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# CHROMATICITY

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**Abstract**
A derivation of chromaticity formulae for circular accelerators is given, including the contribution of quadrupoles and sextupoles and with special attention to the general bending magnet. These are also exact for small radii of curvature.

## 1. INTRODUCTION

In the design of storage rings there are many similarities with the geometry of optics. In analogy to chromatic aberrations in optics, in particle accelerators a parameter called chromaticity is introduced. In optics rays of different wavelength find a different refraction index in a lens and therefore experience a different focal length. Similarly in a storage ring particles of different momentum see a different focusing strength in the quadrupoles and, as a consequence, have a different betatron oscillation frequency.

We define the chromaticity as the variation of the betatron tune $Q$ with the relative momentum deviation $\delta$ ($\delta = \Delta p/p$):

$$Q' = \frac{dQ}{d\delta} \ . \tag{1}$$

Sometimes the relative chromaticity $\xi$ is used:

$$\xi = \frac{Q'}{Q} \ . \tag{2}$$

Let us point out the importance of the chromaticity in circular accelerators. The chromaticity has a deleterious influence on the beam dynamics for two main reasons:

First, a momentum spread $\sigma_p$ is always present in a particle beam, therefore the chromaticity produces a tune spread in the beam:

$$\Delta Q = Q' \, \sigma_p . \tag{3}$$

In large rings, with high tune values, this tune spread is so large that it is impossible to accommodate the beam in the space between the resonance lines in the tune diagram. How dangerous these resonances can be for beam stability has been described by E.J.N. Wilson [1].

Second, in the case of bunched beams the chromaticity produces a transverse instability called "head-tail effect" (see Ref. [2] for a detailed treatment). The wake field produced by the leading part of a bunch (the head) excites an oscillation of the trailing part (the tail) of the same bunch. In half a synchrotron period the head and the tail of the bunch interchange their positions and the oscillation can be anti-damped and may cause a beam loss. A complete mathematical treatment shows that the growth rate of this instability is much faster for negative than for positive chromaticity values and vanishes for zero chromaticity. It may be counteracted by a transverse feedback system, but this makes machine operation much more critical. Therefore most of the storage rings operate with zero or slightly positive chromaticity.

The "natural" chromaticity of a storage ring is that due only to the elements of the linear lattice, i.e. quadrupoles and dipoles. As will be shown later the "natural" chromaticity of a strong focusing storage ring is always negative and cannot be made zero. To correct the chromaticity nonlinear elements, the sextupole magnets, have to be introduced into the lattice.

In strong focusing lattices the main contribution to the chromaticity is due to the quadrupoles, in particular, in large rings with very large radius the contribution of the dipoles can be neglected; for small rings, however, the dipole contribution is important and has to be carefully calculated.

In Sections 2 and 3 it is shown how to calculate the chromaticity due to the quadrupoles and sextupoles respectively. Then, in Section 4, the effects on beam dynamics due to the chromaticity correcting sextupoles are briefly discussed. Finally, in Section 5, a detailed derivation of the chromaticity for a general bending magnet is given, following the approach given by M. Bassetti in Ref. [3], which is very simple and intuitive, avoiding long mathematical derivations.

## 2. QUADRUPOLE

Let us consider the motion in a quadrupole magnet of a charged particle which obeys the betatron equation:

$$y'' + k_y y = 0 \qquad (y = x \text{ or } z) \tag{4}$$

with

$$k_x = -k$$
$$k_z = k \ .$$

Now we consider the dependence of k on the particle momentum $p$:

$$k = \frac{e}{p}\frac{\partial B_z}{\partial x} = \frac{e}{p_0(1+\delta)}\frac{\partial B_z}{\partial x} = \frac{k_0}{(1+\delta)} \simeq k_0\,(1-\delta+\delta^2-\ldots) \tag{5}$$

where $\delta$ is the relative momentum deviation respect to the reference particle. Taking the first order term in $\delta$ it is:

$$k \simeq k_0\,(1-\delta) = k_0 - k_0\delta \ . \tag{6}$$

The chromatic variation has always the opposite sign with respect to the focusing strength, therefore a particle with a larger energy sees a weaker focusing strength. Conversely for light, the variation of the refraction index with the wavelength can be either positive or negative and the chromatic effect can be corrected to first order by combining lenses of different material.

Substituting (6) into the equation of motion for a quadrupole yields:

$$y'' = -k_y\,(1-\delta)\,y \tag{7}$$

which is equivalent to adding to the focusing quadrupole a defocusing one with a strength $-k_y\delta$ and viceversa for the defocusing quadrupole.

