BASIC HAMILTONIAN MECHANICS

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1. INTRODUCTION

At the time of Newton, mechanics was considered mainly in terms of forces, masses and velocities, since these quantities were directly tangible in terms of everyday experience. However, the subsequent evolution of celestial mechanics called for more compact and general methods of handling dynamical systems, and led to the increasing use of potentials as the relevant quantities in the mathematical treatment. Another important development of this period was to free classical mechanics from the constraints of specific co-ordinate systems and to facilitate the transformation from one system to another. This is not only a matter of convenience but a powerful tool for finding invariants of the motion, and a fundamental feature of the Hamiltonian formulation.

Joseph-Louis Lagrange was one of the outstanding pioneers of this development; his Méchanique Analytique laid the foundations of the analytic, in contrast with the geometrical, approach to generalised dynamics. Nevertheless he suggested, apparently as a concession to tradition, that mechanics (with the time dimension included), might be considered as the geometry of a four-dimensional space, which was in a sense a precursor of the structure of special relativity.

Generalised classical mechanics has developed considerably since the time of Lagrange and remains not only a broad and fundamental part of classical physics as a whole but has become an important part of the framework on which quantum mechanics has been formulated. The range of topics is so large that even in the restricted field of particle accelerators our coverage here of Lagrangian and Hamiltonian dynamics can only be rather limited. More detailed presentations of the subjects can be found in the Bibliography and are cited in the text wherever the need is specially manifest.

The Lagrange equations of motion can be presented in a number of different versions, depending on the nature of the problem and the form of the dynamical constraints. A detailed discussion of these topics is given by Goldstein [1] in Chapter 1; for the application to accelerator physics and the Hamiltonian formulation it is sufficient to consider a restricted set of conditions, namely the motion of charged particles in electromagnetic fields, a domain which is thoroughly covered by Jackson [2] in Chapter 12. Furthermore we shall be dealing with the motion of single particles, taking no account of the forces due to space charge.

In the Lagrangian formulation the dynamical behaviour of a system with k degrees of freedom is characterised by a set of generalised "position" co-ordinates q_k, generalised "velocity" co-ordinates $\dot{q}_k = dq_k / dt$ and the independent variable t, which is often, but not necessarily, the time variable. The "Lagrangian" or Lagrange function L of the form:

$$L(q_k, \dot{q}_k, t) \tag{1}$$

is then a function of $2k$ dynamical variables.

2. OUTLINE OF LAGRANGIAN AND HAMILTONIAN FORMALISM

In the simplest, non-relativistic case where the forces can be derived from a scalar potential V, independent of velocity, the Lagrangian takes the specific form:

$$L(q,\dot{q},t) = T(q,\dot{q},t) - V(q,t), \tag{2}$$

where T is the kinetic energy, V is the potential energy and the index k is implicit.

In the presence of electromagnetic fields, which can be time-dependent, a generalised potential, and consequently a Lagrangian, can be formulated by combining the Lorentz equation for the force with Maxwell's equations for the electromagnetic fields.

The Lorentz equation for the force $\boldsymbol{F}$ on a particle of charge e moving with velocity $\boldsymbol{v}$ in an electromagnetic field is given by

$$\boldsymbol{F} = e[\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}] \ , \tag{3}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are respectively the electric and magnetic fields. Now in the case of static fields the part of the force $\boldsymbol{F}$ arising from $\boldsymbol{E}$ can be derived from a scalar potential ϕ, but the presence of a magnetic field and time-variation of either, requires the introduction of a vector potential $\boldsymbol{A}$, which contributes to both electric and magnetic forces.

Since we are not taking account here of space-charge forces, only two of Maxwell's equations are required, namely:

$$\text{curl}\ \boldsymbol{E} + d\boldsymbol{B} / dt = 0 \tag{4}$$

$$\text{div}\ \boldsymbol{B} = 0 \ . \tag{5}$$

From Eq. (5) it follows that one can write:

$$\boldsymbol{B} = \text{curl}\ \boldsymbol{A} \ , \tag{6}$$

where $\boldsymbol{A}$ is the magnetic vector potential. Substituting Eq. (6) into Eq. (4) expanding and rearranging terms, it is then straightforward but somewhat lengthy to show that the non-relativistic Lagrangian for time-dependent electromagnetic fields is:

$$L(q,\dot{q},t) = T(q,\dot{q},t) - U(q,\dot{q},t) \ , \tag{7}$$

where the scalar potential V of Eq. (2) has been replaced by a generalised potential:

$$U = e\phi - \boldsymbol{A} \cdot \boldsymbol{v} \ . \tag{8}$$

The relativistic Lagrangian is not just the difference between kinetic and potential energies, and is more complicated to derive formally. A full discussion is given in Chapter 7 of Goldstein [1]; here we simply present the form appropriate to accelerator dynamics, viz:

$$L = -mc\left[c^2 - \boldsymbol{v} \cdot \boldsymbol{v}\right]^{1/2} - e\phi + e\boldsymbol{A} \cdot \boldsymbol{v} \ . \tag{9}$$

We note that in the non-relativistic limit, $v << c$, this reduces almost to the form of Eq. (7), apart from a constant $-mc^2$, which vanishes on subsequent differentiations.

In the framework of Hamiltonian theory the importance of the Lagrangian lies in the formulation of Hamilton's Principle of Stationary Action (sometimes called "least action" which is usually but not always the case). This principle states that the action integral defined by:

$$S = \int_{t_1}^{t_2} L(q,\dot{q},t)dt \ , \tag{10}$$

is an extremum for the dynamically true path of the time trajectory between t_1 and t_2, i.e.

$$\delta S = \delta \int_{t_1}^{t_2} L dt = 0 \text{ to first order} \ . \tag{11}$$

The evaluation of this by the calculus of variations, which is very clearly explained in Vol. II of the Feynmann Lectures [3], results in the Lagrangian equations of motion for a conservative system:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \ . \tag{12}$$

The quantity $(\partial L / \partial \dot{q}_k)$ is known as the canonical momentum; in the simplest cases where there is no vector potential it is the same as the mechanical momentum, but the presence of an $\boldsymbol{A} \cdot \boldsymbol{v}$ term gives rise to a so-called magnetic momentum. One then recognises that Eq. (12) is Newton's Second Law in a more modern guise.

Summarising, for a system of k degrees of freedom the Lagrangian is a function of $2k$ dynamical variables (generalised co-ordinates and velocities) and the "time" t, k of these variables $(\dot{q}_k)$ being the "time" derivatives of the other k variables (q_k). The corresponding Lagrange equations consist of a set of k second-order differential equations describing the motion of the system.

The Hamiltonian formulation of mechanics describes a system in terms of generalised co-ordinates (q_k) and generalised momenta (p_k), the same as the canonical momenta we identified in Eq. (12) above. The change of basis from the set $(q_k,\dot{q}_k,t)$ to the set (q_k,p_k,t) is obtained through a Legendre transformation, defined by the function:

$$H(q,p,t) = \sum_k p_k \ \dot{q}_k - L(q,\dot{q},t) \ . \tag{13}$$

We can take the differential of H on the left-hand side as

$$dH = \frac{\partial H}{\partial t} dt + \sum_k \frac{\partial H}{\partial q_k} dq_k + \sum_k \frac{\partial H}{\partial p_k} dp_k \tag{14}$$

or alternatively, from the right-hand side of (13) as

$$dH = \sum_k p_k \ d\dot{q}_k + \sum_k \dot{q}_k \ dp_k - \sum_k \frac{\partial L}{\partial q_k} dq_k - \sum_k \frac{\partial L}{\partial \dot{q}_k} \ d\dot{q}_k - \frac{\partial L}{\partial t} dt \ . \tag{15}$$

Designating the canonical momenta of Eq. (12) as

$$p_k = \frac{\partial L}{\partial \dot{q}_k} \ , \tag{16}$$

the first and fourth summations cancel and the remaining terms can be identified with the corresponding terms in (14)

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} \tag{17}$$

$$\dot{q}_k = \frac{\partial H}{\partial p_k} \tag{18}$$

$$\dot{p}_k = -\frac{\partial H}{\partial q_k} \ . \tag{19}$$

The function $H(q,p,t)$ is the Hamiltonian and Eqs. (18) and (19) are the Hamilton equations of motion. They are first order, 2k in number for k degrees of freedom, and show a remarkable symmetry of form between the generalised position co-ordinates q_k and their conjugate momenta p_k. This symmetry leads to very flexible transformation properties between sets of dynamical variables.

