QUANTUM EXCITATION AND EQUILIBRIUM
BEAM PROPERTIES

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Abstract
Effects arising from the discrete, or quantized, nature of the emission of synchrotron radiation are considered. Combined with the results of the previous Chapter on radiation damping, the equilibrium beam distributions and loss of particles due to finite acceptance (quantum lifetime) are derived. The changes in the equilibrium values that are introduced by insertion devices are also considered.

1. INTRODUCTION

In the previous Chapter it was shown that the loss of energy due to the emission of synchrotron radiation (SR) and its replacement in the r.f. cavities can give rise to a *damping* of the betatron and synchrotron oscillations. When this process was first understood it led to speculation that the bunch size would eventually become so compressed that emission of coherent radiation would set a severe limit on the maximum beam intensity [1–3]. However it was later realized, firstly in connection with the energy oscillations [4,5] and later also for the betatron oscillations [6,7], that the emission of SR gives rise to another effect – quantum excitation – that causes a *growth* in the oscillation amplitudes, and that the combination of the two effects can give result in a stable equilibrium.

How can the emission of SR give rise to both a damping and excitation ? As shown in the previous Chapter, radiation damping is related to the *continuous* loss and replacement of energy. However, energy is lost in discrete units or "quanta", i.e. photons, whose energy and time of emission vary randomly. This randomness introduces a type of noise or diffusion, causing growth of the oscillation amplitudes. The damping effect is linearly proportional to the energy loss or gain, $\delta\varepsilon$, and so the total effect depends on the sum of such events, $\Sigma\ (\delta\varepsilon)_i$, and therefore on the total energy loss per turn U_o, independent of how the photons are distributed in energy. However, as will be seen later, the quantum effect depends on $(\delta\varepsilon)^2$ and so the total effect is no longer simply related to the total energy loss per turn – $\Sigma\ (\delta\varepsilon)_i^2 \neq U_o^2$ – but depends on the numbers of photons with different photon energies i.e. the photon distribution function. A further distinguishing feature of expressions that describe the quantum excitation is that they all contain Planck's constant, h, whereas in the expressions for the radiation damping this factor is absent.

In this Chapter we consider the equilibrium distribution of the particles that results from the combined effect of quantum excitation and radiation damping, and derive expressions for the emittance, energy spread and bunch length. An estimate is also made of the rate of loss of particles resulting from the finite acceptance for the betatron and synchrotron oscillations, known as the quantum lifetime. A basic approach has been taken, following closely that of Sands [8]; more sophisticated treatments may be found in Refs. [9,10]. An introduction is also made to the topics of low emittance lattices and the effect on the equilibrium beam properties caused by insertion devices.

2. ENERGY OSCILLATIONS

2.1 Mean-square energy deviation

We recall the basic equations for the energy oscillations, with no damping:

$$\varepsilon(t) = A\cos(\Omega t - \phi) \quad (1)$$

$$\tau(t) = \frac{-\alpha}{E_0\ \Omega}\ A\sin(\Omega t - \phi) \quad (2)$$

where α is the momentum compaction factor and Ω the synchrotron oscillation frequency. The invariate oscillation amplitude is thus given by:

$$A^2 = \varepsilon^2(t) + \left(\frac{E_0\Omega}{\alpha}\right)^2 \tau^2(t)\ . \quad (3)$$

When a photon is emitted the energy deviation changes, $\varepsilon \rightarrow (\varepsilon - u)$, and so the change in A^2 is therefore:

$$\delta A^2 = -2\,\varepsilon\, u + u^2 \quad (4)$$

The first term is linear in u and corresponds to the radiation damping, as can be seen as follows. If the energy loss (u) were independent of energy deviation (ε) then over the synchrotron oscillation period this term would average to zero, i.e. no damping. However, by including the linear part of the variation of the energy loss with energy deviation, and averaging over one turn, we arrive at an equivalent expression for the damping as derived in the previous Chapter but in terms of A^2 rather than A:

$$\frac{dA^2}{dt} = \frac{-A^2}{T_0}\frac{dU}{dE} = \frac{-2A^2}{\tau_\varepsilon} \quad (5)$$

where τ_ε is the synchrotron oscillation damping time.

We will consider now the second term in the above, which being always positive can be seen to give rise to a growth in A^2. Since each emission is independent the average rate of increase is obtained by summing the effect of the $n(u)\ du$ photons emitted in each energy interval du:

$$\left\langle \frac{dA^2}{dt} \right\rangle = \int_0^\infty u^2\, n(u)\, du = N\left\langle u^2 \right\rangle \quad (6)$$

where $n(u)$ is the photon distribution function introduced in the Chapter on Synchrotron Radiation. Both N and $<u^2>$ vary around the orbit, however since the effects that we are interested in occur slowly with respect to the orbit time we may average over many turns. Also, it can be shown that the betatron and synchrotron oscillations have only a small effect, and so we may simply take the average over the design orbit. Including the radiation damping term, we have therefore the following total rate of change of A^2:

$$\frac{dA^2}{dt} = \frac{-2A^2}{\tau_\varepsilon} + \left\langle N\left\langle u^2 \right\rangle \right\rangle \quad (7)$$

An equilibrium is reached when $dA^2/dt = 0$, in which case the mean value of A^2 is given by:

$$\left\langle A^2 \right\rangle = \frac{\tau_\varepsilon}{2}\left\langle N\left\langle \mathrm{u}^2 \right\rangle \right\rangle \quad (8)$$

It follows that the mean-square equilibrium value of the energy deviation is therefore:

$$\left\langle \varepsilon^2 \right\rangle = \frac{\left\langle A^2 \right\rangle}{2} = \frac{\tau_\varepsilon}{4} \left\langle N \left\langle u^2 \right\rangle \right\rangle \tag{9}$$

