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THEORY OF RADIATION GENERATED BY A CHARGED PARTICLE PASSING THROUGH A CONTINUOUS PERIODIC MEDIUM

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1. Introduction

The dynamic effects occurring during the diffraction of free X-rays in mono-crystals are well known (vis e.g. /1/). Similar effects also occur in the case of transient X-rays formed when ultra-relativistic charged particles pass through crystals /2/. Physically speaking, these effects are determined by the dynamic interaction of the Bragg reflected and forward waves, and they mainly occur whenever the wave is propagated in a sufficiently ideal periodic medium with periods similar to the wave's length. In particular, dynamic effects may occur in the long-wave regions of electromagnetic waves, for instance in the sub-millimetric or infrared region. A common feature of all these cases is that the medium has a mean dielectric constant which is substantially different from and mainly greater than unity.

On the other hand, the irradiation of a charged particle in a medium with a continuously and periodically varying density was considered in papers /3 - 6/. Unlike the similar problem involving a system of regularly spaced plates of irregularly varying density where an accurate solution may be found (vis e.g. /7 - 9/), no precise answer was found in the case of a continuous periodic medium. This is because in the latter case the physical problem is much more general and therefore a nonhomogeneous Hill equation of the arbitrary type occurs for which a clear solution may be obtained approximately only when the density of the medium changes slightly. In this case the zero approximation of the perturbation theory is usually obtained /3 - 6/ by using the Hill equation solution which corresponds to a forward wave in the medium allowing for the mean value of the dielectric constant and assuming that the other waves are weak.

However, when there is Bragg reflection, the reflected wave is generally of the same order as the forward wave. Therefore, in this case the perturbation theory must be modified so that both these waves are taken into account in the zero approximation.

This report considers the radiation of a charge in a continuous periodic medium using such a modified perturbation theory. In extreme cases when the radiation frequency is sufficiently far removed from the Bragg frequency, the formulae obtained become the corresponding formulae in the standard perturbation theory. Moreover, close to the Bragg frequencies the formulae differ considerably from known formulae.

One particular result to emerge from our work is that extremely intense radiation occurs near the Bragg frequencies due to the dynamic interaction of the Bragg reflected and forward waves of coherently amplified transient radiation produced at strictly periodic inhomogeneities. This radiation is almost monochromatic and is propogated both forwards in the direction of motion of the charge and also backwards at extremely small angles to the charge trajectory. The maximum intensity of this radiation may not only considerably exceed the intensity of the transient radiation which occurs in the periodic medium away from the Bragg frequencies and which is obtained from the standard perturbation theory /3 - 6/ but may also be higher than the intensity of Cerenkov radiation at the same frequencies (if this radiation occurs).

Moreover, the report also examines the case where the periodic medium has a finite length. In this case, in addition to the radiation which is produced inside the periodic medium, emerges from it and is refracted in the normal way, transient radiation also occurs at the limits of the boundary separating the periodic medium and the vacuum.

Report /10/ describes the first experimental investigation into the radiation from relativistic electron bunches in a wave guide with regularly spaced plates of irregularly varying density.

2. General equations for a finite periodic medium

Let us assume that a fast charged particle with charge e moves uniformly along an axis z with a velocity v in a medium whose dielectric constant is a periodic function of z with a period z_0 ;

$$\varepsilon = \varepsilon(z) = \varepsilon_0\left(1 + q\varphi(z)\right), \tag{1}$$

where $\varphi(z+z_0)=\varphi(z)$, and $\int\limits_0^{z_0}\varphi(z)dz=0$, i.e. ε_0 is the mean dielectric constant of the medium.

If the vectors of the electromagnetic fields are represented as Fourier integrals

$$\begin{aligned}\vec{D}(\vec{r},t)&=\int\limits_{-\infty}^{\infty}\vec{D}(\vec{r},\omega)\exp(-i\omega t)\,d\omega\\ \vec{H}(\vec{r},t)&=\int\limits_{-\infty}^{\infty}\vec{H}(\vec{r},\omega)\exp(-i\omega t)\,d\omega\end{aligned}\tag{2}$$

then from the Maxwell equations we obtain /5/

$$\begin{aligned}&\frac{d^2Y(z)}{dz^2}+\left[\frac{\omega^2}{c^2}\varepsilon-\varkappa^2+\frac{1}{2\varepsilon}\frac{d^2\varepsilon}{dz^2}-\frac{3}{4}\left(\frac{1}{\varepsilon}\frac{d\varepsilon}{dz}\right)^2\right]Y(z)=\\ &=\left(\frac{\varepsilon}{\varepsilon_0}\right)^{1/2}\frac{e}{\pi v}\left(-\frac{i\omega v}{c^2}+\frac{i\omega}{v\varepsilon}-\frac{1}{\varepsilon^2}\frac{d\varepsilon}{dz}\right)\exp\left(i\frac{\omega}{v}z\right)\end{aligned}\tag{3}$$

The function $Y(z)$ is related to the Fourier components of the electromagnetic field's vectors in the following way

$$\begin{aligned}D_z(\vec{r},\omega)&=\int\limits_0^{\infty}(\varepsilon_0\varepsilon)^{1/2}Y(z)\,J_0(\varkappa\rho)\,\varkappa\,d\varkappa\\ D_\rho(\vec{r},\omega)&=\int\limits_0^{\infty}\left[\frac{e}{\pi v}\exp\left(i\frac{\omega}{v}z\right)-(\varepsilon_0\varepsilon)^{1/2}\left(\frac{Y(z)}{2\varepsilon}\frac{d\varepsilon}{dz}+\frac{dY(z)}{dz}\right)\right]J_1(\varkappa\rho)\,d\varkappa\\ H_\varphi(\vec{r},\omega)&=\int\limits_0^{\infty}\left[\frac{e}{\pi v}\exp\left(i\frac{\omega}{v}z\right)-i(\varepsilon_0\varepsilon)^{1/2}\frac{\omega}{c}Y(z)\right]J_1(\varkappa\rho)\,d\varkappa\,,\end{aligned}\tag{4}$$

where $J_0(x)$ and $J_1(x)$ are the Bessel functions of the zeroth and first orders, $\varkappa$, ρ are the values of the transverse components of wave vector $\vec{K}$ and the radius of the vector $\vec{r}$ respectively. From now on calculations will be made for $\omega>0$, so that for $\omega<0$ the expressions for /4/ fields are obtained by complex conjugation.

Let us break down the function $\varphi(z)$ into a Fourier series

$$\varphi(z) = \sum_{n=-\infty}^{\infty} a_{2n} \exp\left(2in\pi \frac{z}{z_0}\right) \quad ; \quad a_0 = 0 \, , \; a_{-2n} = a_{2n}^{*}. \tag{5}$$

We shall assume that the series (5) may be twice differentiated term by term. We should point out that it is wise to distinguish between two cases: (A), when the number of Fourier amplitudes a_{2n} is infinite but they vanish smoothly if the number $|n|$ increases by at least n^{-3} ; (B), when the number of Fourier amplitudes a_{2n} is finite and their values are arbitrary. In the case of a periodic medium consisting of plates, the corresponding Fourier series does not meet all the conditions of case (A) and the Fourier amplitudes vanish like I/n . Nevertheless, as will be seen later, in case (A) the phenomena occurring in the plates retain their main features /11/ .