For non-relativistic motion the Hamiltonian is often, though not necessarily, the sum of potential and kinetic energies

$$H(q,p,t) = T + U \ . \tag{20}$$

A relativistic Hamiltonian for a single particle in an electromagnetic field can be derived from the Lagrangian of Eq. (9). In Cartesian co-ordinates, $k = x, y, z$, the canonical momenta given by (16) are

$$p_k = m_0 \gamma \ v_k + eA_k \tag{21}$$

and differ from the component $m_0 \gamma v_k$ of the mechanical momentum by the contribution eA_k, the electromagnetic momentum. The resulting Hamiltonian is easily shown to be

$$H(q,p,t) = e\phi + c\left[(\boldsymbol{p} - e\boldsymbol{A})^2 + m_0^2 c^2\right]^{1/2} \ , \tag{22}$$

where the p_k have been expressed in vector form.

3. SOME PROPERTIES OF THE HAMILTONIAN

From the Hamiltonian $H(q_k, p_k, t)$ the Hamilton equations of motion are obtained by

$$\dot{q}_k = \frac{dq_k}{dt} = \frac{\partial H}{\partial p_k} \tag{18}$$

$$\dot{p}_k = \frac{dp_k}{dt} = -\frac{\partial H}{\partial q_k} \ . \tag{19}$$

The Poisson bracket of any two dynamical variables $f(q_k,p_k,t)$ and $g(q_k,p_k,t)$ is defined by

$$\{f,g\} \equiv \sum_k \left\{ \frac{\partial f}{\partial q_k}\frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial g}{\partial q_k} \right\} . \tag{23}$$

One sees that

$$\{f,f\} \equiv 0 ,$$
$$\{g,f\} = -\{f,g\} .$$

The time derivative of f

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial t} + \sum_k \left\{ \frac{\partial f}{\partial q_k}\dot{q}_k + \frac{\partial f}{\partial p_k}\dot{p}_k \right\} \\ &= \frac{\partial f}{\partial t} + \{f,H\} \quad \text{(from (18), (19) and (23)) .} \end{aligned} \tag{24}$$

In particular, if $f = H$ we have

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} . \tag{25}$$

If H does not depend *explicitly* on time it is a constant of the motion. Any invariant of the motion not containing t explicitly, has a vanishing Poisson bracket with H.

4. CANONICAL TRANSFORMATIONS

Canonical transformations are of considerable utility in simplifying problems by an advantageous choice of co-ordinate system. In particular they can be used to reduce the number of degrees of freedom of a system by exposing invariant quantities, or quantities that are "almost invariant" apart from a small parameter, permitting perturbation theory to be applied. In non-linear systems they often enable the simple linear part of the motion to be transformed away so that the non-linear part can more easily be handled, without prejudice to the linear motion. Examples of this are treated by Schoch [4] and by Montague [5], the latter case dealing with space-charge forces.

The form of the equations is preserved in transforming between co-ordinate systems (q_k,p_k) and (Q_k,P_k). The necessary and sufficient condition for a transformation to be canonical is

$$\left[\sum_k P_k \; dQ_k - H_1(Q_k,P_k,t)dt \right] - \left[\sum_k p_k \; dq_k - H(q_k,p_k,t)dt \right] = dG \tag{26}$$

where dG is a total differential. This follows from Hamilton's variational principle

$$\delta \int_{t_1}^{t_2} L \; dt = \delta \int_{t_1}^{t_2} \left[\sum_k p_k \, \dot{q}_k - H(q_k, p_k, t) \right] dt = 0 \tag{27}$$

and Eq. (13). The canonical transformation

$$Q_k = Q_k(q_1, q_2 \ldots p_1, p_2 \ldots t) \tag{28}$$

$$P_k = P_k(q_1, q_2 \ldots p_1, p_2 \ldots t) \tag{29}$$

is derived from the generating function G which can have one of four forms:

$$G_1(q,Q,t): \quad p_k = \frac{\partial G}{\partial q_k} \; ; \quad P_k = -\frac{\partial G}{\partial Q_k} \tag{30a}$$

$$G_2(q,P,t): \quad p_k = \frac{\partial G}{\partial q_k} \; ; \quad Q_k = \frac{\partial G}{\partial P_k} \tag{30b}$$

$$G_3(Q,p,t): \quad P_k = -\frac{\partial G}{\partial Q_k} \; ; \quad q_k = -\frac{\partial G}{\partial p_k} \tag{30c}$$

$$G_4(p,P,t): \quad q_k = -\frac{\partial G}{\partial p_k} \; ; \quad Q_k = \frac{\partial G}{\partial P_k} \tag{30d}$$

