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Closed Analytical Expression for the Electric Field Profile in a Loaded RF Structure with Arbitrarily Varying v_g and R'/Q.

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Abstract

The design of a detuned and damped accelerating structure implies variations in the geometry which induce in turn a variation of the group velocity v_g and of the impedance per unit length R', divided by the quality factor Q. The resulting differential equation for the longitudinal electric field (fundamental mode) contains coefficients that depend on the distance z along the structure. This report describes a possible method to solve this nonlinear, first order differential equation analytically and how to obtain approximate closed algebraic forms, by using the sequence of Gauss integration methods. Analytical expressions of the longitudinal field profile in a loaded or unloaded accelerating section is deduced for both linear and arbitrary variations of v_g and R'/Q. Simple relations between the average field $< E >$ and the field at the entrance of the structure $E(0)$ make it possible to provide the dependence of the field function $E(z)$ on the design value for $< E >$ and on the structure parameters. The results are in good agreement with the direct numerical integration. Applications are presented for particular structure designs.

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1 Introduction

In the design of a detuned and damped accelerating structure [1], the variations in the geometry induce a variation of the group velocity v_g and of the shunt impedance per unit length divided by the factor of merit R'/Q [2]. As a consequence, the differential equation for the longitudinal electrical field E is modified into an equation with coefficients that are depending on the coordinate z, i.e. the distance along the structure. Though the resulting equation can be solved numerically, it is always interesting to derive whenever possible an analytical solution in a closed algebraic form. This can provide an insight into the dependence of the result on the input parameters as well as the possibility to use a short symbolic program for a rapid interactive analysis of various structure designs. A method is proposed hereafter to solve the differential equation for the electric field of a loaded or unloaded structure and to find accurate analytical approximations which can be written in a closed form. A comparison with the direct numerical integration of the basic equation (with Cauchy's method) shows a very good agreement with the analytical result in the z-interval of interest. The present analysis provides relationships between, on the one hand, the field profile of the fundamental mode and, on the other hand, the structure length, the average accelerating gradient required, as well as the variations of the group velocity and of R'/Q along the cavity for tapered or detuned damped structures. These variations serve as inputs for the analytical solution of the problem and they are derived from numerical field computation programs. The presented analysis gives a very useful complement to the common relations widely used for constant impedance or constant gradient structures and is applicable in particular to the CLIC tapered damped structure [1]. This structure and a damped detuned structure of NLC type [3] are used to illustrate the application of our formalism and the accuracy of the results.

2 Beam Loading Equation to be Solved

The derivation of the differential equation for the longitudinal electric field as a function of the distance z along an accelerating structure is given in Ref. 2. An exact analytical solution is presented for the case of constant R'/Q and linearly varying group velocity and the equation is solved numerically for the case where both R'/Q and v_g vary linearly. The present paper gives a general class of analytical solutions valid for arbitrarily varying R'/Q and v_g, and for a constant beam-current distribution with respect to z. With the assumption that the power flow is constant except for the dissipation in the wall (depending on the quality factor Q) and the power exchange with the beam (proportional to the factor R'/Q), the basic differential equation writes [2],

$$\frac{d}{dz}\left[E^2(z)\frac{v_g(z)}{R'/Q(z)}\right] + \frac{E^2(z)\cdot\omega}{R'/Q(z)\cdot Q} + E(z)\; I\;\omega = 0 \tag{1}$$

where $v_g(z)$ is the group velocity and ω the frequency of the fundamental mode.

The question arising was how to solve the differential equation (1) with linear variations with respect to z of v_g and R'/Q, over the length L of the structure:

$$\begin{aligned} v_g(z) &= \; v_g(0) - \; \Delta v_g \frac{z}{L} \\ \frac{R'}{Q}(z) &= \; \frac{R'}{Q}(0) + \; \Delta \frac{R'}{Q}\cdot\frac{z}{L} \end{aligned} \tag{2}$$

and for a given initial value (at $z = 0$) of the longitudinal electric field defined by

$$E(z=0) = \; E(0). \tag{3}$$

More generally, we are interested in trying to solve eq. (1) for any relevant functions $F_1[v_g(0), z/L]$ and $F_2[R'/Q(0), z/L]$ of z, representing possible variations of the group velocity and of the impedance factor, and this for a given average longitudinal electric field over the length of the structure. This means solving (1) with

$$\begin{aligned} v_g(z) &= c\ F_1[v_g(0), z/L] = c\ F_1(z) \\ \frac{R'}{Q}(z) &= F_2[\frac{R'}{Q}(0), z/L] = F_2(z) \end{aligned} \tag{4}$$

and assuming that we can find an explicit relation between the initial value of the field and its average $< E >$ taken over the structure (see eq. (10) below)

$$E(0) = \ E(0)[< E >]. \tag{5}$$

In this report, a method is described, that allows the solution of equation (1) in the general case and gives an expression for the field $E(z)$, containing however integrals which can not be fully evaluated analytically for the assumed variations (2) or (4) of the coefficients. Therefore, Gauss' approximations of the integrals are proposed, so as to obtain in both cases explicit analytical solutions which provide very accurate estimates in the parameter interval of interest (confirmed by comparison with numerical quadrature) as well as the dependence of the solutions on the accelerating structure parameters. For any function F_1 and F_2, the approximate solution $E(z)$ obtained for the field is a linear function of its initial value $E(0)$. Therefore the average field $< E >$, obtained by further integration upon z of $E(z)$ will also be a linear function of $E(0)$ which can eventually be solved for $E(0)$ in order to provide the form (5). Again the integration of $E(z)$ is done using Gauss' approximations that provide a very good evaluation of the average as shown in the applications.