By substituting (I) and (5) into equation (3) and by introducing the dimension less variable $\zeta = \pi z / z_0$, we obtain

$$\frac{d^2 Y}{d\zeta^2} + F(\zeta) Y = F_0(\zeta) \quad , \tag{6}$$

where $Y = Y(z_0 \zeta / \pi)$ and

$$F(\zeta) = \sum_{n=-\infty}^{\infty} \theta_{2n} \exp(2in\zeta)$$

$$F_0(\zeta) = \frac{\omega^2 z_0^2}{v^2 \pi^2} \frac{ei}{\pi \omega z_0} \exp\left(i \frac{\omega z_0}{v\pi} \zeta\right) \sum_{n=-\infty}^{\infty} M_{2n} \exp(2in\zeta). \tag{7}$$

The Fourier coefficients θ_{2n} and M_{2n} are unambiguously defined when equation (6) is derived. This equation is a non-homogeneous Hill equation. Using the general theory /12/ its solution may be written out formally. However, in order to obtain the solution in a clear form, we shall assume /12/ that the medium deviates only slightly from the homogeneous, i.e. $|q| \ll 1$. We then obtain the following with accuracy up to terms of higher orders of smallness:

$$\begin{aligned}
&\Theta_0 = \left(\frac{\omega^2\varepsilon_0}{c^2} - \varkappa^2\right)\frac{z_0^2}{\pi^2} \\
&\Theta_{2n} = q\left(\frac{\omega^2 z_0^2\varepsilon_0}{c^2\pi^2} - 2n^2\right) a_{2n} \qquad (n \neq 0) \\
&M_0 = 1 - \beta^2\varepsilon_0 \\
&M_{2n} = -q\left[\tfrac{1}{2}(1+\beta^2\varepsilon_0) + \frac{2\pi v}{\omega z_0} n\right] a_{2n} \quad (n \neq 0).
\end{aligned} \tag{8}$$

If the linearly independent solutions $y_1(\zeta)$ and $y_2(\zeta)$ of the homogeneous Hill equation (6) are known, then the solution of the nonhomogeneous equation may be written in the form

$$Y = y_1(\zeta)\left[A_1 - V_1(\zeta)\right] + y_2(\zeta)\left[A_2 + V_2(\zeta)\right], \tag{9}$$

where

$$\begin{aligned}
&V_{1,2}(\zeta) = \frac{1}{W}\int_0^{\zeta} y_{2,1}(u)\, F_0(u)\, du, \\
&W = y_1 y_2' - y_2 y_1'.
\end{aligned} \tag{10}$$

In order to determine the constants A_1 , and A_2 , we shall use the periodicity condition for the solution of equation (6) (vis /7/)

$$Y(z+z_0) = Y(z)\exp\left(i\,\frac{\omega z_0}{v}\right)$$

We thus obtain

$$A_{1,2} = \mp\frac{V_{1,2}(\pi)\exp\left[i\left(\Gamma_{1,2}\pi - \frac{\omega z_0}{v}\right)\right]}{1 - \exp\left[i\left(\Gamma_{1,2}\pi - \frac{\omega z_0}{v}\right)\right]}, \tag{11}$$

where $\exp(i\gamma_1\pi)$ and $\exp(i\gamma_2\pi)$ are the constants which appear according to the Floquet theorem before the solutions to the homogeneous Hill equation $y_1(\zeta)$ and $y_2(\zeta)$ when the arguments shift to the period π .

3. Solution of the homogeneous equation

In accordance with the general system for solving the homogeneous Hill equation we assume that

$$\zeta(z) = \exp(i\gamma z) \sum_{n=-\infty}^{\infty} C_{2n} \exp(2inz). \tag{12}$$

By substituting (12) into the homogeneous equation corresponding to equation (6), we obtain the recurrent relations

$$\left[\theta_0 - (\gamma + 2n)^2\right] C_{2n} + \sum_{m=-\infty}^{\infty}{}' \theta_{2m} C_{2(n-m)} = 0, \tag{13}$$

where the prime on the sum sign signifies that the term with $m = 0$ must be omitted during summation. These relations give an infinite system of equations for determining the hitherto unknown values C_{2n} which are Fourier coefficients of solution (12) of the homogeneous Hill equation.

The coefficients $\theta_{2m}\ (m \neq 0)$ contain the weak parameter q and the series $\sum\limits_m \theta_{2m}$ must be converging due to the assumption concerning the twofold differentiability of series (5). It may be seen from equation (13) that weak coefficients θ_{2m} precede all $C_{2(n-m)}$ whilst C_{2n} is preceded by the coefficient $\theta_0 - (\gamma + 2n)^2$. If the latter value is not small at any value of the whole number n, then the system of equations (13) has only a trivial solution. This system of equations may have a non-trivial solution only when the value $\theta_0 - (\gamma + 2n)^2 \approx 0$ for a certain whole number n. As the value γ is determined with accuracy up to an arbitrary even number, it may be considered that $n = 0$ in the above condition without violating generality. The system of equations (13) may therefore have a non-trivial solution if $\gamma \approx \pm\sqrt{\theta_0}$.

If the value $\sqrt{\theta_0}$ is not close to a whole number, then the values $\theta_0 - (\gamma + 2n)^2$ at an arbitrary whole number $n \neq 0$ may not be small. It then follows from equations (13) that in this case all coefficients $C_{2n}\ (n \neq 0)$ are much smaller than C_0. In physical terms this corresponds to a situation where the condition for Bragg reflection is not fulfilled and only one forward wave is important /3 - 6/.

Let us now assume that the value $\sqrt{\theta_0}$ is close to a whole number $h > 0$. Previous papers have not discussed this case (vis /12/). The authors of paper /5/ also pointed out the need to make a special study of this case. The expression C_{2n} $(n \neq 0)$ then has one coefficient, namely C_{-2h} , which is of the same order as C_0 whilst the other coefficients remain small. In fact, relationship (13) at $n = -h$ takes the form

$$\left[\theta_0 - (\gamma - 2h)^2\right] C_{-2h} \cdots\cdots + \theta_{-2h} C_0 + \cdots = 0$$

As $\sqrt{\theta_0} \approx \gamma \approx h$, then $\theta_0 - (\gamma - 2h)^2$ is small and it thus follows that C_{-2h} and C_0 may be of the same order. In physical terms this means that there is Bragg reflection and, besides the forward wave C_0 , the reflected wave C_{-2h} also plays an important part.

Taking the above into account, we must retain the following two equations from system (13) as a zero approximation:

$$\begin{aligned} &(\theta_0 - \gamma^2) C_0 + \theta_{2h} C_{-2h} = 0 \\ &\theta_{-2h} C_0 + \left[\theta_0 - (\gamma - 2h)^2\right] C_{-2h} = 0 . \end{aligned} \tag{14}$$

In order to provide equations (14) with a non-trivial solution, the system's determinant must equal zero i.e.

$$\left[\theta_0 - \gamma^2\right]\left[\theta_0 - (\gamma - 2h)^2\right] - \theta_{2h}\theta_{-2h} = 0 \tag{15}$$

This relation is an approximated characteristic equation for determining γ when the value $\sqrt{\theta_0}$ is close to a whole number h which does not equal zero.