For all forms the new Hamiltonian is

$$H_1(Q_k, P_k, t) = H(q_k, p_k, t) + \frac{\partial G}{\partial t} \tag{31}$$

5. INTEGRAL INVARIANTS, LIOUVILLE'S THEOREM

The action integral

$$S = \int_{t_1}^{t_2} L \; dt \tag{32}$$

is invariant under a canonical transformation. So also are the phase space integral invariants (Poincaré invariants):

$$J_1 = \iint \sum_k dp_k dq_k \tag{33a}$$

$$J_2 = \iint \iint \sum_{k,i} dp_i dq_k dq_i dq_k \tag{33b}$$

$$J_n = \int \ldots \int dp_1 \ldots dp_n dq_1 \ldots dq_n \tag{34}$$

where the integrals are taken over any arbitrary phase-space submanifold of appropriate dimension (2, 4 ... 2n respectively). For J_1, corresponding to one degree of freedom, i.e. a two-dimensional phase space, the invariance is that of area conservation in the (q,p) plane of a closed trajectory. For sub-manifolds of dimensions between 4 and (2n-2) the interpretation of the Poincaré invariants is much more complicated and we shall not discuss it further here. For a system with n degrees of freedom, the invariance of J_n is Liouville's theorem.

6. LINEAR OSCILLATOR

The energies are:

kinetic $T = \frac{m}{2}\dot{q}^2$ ($m =$ mass)

potential $U = \frac{k}{2}q^2$ ($k =$ spring constant, not to be confused with index) .

The Lagrangian is:

$$L(q,\dot{q}) = T - U = \frac{m}{2}\dot{q}^2 - \frac{k}{2}q^2$$

and the canonical momentum:

$$p = \frac{\partial L}{\partial \dot{q}} = m\ \dot{q}\ ,$$

whence $\dot{q} = \frac{p}{m}$. Using this to replace $\dot{q}$ in T, the Hamiltonian becomes:

$$H = T + U = \frac{p^2}{2m} + \frac{k\ q^2}{2}$$

and the Hamilton equations of motion are:

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m} \quad \text{and} \quad \dot{p} = -\frac{\partial H}{\partial q} = -k\ q\ .$$

These can be expressed as one second-order equation

$$\ddot{q} = \frac{\dot{p}}{m} = -\frac{k}{m}q \quad \text{or} \quad \ddot{q} + \omega^2 q = 0 \quad \text{where} \quad \omega = \sqrt{\frac{k}{m}}\ .$$

We can transform to a new co-ordinate system of action-angle variables using a canonical transformation with the generating function

$$G(q,Q) = \frac{\sqrt{k\ m}}{2}q^2 \cot Q\ . \tag{35}$$

Then using (30a) one has

$$p = \frac{\partial G}{\partial q} = \sqrt{k\ m} \cdot q \cot Q$$

and

$$P = -\frac{\partial G}{\partial q} = \frac{\sqrt{k\ m}}{2} q^2 \csc^2 Q \ ,$$

whence

$$q^2 = \frac{2P \sin^2 Q}{\sqrt{k\ m}} \quad \text{and} \quad p^2 = 2P\sqrt{k\ m} \cos^2 Q \ .$$

Also,

$$H_1 = H + \frac{\partial G}{\partial t} = H \quad (\text{since } \frac{\partial G}{\partial t} = 0) \ ,$$

so

$$H_1(Q,P) = P\sqrt{\frac{k}{m}} \cos^2 Q + P\sqrt{\frac{k}{m}} \sin^2 Q = P\sqrt{\frac{k}{m}} = \omega \ P \ .$$

The transformed Hamilton equations of motion are then

$$\dot{p} = -\frac{\partial H_1}{\partial Q} = 0 \ ; \quad (P = \text{constant and } H_1 \text{ is cyclic in } Q)$$

and

$$\dot{Q} = \frac{\partial H_1}{\partial p} = \omega \ ; \quad \left(Q = \omega t + \phi \quad \text{and} \quad q = \sqrt{\frac{2P}{m\omega}} \sin(\omega t + \phi) \right) \ .$$

This type of canonical transformation is useful in some non-linear problems to transform away the linear part of the motion and treat the non-linear part by perturbation theory.