In a thin section of a quadrupole of infinitesimal length $ds$ the particle receives an angular kick

$$dy' = y''\,ds = k_y\,\delta\,y\,ds \tag{8}$$

described by a thin lens (defocusing for the focusing quadrupole) matrix:

$$\begin{pmatrix} 1 & 0 \\ k_y \delta ds & 1 \end{pmatrix} \tag{9}$$

To compute the effect of this kick on the betatron tune, the one-turn matrix is obtained by multiplying the unperturbed matrix ($\delta = 0$) by this thin lens.

$$\boldsymbol{M} = \begin{pmatrix} \cos\mu_y & \beta_y \sin\mu_y \\ -\sin\mu_y/\beta_y & \cos\mu_y \end{pmatrix} \begin{pmatrix} 1 & 0 \\ k_y \delta ds & 1 \end{pmatrix} = \begin{pmatrix} \cos\mu_y + \beta_y \sin\mu_y k_y \delta ds & \beta_y \sin\mu_y \\ -\sin\mu_y/\beta_y + \cos\mu_y k_y \delta ds & \cos\mu_y \end{pmatrix} \tag{10}$$

Then we compute the trace of $\boldsymbol{M}$ to get the new value of $\mu_y$ ($\mu_y = 2\pi Q_y$):

$$\frac{1}{2}\ Tr\ \boldsymbol{M} = \cos(\mu_y + d\mu_y) = \cos\mu_y + \frac{1}{2}\ \beta_y \sin\mu_y\ k_y \delta\ ds \tag{11}$$

since:

$$d(\cos\mu_y) = \cos(\mu_y + d\mu_y) - \cos\mu_y = -\sin\mu_y\ d\mu_y \tag{12}$$

we get:

$$d\mu_y = -\frac{1}{2}\ \beta_y\ k_y\ \delta\ ds \tag{13}$$

or

$$dQ_y = -\frac{1}{4\pi}\ \beta_y\ k_y\ \delta\ ds\ . \tag{14}$$

Integrating over all the ring circumference $L$, we obtain the total chromaticity for the two planes, horizontal and vertical:

$$\frac{\partial Q_x}{\partial \delta} = -\frac{1}{4\pi} \int_0^L \beta_x(s)\ k_x(s)\ ds \tag{15}$$

$$\frac{\partial Q_z}{\partial \delta} = -\frac{1}{4\pi} \int_0^L \beta_z(s)\ k_z(s)\ ds\ . \tag{16}$$

*From these formulae we can see why, in a ring with strong focusing lattice, the chromaticity is always negative.* A quadrupole is a focusing lens in one plane, either horizontal or vertical, and defocusing in the other one. The *strength* $k_{x,z}$ is positive in the focusing plane (negative in the defocusing plane) and the chromaticity has the opposite sign, therefore it is negative when the quadrupole is focusing. In a strong focusing lattice the $\beta_{x,z}$ *functions* take the maximum values at the focusing quadrupoles, and the minimum at the defocusing ones, for each plane. Therefore the total chromaticity of a ring is dominated by the contribution of the focusing quadrupoles, negative in both planes.

## 3. SEXTUPOLE

Special magnets, the sextupoles, are inserted in the accelerator's lattice to correct the natural chromaticity produced by the focusing elements. In a sextupole a charged particle passing off-center receives a kick proportional to the square of its displacement from the center, i.e. a sextupole acts like a quadrupole with a focusing strength proportional to the displacement of the closed orbit from the sextupole center. This allows the chomaticity to be corrected because for off-momentumparticles the closed orbit is displaced with respect to the reference one by a quantity $D\delta$, where $D$ is the dispersion function and $\delta$ the momentum deviation.

The field of a sextupole is given by :

$$B_x = g'xz$$
$$B_z = \frac{1}{2}\, g' \,(x^2 - z^2) \qquad (17)$$

with

$$g' = \frac{\partial B_z^2}{\partial x^2} \ .$$

In Fig. 1 an example of the pole shape of a sextupole magnet is given.