In order to solve equation (15) we shall assume

$$\theta_0 = h^2 + a \quad , \quad \gamma = h + \delta , \tag{16}$$

where $|a| \ll h^2$, $|\delta| \ll h$. Then for δ we obtain two values

$$\delta = \delta_{1,2} = \pm \frac{\sqrt{a^2 - \theta_{2h}\theta_{-2h}}}{2h} . \tag{17}$$

In this case we have

$$C_{-2h} = \frac{2h\delta - a}{\theta_{2h}} C_0 . \tag{18}$$

By replacing γ in (12) with the values $\gamma_{1,2} = k + \delta_{1,2}$ and by also taking into account (18), we obtain two linearly independent solutions for the zero approximation of the homogeneous Hill equation

$$y^{(0)}_{1,2}(\xi) = C_0 \exp(i\delta_{1,2}\xi)\Big[\exp(ik\xi) + \frac{2k\delta_{1,2}-a}{\theta_{2k}}\exp(-ik\xi)\Big] \qquad (19)$$

In the zero approximation only two main terms C_0 and C_{-2k} were selected from all the sums in (12). In order to find the next approximation, the contributions from the other C_{2n} terms to the solution must be taken into account. They may be determined using relation (13) in which of the sum in terms of m it is sufficient to keep only the two main terms $m = n$ and $n = k$ corresponding to C_0 and C_{-2k}. We thus obtain

$$y^{(1)}_{1,2}(\xi) = y^{(0)}_{1,2}(\xi) + C_0 \exp[i(k+\delta_{1,2})\xi] \cdot \sum_{n=-\infty}^{\infty}{}'' \frac{\theta_{2n}\theta_{2k} + (2k\delta_{1,2}-a)\theta_{2(n+k)}}{4\theta_{2k}\, n(k+n)} \exp(i2n\xi), \qquad (20)$$

in which the two primes by the sum sign denote that terms with $n = 0$ and $-k$ must be omitted from summation.

It is easy to see that away from the Bragg frequencies when $|a| \gg |\theta_{2k}|$ and $|\theta_{-2k}|$, formula (20) is converted into the expressions

$$\begin{aligned} y^{(1)}_1(\xi) &\approx C_0 \exp(i\gamma_1\xi)\Big\{1 + \sum_{n=-\infty}^{\infty}{}' \frac{\theta_{2n}}{4n(\gamma_1+n)}\exp(2in\xi)\Big\} \\ y^{(1)}_2(\xi) &\approx C_0' \exp(-i\gamma_1\xi)\Big\{1 + \sum_{n=-\infty}^{\infty}{}' \frac{\theta_{-2n}}{4n(\gamma_1+n)}\exp(-2in\xi)\Big\}. \end{aligned} \qquad (21)$$

These formulae provide the solution to the homogeneous equation of the given problem and are obtained in a first approximation of the standard perturbation theory.

4. Solution of nonhomogeneous equation

Having obtained the solution to the homogeneous equation, the solution to the nonhomogeneous equation may be derived in terms of formulae(9)-(11). By using formula (20), we obtain the following after the appropriate calculations.

$$Y = Y^{(0)} + \Delta Y , \tag{22}$$

where $Y^{(0)}$ is obtained by means of the zero approximation formula (19):

$$Y^{(0)} = \frac{i b_0^2 e}{4 h^2 \pi \omega \varepsilon_0} \sum_{n=-\infty}^{\infty} \left\{ \frac{\theta_{2k} M_{2(n-k)}}{(b_n - h)^2 - \delta_1^2} + \frac{4 h^2 (h^2 - b_n^2 - a) M_{2n}}{(b_n^2 - h^2 - \delta_1^2)^2 - 4 h^2 \delta_1^2} + \right.$$
$$\left. + \frac{\theta_{-2k} M_{2(n+k)}}{(b_n + h)^2 - \delta_1^2} \right\} \exp(i b_n z) . \tag{23}$$

Here $b_n = \frac{\omega z_0}{v \pi} + 2n$. The value ΔY is obtained when the first-order corrections made in formula (20) are taken into account, and it has the form

$$\Delta Y = \frac{i b_0^2 e}{8 h^2 \pi \omega \varepsilon_0} \left\{ \frac{a}{h^2} \sum_m \left[\frac{-\theta_{2k} M_{2(m-k)} + (2h(b_m - h) + a) M_{2m}}{(b_m - h)^2 - \delta_1^2} + \right. \right.$$
$$+ \left. \frac{-\theta_{-2k} M_{2(m+k)} + (2h(b_m + h) + a) M_{2m}}{(b_m + h)^2 - \delta_1^2} \right] \exp(i b_m z) -$$
$$- \sum \sum{}'' \frac{1}{2n(k+n)} \left[\frac{-\theta_{2n}\theta_{2k} + (2h(b_{n+m} - h) + a)\theta_{2(n+k)}}{(b_{\ldots} \; \ldots h)^2 - \delta_1^2} M_{2(m-k)} + \right.$$

[illegible]

$$(b_{\ldots} + h)^2 - \delta_1^2$$

$$+ \frac{-\theta_{2n}\theta_{2k} + (2h(b_m - h) + a)\theta_{2(n+k)}}{(b_m - h)^2 - \delta_1^2} M_{2(m-k)} +$$
$$+ \left. \left. \frac{-\theta_{2(n+k)}\theta_{-2k} + (2h(b_m - h) + a)\theta_{2n}}{(b_m - h)^2 - \delta_1^2} M_{2m} \right] \exp(i b_{n+m} z) \right\} .$$

Let us first consider the extreme case of a homogeneous medium when $q = 0$. Then according to (8) the values θ_{2n} and M_{2n} equal zero at $n \neq 0$. We thus obtain

$$Y = -\frac{i\omega e}{v^2 \pi \varepsilon_0} \exp\left(i\frac{\omega}{v} z\right) \frac{1-\beta^2\varepsilon_0}{\omega^2/v^2 - \omega^2\varepsilon_0/c^2 + \varkappa^2} \quad . \tag{25}$$

We shall substitute this expression into (4) and integrate in terms of $\varkappa$ after using the following formulae (vis e.g. /13/)

$$\int_0^\infty \frac{\varkappa J_0(\varkappa\rho)\,d\varkappa}{\varkappa^2 - \zeta^2} = K_0(-i\zeta\rho)$$
$$\int_0^\infty \frac{J_1(\varkappa\rho)\,d\varkappa}{\varkappa^2 - \zeta^2} = -\frac{1}{\zeta^2\rho} + \frac{K_1(-i\zeta\rho)}{i\zeta} \quad , \tag{26}$$

where $K_{0,1}(x)$ are the modified Hankel functions of the zeroth and first orders and $\mathrm{Im}\,\zeta > 0$. As a result of the integration we obtain for instance,

$$D_z(\rho,\omega) = \frac{i\omega e}{v^2\pi} \exp\left(i\frac{\omega z}{v}\right)(\beta^2\varepsilon_0 - 1) K_0\left(-i\frac{\omega\rho}{v}\sqrt{\beta^2\varepsilon_0 - 1}\right) \tag{27}$$

At high absolute values of the argument we have $K_{0,1}(x) \approx \sqrt{\pi/2x}\,\exp(-x)$. When $\beta^2\varepsilon_0 > 1$, the charge field, like the function ρ at large distances, does not vanish exponentially (in the transparent medium) and this corresponds to the production of Cerenkov radiation.