7. SIMPLE PENDULUM

$$U = m\ g\ \ell(1 - \cos\theta)$$

$$T = \frac{m\ v^2}{2} = \frac{m\ \ell^2\ \dot{\theta}^2}{2}$$

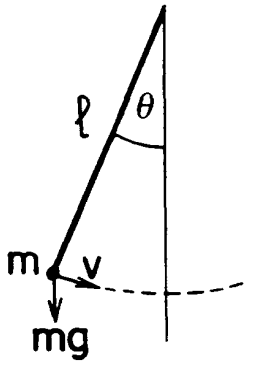
$$L(\theta, \dot{\theta}) = T - U = \frac{m\ \ell^2\ \dot{\theta}^2}{2} - m\ g\ \ell(1 - \cos\theta)$$

$$p = \frac{\partial L}{\partial \dot{\theta}} = m\ \ell^2\ \dot{\theta} \ , \quad \text{so} \quad \dot{\theta} = \frac{p}{m\ \ell^2}$$

$$H(\theta, p) = T + U = \frac{p^2}{2m\ \ell^2} + m\ g\ \ell(1 - \cos\theta)$$

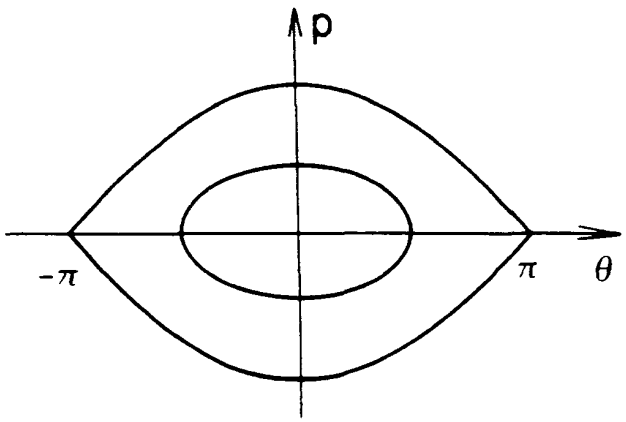
$$\dot{p} = -\frac{\partial H}{\partial \theta} = -m\ g\ \ell \sin\ \theta$$

$$\ddot{\theta} = \frac{\dot{p}}{m\ \ell^2} = -\frac{g}{\ell} \sin\ \theta \qquad \ddot{\theta} + \frac{g}{\ell} \sin\ \theta = 0$$



The Hamiltonian can be written

$$H(\theta,p)=\frac{p^2}{2m\ \ell^2}+2m\ g\ \ell\sin^2\frac{\theta}{2}=\text{ constant }\quad(\text{since }\frac{dH}{dt}=\frac{\partial H}{\partial t}=0)\ .$$



Trajectories of constant H in (θ,p) phase plane.

This example is closely analogous to the problem of phase oscillations in a synchrotron or storage ring.

8. RELATIVISTIC HAMILTONIAN WITH ELECTROMAGNETIC FIELDS

The relativistic Lagrangian for a single particle in an electromagnetic field is, using vector notation

$$L=-m_0c^2\sqrt{1-\beta^2}+e\boldsymbol{A}\cdot\boldsymbol{v}-e\phi \tag{36}$$

where $\boldsymbol{A}$ is the vector potential and ϕ the scalar potential.

Note that $L\neq T-U$ relativistically. However, the canonical momenta p_k are still obtained from

$$p_k=\frac{\partial L}{\partial\dot{q}_k}=m_0\gamma\ v_k+eA_k \tag{37}$$

but are no longer the same as the mechanical momenta $m_0\gamma v_k$ in the presence of a velocity-dependent potential $\boldsymbol{A}$.

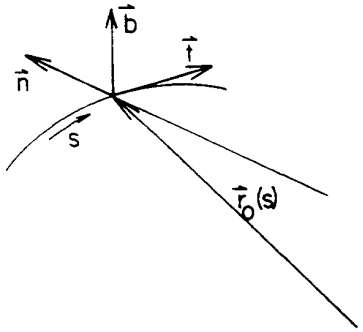
The relativistic single-particle Hamiltonian in the electromagnetic field becomes

$$H(\boldsymbol{q},\boldsymbol{p},t)=e\phi+c\left[(\boldsymbol{p}-e\boldsymbol{A})^2+m_0^2c^2\right]^{1/2}\ . \tag{38}$$

Since $(\boldsymbol{p}-e\boldsymbol{A})^2=(m_0c\ \beta\ \gamma)^2$, $H=e\phi+m_0c^2\gamma$ has the value of the total energy, including the rest energy.