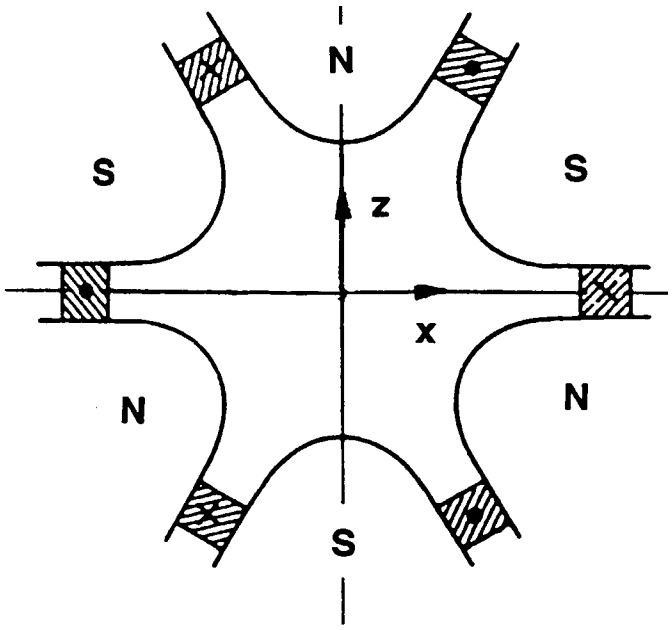


Fig. 1 Schematic representation of a sextupole magnet cross section

The equations of motion become:

$$x'' + \frac{1}{2}\ r\,(x^2 - z^2) = 0 \qquad \text{where } r = \frac{e}{p_0}\, g' \qquad (18)$$

$$z'' - rxz = 0 \ .$$

and the sextupole kick is :

$$dx' = -\frac{1}{2}\ r\,(x^2 - z^2)\,ds$$
$$dz' = r\,x\,z\,ds \ . \tag{19}$$

Substituting the total coordinates for the off-momentum particle

$$x_t = D\delta + x$$
$$z_t = z \tag{20}$$

it becomes:

$$dx' = -\left[ D\,\delta\,x + \frac{1}{2}(D\,\delta)^2 + \frac{1}{2}(\,x^2 - z^2\,) \right] r\,ds$$
$$dz' = [\,D\,\delta\,z + xz\,]\ r\,ds \ . \tag{21}$$

The first term of Eqs. (21) is equivalent to the kick of a quadrupole with gradient $-rD\delta$ and, analogously to Eq. (14), gives a tune shift

$$\Delta Q = \frac{1}{4\pi}\ \beta\, rD\ \delta\, ds \tag{22}$$

and a contribution to the chromaticity:

$$\frac{\partial Q_x}{\partial \delta} = \frac{1}{4\pi}\int_0^L \beta_x(s)\ r(s) D(s)\ ds$$
$$\frac{\partial Q_z}{\partial \delta} = -\frac{1}{4\pi}\int_0^L \beta_z(s)\ r(s) D(s)\ ds \ . \tag{23}$$

## 4 . CHROMATICITY CORRECTION

The most efficient way to correct chromaticity is to perform a localized correction, i.e. to insert a sextupole just in the same position of each quadrupole, where the chromatic effect is produced. In this ideal case the strength required to make the chromaticity zero is minimum and is simply related to the quadrupole strength:

$$r = -\frac{k}{D}\frac{l_Q}{l_S} \tag{24}$$

where $l_q$ and $l_s$ are the lengths of the magnets. In pratical cases one tries to put the chromaticity correcting sextupole as close as possible to each quadrupole but often, for economical reasons, less sextupoles with higher strengths are used.

In many cases, unfortunately, localized correction is not possible. For example collider storage rings have *low-*$\beta$ insertions with very strong quadrupoles and zero dispersion function. Similarly, storage rings for synchrotron light production have many zero dispersion

straight sections for insertion devices, like wigglers and undulators, and strong focusing quadrupoles to get low emittances. In these cases a strong chromaticity produced in the insertions has to be corrected in the arcs and the sextupole strengths become very high.

If the arcs are built up by $N$ periodic cells, two sextupoles are inserted in each cell, one in a high $\beta_x$ place, to correct horizontal chromaticity, and the other in a high $\beta_z$ position to correct the vertical one. The sextupole intensities are obtained by solving the following linear system of equations:

$$\begin{aligned} r_H l_S \beta_x^H \; D^H + r_V l_S \beta_x^V \; D^V &= \frac{Q_x'}{N} \\ -r_H \, l_S \beta_z^H \; D^H - r_V l_S \beta_z^V \; D^V &= \frac{Q_z'}{N} \; . \end{aligned} \tag{25}$$

where $r_H$ and $r_V$ are respectively the strengths of the horizontal and vertical chromaticity correcting sextupoles and $Q_x'$ , $Q_z'$ are the values of the total chromaticity of the ring in the two planes. Since $Q_x'$ *and* $Q_z'$ are, usually, both negative it has to be:

$$r_H < 0 \quad \text{and} \quad r_V > 0$$

and each sextupole corrects the chromaticity in one plane and increases it in the other.

In order to reduce the sextupole strengths, it is important to place them where the dispersion is high and the $\beta$ functions well separated. In fact if the horizontal sextupole is placed where $\beta_x >> \beta_z$ it is very effective in the horizontal plane and gives a negligible contribution in the other plane (viceversa for the vertical sextupole).