In order to calculate the integrals (4) using expressions (22)-(24), the latter must be decomposed into the most common fractions of type (25) with denominators containing $\varkappa^2$. In expressions (23) and (24) the variable $\varkappa^2$ is included only in the values a and δ_1^2. The value δ_1^2 in its turn is expressed via a in terms of formula (17) whilst from (8) and (16) we have

$$a = \frac{\omega^2 z_0^2 \varepsilon_0}{c^2 \pi^2} - k^2 - \frac{\varkappa^2 z_0^2}{\pi^2} \tag{28}$$

Taking the above into account it may be seen that expressions (23) and (24) are decomposed into the most common fractions with denominators of 4 types $a \pm \alpha_+(n)$ and $a \pm \alpha_-(n)$ where

$$\alpha_{\pm}(n) = \sqrt{4h^2(b_n \pm h)^2 + \theta_{2k}\,\theta_{-2k}}\ , \tag{29}$$

n is the corresponding whole number. For $Y^{(o)}$ we have

$$\begin{aligned} Y^{(o)} = \frac{i b_o^2 e}{2\pi\omega\varepsilon_o} \sum_{n=-\infty}^{\infty} \Big\{ & \frac{\theta_{2k} M_{2(n-k)}}{\alpha_-(n)} \left(\frac{1}{a + \alpha_-(n)} - \frac{1}{a - \alpha_-(n)} \right) + \\ & + \frac{h M_{2n}}{b_n} \left[\frac{1}{\alpha_-(n)} \left(\frac{h^2 - b_n^2 + \alpha_-(n)}{a + \alpha_-(n)} - \frac{h^2 - b_n^2 - \alpha_-(n)}{a - \alpha_-(n)} \right) - \right. \\ & \left. - \frac{1}{\alpha_+(n)} \left(\frac{h^2 - b_n^2 + \alpha_+(n)}{a + \alpha_+(n)} - \frac{h^2 - b_n^2 - \alpha_+(n)}{a - \alpha_+(n)} \right) \right] + \\ & + \frac{\theta_{-2k} M_{2(n+k)}}{\alpha_+(n)} \left(\frac{1}{a + \alpha_+(n)} - \frac{1}{a - \alpha_+(n)} \right) \Big\} \exp(i b_n \xi). \end{aligned} \tag{30}$$

Away from the Bragg frequencies when $|a| \gg |\theta_{2k}|$ and $|\theta_{-2k}|$, the above solution Y to the nonhomogeneous equation inevitably becomes the corresponding solution obtained using the standard perturbation theory /3/.

It should, however, be stressed that all the formulae obtained are used under the conditions $|a| \ll h^2$ and $|\delta| \ll h$. If these conditions are not fulfilled, then instead of formulae (17) and (18) we have $\delta_{1,2} = \pm\sqrt{2h^2 + a - \sqrt{4h^4 + 4h^2 a + \theta_{2k}\,\theta_{-2k}}}$ and $C_{-2k} = C_o\,(\delta^2 + 2h\delta - a)/\theta_{2k}$. The appropriate changes must also be made in the subsequent formulae.

5. Radiation occurring inside an infinite continuous periodic medium

Let us first calculate the main contribution to the value $D_\rho(\vec{r}, \omega)$. In order to do this we shall replace the value Y in formula (4) by expression (30) and shall ignore the term containing $d\varepsilon / d\varkappa$. We should point out that expression (23) (and hence (30) and (24)) were obtained under the condition

$$\sqrt{\frac{\omega^2 \varepsilon_0}{c^2} - \varkappa^2} \; \frac{a}{\pi} \approx h \, . \tag{31}$$

When integrating into (4) in terms of $\varkappa$ the condition (31) may be violated. In order to determine under which conditions relation (31) will not be violated when integrating in terms of $\varkappa$, we shall find which values of $\varkappa$ make a substantial contribution to integrals (4). As the Bessel functions $J_{0,1}(\varkappa\rho)$ oscillate and vanish at high values of the argument, the values $\varkappa < \varkappa_0$ where $\varkappa_0 \sim \frac{1}{\rho}$ make a considerable contribution to integrals (4). If we require that $\varkappa_0 \ll \omega\sqrt{\varepsilon_0}/c$, then formula (23) may be used when calculating integrals (4) provided that $\rho \gg c/\omega\sqrt{\varepsilon_0}$ It is clear that the latter condition is easily met. Condition (31) may then be written in the form

$$\frac{\omega a}{c\pi}\sqrt{\varepsilon_0} \approx h \, . \tag{32}$$

which coincides with the Bragg condition in a medium with an average dielectric constant ε_0 at an angle of incidence $\pi/2$ Substituting condition (31) for (32) means in fact that we are restricted to the study of radiation emitted at small angles to the charge trajectory.

Taking the above into account, after integrating in terms of $\varkappa$ we obtain the following using formulae (26)

$$D_\rho^{(0)}(\vec{z},\omega)=\frac{\exp(i\frac{7\pi}{4})}{2}\,\frac{\omega e}{v^2 z_0}\left(\frac{\pi}{2\rho}\right)^{1/2}\sum_{n=-\infty}^{\infty}\exp(i b_n \pi \tfrac{z}{z_0})\;\cdot$$
$$\cdot\left\{\frac{b_n\theta_{2k}M_{2(n-k)}+(k^2-b_n^2-\alpha_-(n))kM_{2n}}{\varkappa_{n1}^{3/2}\,\alpha_-(n)}\exp(i\varkappa_{n1}\rho)-\right.$$
$$-\frac{b_n\theta_{2k}M_{2(n-k)}+(k^2-b_n^2+\alpha_-(n))kM_{2n}}{\varkappa_{n2}^{3/2}\,\alpha_-(n)}\exp(i\varkappa_{n2}\rho)\qquad(33)$$
$$+\frac{b_n\theta_{-2k}M_{2(n+k)}-(k^2-b_n^2-\alpha_+(n))kM_{2n}}{\varkappa_{n3}^{3/2}\,\alpha_+(n)}\exp(i\varkappa_{n3}\rho)-$$
$$\left.-\frac{b_n\theta_{-2k}M_{2(n+k)}-(k^2-b_n^2+\alpha_+(n))kM_{2n}}{\varkappa_{n4}^{3/2}\,\alpha_+(n)}\exp(i\varkappa_{n4}\rho)\right\},$$

where

$$\varkappa_{n1,2}^2=\frac{\omega^2}{c^2}\varepsilon_0-\frac{k^2\pi^2}{z_0^2}\mp\frac{\pi^2}{z_0^2}\alpha_-(n)$$
$$\varkappa_{n3,4}^2=\frac{\omega^2}{c^2}\varepsilon_0-\frac{k^2\pi^2}{z_0^2}\mp\frac{\pi^2}{z_0^2}\alpha_+(n)\,.\qquad(34)$$

After carrying out similar calculations using formula (4), (30) and (26) we obtain

$$D_z^{(0)}(\vec{z},\omega)=\frac{\exp(i\frac{3\pi}{4})}{2}\,\frac{\omega e}{v^2\pi}\left(\frac{\pi}{2\rho}\right)^{1/2}\sum_{n=-\infty}^{\infty}\frac{\exp(i b_n\pi\frac{z}{z_0})}{b_n}\;\cdot$$
$$\cdot\left\{\frac{b_n\theta_{2k}M_{2(n-k)}+(k^2-b_n^2-\alpha_-(n))kM_{2n}}{\varkappa_{n1}^{1/2}\,\alpha_-(n)}\exp(i\varkappa_{n1}\rho)-\right.$$
$$-\frac{b_n\theta_{2k}M_{2(n-k)}+(k^2-b_n^2+\alpha_-(n))kM_{2n}}{\varkappa_{n2}^{1/2}\,\alpha_-(n)}\exp(i\varkappa_{n2}\rho)+\qquad(35)$$
$$+\frac{b_n\theta_{-2k}M_{2(n+k)}-(k^2-b_n^2-\alpha_+(n))kM_{2n}}{\varkappa_{n3}^{1/2}\,\alpha_+(n)}\exp(i\varkappa_{n3}\rho)-$$
$$\left.-\frac{b_n\theta_{-2k}M_{2(n+k)}-(k^2-b_n^2+\alpha_+(n))kM_{2n}}{\varkappa_{n4}^{1/2}\,\alpha_+(n)}\exp(i\varkappa_{n4}\rho)\right\}.$$

The value $H_\varphi(\vec{z},\omega)$ is expressed by the formula obtained from (33) if the factor $z_0\omega/\pi c b_n$ is included under the sum sign.