2.2 Distribution of the energy deviation

The above calculation results in a value of the mean-square energy deviation, but tells us nothing about the distribution function of the energy deviation, which is also of interest. In the approximation used so far that the energy oscillations are linear with respect to energy deviation, we can write an expression for the energy deviation at a given time as a sum of all the previous photon emissions, including the damping term:

$$\varepsilon(t) = \sum_{i,t>t_i} u_i \exp\left[\frac{-(t - t_i)}{\tau_\varepsilon}\right] \cos\left[\Omega\left(t - t_i\right)\right] \tag{10}$$

Since the typical energy deviation far exceeds the typical photon energy, the sum therefore contains a large number of small terms. These terms are also statistically independent and equally positive or negative (due to the phase factor). Therefore, according to the Central Limit Theorem of probability theory [11], the resulting distribution of the energy deviation is Gaussian, *independent* of the probability distribution function for u. Furthermore, the variance of the distribution is equal to the sum of the variances of the individual terms:

$$\left\langle \varepsilon^2 \right\rangle = \sum_i \frac{\left\langle u^2 \right\rangle}{2} \exp\left[\frac{-2\left(t - t_i\right)}{\tau_\varepsilon}\right] \tag{11}$$

Approximating as an integral and evaluating then gives:

$$\left\langle \varepsilon^2 \right\rangle = \frac{N\left\langle u^2 \right\rangle}{2} \int_{-\infty}^{t} \exp\left[\frac{-2\left(t - t_i\right)}{\tau_\varepsilon}\right] dt_i = \frac{N\left\langle u^2 \right\rangle \tau_\varepsilon}{4} \tag{12}$$

in agreement with Eq. (9). Thus, we can relate the previous mean-square deviation to the standard deviation of the Gaussian distribution for the energy deviation:

$$\sigma_e^2 = \left\langle \varepsilon^2 \right\rangle = \left\langle N \left\langle u^2 \right\rangle \right\rangle \frac{\tau_\varepsilon}{4} \tag{13}$$

It is interesting to note that, as one might expect, the resulting value corresponds closely to the statistical uncertainty in the number of photon emissions that occur in one damping time, multiplied by the typical photon energy:

$$\sigma_\varepsilon \cong \sqrt{N \tau_\varepsilon}\, u_c \tag{14}$$

2.3 Equilibrium energy spread

We use the following results from the previous Chapters:

$$\left\langle u_c^2 \right\rangle = \frac{11}{27} u_c^2 \quad N = \frac{15\sqrt{3}}{8} \frac{P}{u_c} \tag{15}$$

and therefore:

$$\left\langle N \left\langle u^2 \right\rangle \right\rangle = \frac{55}{24\sqrt{3}} \left\langle P\, u_c \right\rangle \tag{16}$$

where,

$$P = \frac{e^2\, c}{6\pi\, \varepsilon_0} \frac{\gamma^4}{\rho^2} \qquad\qquad u_c = \frac{3}{2} \frac{\hbar\, c\, \gamma^3}{\rho} \tag{17}$$

Also,

$$\tau_\varepsilon = \frac{2\, E_0\, T_0}{J_\varepsilon\, U_0} = \frac{2\, E_0}{J_\varepsilon \left\langle P \right\rangle} \tag{18}$$

Inserting in Eq. (13) results in:

$$\sigma_\varepsilon^2 = \frac{55}{32\sqrt{3}} \hbar c\, \gamma^3 \frac{\left\langle 1/\rho^3 \right\rangle}{\left\langle 1/\rho^2 \right\rangle} \frac{E_0}{J_\varepsilon} \tag{19}$$

The relative energy deviation is then:

$$\left(\frac{\sigma_\varepsilon}{E_0}\right)^2 = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} \frac{\gamma^2}{J_\varepsilon} \frac{\left\langle 1/\rho^3 \right\rangle}{\left\langle 1/\rho^2 \right\rangle} \tag{20}$$

which simplifies in the isomagnetic case to:

$$\left(\frac{\sigma_\varepsilon}{E_0}\right)^2 = C_q \frac{\gamma^2}{J_\varepsilon \rho} \tag{21}$$

where the constant C_q is defined by:

$$C_q = \frac{55}{32\sqrt{3}} \frac{\hbar}{mc} = 3.84\; 10^{-13} \; m \qquad \text{(electrons)} \tag{22}$$

Since in most existing rings the chosen bending radius increases roughly as energy [2], the resulting energy spread values are very similar, typically about 0.1% as can be seen from the examples given in the table below.

Table 1
Energy spread in various electron storage rings.

Ring	E (GeV)	ρ (m)	σ_ε/E (%)
EPA	0.6	1.43	0.06
ESRF	6.0	25	0.10
PEP	18.0	166	0.12
LEP	55.0	3100	0.08

2.4 Equilibrium bunch length

A Gaussian distribution in energy results in a similar distribution in the time deviation τ, and hence a Gaussian bunch shape in the longitudinal direction with standard deviation given by:

$$\sigma_\tau = \frac{\alpha}{\Omega\, E_0}\ \sigma_\varepsilon \tag{23}$$

In the isomagnetic case therefore:

$$\sigma_\tau^2 = \left(\frac{a}{\Omega}\right)^2 C_q\ \frac{\gamma^2}{J_\varepsilon\, \rho} \tag{24}$$

Inserting the expression for the synchrotron oscillation frequency,

$$\Omega^2 = \frac{\alpha}{T_0\, E_0}\ e\dot{V}_0 \tag{25}$$

we obtain:

$$\sigma_\tau^2 = \frac{C_q}{\left(mc^2\right)^2}\ \frac{\alpha\, T_0}{J_\varepsilon\, \rho}\ \frac{E_0^3}{e\dot{V}_0} \tag{26}$$

Expressing the slope of the r.f. voltage in terms of the overvoltage, $q = e\hat{V} / U_0$, for the case of a sinusoidal variation in voltage with peak value $\hat{V}$:

$$e\dot{V}_0 = \left(q^2 - 1\right)^{1/2}\ U_0\ \omega_{r.f.} \tag{27}$$

gives the following:

$$\sigma_\tau^2 = \frac{E_1}{2\pi}\ \frac{\alpha\, T_0}{J_\varepsilon\, E_0}\ \frac{1}{\left(q^2 - 1\right)^{1/2}\ \omega_{r.f.}} \tag{28}$$

where $\omega_{r.f.}$ is the angular r.f. frequency, E_1 is a constant = $(55\sqrt{3}\ / 64)\ \hbar c / r_0$ = 1.042 10^8 eV, and r_0 is the classical electron radius (2.818 10^{-15} m).