Generally the vertical sextupole has a strength higher than the horizontal one because the dispersion function $D$ follows the behaviour of the horizontal $\beta$-function, i.e. it is higher at the horizontal sextupole and lower at the vertical one. This is specially true for collider storage rings, where, due to the low value of $\beta_z$ at the interaction region, the vertical chromaticity is generally higher than the horizontal one.

The sextupoles necessary to correct the chromaticity introduce unwanted effects due to the other two terms in Eqs. (21):

- the chromatic aberration term $(D\delta)^2$
- the geometric aberration term $(x^2 - z^2)$, $xz$ .

The geometric aberration term introduces higher-order terms in the equations of motion. In fact each sextupole inserted into the linear lattice, also in thin-lens approximation, doubles the order of the polynomial which links the initial and final coordinates for one turn in the ring. With $N$ sextupoles in the ring the final coordinates depend on the $2^N$-th power of the initial one:

$$x(L) = a_{11} \, x(0) + a_{12} \, x'(0) + a_{13} \, \delta + ... a_{1j} \, x(0)^{2^N} .$$

When the nonlinear terms become important the stability of the particle trajectories in a circular accelerator is no longer obtained from the one-turn matrix $\boldsymbol{M}$ ($|\mathrm{tr}\, \boldsymbol{M}| = |2 \cos \mu| < 2$), but depends on the amplitude of the betatron and synchrotron oscillations so that the beam dynamics becomes much more complicated.

In some very simplified cases an analytical calculation of the stability region is possible, for example in the unidimensional case ($x,x'$ or $z,z'$ phase plane) in the vicinity of a single resonance. In this case a closed curve, called the separatrix, can be found which divides the phase plane in two regions, a stable one inside the separatrix and an unstable region outside.

In more general cases tracking is used, i.e. a computer code which, given the initial coordinates for a particle in phase space, follows the evolution of a trajectory with the mathematical model chosen for the ring. A trajectory is considered stable if it remains confined in a certain phase space region for a given number of turns. The initial coordinates of the particle's tracking are changed to determine the largest region of phase space which contains all stable trajectories. This region is called the *dynamic aperture.*

This procedure is limited by computer time and precision, in fact the range of initial coordinates which can be explored in a six-dimensional phase space is very poor and the number of turns is always much smaller with respect to the beam lifetime or damping time.

After the linear lattice design a dynamic aperture optimization has to be carried out by choosing the distribution and the strengths of the sextupoles, the working point in the tune diagram, and even modifying the linear lattice to reduce chromaticities and sextupole strengths. A discussion of the methods for the determination and optimization of the dynamic aperture, a very important problem in the design of new accelerators, is given by A. Ropert [4].

## 5. GENERAL BENDING MAGNET

In rings with large radius of curvature and small dispersion function the contribution to the chromaticity due to the bending magnets is negligible with respect to that of the quadrupoles. This is not the case for small rings. In the following is presented a derivation of the cromaticity formulae for the bending magnet valid also for small radii of curvature and taking into account the variation of the magnetic field in the ends.

In a bending magnet the betatron motion is given by the following equations:

$$y'' + k_y(s)y = 0 \qquad (y = x \text{ or } z) \tag{27}$$

with

$$k_x = -k+h^2$$

$$k_z = k$$

$$h = -\frac{e}{p} B_z \quad ; \qquad k = \frac{e}{p} \frac{\partial B_z}{\partial x} \ .$$

The solution of these equations is represented, in each plane, by the two-by-two betatron matrix $\boldsymbol{A}$. This matrix can be written as the product of $N$ matrices $\boldsymbol{A}_i$:

$$\boldsymbol{A} = \prod_{i=1}^{N} \boldsymbol{A}_i\left(\frac{l_B}{N}\right) \tag{28}$$

where $l_B$ is the length of the bending magnet. We choose $N$ to be large so that

$$\Delta s = \frac{l_B}{N} \ \longrightarrow \ 0 \ .$$

This is equivalent to subdividing the magnet into $N$ thin pieces of length $\Delta s$. To first

order in $\Delta s$, $A_i$ can be written as the product of a thin lens and a drift space:

$$\boldsymbol{A}_i = \begin{pmatrix} 1 & 0 \\ -k_y(s)\Delta s & 1 \end{pmatrix} \begin{pmatrix} 1 & \Delta s \\ 0 & 1 \end{pmatrix} \tag{29}$$

Now we consider the changes that occur in the betatron motion (i.e. in the matrix $\boldsymbol{A}_i$) for a particle with a relative momentum deviation $\delta$ oscillating around the off-momentum closed orbit.