If Cerenkov radiation is likely to occur in the medium, i.e. when $\beta^2\varepsilon_0>1$, and we are not near the threshold, the radiation described by formula (33) must be supplemented by the conventional Cerenkov radiation in a medium with an average dielectric constant. This radiation is determined in the zero approximation by formula

(25). This is because the transverse component of the wave vector of Cerenkov radiation $\omega\sqrt{\beta^2\varepsilon_0-1}\,/v$ away from the threshold is large and therefore conventional Cerenkov radiation is not covered by formula (33) which is obtained when condition (32) is met.

It may be seen from formula (33) that only those terms for which the (34) values are positive contribute to the field at large ρ distances. Moreover, it may also be seen from (33) that these terms will increase as the (34) values decrease. As we are considering a case where the Bragg condition (32) is met, the (34) values will be small only if $\alpha_\pm(n) \ll h^2$. It may be seen from the expressions for $\alpha\pm(n)$ and b_n that this may be achieved if $|\omega\varepsilon_0/v\pi + 2n \pm h| \ll h$.

With this in view, we shall require that

$$\frac{\omega\varepsilon_0}{v\pi} = h' + d \quad , \tag{36}$$

where $|d| \ll h'$ and h' is a positive whole number of the same parity as h. Then, by substituting (36) into (29) and assuming $n = n_0 + n_1$, where n_0 meets the conditions

$$2n_0 = \mp h - h' \tag{37}$$

for $\alpha\pm(n)$ we obtain

$$\alpha_\pm(n) = \sqrt{4h^2(d+2n_1)^2 + \theta_{2h}\,\theta_{-2h}} \; . \tag{38}$$

The order of magnitude below the root in formula (38) is determined by the first term. Let us compare the values $\alpha\pm(n_0)$, $\alpha\pm(n_0 \pm 1)$, $\alpha\pm(n_0 \pm 2)$, ... If h' is a small number, then the two values $\alpha\pm(n_0)$ are always much smaller than the remaining values. If $h' \gg 1$, then these values are small and of one order at $|n_1| \ll h'$.

It follows from conditions (36) and (32), under which radiation is intensified, that the difference between the time of flight of the charge and the average time for the propagation of radiation through the medium's inhomogeneity period must equal a whole multiple period of oscillation of the radiation.

We should point out that if the value $|d| \gg \frac{\sqrt{\theta_{2k}\theta_{-2k}}}{h}$ in condition (36), then it follows from formula (38) that $\alpha_{\pm}(n) \approx 2h\,|d+2n_1|$ and formula (33) becomes the corresponding formula obtained from the standard perturbation theory.

Thus, if h' (and consequently h) is a small number, i.e. if the radiation wave-length is comparable to the medium's period of inhomogeneity, then the main contribution to formula (33) comes from terms with $n = n_0$. We thus obtain for instance,

$$
\begin{aligned}
D_{\rho}^{(0)}(\vec{z},\omega) = & -\frac{\exp\left[i\left(\frac{3}{4}+\frac{hd}{z_0}\right)\pi\right]}{2\alpha}\,\frac{\omega^2 e}{v^2 c}\left(\frac{\varepsilon_0}{2\rho\pi}\right)^{1/2} \cdot \\
& \cdot\Big\{\Big[\frac{-(2hd+\alpha)M_{k-k'}+\theta_{2k}M_{-k-k'}}{\varkappa_1^{3/2}}\exp(i\varkappa_1\rho)+ \\
& + \frac{(2hd-\alpha)M_{k-k'}-\theta_{2k}M_{-k-k'}}{\varkappa_2^{3/2}}\exp(i\varkappa_1\rho)\Big]\exp\left(\frac{ih\pi z}{z_0}\right)- \\
& - \Big[\frac{\theta_{-2k}M_{k-k'}+(2hd-\alpha)M_{-k-k'}}{\varkappa_1^{3/2}}\exp(i\varkappa_1\rho)- \\
& - \frac{\theta_{-2k}M_{k-k'}+(2hd+\alpha)M_{-k-k'}}{\varkappa_2^{3/2}}\exp(i\varkappa_2\rho)\Big]\exp\left(-\frac{ih\pi z}{z_0}\right)\Big\},
\end{aligned}
\qquad (39)
$$

where

$$
\begin{aligned}
\alpha &= \sqrt{4h^2d^2+\theta_{2k}\theta_{-2k}} \\
\varkappa_{1,2}^2 &= \frac{\omega^2}{c^2}\varepsilon_0 - \frac{h^2\pi^2}{z_0^2} \mp \frac{\pi^2}{z_0^2}\alpha .
\end{aligned}
\qquad (40)
$$

We should point out that in the particular case of the ultra-relativistic particle, i.e. $1-\beta^2 \ll 1$, and in the frequency region where $\varepsilon_0 - 1 = g_0 < 0$ and $|g_0| \ll 1$, expression (39) must coincide with the expression $E_{pac}(\vec{z},\omega)$ for the case of almost exactly backward Bragg reflection obtained in /2/. In fact, if the appropriate integration in terms of the wave vector $\vec{k}$ is made using formula (29) from /2/, we then obtain an expression coinciding with expression (39) considered in the above particular case.

Let us now find the intensity of radiation occurring per unit length of the charged particle's trajectory. In order to do this we shall calculate the flux of the Pointing vector passing through the circular region $\rho_1 \leq \rho \leq \rho_2$ in the plane perpendicular to the z axis (vis Fig. 1),

$$dW = \frac{c}{4\pi}\int_{-\infty}^{\infty} dt \int_{\rho_1}^{\rho_2} 2\pi\rho \, d\rho \, E_\rho(\vec{r},t) H_\varphi(\vec{r},t). \tag{41}$$

In this case $\rho_2 = z \, tg \, \vartheta$, $\rho_1 = (z - dz) \, tg \, \vartheta$, where $\vartheta = arctg(\pm k_z \varkappa_i / \omega\sqrt{\varepsilon_0})$

is the radiation angle. The values $E_\rho(\vec{r},t)$ and $H_\varphi(\vec{r},t)$ are easily obtained from (39) and (2). When calculating integral (41) interference terms containing multiples of the type $exp\, i(\varkappa_1 - \varkappa_2)\rho$ or $exp\, 2ik\pi z/z_0$ occur and must be omitted.

We thus obtain for forward radiation,

$$\frac{dW}{dz} = \frac{e^2}{4v^4}\int \frac{\omega^3}{|\alpha|^2 \varepsilon_0} \left\{ \frac{|-(2kd+\alpha)M_{k-k'} + \theta_{2k} M_{-k-k'}|^2}{|\varkappa_1|^3} Re\, \varkappa_1 + \right.$$
$$\left. + \frac{|(2kd-\alpha)M_{k-k'} - \theta_{2k} M_{-k-k'}|^2}{|\varkappa_2|^3} Re\, \varkappa_2 \right\} d\omega , \tag{42}$$

and for backward radiation,

$$\frac{dW}{dz} = \frac{e^2}{4v^4}\int \frac{\omega^3}{|\alpha|^2 \varepsilon_0} \left\{ \frac{|\theta_{-2k} M_{k-k'} + (2kd-\alpha)M_{-k-k'}|^2}{|\varkappa_1|^3} Re\, \varkappa_1 + \right.$$
$$\left. + \frac{|\theta_{-2k} M_{k-k'} + (2kd+\alpha)M_{-k-k'}|^2}{|\varkappa_2|^3} Re\, \varkappa_2 \right\} d\omega . \tag{43}$$

It may be seen from formulae (42) and (43) that the radiation intensities are determined by the Fourier amplitudes a_{2k}, $a_2(k-k')$ and $a_{-2}(k+k')$ regardless of the presence of other amplitudes. As we are considering the small numbers k and k' , cases A and B mentioned in section 2 do not differ greatly from one another. We should emphasize that these formulae are correct only in the neighbourhood of the Bragg frequencies $\omega_B = k\pi c / z_0\sqrt{\varepsilon_0}$ and at $\omega > 0$.