3 Closed Expression of the Beam Loading Voltage

3.1 Solution for linear variations of v_g and R'/Q

Solving equation (1) for the longitudinal electric field in the case of the linear variations defined by (2) has been done according to the derivation described in Appendix A. It provides the following expression for the beam loading voltage profile as a function of z, to the second order of the Gauss' approximation

$$\begin{aligned} E(z) = &\sqrt{\frac{R'/Q(0) + \Delta(R'/Q)\cdot z/L}{v_g(0) - \Delta v_g \cdot z/L}} \left(1 - \frac{\Delta v_g}{v_g(0)} z/L\right)^p \left[\sqrt{\frac{v_g(0)}{R'/Q(0)}} E(0) - \right. \\ &\frac{I\omega}{4} z \left(\left(1 - \frac{\Delta v_g \alpha_1}{v_g(0) z/L}\right)^{-p} \sqrt{\frac{R'/Q(0) + \alpha_1 \Delta(R'/Q)\cdot z/L}{v_g(0) - \alpha_1 \Delta v_g z/L}} \right. \\ &\left.\left. + \left(1 - \frac{\Delta v_g \alpha_2}{v_g(0)} z/L\right)^{-p} \sqrt{\frac{R'/Q(0) + \alpha_2 \Delta(R'/Q)\cdot z/L}{v_g(0) - \alpha_2 \Delta v_g z/L}} \right) \right] \end{aligned} \tag{6}$$

with

$$p = \frac{\omega L}{2Q\Delta v_g} \tag{7}$$

and

$$\alpha_{1,2} = \frac{1}{2} \mp \frac{1}{6}\sqrt{3} \tag{8}$$

remembering that z remains smaller than the structure length L

$$0 \;\leq\; z \;\leq\; L. \tag{9}$$

The next step consists of finding a relation between the initial value $E(0)$ of the electric field which appears in (6) and its average $< E >$ obtained by integration over the structure length.

$$< E >= \frac{1}{L}\int_0^L E(z)dz. \tag{10}$$

Using Gauss' approximation of this integral to second order gives the following expression,

$$< E >= \frac{1}{2}[E(\alpha_1 L) + E(\alpha_2 L)]. \tag{11}$$

Since the field $E(z)$ in eq. (6) is a linear function of $E(0)$, its average $< E >$ in (11) will also be a linear function of $E(0)$. This linear relation can easily be inverted, in the way shown in Appendix A, in order to provide the necessary expression for the initial field which corresponds to a given average gradient. The initial value so obtained can then be plugged into the solution for the voltage profile which is eventually expressed as a function of the average $< E >$, the quantity that is relevant for the structure design.

$$\begin{aligned}
E(0) = {} & \sqrt{\frac{\frac{R'}{Q}(0)}{v_g(0)}}\frac{2}{\sqrt{\frac{\frac{R'}{Q}(0)+\alpha_1\Delta\frac{R'}{Q}}{v_g(0)-\Delta v_g\alpha_1}}(1-\frac{\Delta v_g}{v_g(0)}\alpha_1)^p + \sqrt{\frac{\frac{R'}{Q}(0)+\alpha_2\Delta\frac{R'}{Q}}{v_g(0)-\Delta v_g\alpha_2}}(1-\frac{\Delta v_g}{v_g(0)}\alpha_2)^p} \\
& \left[< E > + \frac{L}{8}I\omega\alpha_1\left((1-\alpha_1^2\frac{\Delta v_g}{v_g(0)})^{-p}\sqrt{\frac{\frac{R'}{Q}(0)+\Delta\frac{R'}{Q}\alpha_1\alpha_2}{v_g(0)-\Delta v_g\alpha_1^2}} + \right.\right. \\
& \left.(1-\alpha_2\alpha_1\frac{\Delta v_g}{v_g(0)})^{-p}\sqrt{\frac{\frac{R'}{Q}(0)+\Delta\frac{R'}{Q}\alpha_2\alpha_1}{v_g(0)-\Delta v_g\alpha_2\alpha_1}}\right)\sqrt{\frac{\frac{R'}{Q}(0)+\alpha_1\Delta\frac{R'}{Q}}{v_g(0)-\alpha_1\Delta v_g}}(1-\alpha_1\Delta v_g/v_g(0))^p \\
& + \frac{L}{8}I\omega\alpha_2\left((1-\alpha_1\alpha_2\frac{\Delta v_g}{v_g(0)})^{-p}\sqrt{\frac{\frac{R'}{Q}(0)+\Delta\frac{R'}{Q}\alpha_1\alpha_2}{v_g(0)-\Delta v_g\alpha_1\alpha_2}} + \right. \\
& \left.\left.(1-\alpha_2^2\frac{\Delta v_g}{v_g(0)})^{-p}\sqrt{\frac{\frac{R'}{Q}(0)+\Delta\frac{R'}{Q}\alpha_2^2}{v_g(0)-\Delta v_g\alpha_2^2}}\right)\sqrt{\frac{\frac{R'}{Q}(0)+\alpha_2\Delta\frac{R'}{Q}}{v_g(0)-\alpha_2\Delta v_g}}(1-\Delta v_g/v_g(0)\alpha_2)^p\right]
\end{aligned} \tag{12}$$

where the exponent p is equal to

$$p = \frac{\omega L}{2Q\Delta v_g}. \tag{13}$$

3.2 Solution for arbitrary variations of v_g and R'/Q

Solving equation (1) for the longitudinal electric field in the case of the arbitrary variations defined by (4) has been done according to the derivation summarised in Appendix B. The second order approximation used in Appendix B gives a good accuracy for the final result, provided that the functions F_1 and F_2 have sufficiently small variations, so that they have a behaviour close to polynomials of a degree three. For functions with larger variations, the result should be cross checked by numerical integration before making general use of it and, if the accuracy is judged to be inadequate, the next member in the sequence of Gauss approximations should be tried. Taking this caveat into account, the solution (63) gives a closed expression for the beam-loaded voltage profile as a function of z, which now contains second order Gauss' approximations of all the definite integrals including the one giving the exponent of the solution of the homogenous equation.

$$E(z) = \sqrt{\frac{F_2(z)}{F_1(z)}} \cdot exp\Big[-\frac{\omega\, z}{4Q}\left(\frac{1}{F_1(\alpha_1 z)} + \frac{1}{F_1(\alpha_2 z)}\right)\Big]$$
$$\Bigg[\sqrt{\frac{F_1(0)}{F_2(0)}} \cdot E(0) \quad - \frac{I\omega}{4} z \left(\sqrt{\frac{F_2(\alpha_1 z)}{F_1(\alpha_1 z)}} \cdot exp\Big[\frac{\omega\,\alpha_1 z}{4Q}\left(\frac{1}{F_1(\alpha_1^2 z)} + \frac{1}{F_1(\alpha_1\alpha_2 z)}\right)\Big]\right.$$
$$\left.+ \quad \sqrt{\frac{F_2(\alpha_2 z)}{F_1(\alpha_2 z)}} \cdot exp\Big[\frac{\omega\,\alpha_2 z}{4Q}\left(\frac{1}{F_1(\alpha_1\alpha_2 z)} + \frac{1}{F_1(\alpha_2^2 z)}\right)\Big]\right)\Bigg] \tag{14}$$