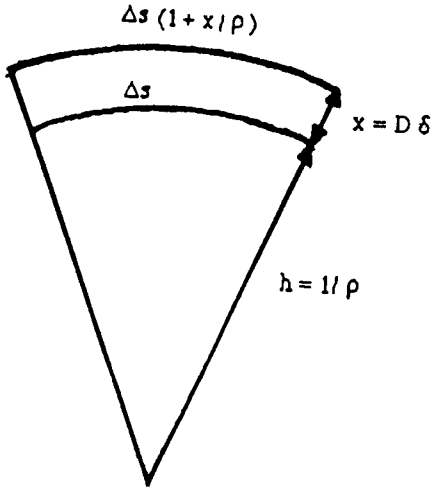


Fig. 2 Orbit lengthening

Two changes occur in the matrix $\boldsymbol{A}_i$:

i) an orbit lengthening (see Fig. 2)

$$\Delta s(\delta) = \Delta s(1 + hD\delta) \tag{30}$$

ii) a change in the focusing strength of the thin lens due to:

- momentum dependence of the focusing functions

- variation of the length

$$- k_y(s,\delta)\, \Delta s(\delta) = - [\, k_y(s) + k_{1y}(s)\delta + k_y(s)hD\delta]\, \Delta s \tag{31}$$

where $k_{1y}$ is the derivative: $k_{1y} = \partial k_y / \partial \delta$ .

As already seen, a change $\Delta k$ in the focusing function at the position $s$ gives a tune shift :

$$\Delta Q = - (1/4\pi)\, \beta(s)\, \Delta k \tag{32}$$

and, similarly, a change $\Delta s$ in the length of a drift space gives:

$$\Delta Q = (1/4\pi)\,\gamma(s)\Delta s \tag{33}$$

where $\gamma(s)$ is the Twiss function.

Integrating over all the circumference gives

$$\frac{\partial Q_y}{\partial \delta} = \frac{1}{4\pi}\int_0^L \{\beta_y\,[\,k_{1y} + k_y h D\,] + \gamma_y h D\,\}\,ds\ . \tag{34}$$

This formula is a generalization of that for a quadrupole, in fact for a quadrupole we have $h = 0$ and $k_{1y} = dk_y/d\delta = -k_y$ and we obtain again the formulae of Eqs. (15) and (16).

In order to calculate $k_{1y}$ for the general bending magnet we need to know the fields seen by an off-momentum particle. First we write the second-order magnetic field expansion in the reference system of the design orbit for zero momentum deviation. The formulation of the field equations is that given by K. Steffen [5] with the only difference that $h(s)$ has the opposite sign and its dependence on s is explicitly given, i.e:

$$h(s) = [h + h's + \frac{1}{2}\,h''s^2 + 0(3)]\ . \tag{35}$$

As it will be useful in the following to distinguish the second-order terms they have been enclosed in square brackets:

$$B_z = \frac{p}{e}\left\{-h - h's + kx + \left[-\frac{1}{2}h''s^2 + k'xs + \frac{1}{2}rx^2 + \frac{1}{2}(h'' - hk - r)\,z^2\right] + 0(3)\right\}$$

$$B_x = \frac{p}{e}\{\,kz + [\,k'zs + rxz\,] + 0(3)\} \tag{36}$$

$$B_s = \frac{p}{e}\{\,-h'z + [\,(hh' + k')xz\,] + 0(3)\}\quad .$$

The previous equations are completely general, they are only based on the assumption of a field symmetry with respect to the median plane ($z = 0$). Therefore, if we change the momentum of the particle, the origin and the orientation of the axis in the $z = 0$ plane, the magnetic field has always the same form, but different values of the coefficients.

Now we make a transformation to the reference system of the off-momentum particle, as shown in Fig. 3:

$$\begin{aligned} p &= p^*/(1+\delta) \\ z &= z^* \\ x &= d + x^*\cos\theta + s^*\sin\theta \\ s &= -x^*\sin\theta + s^*\cos\theta \end{aligned} \tag{37}$$

where $d = D\delta$ and $\theta = D'\delta$.