Let us now examine these expressions when the value is arbitrary in two separate cases, when the value $\beta^2 \varepsilon_0$ differs considerably from unity (away from the Cerenkov radiation threshold) and when $\beta^2 \varepsilon_0 \approx 1$ (near the threshold). This separation is made because in the first case, as was pointed out above, Cerenkov radiation is not covered by formula (39) whereas in the second case the

Čerenkov radiation is propagated at small angles and is automatically taken into account in formula (39).

Let us introduce a slight deviation into the frequency $\nu = (\omega - \omega_B) / \omega_B$. Then $d = h'(d_0 + \nu)$ where

$$d_0 = \frac{h}{h' \beta \sqrt{\varepsilon_0}} - 1 \tag{44}$$

and, moreover, we obtain from (40)

$$\varkappa_{1,2}^2 = \frac{2\pi^2 h^2}{z_0^2} \left\{ \nu \mp \frac{1}{h} \sqrt{h'^2 (d_0 + \nu)^2 + \frac{\theta_{2h} \theta_{-2h}}{4h^2}} \right\} . \tag{45}$$

Let us examine the first case where $\beta^2 \varepsilon_0 \not\approx 1$. It may be seen from (32) and (36) that in this case $h \neq h'$ and $h > h'$ corresponds to $\beta^2 \varepsilon_0 > 1$ whilst $h < h'$ corresponds to $\beta^2 \varepsilon_0 < 1$. Taking into account the medium's absorbent properties, i.e. if we assume that $\varepsilon_0 = \varepsilon_0' + i\varepsilon_0'' (|\varepsilon_0''| \ll |\varepsilon_0'|)$, we then obtain from expression (40) for $\varkappa_i^2$

$$\begin{aligned} Re\, \varkappa_i &= \frac{\pi}{z_0} \sqrt{\frac{2h^2\nu \mp \alpha + \sqrt{(2h^2\nu \mp \alpha)^2 + h^4 (\varepsilon_0''/\varepsilon_0')^2}}{2}} \\ Im\, \varkappa_i &= \frac{\pi^2 h^2 \varepsilon_0''}{2 Re\, \varkappa_i z_0^2 \varepsilon_0'} . \end{aligned} \tag{46}$$

The denominators $|\varkappa_i|^3$ in formula (42) and (43) take the form

$$\pi^3 \left[\left(2h^2\nu \mp \sqrt{4h^2 d^2 + \theta_{2h} \theta_{-2h}}\right)^2 + h^4 (\varepsilon_0''/\varepsilon_0')^2 \right]^{3/4} / z_0^3 .$$

It may be seen that the radiation intensities (42) and (43) reach maximum values at

$$2h^2 \nu \mp \sqrt{4h^2 d^2 + \theta_{2h} \theta_{-2h}} \approx 0 .$$

By expressing d in terms of ν and d_0, we find that the maximum occurs at

$$\nu = \nu_0 = \frac{d_0 k' \pm \sqrt{k^2 k'^2 d_0^2 + (k^2 - k'^2)\,\theta_{2\ell}\,\theta_{-2\ell}/4k^2}}{k^2 - k'^2}$$

When the radiation intensities (42) and (43) are at a maximum, they are inversely proportional to $k^2 \varepsilon_0'' / \varepsilon_0'$.

Let us estimate the ratio of the maximum forward radiation intensity to the intensity of Cerenkov radiation at the same frequency (when the latter occurs). This ratio is of the order

$$q^2 a^2_{2(\ell-\ell')} \left[(\sqrt{\varepsilon_0'} + 1)^2 - 2 \right]^2 \omega_B^2\, \varepsilon_0'\, z_0^2 \,/\, 16 \; [\text{illegible}]$$

(in this case we consider that $|a_{-\ell}(k + k')| \ll |a_\ell(k - k')|$). When the absorption is sufficiently low, i.e. the value ε_0'' is sufficiently low, this ratio may be of the order of unity or greater.

The band-width of the maximum may be estimated as the deviation $\Delta\nu = \nu - \nu_0$ at which $\mathrm{Re}\,\varkappa_i \sim \mathrm{Im}\,\varkappa_i$. It is easy to find that $\Delta\nu \sim k^2 \varepsilon_0'' / (k^2 - k'^2)\, \varepsilon_0'$. The radiation angle is of the order $k\sqrt{|\varepsilon_0''/\varepsilon_0'|}$, i.e. extremely small for media of low absorptivity.

At sufficiently high $|\nu| \gg d_0$ and $\sqrt{\theta_{2k}\,\theta_{-2k}}/k^2$ values, the value $|\varkappa_i|^2$ may increase in proportion to $|\nu|$ and therefore the radiation intensity defined by formulae (42) and (43) will be low.

A similar situation occurs close to the Cerenkov threshold when $\beta^2 \varepsilon_0 \approx 1$, i.e. $k \simeq k'$. The only difference is that in this case close to each Bragg frequency ω_B only one maximum occurs at

$$\nu_0 = -\frac{d_0}{2} - \frac{\theta_{2\ell}\,\theta_{-2\ell}}{8 d_0 k^4}.$$

The width of the maximum is of the order $\varepsilon_0'' / \varepsilon_0'$. Away from the Bragg frequency backward radiation, as in the case $h \neq h'$, vanishes rapidly. As for forward radiation, above the threshold it becomes normal Cerenkov radiation and below the threshold it becomes weak.

Thus, by analysing formulae (42) and (43) we see that when a charged particle passes through a periodic medium of low absorptivity, the most noteworthy feature is the appearance close to the Bragg frequencies of extremely intense and almost monochromatic radiation which is propagated both forwards and backwards in relation to the direction of motion of the charge. Of course, there is also Cerenkov radiation (if $\beta^2 \varepsilon_0 > 1$) and the usual weak transient radiation away from the Bragg frequencies which is formed on the inhomogeneities of the medium and may be described by formulae obtained not only from the standard perturbation theory /3 - 6/ but also from formula (33) in this paper. The intense and almost monochromatic radiation near the Bragg frequencies is by nature the result of the dynamic interaction between the Bragg reflected and the forward waves of coherently amplified transient radiation which is formed at strictly periodic inhomogeneities of the medium. As the medium's inhomogeneities are continuously distributed along all the paths of motion of the charge, the intensity of the transient radiation both near to and away from the Bragg frequencies is proportional to the path length.

In order to illustrate this, Figs. 2 and 3 show the curves of the spectral dependence of the number of quanta $dN / d\nu$ emitted from a path equal to one radiation wave-length. Figs. 2 and 3 relate to forward and backward radiation respectively. These curves are calculated using formulae (42) and (43) in which only the Fourier amplitudes $a_{2(h-h')}$ and a_{2h} at $h = 5$ and $h' = 3$ are considered to be non-zero. Moreover, $\varepsilon_0' = 2.778$, $\varepsilon_0'' / \varepsilon_0' = 3 \cdot 10^{-4}$, $(1 - \beta^2)/2 = 1 \cdot 10^{-4}$, and $q = 0.15$. Figures 1, 2 and 3 denote that a_4 and a_{10} respectively equal 0.25 and 0.25; 0.45 and 0.05; 0.4995 and 0.0005. The hatched line in Fig. 2 relates to Cerenkov radiation.