The bunch length, $c\sigma_\tau$, thus depends on many parameters such as energy, r.f. frequency and voltage, and the momentum compaction factor, which depends on the lattice design. Typical bunch lengths lie in the range 1–5 cm, however there are wide variations as can be seen from the examples in the table below:

Table 2
Natural bunch lengths in various electron storage rings

Ring	E (GeV)	r.f. (MHz)	$c\sigma_\tau$ (cm)
EPA	0.6	19.3	25
SLC damping ring	1.2	714	0.5
ESRF	6.0	352	0.5
LEP	55.0	352	1.6

In a given ring the bunch length is most commonly adjusted by changing the r.f. voltage. If the overvoltage (q) is large, then Eq. (28) shows that the bunch length is inversely proportional to $\sqrt{q}$. Another possibility is to change the r.f. frequency, by using a separate set of accelerating cavities. For example, using two r.f. systems (62.4 and 500 MHz) and r.f.

voltage adjustment the bunch length in the BESSY storage ring was varied over a wIDe range between 0.7 and 8 cm [12]. A further technique that has been used in some cases is an additional r.f. cavity operating on a higher harmonic of the r.f. frequency in order to change the slope of the r.f. voltage, V_0 in Eq. (26).

It should be noted that the bunch length calculated above is usually only obtained in practice with very small beam currents. Most rings exhibit the phenomenum of "bunch lengthening" as a function of the beam current, due a collective interaction of the beam with its surroundings.

3. BETATRON OSCILLATIONS

3.1 Horizontal plane

We recall from the previous Chapter that photon emission at a point with non-zero dispersion gives rise to a change in the off-energy orbit, and hence introduces a change in the betatron motion. For an individual photon of energy u therefore:

$$\delta x_\beta = - D(s) \frac{u}{E_0} \qquad\qquad \delta x_\beta' = - D'(s) \frac{u}{E_0} \tag{29}$$

The betatron oscillation invariant is given by:

$$A^2 = \gamma x^2 + 2\alpha x x' + \beta x'^2 \tag{30}$$

and hence the change due to the photon emission is therefore:

$$\delta A^2 = (\gamma D^2 + 2\alpha D D' + \beta D'^2) \frac{u^2}{E_0{}^2} \tag{31}$$

Only terms in u^2 have been included, since the linear terms correspond to the radiation damping, as was the case with the energy oscillations.

Defining the important quantity, H:

$$H(s) = \gamma D^2 + 2\alpha D D' + \beta D'^2 \tag{32}$$

and following the same procedure as for the energy oscillations, the average rate of increase of A^2 is then given by:

$$\left\langle \frac{dA^2}{dt} \right\rangle = \frac{\left\langle N \left\langle u^2 \right\rangle H \right\rangle}{E_0^2} \tag{33}$$

where as before the average is taken around the design orbit. Including the radiation damping term:

$$\frac{dA^2}{dt} = \frac{-2A^2}{\tau_x} \tag{34}$$

results in an equilibrium with mean-square value given by:

$$\frac{\langle A^2 \rangle}{2} = \frac{\tau_x}{4} \frac{\langle N \langle u^2 \rangle H \rangle}{E_0^2} \tag{35}$$

This defines the important quantity known as the (horizontal) beam emittance, ε_x, which by analogy with the earlier result for the energy oscillations is given as follows:

$$\varepsilon_x = \frac{\langle A^2 \rangle}{2} = C_q \frac{\gamma^2}{J_x} \frac{\langle H / \rho^3 \rangle}{\langle 1/\rho^2 \rangle} \tag{36}$$

The same argument about the cumulative affect of a large number of small deviations can be applied in this case also, leading to the conclusion that there is a Gaussian distribution in the conjugate variables x, x'. Figure 1 illustrates this distribution, which consists of a series of ellipses each with a constant value of A^2. The ellipse with $A^2 = \varepsilon_x$ defines the "1 σ" contour and hence the r.m.s. beam size (σ_x) and divergence (σ_x') of the distributions projected on the x, x' axes respectively.