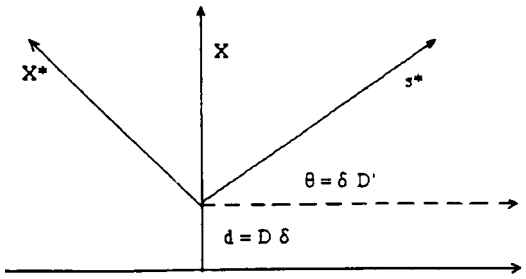


Fig. 3 Transformation of the reference system

The field equations change in the following way:

$$\begin{aligned} B_z^* &= B_z[\vec{x}(\vec{x}^*)] \\ B_x^* &= B_x[\vec{x}(\vec{x}^*)]\cos\theta - B_s[\vec{x}(\vec{x}^*)]\sin\theta \\ B_s^* &= B_x[\vec{x}(\vec{x}^*)]\sin\theta + B_s[\vec{x}(\vec{x}^*)]\cos\theta \,. \end{aligned} \tag{38}$$

We are interested in the first-order field expansion, therefore we take only the first-order terms in Eqs. (36) and make the substitution :

$$\begin{aligned} B_z^* &= \frac{p}{e(1+\delta)}[-h + kd + x^*(k\cos\theta + h'\sin\theta) - s^*(h'\cos\theta - k\sin\theta)] \\ B_x^* &= \frac{p}{e(1+\delta)} z[k\cos\theta + h'\sin\theta] \\ B_s^* &= \frac{p}{e(1+\delta)} z[k\sin\theta - h'\cos\theta] \quad . \end{aligned} \tag{39}$$

As already said, the various terms in the field equations have to be the same as in Eqs. (36), therefore equating the corresponding first-order terms we get the new coefficients:

$$\begin{aligned} h^* &= \frac{h-kd}{1+\delta} \\ k^* &= \frac{k\cos\theta + h'\sin\theta}{1+\delta} \\ h'^* &= \frac{h'\cos\theta - k\sin\theta}{1+\delta} \quad . \end{aligned} \tag{40}$$

Using

$$\sin\theta \sim D'\delta \,; \qquad \cos\theta \sim 1$$

and keeping only first-order terms in $\delta$ we get:

$$\begin{aligned} k_x^* &= h^{*2} - k^* = (h^2 - k) + \delta(-2h^2 - 2hkD + k - h'D') \\ k_z^* &= k^* = k + \delta(-k + h'D') \,. \end{aligned} \tag{41}$$

We obtain the values of $k_{1y}(s)$ as:

$$k_{1x} = \frac{\partial k_x^*}{\partial \delta} = k - 2h^2 - 2hkD - h'D'$$
$$k_{1z} = \frac{\partial k_z^*}{\partial \delta} = -k + h'D' \,. \tag{42}$$

Inserting these values into Eq. (34) we obtain the final formulae:

$$\frac{\partial Q_x}{\partial \delta} = \frac{1}{4\pi} \int_0^L \{\beta\,(k - 2h^2 - 2hkD - h'D') + \beta hD\,(h^2 - k) + \gamma hD\}\,ds$$
$$\frac{\partial Q_z}{\partial \delta} = \frac{1}{4\pi} \int_0^L \{\beta\,(-k + hkD + h'D') + \gamma hD\}\,ds \,. \tag{43}$$

As we used only first-order terms in this derivation the contribution of the sextupole term $\beta rD$, calculated in Section 3, does not appear in Eqs. (43). In Appendix I a similar derivation using the second-order field expansion is given. The final formulae contain the same terms as Eqs. (43) plus the sextupolar terms coming from the second-order terms in the field expansion which are linear in $x$ and, applying the translation $x = x^* + D\delta$ of Eqs. (37), produce linear terms.

## 5. END-FIELD EFFECTS

From Eqs. (43) it is possible to calculate the contribution of the fringing fields to the chromaticity, once an expression for $h'(s)$ is known. In a paper by W. Hardt, J. Jä ger and D. Mö hl [6] the same formulae are obtained with a different derivation, moreover a detailed calculation of the fringing field effects is given. For completeness we report the final formula obtained there. In Fig. 4 is reproduced the illustration of Ref. [6] which shows the schematization used for the end fields, the corresponding parameter definitions are listed below.

| | |
|---|---|
| $s_1$ | beginning of the central part |
| $s_2$ | end of the central part |
| "1" | entrance of the fringing region |
| "2" | exit of the fringing region |
| $\theta$ | entrance or exit angle of the trajectory |
| $\frac{1}{\tau \cos^3\theta}$ | radius of curvature of the end faces |
| $h = \frac{1}{\rho}$ | curvature of the reference orbit |
| $k = -\frac{1}{B\rho}\frac{\partial B_z(0,0,s)}{\partial x}$ | quadrupole component |

| | |
|---|---|
| $r = -\frac{1}{B\rho}\frac{\partial^2 B_z(0,0,s)}{\partial x^2}$ | sextupole component |
| $D,\ D'$ | dispersion function and its derivative |
| $\alpha,\ \beta,\ \gamma$ | Twiss functions. |