Let us now assume that h and h' are much greater than unity, i.e. the radiation wave-length is much smaller than the inhomogeneity period of the medium.

Let us first examine case A mentioned in section 2. Besides the term (39), terms with $n = n_0 + n_1$, where $n_1 = \pm 1, \pm 2, \ldots\ldots$, and $|n_1| \ll h'$ also made a considerable contribution in formula (33). Let us write out these terms in a clear form for the particular case of the ultra-relativistic particle ($1 - \beta^2 \ll 1$) and in the frequency region where $|\varepsilon_0 - 1| = |g_0| \ll 1$. In this case, as can be seen from conditions (32) and (36) $h = h'$. As $h \gg 1$, the term $\theta_{2h}\,\theta_{-2h}$ compared with the term $4h^2(d + 2n_1)^2$ may be ignored in formula (38), except for certain ν values for which $d + 2n_1 = 0$. As $d = h(1 - \beta + \nu - g_0/2)$, we have

$$\varkappa_{ni}^2 \approx \frac{2h^2\pi^2}{z_0^2}\left\{\nu \mp \left|1 - \beta - \frac{g_0}{2} + \nu + \frac{2n_1}{h}\right|\right\}. \tag{47}$$

where $n = n_0 + n_1$, and the signs $\pm$ correspond to $i = 1.3$ and $i = 2.4$.

From the expression for b_n it is easy to find that $b_n = h + d + 2n_1$, for the first two terms in formula (33) and $b_n = -h + d + 2n_1$, for the last two terms. This means that the first two terms correspond to waves which are propagated forwards in relation to the direction of motion of the charge whereas the last two terms relate to the opposite direction.

Bearing in mind that at $h \gg n_1$, the value M_{-2h+2n_1}, is much smaller than the value M_{2n_1}, we find from formula (33) that the part of the wave $D_\rho(\vec{r}, \omega)$ which is propagated forwards takes the form

$$\frac{\exp\left(i\frac{3\pi}{4}\right)}{2}\,\frac{\omega e}{v^2 z_0}\left(\frac{\pi}{2\rho}\right)^{1/2}\sum_{n_1}\exp\left[(h + d + 2n_1)\frac{\pi z}{z_0}\right]\cdot \\ \cdot\frac{2h\,M_{2n_1}}{\varkappa_{nj}^{3/2}}\exp\left(i\varkappa_{nj}\rho\right), \tag{48}$$

where j = 1 or 2 depending on whether the value $d+2n_1$, is positive or negative and summation is done in terms of the whole numbers n_1, so that $\varkappa^2_{nj} > 0$. It may easily be seen from formula (47) that whatever the sign of the value $d+2n_1$ we obtain

$$\varkappa^2_{nj} = -\frac{k^2\pi^2}{z_0^2}\left(1-\beta^2-g_0+\frac{4n_1}{k}\right) \tag{49}$$

If the angle of radiation $\vartheta = Re\, \varkappa_{nj}\, c/\omega$ is introduced, then it is easy to obtain from formula (49)

$$\vartheta^2 = -\frac{4\pi n_1 c}{\omega z_0} - (1-\beta^2) + g_0 . \tag{50}$$

The radiation defined by formula (48) is the normal transient radiation formed at the periodic inhomogeneities of the medium when the wave-length is much smaller than the inhomogeneity period. As $|a| \gg |\theta_{2k}|$ in the case in question, then, as was stated at the end of section 3, this radiation may be calculated using the "single-wave" perturbation theory /3 - 6/. In particular, the angles of radiation (50) coincide with the corresponding angles obtained in /4/ (vis. also /14/).

As can be seen from (33) waves propagated backwards are extremely weak in this case due to the smallness of values θ_{-2k} and $M_{-2k}+2n_1$.

In case B, as the Fourier amplitudes a_{2n} are arbitrary, the radiation must be calculated by the "two-wave" theory method described above if the value $\theta_{2k}\,\theta_{-2k}$ is sufficiently large (vis. formulae (33) - (35)).

5. Radiation in the case of a limited periodic medium

Let the periodic medium be limited and situated, for instance, between the planes $z = 0$ and $z = l = Nz_0$, where N is a whole number and assume that the medium is surrounded by a vacuum. The radiation outside the periodic medium may be obtained by using the results from the previous section and also the condition for the continuity of fields at the limits of the boundary between the medium and the vacuum.

The transverse component of the field $E_\rho(\vec{r}, \omega)$ outside the periodic medium may be written in the form

$$
\begin{aligned}
E_\rho(\vec{r},\omega) &= \int_0^\infty \left\{ \frac{e}{\pi v} \frac{\varkappa^2 \exp\left(i \frac{\omega z}{v}\right)}{\frac{\omega^2}{v^2} - \frac{\omega^2}{c^2} + \varkappa^2} + G_1 \exp(-i\lambda_0 z_0) \right\} J_1(\varkappa\rho)\, d\varkappa \\
E_\rho(\vec{r},\omega) &= \int_0^\infty \left\{ \frac{e}{\pi v} \frac{\varkappa^2 \exp\left(i \frac{\omega z}{v}\right)}{\frac{\omega^2}{v^2} - \frac{\omega^2}{c^2} + \varkappa^2} + G_2 \exp(i\lambda_0 z_0) \right\} J_1(\varkappa\rho)\, d\varkappa
\end{aligned}
\tag{51}
$$

for regions $z < 0$ and $z > l$ respectively. G_1, and G_2 are arbitrary constants and $\lambda_0^2 = \omega^2 / c^2 - \varkappa^2$. Inside the periodic medium, in addition to the waves found in the previous section, there are free fields which are due to the presence of boundaries and which are conditioned by the solution to the homogeneous Hill equation. Taking into account relations (16) - (18), these fields may be represented in the form

$$
\begin{aligned}
E_\rho^{c\delta}(\vec{r},\omega) = -\int_0^\infty \Big\{ & G_3 \Big\{ \exp\Big[i(h+\delta_1)\frac{\pi z}{z_0}\Big] - \frac{2h\delta_1 - a}{\theta_{2h}} \exp\Big[i(-h+\delta_1)\frac{\pi z}{z_0}\Big] \Big\} + \\
& + G_4 \Big\{ \exp\Big[i(h-\delta_1)\frac{\pi z}{z_0}\Big] + \frac{2h\delta_1 + a}{\theta_{2h}} \exp\Big[-i(h+\delta_1)\frac{\pi z}{z_0}\Big] \Big\} \Big\} J_1(\varkappa\rho)\, d\varkappa .
\end{aligned}
\tag{52}
$$