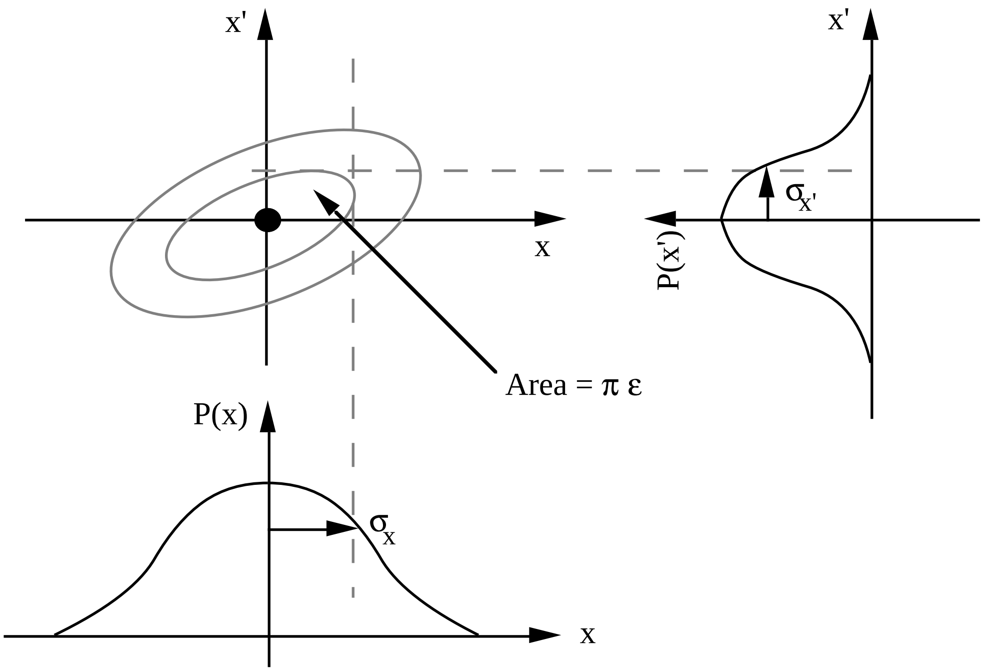

Fig. 1 Gaussian distribution of electron density in the (x,x') plane

It follows from the properties of the ellipse defined in Eq. (30) that:

$$\sigma_x = \sqrt{\varepsilon\, \beta(s)} \quad \sigma_{x'} = \sqrt{\varepsilon\, \gamma(s)} \tag{37}$$

Thus, although the emittance is a constant for a given lattice and energy, the beam size and divergence vary around the design orbit. At a symmetry point in the lattice therefore, where $\alpha = 0$ and $\gamma(s) = 1/\beta(s)$, we have the simple interpretation that the emittance is the product of the beam size and divergence, $\varepsilon = \sigma_x \sigma_x'$.

At a point in the lattice where there is finite dispersion the total horizontal beam size and divergence includes also a contribution from the energy spread. Since the betatron and synchrotron motions are uncorrelated the two widths add quadratically, and hence:

$$\sigma_x = \left\{ \varepsilon_x \, \beta_x(s) + D^2(s) \left(\frac{\sigma_\varepsilon}{E_0} \right)^2 \right\}^{1/2}$$

$$\sigma_x' = \left\{ \varepsilon_x \, \gamma_x(s) + D'^2(s) \left(\frac{\sigma_\varepsilon}{E_0} \right)^2 \right\}^{1/2} \quad (38)$$

In a given ring both the emittance and energy spread vary with E^2, and so the beam size and divergence vary linearly with energy. This is in sharp contrast to the situation for heavier particles for which radiation effects are negligible. In that case, the normalized phase-space area occupied by the beam is constant, in other words emittance is inversely proportional to energy.

3.2 Vertical plane

In the usual case of no bending and hence no dispersion in the vertical plane, the previous calculation would predict no quantum excitation and hence zero emittance in the vertical plane. A small effect arises due to the fact that the photons are not emitted exactly in the direction of the electron motion, which was neglected in the previous Section. If a photon of energy u is emitted at angle θ_z with respect to the median plane, the change in angle of the electron is given by:

$$\delta z' = \frac{u}{E_0} \, \theta_z \quad (39)$$

and hence the change in vertical oscillation amplitude is:

$$\delta A^2 = \frac{u^2}{E_0^2} \, \theta_z^2 \, \beta_z(s) \quad (40)$$

By comparison with the previous formulae, and approximating as follows:

$$\left\langle u^2 \, \theta_z^2 \right\rangle \approx \left\langle u^2 \right\rangle \left\langle \theta_z^2 \right\rangle \; ; \; \left\langle \theta_z^2 \right\rangle \approx 1/2 \, \gamma^2 \quad (41)$$

the resulting equilibrium emittance becomes:

$$\varepsilon_z = \frac{C_q}{2} \, \frac{1}{J_z} \, \frac{\left\langle \beta_z / \rho^3 \right\rangle}{\left\langle 1/\rho^2 \right\rangle} \quad (42)$$

which in the isomagnetic case is:

$$\varepsilon_z = C_q \frac{\langle \beta_z \rangle}{2\rho} \qquad \text{bending magnets} \quad (43)$$

where the average is taken over the bending magnets. Taking into account the value of C_q, Eq. (22), it can be seen that this value is very small indeed.

In practice the vertical emittance is not given by the value above, but arises from other processes:

- coupling of the horizontal and vertical betatron motion, arising from skew-quadrupole field errors. The latter can arise from angular positioning errors of the quadrupole magnets, and also from vertical closed orbit errors in sextupole magnets.
- vertical dispersion errors, arising from vertical bending fields produced by angular positioning errors of the dipoles, and vertical positioning errors of the quadrupoles.

The resulting vertical emittance thus depends only on errors, which can only be estimated statistically. It is common to describe the effect in terms of a coupling coefficient, κ, defined such that the sum of the horizontal and vertical emittances is constant:

$$\varepsilon_x = \frac{1}{1+\kappa}\ \varepsilon_{x_0} \qquad\qquad \varepsilon_z = \frac{\kappa}{1+\kappa}\ \varepsilon_{x_0} \tag{44}$$

The quantity ε_{x_0}, calculated with Eq. (36), is often called the "natural beam emittance". The vertical beam size and divergence are then calculated as follows:

$$\sigma_z(s) = \sqrt{\varepsilon_z\ \beta_z(s)} \qquad\qquad \sigma_z'(s) = \sqrt{\varepsilon_z\ \gamma_z(s)} \tag{45}$$

Typically, without correction, the coupling has a value of 1–10 %. In a given ring the coupling can be adjusted by means of an appropriate distribution of skew-quadrupole magnets, which excite a linear coupling resonance [13].