(15)

with

$$\alpha_{1,2} = \frac{1}{2} \mp \frac{1}{6}\sqrt{3} \tag{16}$$

and for

$$0 \;\le\; z \;\le\; L. \tag{17}$$

As in the preceding case, it is now necessary to express the initial field value as a function of the average $< E >$ required. This has been done and is documented in Appendix B. The result, valid for arbitrary smooth functions F_1 and F_2, is given here

$$E(0) = \frac{\sqrt{\frac{F_2(0)}{F_1(0)}}}{\sqrt{\frac{F_2(\alpha_1 L)}{F_1(\alpha_1 L)}}e^{-\frac{\omega}{2Q}G_1(\alpha_1 L)} + \sqrt{\frac{F_2(\alpha_2 L)}{F_1(\alpha_2 L)}}e^{-\frac{\omega}{2Q}G_1(\alpha_2 L)}} \times$$
$$\times \Bigg[2 < E > + \frac{I\omega}{4}\sqrt{\frac{F_2(\alpha_1 L)}{F_1(\alpha_1 L)}}e^{-\frac{\omega}{2Q}G_1(\alpha_1 L)}\alpha_1 L \left(e^{\frac{\omega}{2Q}G_1(\alpha_1^2 L)}\sqrt{\frac{F_2(\alpha_1^2 L)}{F_1(\alpha_1^2 L)}} + e^{\frac{\omega}{2Q}G_1(\alpha_1\alpha_2 L)}\sqrt{\frac{F_2(\alpha_1\alpha_2 L)}{F_1(\alpha_1\alpha_2 L)}}\right) +$$
$$+ \frac{I\omega}{4}\sqrt{\frac{F_2(\alpha_2 L)}{F_1(\alpha_2 L)}}e^{-\frac{\omega}{2Q}G_1(\alpha_2 L)}\alpha_2 L \left(e^{\frac{\omega}{2Q}G_1(\alpha_1\alpha_2 L)}\sqrt{\frac{F_2(\alpha_1\alpha_2 L)}{F_1(\alpha_1\alpha_2 L)}} + e^{\frac{\omega}{2Q}G_1(\alpha_2^2 L)}\sqrt{\frac{F_2(\alpha_2^2 L)}{F_1(\alpha_2^2 L)}}\right)\Bigg] \tag{18}$$

with the following expression for G_1 (Appendix B)

$$G_1(z) = \frac{z}{2}\left[\frac{1}{F_1(\alpha_1 z)} + \frac{1}{F_1(\alpha_2 z)}\right] \tag{19}$$

and the numerical values of α_1 and α_2 recalled above and in both Appendices.

All the preceding results are strictly correct for a constant Q-value. When the quality factor Q is not constant, the results are formally the same, as shown in Appendix B, after replacing F_1 by F_3 (see below) and Q by its average $\bar{Q}$, in the three exponential functions of equation (14) as well as in (19). The function F_3 includes the variation of Q with respect to z around its average value $\bar{Q}$, according to the following definitions

$$\begin{aligned} Q &= \bar{Q} f(z) \\ F_3 &= f \cdot F_1 \end{aligned} \tag{20}$$

These slight modifications make it possible to fully include variations of the quality factor in strongly detuned structures.

4 Applications to Accelerating Structure Designs

4.1 The CLIC tapered damped structure

The parameter values (21) retained for the application discussed in this section are those of the CLIC design of a tapered damped structure (TDS) [1,2]. This case can be treated by considering linear variations of the group velocity and of the shunt impedance per unit length such as the relations given in Section 3.1 apply. The numerical values actually introduced in the equations (2), (6) (7) and (12) and corresponding to the CLIC TDS [2] are listed hereafter;

$$\begin{aligned} & v_g(0) = 3.240 \times 10^7 \quad m/s \\ & \Delta v_g = 1.619 \times 10^7 \quad m/s \\ & \frac{R'}{Q}(0) = 2.23 \times 10^4 \quad \Omega/m \\ & \Delta\frac{R'}{Q} = 0.78 \times 10^4 \quad \Omega/m \\ & \frac{\omega}{Q} = 5.118 \times 10^7 \quad s^{-1} \\ & I\omega = 1.811 \times 10^{11} \quad A/s \\ & E(0) = 1.866 \times 10^8 \quad V/m \\ & L = 0.5\ m \end{aligned} \tag{21}$$

The decimal values of the coefficients α_1 and α_2 are

$$\begin{aligned} \alpha_1 &= 0.211325 \\ \alpha_2 &= 0.788675 \end{aligned} \tag{22}$$

This application allows the comparison of the results of a direct numerical integration with the analytical approximation of the solution (6), in the case of a linear variation of v_g and R'/Q. The curves of Fig. 1 indicate that the analytic expressions (6) depict extremely well the voltage profile in the structures either unloaded ($I = 0$) or loaded with the assumed beam current ($I = 0.96A$). The actual deviation never exceeds 0.2 % in this particular case. In addition, the average value given by (11) and equal to 163.50 MV/m agrees very well with the one obtained by numerical integration, i.e. 163.47 MV/m.

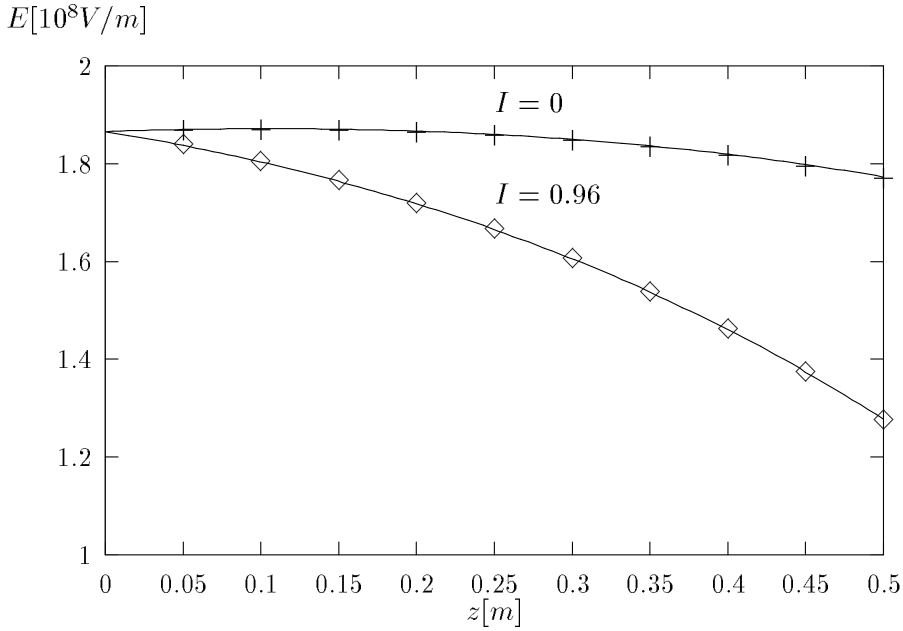

Fig. 1 Voltage profile for the CLIC structure with a field of 186.6 MV/m at the entrance. The full curves are given by the formulae of Section 3.1 for linear variations of v_g and R'/Q, while crosses and diamonds result from numerical integration of the differential equation.

