical motion of the experimental apparatus or microphonic contaminants with well defined frequency and phase. Once again this can be dealt with in the way described in Section 4 albeit with obvious modifications to Eqs. 45 and 46 to include temporal templates.

(v) For ground-based and balloon-borne experiments an essential problem is atmospheric contamination (De Bernardis & Masi 1998). The template is now spatially and temporally varying and therefore quite difficult to characterize in detail. A possible way of dealing with this contaminant is to have a channel dedicated to measuring atmospheric emission at high frequency during the time of observation; one then uses the data stream of this channel as a template. Of course this is an open problem and must be dealt with carefully in current CMB experiments.

The techniques we have described here will be applied to the MAXIMA (Lee *et al* 1999) and BOOMERanG (De Bernardis 1999) experiments.

APPENDIX A1: ANALYTICAL MODEL OF GREAT CIRCLE SCAN

One would like to relate the signal-to-noise in the map to the signal-to-noise in the time series. To do so we will work out a simple analytical model of a one dimensional experiment (Delabrouille, Gorski & Hivon 1998). We shall consider a great circle scan *i.e.*, choosing $\mathbf{x}_p = (\theta_p, \phi_p)$ with $\theta = \pi/2$ and $0 < \phi < 2\pi$. If we assume a scan speed $\dot{\phi} = 2\pi/T_s$ we can define the pointing matrix to be

$$A_{ip} = \delta[(i\delta t\dot{\phi})\mathrm{mod}(2\pi) - \phi_p].\tag{A1}$$

The periodogram of the data stream is

$$\left|\tilde{d}(\omega_k)\right|^2 = \left|\tilde{A}_{kp}T_p\right|^2 + \widetilde{N}(\omega_k)\tag{A2}$$

where

$$\tilde{A}_{kp} = \frac{2\pi}{\dot{\phi}}\frac{\sin\left(\frac{n_s\pi\omega_k}{\dot{\phi}}\right)}{\sin\left(\frac{\pi\omega_k}{\dot{\phi}}\right)}e^{i(n_s-1)\pi\omega_k/\dot{\phi}}e^{i\phi_p\omega_k/\dot{\phi}}\tag{A3}$$

and we have assumed that the total time of the scan is $T = n_s T_s$, corresponding to n_s consecutive scans of the great circle.

As we can see from Figure A1, $|\tilde{A}_{\omega\phi}|^2$ is highly peaked at multiples of the scan frequency $w_s = 2\pi/T_s$ (with peak amplitude $(n_s T_s)^2$); this means that the signal in the time series is concentrated at these frequencies. A natural measure of the signal-to-noise in the time series is to compare the integrated power in signal for a band around a multiple $k\omega_s$ of the scan frequency with the integrated power in noise within the same frequency band, *i.e.*,

$$\left.\frac{S}{N}\right|_{\mathrm{band}} = \frac{\int_{(k-1/2)\omega_s}^{(k+1/2)\omega_s} d\omega\langle|\tilde{s}|^2(\omega)\rangle}{\int_{(k-1/2)\omega_s}^{(k+1/2)\omega_s} d\omega\langle\widetilde{N}(\omega)\rangle}.\tag{A4}$$
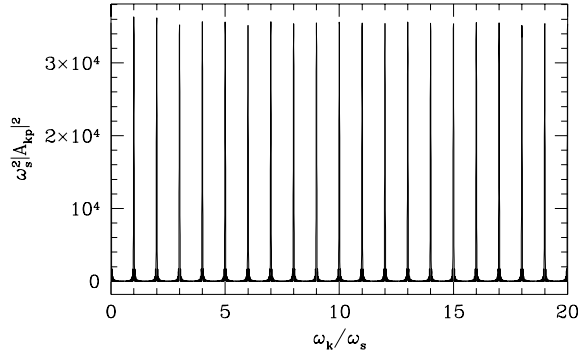
Figure A1. Dependence of $\omega_s^2|\tilde{A}_{kp}|^2$ on ω

Naturally for arbitrary noise and signal, this ratio will depend on the multiple of the scan frequency one is looking at. For example if the noise has a $1/\omega$ the $\frac{S}{N}|_{\text{band}}$ will be much smaller at low ω than at high ω. Let us consider a simple case where the comparison is independent of k: $\langle T_p T_{p'} \rangle = \sigma_T^2 \delta_{pp'}$ and $N_{tt'} = \sigma_N^2 \delta_{tt'}$. We then find that (looking at $k = 0$):

$$
\int_{(k-1/2)\omega_s}^{(k+1/2)\omega_s} d\omega \langle |\tilde{s}|^2(\omega) \rangle = 2\pi n_s T_s \sigma_T^2
$$
$$
\int_{(k-1/2)\omega_s}^{(k+1/2)\omega_s} d\omega \langle \widetilde{N}(\omega) \rangle = 2\pi \sigma_N^2. \tag{A5}
$$

The signal-to-noise in a given band is then

$$
\frac{S}{N}\Big|_{\text{band}} = \frac{n_s T_s \sigma_T^2}{\sigma_N^2} = \frac{S}{N}\Big|_{\text{beam}}. \tag{A6}
$$

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