9. APPLICATION TO AN ACCELERATOR ORBIT

We take as reference curve $r_0(s)$ and express any neighbouring point $r(x,y,s)$ in terms of the Frenet unit vectors



$t(s)$ = tangent

$n(s)$ = normal

$b(s)$ = binormal

$$r(x,y,s) = r_0(s) + x\ n(s) + y\ b(s)$$

The following properties hold

$$\begin{aligned} &\frac{dr_0}{ds} = t\ , \qquad \frac{dt}{ds} = \kappa\ n\ , \\ &\frac{dn}{ds} = \tau\ b - \kappa\ t\ , \qquad \frac{db}{ds} = -\tau\ n\ , \end{aligned} \tag{39}$$

where $\kappa = \dfrac{1}{\rho}$ is the curvature and τ the torsion of the reference curve.

The canonically-conjugate momenta p in the co-ordinate system r are obtained from the generating function

$$\begin{aligned} G_2(r,p) &= p \cdot r \\ &= p \cdot [r_0(s) + xn(s) + yb(s)] \end{aligned} \tag{40}$$

which generates the identity transformation

$$p_k = \frac{\partial G}{\partial r_k} = P_k \tag{41a}$$

$$Q_k = \frac{\partial G}{\partial P_k} = r_k \tag{41b}$$

and since $\dfrac{\partial G}{\partial t} = 0$, the Hamiltonian is unchanged. From Eqs. (40) and (41) we have, using (32)

$$\begin{aligned} p_s &= \frac{\partial G}{\partial s} \\ &= p \cdot \left[\frac{dr_0}{ds} + x\frac{dn}{ds} + y\frac{db}{ds} \right] \\ &= p \cdot [t + x(\tau\ b - \kappa\ t) - y\ \tau\ n] \\ &= p \cdot [(1 - \kappa x)t + \tau(x\ b - y\ n)] \end{aligned} \tag{42}$$

If we now define the reference curve to be plane, the torsion τ vanishes. Taking also the other canonical momenta from (41a) we have

$$\begin{aligned} p_x &= \boldsymbol{p}\cdot\boldsymbol{n} \\ p_y &= \boldsymbol{p}\cdot\boldsymbol{b} \\ p_s &= \boldsymbol{p}\cdot\boldsymbol{t}(1-\kappa\ x) \end{aligned} \tag{43}$$

The vector potential $\boldsymbol{A}$ transforms similarly

$$\begin{aligned} A_x &= \boldsymbol{A}\cdot\boldsymbol{n} \\ A_y &= \boldsymbol{A}\cdot\boldsymbol{b} \\ A_s &= \boldsymbol{A}\cdot\boldsymbol{t}(1-\kappa\ x) \end{aligned} \tag{44}$$

One notes that p_S, A_S are generally *not* the components in the tangential direction of the reference curve.

The Hamiltonian (38) can now be written explicitly in terms of the canonical momenta (43) and the vector potential components (44). We choose the Coulomb gauge for Maxwell's equations to make $\phi = 0$ and obtain

$$H(q_k,p_k,t) = c\left[\left(p_x - eA_x\right)^2 + \left(p_y - eA_y\right)^2 + \left\{\frac{p_s - eA_s}{1-\kappa\ x}\right\}^2 + m_0^2c^2\right]^{1/2} \tag{45}$$

where the A_k are functions of position and time in general.

For a cyclic accelerator it is convenient to use s as the independent variable instead of t, since the fields arising from $\boldsymbol{A}$ can readily be expanded in Fourier components periodic around the circumference. Hamilton's principle in Eq. (27) can be re-expressed in terms of an integral along a space trajectory s rather than a time trajectory t, since we are considering the motion of a single particle with a one-one correspondence between time and position. The change of independent variable from t to s is then trivial and we can write Hamilton's principle in the form:

$$\delta\int_{s_1}^{s_2}\left[p_x\frac{dx}{ds} + p_y\frac{dy}{ds} - H\frac{dt}{ds} + p_s\right]ds = 0 \tag{46}$$

where we now consider the new Hamiltonian to be

$$-p_s\left(x,y,t,p_x,p_y,-H,s\right) = F \text{ say }, \tag{47}$$

and where $(t, -H)$ is a new pair of canonically-conjugate variables. The satisfying of Hamilton's variational principle (11) guarantees the validity of the transformation, and the previous results may be taken over with the appropriate changes in notation.