In practice, one would rather start from an average field value, e.g. 150 MV/m, compute the corresponding initial value $E(0)$ with (12), and then deduce the voltage profile with and without beam loading as illustrated in Fig. 2. This provides a very direct and precise way to obtain the electric field along the structure for a wanted average accelerating gradient.

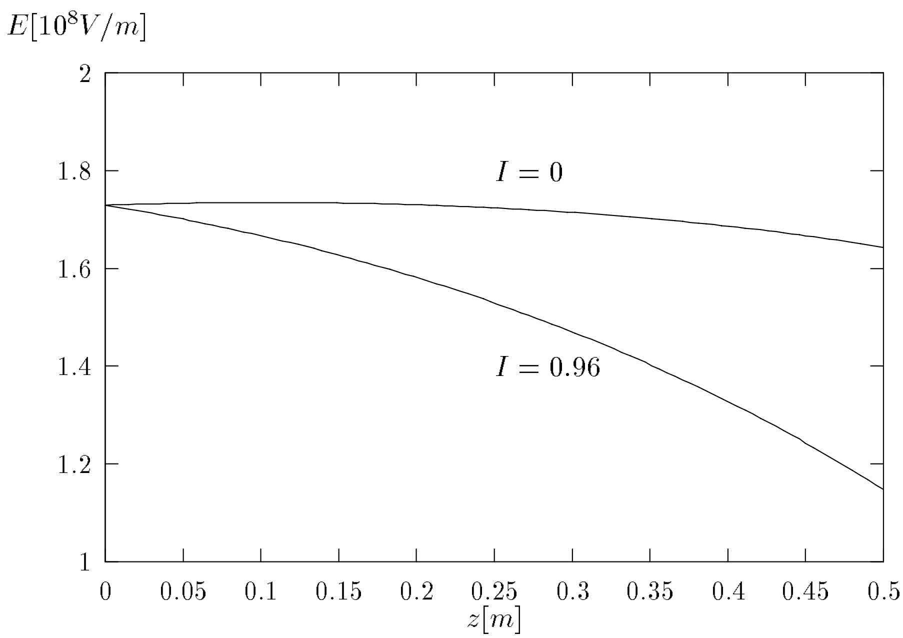

Fig. 2 Voltage profile for the CLIC structure with an average accelerating field of 150 MV/m.

4.2 Damped detuned structure of the NLC type

In order to check the analytical expressions of Section 3.2, which are valid for non-linear variations of the group velocity and of the shunt impedance, we would like to now consider a structure of the type studied at SLAC and known under the name of RDDS [3]. In such a structure, the variation of v_g can be large and strongly non-linear while the shunt impedance equally varies non-linearly. In the selected example, v_g decreases from about $0.11c$ to $0.03c$ and the impedance increases from 7.7×10^7 to 1.03×10^8 Ω/m. For getting the functions $v_g(z)$ and $R'/Q(z)$ generally defined by the equations (4), polynomial fits of the curves $v_g(z)$ and $R'(z)$ provided to us [4] were made and the factor of quality Q was assumed to be constant and equal to 7875. Retaining the average of Q represents at this stage a good approximation (in fact Q may vary from about 8250 to 7500). This approximation can however be removed at any time by applying the relations (20) and using the actual function $Q(z) = \bar{Q}f(z)$ to generate $R'/Q(z)$ before doing the fit.

The results of the fits give the following functions $F1$ and $F2$.

$$F_1(z) = \left[2.325 - 0.5z + 0.345(1.111z - 1.0)^2 - 0.78(1.111z - 1.0)^3\right]10^7 \tag{23}$$

$$F_2(z) = \left[10.857 + 0.705z - 0.222(1.111z - 1.0)^2 + 0.857(1.111z - 1.0)^3\right]1000 \tag{24}$$

Figures 3 and 4 illustrate the quality of the fits (23) and (24) made for the relative group velocity and the shunt impedance per unit length, respectively. In these graphs, the full lines correspond to the polynomial fits of degree three while the diamonds correspond to the initial data.

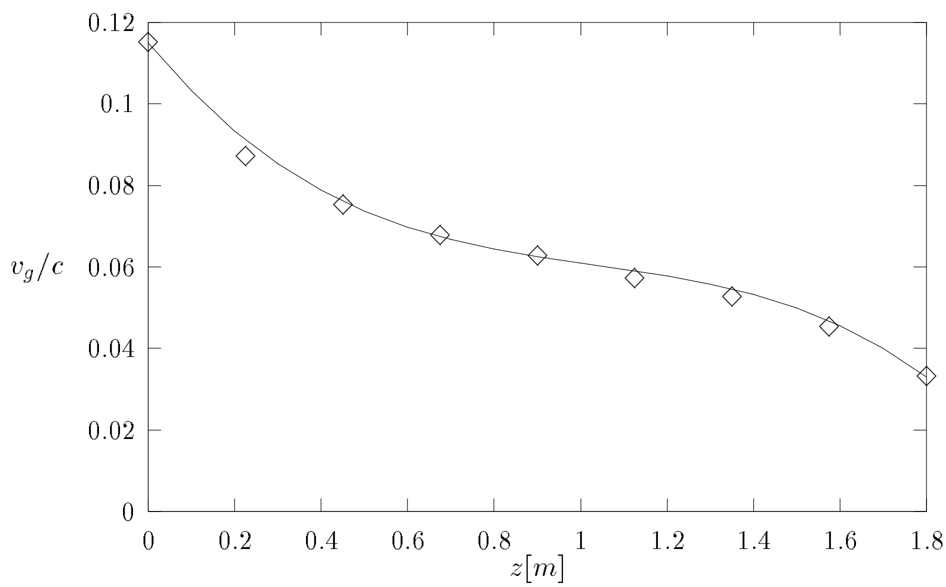

Fig. 3 Group velocity of the RDDS structure of the NLC type.
The full curve is the fit obtained through the initial data points.