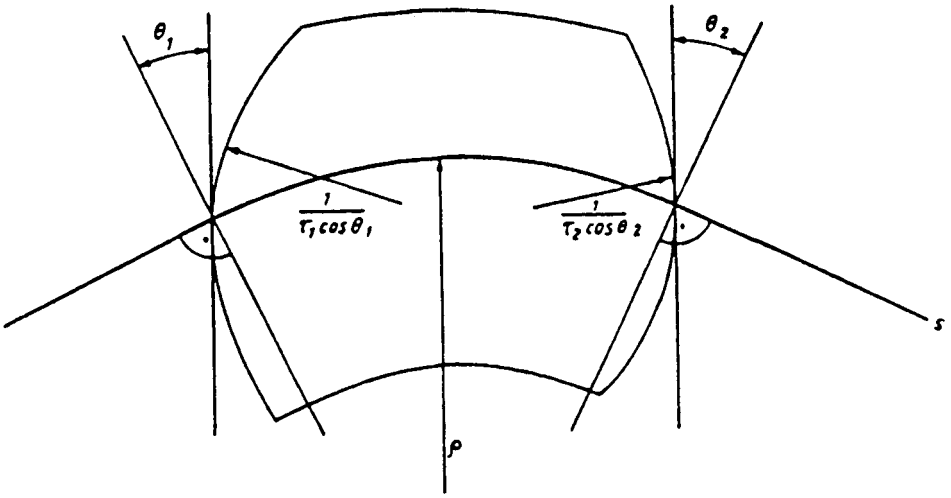


Fig. 4 Field boundaries for a bending magnet

The formulae to calculate the chromaticity of a magnet in terms of the lattice functions at the reference orbit are:

$$\frac{\partial Q_x}{\partial \delta} = -\frac{1}{4\pi}\int_{s_1}^{s_2}[(h^2-k)\beta + rD\beta + h(2kD\beta + 2D'\alpha - D\gamma)]ds$$

$$+[-\mathrm{tg}\theta(h\beta + 2Dk\beta) + h\mathrm{tg}^2\theta(\beta D' - 2\alpha D + hD\beta\mathrm{tg}\theta) + \tau h\beta D]_{"1"}$$

$$+[-\mathrm{tg}\theta(h\beta + 2Dk\beta) - h\mathrm{tg}^2\theta(\beta D' - 2\alpha D - hD\beta\mathrm{tg}\theta) + \tau h\beta D]_{"2"} \qquad (44)$$

$$\frac{\partial Q_z}{\partial \delta} = -\frac{1}{4\pi}\int_{s_1}^{s_2}[k\beta - rD\beta - h(kD\beta + D\gamma)]ds$$

$$+[\mathrm{tg}\theta(h\beta + 2Dk\beta) - h\mathrm{tg}^2\theta(\beta D' - 2\alpha D - hD\beta\mathrm{tg}\theta) - \beta hD' - \tau h\beta D]_{"1"}$$

$$+[\mathrm{tg}\theta(h\beta + 2Dk\beta) + h\mathrm{tg}^2\theta(\beta D' - 2\alpha D + hD\beta\mathrm{tg}\theta) + \beta hD' - \tau h\beta D]_{"2"}$$

## ACKNOWLEDGEMENTS

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# APPENDIX 1

## CHROMATICITY FOR A BENDING MAGNET TAKING INTO ACCOUNT SECOND-ORDER TERMS

### A1.1 DERIVATION OF THE CHROMATICITY FORMULAE

The chromaticity formulae in Section 5 are obtained using the first-order expansion of the magnetic field. To obtain an expression for the chromaticity which contains also the sextupolar terms, the same derivation is repeated here using also the second-order terms in the field expansion given by Eqs. (36). To get the new expressions for the magnetic field we apply the reference system transformation described in Section 5 to the second-order field expansion of Eqs. (36).

Following the derivation given in Section 5 and using :

$$\sin\theta \sim D'\delta \qquad ; \qquad \cos\theta \sim 1$$

we get the expressions for the magnetic field in the new reference system:

$$B_z^* = \frac{p}{e(1+\delta)} \; [-h - h'(-x^*D'\delta + s^*) + k(D\delta + x^* + s^*D'\delta) + k'(D\delta + x^* + s^*D'\delta)(-x^*D'\delta + s^*)$$

$$+ \frac{1}{2} r (D\delta + x^* + s^*D'\delta)^2 + \frac{1}{2}(h'' - hk - r) z^2]$$

$$B_x^* = \frac{p}{e(1+\delta)} z\{k + k'(-x^*D'\delta + s^*) + r (D\delta + x^* + s^*D'\delta) - [-h' - h''(-x^*D'\delta + s^*)$$