4. SYNCHROTRON RADIATION INTEGRALS

The equations derived above and in the previous Chapter can be expressed in a general form that is valID also in the case of a non isomagnetic lattice, using the following Synchrotron Radiation Integrals [14]:

$$I_2 = \oint \frac{1}{\rho^2}\ ds \qquad\qquad I_3 = \oint \frac{1}{|\rho^3|}\ ds$$

$$I_4 = \oint \frac{D}{\rho}\left(\frac{1}{\rho^2} - 2k\right) = \oint \frac{(1-2n)\,D}{\rho^3} \qquad\qquad I_5 = \oint \frac{H}{|\rho^3|}\ ds \tag{46}$$

It should be noted that a modulus sign has been included in some cases, in order that the correct values are obtained for elements with an opposite curvature to that of the main bending magnets e.g. in insertion devices. The beam parameters that can be calculated using the integrals are as follows:

Energy loss per turn:

$$U_0 = \frac{e^2}{6\pi\ \varepsilon_0}\ \gamma^4\, I_2 \tag{47}$$

Damping partition numbers:

$$J_x = 1 - \frac{I_4}{I_2} \qquad\qquad J_z = 1 \qquad\qquad J_\varepsilon = 2 + \frac{I_4}{I_2} \tag{48}$$

Damping times:

$$\tau_x = \frac{3\,T_0}{r_0\,\gamma^3}\,\frac{1}{I_2 - I_4} \qquad \tau_z = \frac{3\,T_0}{r_0\,\gamma^3}\,\frac{1}{I_2} \qquad t_\varepsilon = \frac{3\,T_0}{r_0\,\gamma^3}\,\frac{1}{2I_2 + I_4} \tag{49}$$

Energy spread:

$$\left(\frac{\sigma_\varepsilon}{E_0}\right)^2 = C_q\,\gamma^2\,\frac{I_3}{2I_2 + I_4} = \frac{C_q\,\gamma^2}{J_\varepsilon}\,\frac{I_3}{I_2} \tag{50}$$

Natural emittance:

$$\varepsilon_{x_0} = C_q\,\gamma^2\,\frac{I_5}{I_2 - I_4} = \frac{C_q\,\gamma^2}{J_x}\,\frac{I_5}{I_2} \tag{51}$$

where C_q is defined in Eq. (22). It can be noticed in the above that I_2 and I_4 are related to the radiation damping, whereas I_3 and I_5 are related to the quantum excitation.

At this point it is worth mentioning that the above equations are val*ID* also for *protons*, with appropriate numerical values for r_0 and C_q. In a given lattice therefore, the energy loss and damping times are reduced as the fourth power of the ratio of the masses of the particles (m_p/m_e = 1823) and the energy spread and emittance as the third power. Thus, even in the case of the next generation of high energy proton machines such as LHC and SSC, the equilibrium values calculated from the above formulae are so small that in practice the beam dimensions will be limited by other processes such as intra-beam scattering.

5. QUANTUM LIFETIME

5.1 Betatron oscillations

The distribution of beam intensity, both radially and vertically, is Gaussian and therefore in principle extends to infinity. However, the aperture defined by the vacuum chamber is finite and so there will be a constant loss of those electrons that approach the vacuum chamber walls. To calculate the effect we cannot simply use the probability distribution of the beam displacement (x) at a given point in the ring, since the particles at a given x but lying on different phase ellipses will arrive eventually at different maximum values (see Fig. 1). We need therefore the probability distribution for the maximum value reached at that point in the ring, x_{max}, or equivalently the invariant A^2, since $x_{max}^2 = A^2\beta$.

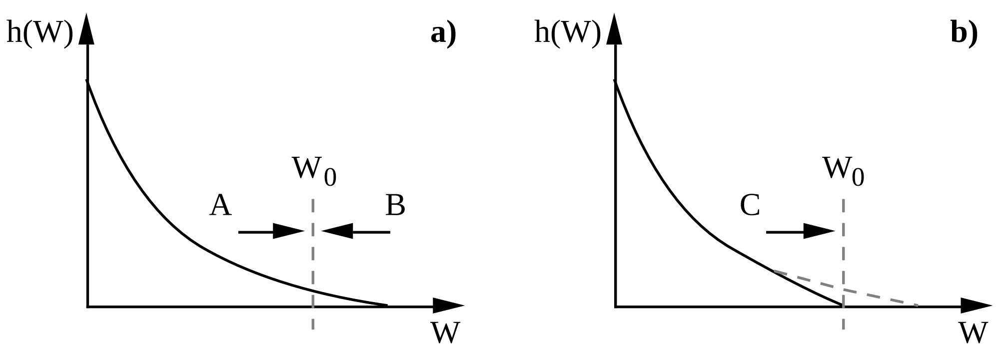

Fig. 2 Distribution of oscillation energies with no aperture (a) and with an aperture (b)

A^2, which we shall call W, is a kind of "oscillation energy". It can be shown that the probability distribution for W is as follows:

$$h(W) = \frac{1}{\langle W \rangle}\,exp\left(\frac{-W}{\langle W \rangle}\right) \tag{52}$$

Consider the number of electrons with oscillation energies increasing above or decreasing below a value W_o which corresponds to the aperture limit, i.e. $W_o = x_{max}^2/\beta$. Figure 2 illustrates the situation. With no aperture, and in the steady state, the number of electrons crossing W_o in each direction must be equal ($A = B$). If the aperture is sufficiently far from the centre of the beam distribution the number of electrons which increase in amplitude, and hence are lost on the aperture limit, will be very nearly the same as if there were no aperture i.e. $A \simeq C$. Hence the loss rate (C) can be estimated by the rate at which particles cross the limit W_o due to radiation damping, in the steady state (B). The rate is therefore:

$$\left(\frac{dN}{dt}\right)_{W_0} = \left(\frac{dN}{dW} \; \frac{dW}{dt}\right)_{W_0} \tag{53}$$

where:

$$\frac{dN}{dW} = N\,h(W) = \frac{N}{\langle W\rangle} \exp\left(\frac{-W_0}{\langle W\rangle}\right) \tag{54}$$

and:

$$\frac{dW}{dt} = \frac{-2W_0}{\tau_x}\,, \quad \text{since } W = \hat{W} \exp\left(-2t / \tau_x\right) \tag{55}$$