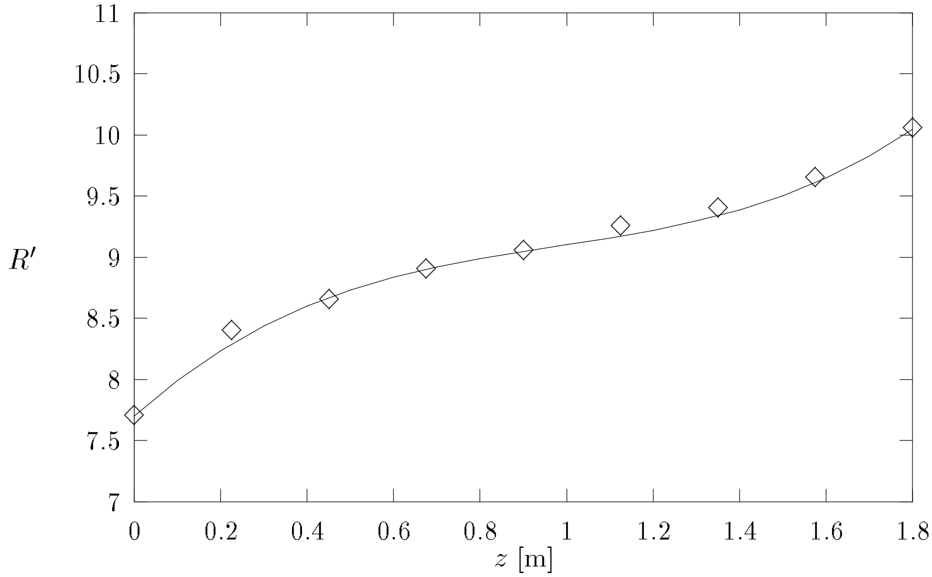

Fig. 4 Shunt impedance per unit length of the RDDS of NLC type in units of $10^7 \Omega/m$. The full curve is the fit obtained through the initial data points.

For the other necessary parameters of the structure, the following values [4] were taken

$$
\begin{aligned}
&\frac{\omega}{Q} = 0.9096 \times 10^7 \quad s^{-1} \\
&I\omega = 0.7163 \times 10^{11} \quad A/s \\
&< E >= 50 \times 10^6 \quad V/m \\
&L = 1.8\ m
\end{aligned}
\tag{25}
$$

It is first necessary to use the relation (18) for deducing the initial field value from its average, using the particular functions (23) and (24) and the parameters (25). The value so obtained is 55.62 MV/m. Equation (14) then gives the electric field profile (to second order in the Gauss approximation) with and without beam loading along the structure (Fig. 5). Comparison with numerical integration of the differential equation indicates a very good agreement (Fig. 5). In this nonlinear example, the maximum deviation which takes place at the end of the strongly loaded structure reaches 4.5 % approximately. If necessary, this deviation could be further reduced to a level comparable to the one of the first application by working to the third order.

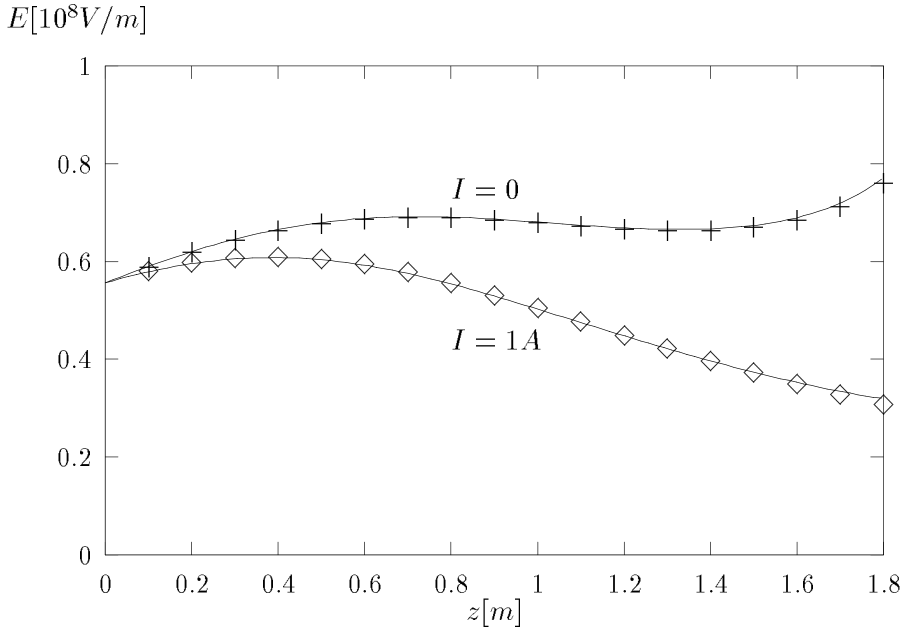

Fig. 5 Voltage profiles of an NLC type cavity.
Full curves result from the formulae of Section 3.2 for non-linear variations of v_g and R'/Q, while crosses and diamonds come from numerical integration of the differential equation.

5 Conclusions

This paper describes the method proposed by the authors to solve analytically the differential equation for the longitudinal electric field as a function of the coordinate z along an accelerating structure and to extend the range of solutions. The method provides a closed expression of the field profile for arbitrary but smooth variations of the group velocity v_g and of the impedance per unit length, divided by the quality factor, R'/Q. This expression results from an approximation that is required to achieve the final quadrature explicitly, but can be made as accurate as desired by raising the order of this approximation. When dealing with the electric field profile in an RF cavity, it is shown that a second order approximation is already very good.

The first step consists of changing variables in order to write the equation of the field in the form of Bernouilli's equation, which can then be transformed into a linear, inhomogeneous equation by a standard substitution. The latter equation is solved in the usual manner (Green's method) and the result is an expression for the field which contains a double quadrature. This last quadrature can only be evaluated in a closed form with some approximation. For linear variations of v_g and R'/Q, one integral can be resolved and the double quadrature replaced by a single one, while for non-linear variations this is not possible. In all cases, the remaining single or double quadrature is achieved by using the Gauss integration sequence most frequently introduced in numerical applications. Provided the integrand-function shows a sufficiently smooth variation with the independent variable numerical integration formulae can be applied for an analytical description of a quadrature operation by just using **one** discretization step. The remarkable result is that the second member of the Gaussian sequence of approximations applied over the entire interval of integration not only gives excellent estimations in the single quadrature case (within 0.2 %) but also provides very good evaluations of the double quadrature (within better than 5 %). This accuracy can of course always be improved by going to the next order of the Gauss approximation though at the expense of a more complex expression for the solution.