$$+ (hh' + k') (D\delta + x^* + s^*D'\delta)] D'\delta\} \qquad (45)$$

$$B_s^* = \frac{p}{e(1+\delta)} z\{[k + k'(-x^*D'\delta + s^*) + r (D\delta + x^* + s^*D'\delta)] D'\delta - h' - h''(-x^*D'\delta + s^*)$$

$$+ (hh' + k')(D\delta + x^* + s^*D'\delta)\} .$$

Neglecting the second-order terms, except for the chromatic ones, i.e. the terms $x\delta$, $z\delta$ and $s\delta$, we obtain:

$$B_z^* = \frac{p}{e(1+\delta)} \; [-h + kD\delta + x^*(h'D'\delta + k + rD\delta) + s^*(-h' + kD'\delta + k'D\delta) ]$$

$$B_x^* = \frac{p}{e(1+\delta)} z[k + rD\delta + h'D'\delta] \qquad (46)$$

$$B_S^* = \frac{p}{e(1+\delta)}\, z[-h' + kD'\delta + (hh'+k')D\delta] \,.$$

Comparing these equations with Eqs. (36) and equating the corresponding first-order terms, we get the new coefficients:

$$h^* = \frac{h-kD\delta}{1+\delta}$$

$$k^* = \frac{k + h'D'\delta + rD\delta}{1 + \delta} \tag{47}$$

$$h'^* = \frac{h' + kD'\delta + (hh'+k')D\delta}{1 + \delta} \,.$$

Now, following the same procedure as in Section 5, we use the coefficients $h^*$ and $k^*$ given by Eqs. (47) to obtain the values of the focusing strength for the off-momentum particle:

$$\begin{aligned} k_x^* &= h^{*2} - k^* = (h^2 - k) + \delta(-2h^2 - 2hkD + k - h'D' - rD) \\ k_z^* &= k^* = k + \delta(-k + h'D' + rD). \end{aligned} \tag{48}$$

Then, we get the variation of the focusing strength with momentum, $k_{1y}(s)$ :

$$\begin{aligned} k_{1x} &= \frac{\partial k_x^*}{\partial \delta} = k - 2h^2 - 2hkD - h'D' - rD \\ k_{1z} &= \frac{\partial k_z^*}{\partial \delta} = -k + h'D' + rD \,. \end{aligned} \tag{49}$$

The variation of the orbit length with momentum has been already taken into account in Eq. (34), therefore inserting Eqs. (49) into (34) we obtain the final formulae for the chromaticity, which are more complete than those of Eqs. (43) because they contain also the sextupolar terms.

$$\begin{aligned} \frac{\partial Q_x}{\partial \delta} &= \frac{1}{4\pi} \int_0^L \{\beta\,(k - 2h^2 - 2hkD - h'D' - rD) + \beta hD\,(h^2 - k) + \gamma hD\}\, ds \\ \frac{\partial Q_z}{\partial \delta} &= \frac{1}{4\pi} \int_0^L \{\beta\,(-k + hkD + h'D' + rD) + \gamma hD\}\, ds \,. \end{aligned} \tag{50}$$

### A1.1.1 An observation on Eqs. (47)

Let us notice that the coefficient $h'^*$ given by Eqs. (47) is obtained as the coefficient of the variable $z$ in the equation (46) for $B_S$, and that it is different from the coefficient of s which appears in the expression for $B_z$. This ambiguity comes from the fact that, while for Eqs. (36) the relation:

$$\frac{\partial B_z}{\partial s} = \frac{\partial B_s}{\partial z} \tag{51}$$

is valid, this is not true for Eqs. (46), for which it is:

$$\frac{\partial B_s^*}{\partial z} \neq \frac{\partial B_z^*}{\partial s^*} \ . \tag{52}$$

Equations (46) are anyway correct, but the new variable $s^*$ has to be modified. In cylindrical coordinates $(z,x,\phi)$, the radial component of the Maxwell equation is written:

$$\frac{1}{\rho}\frac{\partial Bz}{\partial \phi} = \frac{\partial B_\phi}{\partial z} \ . \tag{53}$$

When making the transformation given by Eqs. (37), which is essentially a translation in the radial direction, in Eqs. (53) $\rho$ has to be replaced by $\rho + D\delta$. As a consequence, the Maxwell equation is written:

$$\frac{1}{\rho+D\delta}\frac{\partial B_z^*}{\partial \phi} = \frac{1}{1+hD\delta}\frac{\partial B_z^*}{\partial s^*} = f(\partial B_s^*, \partial z) \ ; \qquad h = \frac{1}{\rho} \ . \tag{54}$$

This relation is in effect verified by Eqs. (46) to first order in $\delta$.