Hence:

$$\frac{dN}{dt} = -N \frac{2}{\tau_x} \; \frac{W_0}{\langle W\rangle} \exp\left(\frac{-W_0}{\langle W\rangle}\right) \tag{56}$$

The loss is therefore exponential:

$$N = N_0 \exp\left(-t / \tau_q\right); \; \frac{dN}{dt} = \frac{-N}{\tau_q} \tag{57}$$

where τ_q is the "quantum lifetime" defined by:

$$\tau_q = \frac{\tau_x}{2} \; \frac{\langle W\rangle}{W_0} \exp\left(\frac{W_0}{\langle W\rangle}\right) \tag{58}$$

We can write this in the standard form:

$$\tau_q = \frac{\tau_x}{2} \; \frac{\exp\left(\xi\right)}{\xi} \tag{59}$$

where:

$$\xi = \frac{W_0}{\langle W\rangle} = \frac{x_{max}^{\;2}}{2\sigma_x^{\;2}} \tag{60}$$

The lifetime is determined by the minimum value of x_{max}/σ_x, or equivalently $x_{max}/\sqrt{\beta}$, that occurs at some point around the ring (neglecting the effect of any closed orbit errors). This is known as the limiting acceptance, which because of the dependence on β may not correspond to the smallest physical aperture. Because of the exponential factor the quantum lifetime increases rapidly with ξ and hence x_{max}/σ_x. To illustrate this fact, the table below gives values of the quantum lifetime for a range of values of x_{max}/σ_x for a typical damping time of 10 ms.

Table 3
Quantum lifetime as a function of limiting aperture.

x_{max}/σ_x	5	5.5	6.0	6.5	7.0
τ_q	1.8 min	20.4 min	5.1 h	98.3 h	103 days

Hence, one arrives at the 'Golden Rule' for long lifetime:

$$x_{max}/\sigma_x \;\geq\; 6.5 \qquad (61)$$

i.e. the beam aperture should be at least 13 times the r.m.s. beam size in order that the quantum lifetime does not play a significant part in determining the overall beam lifetime. Other processes will then dominate, such as scattering off resIDual gas molecules, Touschek scattering etc. [15].

5.2 Energy oscillations

There is also a quantum lifetime resulting from the finite r.f. acceptance for the energy oscillations. The synchrotron oscillations become non-linear at large energy deviations, but if we assume that the maximum possible energy deviation, ε_{max}, is large compared to the r.m.s. energy deviation, the calculated loss rate should be approximately correct. We can estimate this loss rate in the same way as for the betatron oscillations. With the oscillation amplitude given by:

$$W = A^2 \;=\; \varepsilon^2 + \left(\frac{E_0\,\Omega}{\alpha}\right)^2 \tau^2 \qquad (62)$$

we obtain a similar result to Eq. (59):

$$\tau_q \;=\; \frac{\tau_\varepsilon}{2}\,\frac{\exp\xi}{\xi} \qquad (63)$$

where:

$$\xi = \frac{W_0}{\langle W\rangle} \;=\; \frac{\varepsilon_{max}^2}{2\sigma_\varepsilon^2} \qquad (64)$$

Inserting expressions for ε_{max} and σ_ε derived earlier we can write:

$$\xi \;=\; E_1\,\frac{J_\varepsilon E_0}{\alpha h}\,F(q) \qquad (65)$$

where $F(q)$ is the 'energy aperture' function defined by [8]:

$$F(q) = 2\left\{\left(q^2 - 1\right)^{1/2} - \cos^{-1}(1/q)\right\} \tag{66}$$

and *h* is the harmonic number of the r.f. system (r.f. frequency divided by orbit frequency).

The overvoltage (q) required to ensure adequate quantum lifetime therefore is smallest in a lattice with small momentum compaction (α) and high harmonic number (h). As an example, the 2 GeV SRS storage ring in its first phase had a relatively large momentum compaction ($\alpha = 0.135$) which gave rise to a high overvoltage requirement of 7.2, for 100 hours quantum lifetime. After changing the magnet lattice in order to reduce beam emittance this also resulted in smaller momentum compaction (α = 0.029) and hence a significantly smaller overvoltage requirement of 2.7, with a consequent reduction in r.f. power demands.

It can be seen that similar terms appear in the expression above for ξ as in the expression for the bunch length, Eq. (28). We can use this fact to make a rough estimate for the bunch length under the conditions that the quantum lifetime is large. Combining the relevant equations we obtain:

$$\left(\frac{\sigma_\tau}{T_{\mathrm{r.f.}}}\right)^2 = \frac{F(q)}{\xi\left(q^2-1\right)^{1/2}} \frac{1}{(2\pi)^2} \tag{67}$$

where $T_{\mathrm{r.f.}}$ is the r.f. period, i.e. the time interval between r.f. buckets. Approximating $F(q)/(q^2 - 1)^{1/2}$ by its limiting value for large overvoltage (=2) then gives:

$$\left(\frac{\sigma_t}{T_{\mathrm{r.f.}}}\right)^2 \approx \frac{1}{2\pi^2\xi} \tag{68}$$

Thus, with a value of ξ = 21 for good lifetime (equivalent to a value of $\varepsilon_{\mathrm{max}}/\sigma_\varepsilon$ = 6.5), we obtain the simple result that the total bunch length (for example, the full w*ID*th half maximum) is about 10% of the bunch separation.