The closed, analytical expressions of the field obtained have been applied firstly to the tapered, damped structure of CLIC (30 GHz) where linear variations of the key quantities can be assumed and secondly to a damped, detuned structure of NLC type, with strong non-linear variations of these same quantities. In both cases, checking with a direct numerical integration of the differential equation proves the noteworthy validity of the proposed solution. Furthermore, this solution is by nature of the problem a linear function of the field at the entrance of the cavity. Therefore, an additional integration over the cavity length, which is again done following the same method, provides an explicit relation between this initial field and the field average in the cavity. This allows the direct expression of the voltage profile as a function of the average accelerating field, which is one of the main characteristics of the design.

It is important to underline that all the obtained field-profile expressions valid for a wide range of detuned accelerating structures can be introduced, in their symbolic form, into executable files of mathematical computation applications such as MapleV, Mathcad and Excel. This makes possible a rapid, interactive evaluation and optimisation of the characteristics of specific structures for various design parameters, without resorting to any numerical integration.

The method described here for analytically solving the nonlinear, first-order differential equation associated with the field distribution in an RF structure is sufficiently general to be applied to other problems of physics or engineering provided the coefficients appearing in the differential equation of the phenomenon vary smoothly enough with the independent variable. It has proven to be very successful in predicting the longitudinal field profiles of different structures, with and without beam loading.

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A Analytical Solution with linear variation of v_g and R'/Q

For the authors' convenience, the nonlinear first-order ordinary differential equation (1) has first been re-written by using the following definitions and changes of variables:

$$\begin{aligned}
a_0 &= v_g(0) & a_1 &= \Delta v_g \\
b_0 &= \tfrac{R'}{Q}(0) & b_1 &= \Delta \frac{R'}{Q} \\
C_1 &= \omega/Q & C_2 &= I\,\omega \\
x &= z & y &= E(z).
\end{aligned} \tag{26}$$

These variable definitions (26) will be used, in the limited interval $0 < x < L$, to express the solution $y(x)$ of the differential equation

$$\frac{d}{dx}\left(y^2 \frac{F_1}{F_2}\right) + C_1 y^2 \frac{1}{F_2} + C_2 y = 0 \tag{27}$$

for the given initial condition

$$y(0) = y_0. \tag{28}$$

The quantities C_1 and C_2 are constant (if Q is not constant, its average value has to be introduced into C_1, as an approximation) and the functions F_1 and F_2 vary linearly with x, in agreement with (2)

$$F_1 = a_0 - \frac{a_1}{L} x \tag{29}$$

$$F_2 = b_0 + \frac{b_1}{L} x. \tag{30}$$

In a first step a simplification of the equation can be obtained by performing a substitution of the dependent variable y:

$$y^2 \frac{F_1}{F_2} = z. \tag{31}$$

This leads to the new equation:

$$\frac{dz}{dx} + \frac{C_1}{F_1} z + C_2 \sqrt{\frac{F_2}{F_1}} z^{1/2} = 0 \tag{32}$$

which can be identified as Bernoulli' equation [5] with exponent 1/2 in the new variable $z = z(x)$. As usual for the analytic solution of this type of equation, we now apply a second substitution of polynomial type which is defined by:

$$z = u^{1-q} \tag{33}$$

where q is yet to be determined. The multiplication of the equation thus obtained by u^q gives:

$$(1-q)\frac{du}{dx} + \frac{C_1}{F_1} u + C_2 \sqrt{\frac{F_2}{F_1}} u^{\frac{1+q}{2}} = 0. \tag{34}$$

The resulting equation becomes linear and inhomogeneous if $q = -1$ and can be written as:

$$\frac{du}{dx} + \frac{C_1}{2F_1} u = -\frac{1}{2} C_2 \sqrt{\frac{F_2}{F_1}} \tag{35}$$

with the solution

$$u(x) = e^{-\int_0^x \frac{C_1}{2F_1}dx}\left[u_0 - C_2\frac{1}{2}\int_0^x e^{\int_0^x \frac{C_1}{2F_1}dx}\sqrt{\frac{F_2}{F_1}}dx\right]. \tag{36}$$

Using the definitions for F_2 and F_1 as listed above, two of the three integrals are:

$$-\int_0^x \frac{C_1}{2F_1}dx = ln\left(1 - \frac{a_1 x}{a_0 L}\right)^{\frac{C_1 L}{2a_1}} \tag{37}$$

$$\int_0^x \frac{C_1}{2F_1}dx = ln\left(1 - \frac{a_1 x}{a_0 L}\right)^{-\frac{C_1 L}{2a_1}} \tag{38}$$

and using the notation $\xi = x/L$ and $p = C_1 L/(2a_1)$, $u(x)$ becomes

$$u(x) = \left(1 - \frac{a_1}{a_0\xi}\right)^p\left[u_0 - \frac{L}{2}C_2\int_0^\xi\left(1 - \frac{a_1}{a_0\xi}\right)^{-p}\sqrt{\frac{b_0 + b_1\xi}{a_0 - a_1\xi}}d\xi\right]. \tag{39}$$

Since the remaining quadrature cannot be evaluated in closed form, the function (39) provides the most general expression for the solution of (35). Having in mind an interest in a simplified, closed analytical expression giving an accurate estimate of the function (39), we use the following two approximations of a general integral:

$$\int_0^x f(t)dt \approx x f(\frac{x}{2}) \tag{40}$$

and

$$\int_0^x f(t)dt \approx \frac{x}{2}\left[f(\alpha_1 x) + f(\alpha_2 x)\right] \tag{41}$$

where

$$\alpha_{1,2} = \frac{1}{2} \mp \frac{1}{6}\sqrt{3}. \tag{42}$$

While Eq. (40) represents the well known "mean value approximation" of an integral, Eq. (40) and (41) are generally known as the first two members of the sequence of Gauss integration approximations [6]. While (40) is the exact integral of linear functions, the approximation (41) turns out to be exact up to third-order polynomials. The constants $\alpha_{1,2}$ are the zeros of the second order Legendre polynomial [4]. In this way and after transforming back the dependent variable from u to y, we finally obtain two approximations for the actual solution of eq. (27):

- Using the first-order Gauss approximation (40):

$$y_1(\xi) = \sqrt{\frac{b_0 + b_1\xi}{a_0 - a_1\xi}}\left(1 - \frac{a_1}{a_0}\xi\right)^p\left[\sqrt{\frac{a_0}{b_0}}y_0 - \frac{L}{2}C_2\xi\left(1 - \frac{a_1}{2a_0}\xi\right)^{-p}\sqrt{\frac{b_0 + b_1\frac{\xi}{2}}{a_0 - a_1\frac{\xi}{2}}}\right] \tag{43}$$