Rewriting (45) in the new form

$$F\left(x,y,t,p_x,p_y,-H,s\right)=-p_s$$
$$=-eA_s-(1-\kappa\ x)\left[\frac{1}{c^2}\left(H^2-m_0^2c^4\right)-\left(p_x-eA_x\right)^2-\left(p_y-eA_y\right)^2\right]^{1/2} \tag{48}$$

where numerically

$$\frac{1}{c^2}\left(H^2-m_0^2c^4\right)=m_0^2c^2\left(\gamma^2-1\right)=m_0^2c^2\beta^2\gamma^2=\boldsymbol{p}\cdot\boldsymbol{p} \tag{49}$$

is the square of the total momentum.

The Hamilton equations of motion are, following Eqs. (18) and (19)

$$x'=\frac{\partial F}{\partial p_x}\ ;\quad y'=\frac{\partial F}{\partial p_y}\ ;\quad t'=-\frac{\partial F}{\partial H} \tag{50a}$$

$$p_x'=-\frac{\partial F}{\partial x}\ ;\quad p_y'=-\frac{\partial F}{\partial y}\ ;\quad H'=-\frac{\partial F}{\partial t} \tag{50b}$$

where the primes denote $\frac{d}{ds}$.

In general the components of the vector potential A are functions of all the dynamical variables and of the independent variable s (the distance along the reference trajectory). The curvature κ, which introduces a kinematic non-linear term (usually small), is generally a function of s. With a given accelerator or storage-ring lattice these terms can be calculated for all the elements (dipoles, quadrupoles, etc.) and expanded as Fourier series in s around the circumference of the machine. The Hamiltonian F in Eq. (48) can then be expressed as a power series expansion in the dynamical variables up to any order desired. The equations of motion may be obtained from Eqs. (50); however, the Hamiltonian can conveniently be used directly for the study of resonance behaviour by selection of near-resonant terms in the machine periodicities and by judicious approximations. Although these procedures are well defined and straightforward in principle, the calculation of the power-series coefficients is in practice rather laborious for all but the simplest cases.

Certain simplifications can often be made in Eq. (48). If A is not an explicit function of t then $H' = 0$ and H is an invariant of the motion and is called a cyclic variable. This corresponds in (49) to a constant value of $m_0 \beta \gamma$, t is an ignorable co-ordinate and the degrees of freedom are reduced from three to two. Such a situation arises for a coasting beam in a storage ring, and as an approximation to the motion in an accelerator on a sufficiently short time scale. Also it is sometimes permissible to approximate bending magnets, quadrupoles, etc. by piecewise constant elements with $A_x = A_y = 0$, except at the ends where they can be expressed as δ-functions. The only component of the vector potential is

$$A_s=\frac{p_0K}{2e}\left(x^2-y^2\right) \tag{51}$$

where $p_0 = m_0 c \beta \gamma$ and K is the gradient parameter. The reference orbit is the axis of the quadrupole and the curvature $\kappa = 0$. Then Eq. (48) simplifies to

$$F(x,y,p_x,p_y) = -\frac{p_0 K}{2}(x^2 - y^2) - \left[p_0^2 - p_x^2 - p_y^2\right]^{1/2}$$

The equations of motion (50) then become, assuming $p_0^2 >> p_x^2 + p_y^2$

$$\dot{p_x} = -\frac{\partial F}{\partial x} = p_0 \; K \; x \tag{52a}$$

$$\dot{p_y} = -\frac{\partial F}{\partial y} = p_0 \; K \; y \tag{52b}$$

$$x' = \frac{\partial F}{\partial p_x} = \frac{p_x}{\left[p_0^2 - p_x^2 - p_y^2\right]^{1/2}} \approx \frac{p_x}{p_0} \tag{52c}$$

$$y' = \frac{F}{p_y} = \frac{p_y}{\left[p_0^2 - p_x^2 - p_y^2\right]^{1/2}} \approx \frac{p_y}{p} \tag{52d}$$