6. LOW EMITTANCE LATTICES

Here we examine two applications of an electron storage ring in which a low emittance is a particular requirement.

6.1 Synchrotron radiation sources

A small beam size and divergence, i.e. a small beam emittance, is a general requirement of synchrotron radiation sources, in order to increase the brightness of the emitted radiation. From the expressions derived earlier, e.g. Eq. (51), it can be seen that to obtain low emittance requires a lattice design which minimizes the average H function in the bending magnets. In particular a small dispersion is required, which is not achieved in the classic separated function lattice, the FODO. Various types of lattice have therefore been developed in order to achieve this [16,17]. The first was the Chasman-Green (CG) structure, which is based on an achromatic arc composed of a pair of bending magnets with a focusing quadrupole in between [18]. Such a design results in zero dispersion in the straight sections between achromats which are therefore suitable locations for insertion devices (see Section 7). The limited flexibility of this lattice has led to the extended CG or double-bend achromat (DBA) and triple-bend achromat (TBA). For each lattice type there is a minimum achievable emittance which is given by an expression of the form:

$$\varepsilon_{x_0,\,\mathrm{min}} = f\ \frac{C_q\,\gamma^2}{J_x}\ \theta_b^3 \qquad (69)$$

where θ_b is the bending angle, assumed equal for all magnets. The factor f varies depending on the lattice type from 0.05 in the case of a DBA lattice to 0.36 for a FODO lattice. It should be noted however that in all cases the emittance increases as the square of the energy and varies inversely with the third-power of the number of bending magnets. In the case of several of the third generation synchrotron radiation sources that are under construction, the typical natural emittance is 7 10^{-9} m rad even though the rings vary widely in energy from 1.5 to 8 GeV. This is achieved by adjusting the number of achromats (and hence θ_b) between 10 and 44.

It can be seen from the equations in Section 4 that some reduction in the emittance can be obtained by increasing J_x which can be achieved by adding a vertically focusing field gradient (k and n positive) in the dipole magnets [19]. Some new storage rings (e.g. ALS, Berkeley, and ELETTRA, Trieste) employ such a gradient field both for emittance reduction and for optimization of the lattice β functions. In the latter case for example, the bending magnet has a field index of 13, giving $J_x = 1.3$.

6.2 Damping rings

A damping ring serves as a temporary storage ring to reduce the emittance of an injected beam by means of radiation damping. It can be seen from Eqs. (33) and (34) that a combination of quantum excitation and radiation damping processes leads to a general equation for the emittance, of the form:

$$\frac{d\varepsilon}{dt} = \text{constant} - \frac{2\varepsilon}{\tau_x} \qquad (70)$$

The emittance (ε) therefore varies in time as follows:

$$\varepsilon(t) = \varepsilon_i \exp\left(-\frac{2t}{\tau_x}\right) + \varepsilon_0\left[1 - \exp\left(-\frac{2t}{\tau_x}\right)\right] \qquad (71)$$

where ε_0 is the equilibrium emittance ($t = \infty$) and ε_i is the injected beam emittance ($t = 0$). For a given storage time, the optimum ring energy is thus a compromise between the need for both small equilibrium emittance (low energy) and fast damping (high energy). Since from Eq. (49) the damping time is proportional to $T_0\rho/E_0{}^3$ it follows that a fast damping requires also a small orbit circumference and small bending radius (high field strength). An example is the 1.21 GeV SLC positrondamping ring [20], which has a small circumference (35 m) and bending radius (2 m, corresponding to a 2 T magnetic field) resulting in a small damping time of 3.1 ms, sufficient to reduce the initial positron beam emittance by about a factor of 300 with a storage time of 11.1 ms.

7. CHANGES IN BEAM PROPERTIES DUE TO INSERTION DEVICES

An insertion device (ID) is a magnetic device located in a straight section of a ring that produces a transverse field component that alternates in polarity along the beam direction. Such devices are used both as special sources of synchrotron radiation [see Chapter on Synchrotron Radiation] and as a means of controlling various beam parameters [21]. In general, IDs give rise to both additional radiation damping and quantum excitation, and so result in different equilibrium values of damping times, emittance and energy spread etc. which depend on the ID parameters and on the lattice functions at the ID location.

The Robinson, or gradient, wiggler introduced in the previous Chapter was the first type of insertion device, and was developed as a means of overcoming the radial anti-damping of combined function lattices. Such a device can also be used in separated function lattices that are already damped in all 3 planes as a means of reducing beam emittance. The dominating effect is the change introduced in the I_4 integral, which affects the damping partition numbers. It can be seen directly from Eq. (51) that an increase in J_x from its usual value of 1 to 2 can reduce the emittance by a factor of 2, while still allowing damping of all oscillation modes.

The more common type of insertion device is the dipole or damping wiggler, which in general contributes to all of the Synchrotron radiation integrals. It can be seen from the equations in Section 4 that I_2 always increases and hence the damping times all reduce, as described in the previous Chapter. The effects on energy spread and emittance are however more complicated. It follows from Eq. (50) that the ratio of the modified to the original (no *ID*) equilibrium values can in general be written as follows:

$$\frac{\sigma_\varepsilon'}{\sigma_\varepsilon} = \left[\frac{1 + I_3^{ID}/I_3}{I + \left(2I_2^{ID} + I_4^{ID}\right)/\left(2I_2 + I_4\right)}\right]^{1/2} \qquad (72)$$

where the contributions of the insertion device to the integrals are labelled *ID*. In the common case of a sinusoidal field variation these contributions can be written as follows:

$$I_2^{ID} = \frac{L}{2\,\rho_{ID}^2} \qquad I_3^{ID} = \frac{4}{3\pi}\frac{L}{\rho_{ID}^3} \qquad I_4^{ID} = -\frac{1}{32\,\pi^2}\frac{\lambda_o^2}{\rho_{ID}^4}L \qquad (73)$$

where L is the length of the ID, λ_0 the period length and $\rho_{I\Delta}$ is the bending radius corresponding to the peak field of the ID. The I_4^{ID} term arises from the dispersion generated by the device itself, the so called self-dispersion. In most cases however it is negligible compared to the larger I_2^{ID} term. Simplifying for the isomagnetic lattice case, and also neglecting the I_4 term, i.e. assuming $J_x = 1$, results finally in the following:

$$\frac{\sigma_e'}{\sigma_e} = \left[\frac{1 + \frac{4}{3\pi}\frac{L}{2\pi\rho}\left(\frac{\rho}{\rho_{ID}}\right)^3}{1 + \frac{1}{2}\frac{L}{2\pi\rho}\left(\frac{\rho}{\rho_{ID}}\right)^2}\right]^{1/2} \qquad (74)$$

It can be seen from the above that if the peak field in the ID is less than that of the bending magnets ($\rho_{ID} > \rho$) there is a *reduction* in energy spread, whereas if the peak ID field exceeds the bending magnet field ($\rho_{ID} > \rho$)the energy spread is *increased*.

In the case of the emittance the effect is complicated by the fact that the ID self-dispersion must be added to the dispersion that is present in the straight section without the ID:

$$D(s) = \left(\frac{\lambda_0}{2\pi}\right)^2 \frac{1}{\rho_{ID}} \cos\left(\frac{2\pi s}{\lambda_0}\right) \qquad D'(s) = \left(\frac{\lambda_0}{2\pi}\right) \frac{1}{\rho_{ID}} \sin\left(\frac{2\pi s}{\lambda_0}\right) \qquad (75)$$

The net result must in general be evaluated numerically, however some limiting cases can be examined. In the case that the dispersion in the straight section is large we can write (in the isomagnetic case):

$$\frac{\varepsilon_{x_0}'}{\varepsilon_{x_0}} = \left[\frac{1 + \frac{4}{3\pi}\ \frac{L}{2\pi\rho} \left(\frac{H_{ID}}{H}\right) \left(\frac{\rho}{\rho_{ID}}\right)^3}{1 + \frac{1}{2}\ \frac{L}{2\pi\rho} \left(\frac{\rho}{\rho_{ID}}\right)^2} \right] \tag{76}$$

Thus the emittance can be increased or decreased depending on the relative values of H and ρ in the insertion device and in the bending magnets. It can be seen that if H/ρ in the ID exceeds H/ρ in the bending magnets then the emittance will be increased. In modern synchrotron radiation sources therefore, where it is usually wanted to preserve the low emittance, the insertion devices are usually placed in straight sections with zero dispersion. If however they are placed in a straight section with finite dispersion and if it is wanted to minimize the emittance increase then it can be seen from the above that the quantity $\int 1/\rho^3\, ds$ should be minimized in the magnet design.

In the case where there is zero dispersion in the straight section, the self-generated dispersion in the *ID* dominates. The largest term involved is:

$$I_5^{ID} = \frac{\lambda_0^2}{15\pi^3 \rho_{ID}^5} \langle \beta_x \rangle\, L \tag{77}$$

A similar expression can be derived in the case of a rectangular, rather than sinusoidal, field model. From the above a condition for the emittance not to be increased can be derived as follows:

$$\lambda_0^2\, B^3 \leq\ 5.87\ 10^9\ \frac{E\,[\mathrm{GeV}]\ \varepsilon_{x_0}}{\langle \beta_x \rangle} \tag{78}$$

It can be seen therefore that except for very high field devices in low emittance and low energy rings the emittance is generally reduced by the *ID*.

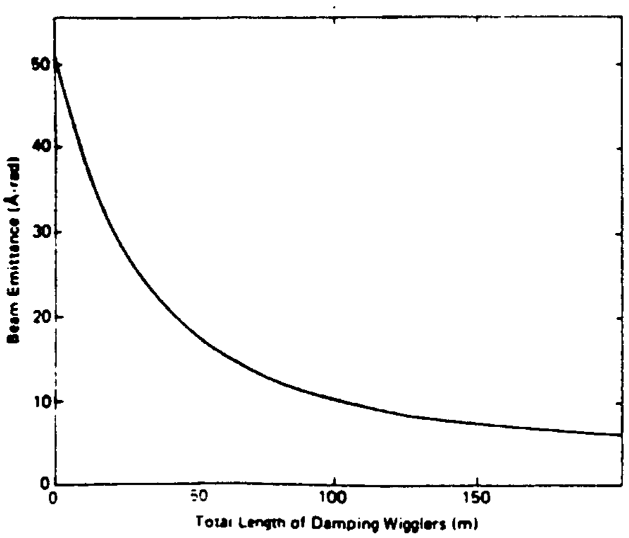

Fig. 3 Beam emittance in PEP as a function of the total damping wiggler length

It follows from the above that dipole wigglers may be used as a means of either increasing or decreasing the beam emittance. For example, they have been proposed as a means of obtaining a very low emittance in PEP, for operation as a synchrotron radiation source [22]. With a 1.26 T wiggler with a period length of 12 cm, Fig. 3 shows that a reduction in emittance of nearly a factor of 10 can be achieved, albeit with a total wiggler length of some 200 m. The possibility of including dipole wigglers in the design of damping rings for the next generation of linear colliders has also been considered [23].

Two sets of dipole wigglers are in routine operation in LEP [24]. One set is located in a dispersion free region and is used at injection to increase the energy spread and bunch length by 5–6 times the normal value in order to improve beam stability. A second set is in a finite dispersion region and is used to increase the emittance in order to optimize the beam luminosity.

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