- Using the second-order Gauss approximation (41):

$$y_2(\xi) = \sqrt{\frac{b_0 + b_1\xi}{a_0 - a_1\xi}}\left(1 - \frac{a_1}{a_0}\xi\right)^p\Bigg[\sqrt{\frac{a_0}{b_0}}y_0 -$$
$$-\frac{L}{4}C_2\xi\left(\left(1 - \frac{a_1\alpha_1}{a_0}\xi\right)^{-p}\sqrt{\frac{b_0 + b_1\alpha_1\xi}{a_0 - a_1\alpha_1\xi}} + \left(1 - \frac{a_1\alpha_2}{a_0}\xi\right)^{-p}\sqrt{\frac{b_0 + b_1\alpha_2\xi}{a_0 - a_2\alpha_2\xi}}\right)\Bigg] \tag{44}$$

where

$$\alpha_{1,2} = \frac{1}{2} \mp \frac{1}{6}\sqrt{3} \tag{45}$$

$$\xi = \frac{x}{L} \qquad 0 < \xi < 1 \tag{46}$$

$$p = \frac{C_1 L}{2a_1}. \tag{47}$$

In Fig. 1, a comparison is presented of the two analytic approximations $y_1(x)$ and $y_2(x)$ given in Eqs. (43) and (44) with a direct numerical integration of Eq. (27), using the numerical values listed in (21). The upper full line corresponds to $y_1(x)$ and the lower one to $y_2(x)$. As can be seen the numerical solution represented by the diamonds is indistinguishable from the approximation $y_2(x)$ within the entire interval of integration. However, even $y_1(x)$ only differs from the numerical integration results by an amount which never exceeds about 3 % (value reached at the end of the interval, when $x = L$)

As mentioned in the Section 2, it is then necessary to express the field $y_2(\xi)$ as a function of the average field $< y >$ instead of its initial value y_0 as in (44). The average is simply given by the following integral

$$< y > = \int_0^1 y_2(\xi) d\xi. \tag{48}$$

Considering the curve shown in Fig. 1 for the voltage profile, it is evident that the function y_2 is smooth with no zeros in the interval [0,1]. As a consequence, the Gauss appromation described above applies to the integral (48). Using it to second order, with the special value $x = 1$ according to (48), we obtain

$$< y_2 > = \frac{1}{2}\left[y_2(\alpha_1) + y_2(\alpha_2) \right]. \tag{49}$$

Since the differential equation for the dependent variable u is linear, the solution is always a linear function of the initial condition. This is obviously satisfied by the approximate solution which takes the form

$$y_2(\xi) = g(\xi) y_0 - h(\xi). \tag{50}$$

The last equation gives the definition of the functions $g(\xi)$ and $h(\xi)$, by direct comparison with (44). Introducing (50) into (49), the result for the average estimate becomes

$$< y_2 > = \frac{1}{2}\left[(g(\alpha_1) + g(\alpha_2)) y_0 - (h(\alpha_1) + h(\alpha_2)) \right]. \tag{51}$$

This relation can of course easily be solved for the initial condition y_0

$$y_0 = \frac{2 < y > + (h(\alpha_1) + h(\alpha_2))}{g(\alpha_1) + g(\alpha_2)} \tag{52}$$

where the notation $< y_2 >$, valid for the second order approximation, is replaced by the more general notation $< y >$. Hence, when designing a structure for a given average field, the relation (52) can be used to calculate the initial value corresponding to the design characteristics. Once the initial value y_0 is known, then the general expression (44) is applicable to find out the voltage profile related to the specific linear variations assumed for v_g and R'/Q.

Inserting the explicit form of the functions $g(\xi)$ and $h(\xi)$ into the relation (52) gives the full expression for the initial value associated with a particular average. In the case of linear variations treated in this appendix, this expression is

$$y_0 = \sqrt{\frac{b_0}{a_0}} \frac{2}{\sqrt{\frac{b_0+\alpha_1 b_1}{a_0-a_1\alpha_1}}(1-\frac{a_1}{a_0}\alpha_1)^p + \sqrt{\frac{b_0+\alpha_2 b_1}{a_0-a_1\alpha_2}}(1-\frac{a_1}{a_0}\alpha_2)^p}$$
$$\Bigg[< y > + \frac{L}{8} C_2 \alpha_1 \left((1-\alpha_1^2 \frac{a_1}{a_0})^{-p} \sqrt{\frac{(b_0+b_1\alpha_1\alpha_2)}{(a_0-a_1\alpha_1^2)}} + \right.$$
$$\left. (1-\alpha_2\alpha_1\frac{a_1}{a_0})^{-p}\sqrt{\frac{(b_0+b_1\alpha_2\alpha_1)}{(a_0-a_1\alpha_2\alpha_1)}} \right) \frac{\sqrt{b_0+\alpha_1 b_1}}{\sqrt{a_0-a_1\alpha_1}} (1-\alpha_1 a_1/a_0)^p$$
$$+ \frac{L}{8} C_2 \alpha_2 \left((1-\alpha_1\alpha_2 \frac{a_1}{a_0})^{-p} \sqrt{\frac{(b_0+b_1\alpha_1\alpha_2)}{(a_0-a_1\alpha_1\alpha_2)}} + \right.$$
$$\left. (1-\alpha_2^2\frac{a_1}{a_0})^{-p}\sqrt{\frac{(b_0+b_1\alpha_2^2)}{(a_0-a_1\alpha_2^2)}} \right) \frac{\sqrt{b_0+\alpha_2 b_1}}{\sqrt{a_0-a_1\alpha_2}} (1-a_1/a_0\alpha_2)^p \Bigg] \tag{53}$$

where the coefficients are defined in (26), the parameters α_1 and α_2 in (45) and the exponent p in (47).