leading to the familiar form of the equations of motion

$$x'' - K \; x = 0$$
$$y'' + k \; y = 0$$

10. ADIABATIC INVARIANTS

In Section 5 we briefly discussed integral invariants, the invariance property being with respect to canonical transformations of phase-space integrals. There is another type of invariance related to the behaviour of a system under slow changes of a parameter normally considered as constant. The original classical model of this Adiabatic Principle, formulated by Einstein around 1911, supposed a weight suspended by a string passing through a hole in the ceiling, forming a pendulum whose frequency could be slowly changed by raising or lowering the string. Einstein showed that, provided the fractional change in frequency during one oscillation period is sufficiently small, the energy divided by the frequency, which is nothing else but the action variable S, is a constant. This principle was of immediate consequence for the understanding of quantum theory and the stability of atoms under changes of environmental fields; subsequently it became of great practical significance in many other branches of physics. In accelerator theory adiabatic invariance is of fundamental importance in ensuring the stability of beams under conditions of changing parameters, as happens during the acceleration cycle in a synchrotron. In particular, an analysis of the phase-space stacking process in proton storage rings requires a full understanding of the adiabaticity conditions.

The basic principle of adiabatic invariance, as formulated in the pendulum with slowly varying frequency, is intuitively rather simple to grasp. For a constant frequency, and in the absence of damping, it is obvious that the sum of potential and kinetic energies is constant. If now the pendulum is gradually shortened, gravitational energy is being supplied to the system by an external force at the same time as the frequency is slowly changing. It seems intuitively plausible that for sufficiently slow changes something must be conserved and the main questions that arise are (i) what is conserved? and (ii) what constitutes "sufficiently slow" in this context?

Surprisingly, these questions are not always very simple to answer, and in many cases need to be examined in the framework of canonical perturbation theory. The subject of adiabatic invariance is covered at some length in Section II-7 of Goldstein [1]; here we limit the discussion to an outline of the method and its application to a simple example.

Since we are interested in systems which involve some kind of periodic or quasi-periodic behaviour, the first step is to apply canonical transformations to the Hamiltonian in order to bring it into the form of action-angle variables, as we did in Section 6 for the linear harmonic oscillator. When we derived the new Hamiltonian H_1 using the generating function $G(q,Q)$ of Eq. (35), it had the same value as the old H, since $\partial G / \partial t = 0$, G having no explicit time-dependence. If we now consider ω to be a slowly-varying function of time, such that in any particular period of oscillation $(1/\omega)(d\omega/dt) = \varepsilon << 1$, an extra term is introduced into H_1; however the transformation remains canonical and the contribution $\partial G / \partial t$ amounts to a perturbation of the original Hamiltonian, which can be handled by perturbation theory.

Supposing that in any particular period τ of oscillation $(1/\omega)(d\omega/dt) = \varepsilon$ is constant, the variation in ω over this period is $\omega = \omega_0 \exp(\varepsilon_\tau) \equiv (1 + \varepsilon\tau)$ for $\varepsilon_\tau << 1$. Then it can be shown by a straightforward but rather lengthy procedure that the fractional secular change in the action variable S over the period is $\Delta S / S = 1/2(\varepsilon\tau)^2$. Thus if $(1/\omega)(d\omega/dt) = \varepsilon$ is already small in first order, the long-term changes in S are very small.

The importance of adiabatic invariance of the action variables is of great significance in many fields, some of which are discussed in detail in Section II-7 of Goldstein [1]. It is in fact the mechanism which enables a ballerina to execute an accelerating pirouette by drawing in the outstretched arms, adiabatically of course!

REFERENCES

[1] H. Goldstein, Classical Mechanics, Second Edition, Addison-Wesley (1980).

[2] J.D. Jackson, Classical Electrodynamics, Second Edition, John Wiley & Sons (1975).

[3] R.P. Feynman, R.B. Leighton and M. Sands, The Feynman Lectures on Physics, Vol. II, ch. 19, Addison-Wesley (1965).

[4] A. Schoch, Report CERN 57-21 (1958). [An early systematic treatment of linear and non-linear resonances in the CERN PS.]

[5] B.W. Montague, Non-Linear Coupling from The Beam-Beam Force in Electron-Positron Storage Rings. Nuclear Instruments and Methods 187 (1981) 335-340, North-Holland.