B Analytical Solution with arbitrary variation of v_g and R'/Q

Let us start again from the general form (36) of the solution obtained in Appendix A.

$$u(x) = e^{-\int_0^x \frac{C_1}{2F_1}dx}\left[u_0 - C_2\frac{1}{2}\int_0^x e^{\int_0^x \frac{C_1}{2F_1}dx}\sqrt{\frac{F_2}{F_1}}dx\right]. \quad (54)$$

The functions F_1 and F_2 are now arbitrary and not explicitely defined, though assumed to have small enough variations for using Gauss' approximations of the integrals. The last condition means that the functions F_1 and F_2 must not have too many oscillations and zeros in the interval of interest. More precisely, the use of the second order Gauss' approximation gives exact results for polynomials of up to degree 3. For the integral, which appears in the exponential functions of (54), we can write for constant Q

$$G_1(x) = \int_0^x \frac{1}{F_1}dx = \frac{x}{2}\left[\frac{1}{F_1(\alpha_1 x/L)} + \frac{1}{F_1(\alpha_2 x/L)}\right] \quad (55)$$

applying the second-order approximation defined in eq. (41). This form of G_1 strictly applies for a constant Q. When Q varies, it is sufficient to modify (55) according to the following description. Let us first define the variation of Q around its average value $\bar{Q}$ by

$$Q = \bar{Q}f(z) \quad (56)$$

and the corresponding constant C_1 by

$$C_1 = \frac{\omega}{\bar{Q}} \quad (57)$$

With these definitions and the introduction of the function $f(z)$ in the development of Appendix A leading to the equation (36), the function G_1 is mofified as follows

$$G_1(x) = \int_0^x \frac{1}{fF_1}dx = \int_0^x \frac{1}{F_3}dx \quad (58)$$

and (58) defines F_3 as the product fF_1. The form of G_1 remains unchanged with simply F_3 replacing F_1, i.e. for varying Q

$$G_1(x) = \frac{x}{2}\left[\frac{1}{F_3(\alpha_1 x/L)} + \frac{1}{F_3(\alpha_2 x/L)}\right] \quad (59)$$

and the whole subsequent treatment applies with either (55) or (59).

The next step consists of finding an approximation of the second integral in (54) which represents a particular solution of the inhomogenous differential equation and contains the definite integral G_1

$$G_2(x) = \int_0^x e^{\int_0^x \frac{C_1}{2F_1}dx}\sqrt{\frac{F_2}{F_1}}dx = \int_0^x e^{\frac{C_1}{2}G_1(x)}\sqrt{\frac{F_2(x)}{F_1(x)}}dx. \quad (60)$$

Having included the approximation (55) into (60), the expression of G_2 has been reduced to a single integral containing the three functions $G_1(x)$, $F_1(x)$ and $F_2(x)$. At this point it is once more possible to apply the second order Gauss' approximation (41) to the last form of G_2 in(60) and get

$$G_2(x) = \frac{x}{2}\left[e^{\frac{C_1}{2}G_1(\alpha_1 x)}\sqrt{\frac{F_2(\alpha_1 x)}{F_1(\alpha_1 x)}} + e^{\frac{C_1}{2}G_1(\alpha_2 x)}\sqrt{\frac{F_2(\alpha_2 x)}{F_1(\alpha_2 x)}}\right]. \quad (61)$$

The solution (54) now becomes

$$u(x) = e^{-\frac{C_1}{2}G_1(x)}\left[u_0 - \frac{C_2}{2}G_2(x)\right] =$$
$$= e^{-\frac{C_1}{2}G_1(x)}\left[\sqrt{\frac{F_1(0)}{F_2(0)}}y(0) - \frac{C_2}{2}G_2(x)\right]. \tag{62}$$

Introducing (55) and (61) into (62) and back transforming the variable u into y provides the approximate expression sought for the solution of (27) in the case of a general variation of v_g and R'/Q

$$y(x) = \sqrt{\frac{F_2(x)}{F_1(x)}}e^{-\frac{\omega}{2Q}G_1(x)}\left[\sqrt{\frac{F_1(0)}{F_2(0)}}y_0 - \frac{I\omega}{4}x\left(e^{\frac{\omega}{2Q}G_1(\alpha_1 x)}\sqrt{\frac{F_2(\alpha_1 x)}{F_1(\alpha_1 x)}} + e^{\frac{\omega}{2Q}G_1(\alpha_2 x)}\sqrt{\frac{F_2(\alpha_2 x)}{F_1(\alpha_2 x)}}\right)\right]. \tag{63}$$

As expected, the solution is again a linear function of the initial condition y_0 and the functions $g(x)$ and $h(x)$ defined in equation (50) take the following forms

$$g(x) = \sqrt{\frac{F_2(x)}{F_1(x)}}e^{-\frac{\omega}{2Q}G_1(x)}\sqrt{\frac{F_1(0)}{F_2(0)}}$$
$$h(x) = \sqrt{\frac{F_2(x)}{F_1(x)}}e^{-\frac{\omega}{2Q}G_1(x)}\frac{I\omega}{4}x\left(e^{\frac{\omega}{2Q}G_1(\alpha_1 x)}\sqrt{\frac{F_2(\alpha_1 x)}{F_1(\alpha_1 x)}} + e^{\frac{\omega}{2Q}G_1(\alpha_2 x)}\sqrt{\frac{F_2(\alpha_2 x)}{F_1(\alpha_2 x)}}\right). \tag{64}$$

Having these two functions and following the deduction made in Appendix A, the application of the equation (52) gives the explicit relation between the initial value y_0 and the average value $< y >$, required for designing a structure.

$$y_0 = \frac{\sqrt{\frac{F_2(0)}{F_1(0)}}}{\sqrt{\frac{F_2(\alpha_1 L)}{F_1(\alpha_1 L)}}e^{-\frac{\omega}{2Q}G_1(\alpha_1 L)} + \sqrt{\frac{F_2(\alpha_2)}{F_1(\alpha_2)}}e^{-\frac{\omega}{2Q}G_1(\alpha_2)}} \times$$
$$\times\left[2 < y > + \sqrt{\frac{F_2(\alpha_1 L)}{F_1(\alpha_1 L)}}e^{-\frac{\omega}{2Q}G_1(\alpha_1 L)}\frac{I\omega}{4}\alpha_1 L\left(e^{\frac{\omega}{2Q}G_1(\alpha_1^2 L)}\sqrt{\frac{F_2(\alpha_1^2 L)}{F_1(\alpha_1^2 L)}} + e^{\frac{\omega}{2Q}G_1(\alpha_1\alpha_2 L)}\sqrt{\frac{F_2(\alpha_1\alpha_2 L)}{F_1(\alpha_1\alpha_2 L)}}\right) + \right.$$
$$\left. + \sqrt{\frac{F_2(\alpha_2 L)}{F_1(\alpha_2 L)}}e^{-\frac{\omega}{2Q}G_1(\alpha_2)}\frac{I\omega}{4}\alpha_2 L\left(e^{\frac{\omega}{2Q}G_1(\alpha_1\alpha_2 L)}\sqrt{\frac{F_2(\alpha_1\alpha_2 L)}{F_1(\alpha_1\alpha_2 L)}} + e^{\frac{\omega}{2Q}G_1(\alpha_2^2 L)}\sqrt{\frac{F_2(\alpha_2^2 L)}{F_1(\alpha_2^2 L)}}\right)\right]. \tag{65}$$