The arbitrary constants G_1 , ... , G_4 may be determined from the conditions for the matching of corresponding fields at the boundaries $z = 0$ and $z = \ell$:

$$\begin{aligned}
&G_1 + p_1 G_3 + p_2 G_4 = q_1 \\
&\frac{h\pi}{\lambda_0 z_0} G_1 - \varepsilon_0 p_3 G_3 - \varepsilon_0 p_4 G_4 = q_2 \\
&\exp\left[i\left(\lambda_0 - \frac{h\pi}{z_0}\right)\ell\right] G_2 + p_1 \exp\left(i\frac{\pi\ell\delta_1}{z_0}\right) G_3 + p_2 \exp\left(-i\frac{\pi\ell\delta_1}{z_0}\right) G_4 = q_1 \exp\left[i\left(\frac{\omega}{v} - \frac{h\pi}{z_0}\right)\ell\right] \\
&-\frac{h\pi}{z_0\lambda_0} \exp\left[i\left(\lambda_0 - \frac{h\pi}{z_0}\right)\ell\right] G_2 - \varepsilon_0 p_3 \exp\left(i\frac{\pi\ell\delta_1}{z_0}\right) G_3 - \varepsilon_0 p_4 \exp\left(-i\frac{\pi\ell\delta_1}{z_0}\right) G_4 = q_2 \exp\left[i\left(\frac{\omega}{v} - \frac{h\pi}{z_0}\right)\ell\right]
\end{aligned} \tag{53}$$

the following notations are introduced here

$$p_1 = \frac{\theta_{2k} - 2h\delta_1 + a}{\theta_{2k}}, \quad p_2 = \frac{\theta_{2k} + 2h\delta_1 + a}{\theta_{2k}}$$
$$p_3 = \frac{\theta_{2k} + 2h\delta_1 - a}{\theta_{2k}}, \quad p_4 = \frac{\theta_{2k} - 2h\delta_1 - a}{\theta_{2k}} \tag{54}$$

with regard to the values q_1 and q_2 on the right-hand side of equations (53), we shall write them out clearly only for the most interesting case when the radiation wave-length is of the order of the medium's inhomogeneity period :

$$q_1 = -\frac{e}{v\pi}\left[\frac{\varkappa^2}{\frac{\omega^2}{v^2} - \frac{\omega^2}{c^2} + \varkappa^2} - \frac{4(d^2-\delta_1^2) + M_{k-k'}(-2hd - a - \theta_{-2k}) + M_{-k-k'}(-2hd + a + \theta_{2k})}{4\varepsilon_0(d^2 - \delta_1^2)}\right]$$

$$q_2 = -\frac{eh}{\omega z_0}\left[\frac{\frac{\omega^2}{v^2} - \frac{\omega^2}{c^2}}{\frac{\omega^2}{v^2} - \frac{\omega^2}{c^2} + \varkappa^2} + \frac{M_{k-k'}(-2hd - a + \theta_{-2k}) + M_{-k-k'}(2hd - a + \theta_{2k})}{4(d^2 - \delta_1^2)}\right]. \tag{55}$$

By solving the system of equations (53), the following constants may be expressed clearly

$$
\begin{aligned}
G_1 = \Big\{ & \left(p_2 - \varepsilon_0 p_4 \frac{\lambda_0 z_0}{h\pi}\right)\left(\varepsilon_0 q_1 p_3 + q_2 p_1\right)\frac{\lambda_0 z_0}{h\pi} \exp\left(-\frac{i\pi\ell\delta_1}{z_0}\right) - \\
& -\left(p_1 - \varepsilon_0 p_3 \frac{\lambda_0 z_0}{h\pi}\right)\left(q_2 p_2 + \varepsilon_0 q_1 p_4\right)\frac{\lambda_0 z_0}{h\pi} \exp\left(\frac{i\pi\ell\delta_1}{z_0}\right) + \\
& + \varepsilon_0\left(q_1 + \frac{\lambda_0 z_0}{h\pi} q_2\right)\left(p_1 p_4 - p_2 p_3\right) \exp\left[i\left(\frac{\omega}{v} - \frac{h\pi}{z_0}\right)\ell\right]\Big\} \Delta^{-1}, \\
G_2 = \Big\{ & -\left(p_1 + \varepsilon_0 p_3 \frac{\lambda_0 z_0}{h\pi}\right)\left(\varepsilon_0 q_1 p_4 + q_2 p_2\right) \exp\left[i\left(\frac{\omega}{v} - \frac{h\pi}{z_0} - \frac{\pi\delta_1}{z_0}\right)\ell\right] + \\
& + \left(p_2 + \varepsilon_0 p_4 \frac{\lambda_0 z_0}{h\pi}\right)\left(q_2 p_1 + \varepsilon_0 q_1 p_3\right) \exp\left[i\left(\frac{\omega}{v} - \frac{h\pi}{z_0} + \frac{\pi\delta_1}{z_0}\right)\ell\right] + \\
& + \varepsilon_0\left(p_1 p_4 - p_2 p_3\right)\left(q_1 - \frac{\lambda_0 z_0}{h\pi} q_2\right)\Big\} \frac{\lambda_0 z_0}{h\pi} \Delta^{-1} \exp\left[-i\left(\lambda_0 - \frac{h\pi}{z_0}\right)\ell\right], \\
G_3 = \Big\{ & \left(q_1 - \frac{\lambda_0 z_0}{h\pi} q_2\right)\left(p_2 - \varepsilon_0 p_4 \frac{\lambda_0 z_0}{h\pi}\right) \exp\left(-i\frac{\pi\ell\delta_1}{z_0}\right) - \\
& - \left(q_1 + \frac{\lambda_0 z_0}{h\pi} q_2\right)\left(p_2 + \varepsilon_0 p_4 \frac{\lambda_0 z_0}{h\pi}\right) \exp\left[i\left(\frac{\omega}{v} - \frac{h\pi}{z_0}\right)\ell\right]\Big\} \Delta^{-1}, \\
G_4 = \Big\{ & -\left(q_1 - \frac{\lambda_0 z_0}{h\pi} q_2\right)\left(p_1 - \varepsilon_0 p_3 \frac{\lambda_0 z_0}{h\pi}\right) \exp\left(i\frac{\pi\ell\delta_1}{z_0}\right) + \\
& + \left(q_1 + \frac{\lambda_0 z_0}{h\pi} q_2\right)\left(p_1 + \varepsilon_0 p_3 \frac{\lambda_0 z_0}{h\pi}\right) \exp\left[i\left(\frac{\omega}{v} - \frac{h\pi}{z_0}\right)\ell\right]\Big\} \Delta^{-1},
\end{aligned}
\tag{56}
$$

where

$$
\begin{aligned}
\Delta = & \left(p_1 + \varepsilon_0 p_3 \frac{\lambda_0 z_0}{h\pi}\right)\left(p_2 - \varepsilon_0 p_4 \frac{\lambda_0 z_0}{h\pi}\right) \exp\left(-i\frac{\pi\ell\delta_1}{z_0}\right) - \\
& - \left(p_1 - \varepsilon_0 p_3 \frac{\lambda_0 z_0}{h\pi}\right)\left(p_2 + \varepsilon_0 p_4 \frac{\lambda_0 z_0}{h\pi}\right) \exp\left(i\frac{\pi\ell\delta_1}{z_0}\right).
\end{aligned}
\tag{57}
$$

By analysing formula (56) it may be seen that the radiation in the vacuum is a superposition, firstly of the transient radiation occurring at the limits of the boundary between the medium and the vacuum, secondly of the transient radiation occurring at the medium's periodic inhomogeneities throughout the length of the charged particle's trajectory and thirdly of the Cerenkov radiation (if it occurs). When the latter two types of radiation emerge from the periodic medium, they naturally undergo reflection and refraction if the mean dielectric constant ε_0 of the medium is not close to unity.

An analysis of the formulae has shown that, as occurred in /2/, the radiation maxima close to the Bragg frequencies which were found in the previous section occur only if the medium is sufficiently extended, when $\left| q\sqrt{a_{2h}\, a_{-2h}}\, \ell h / z_0 \right| \gg 1$. In this case the given maxima will also occur in the vacuum both behind and infront of the periodic medium.

R e f e r e n c e s

1. W.H. Zachariasen. Theory of the Diffraction of X-Rays by Cristals, N.Y. 1967;
   B. Batterman, H. Kole. Rev. Mod. Phys. 36, 681 (1964).

2. G.M. Garibyan, Yan Shi. Zh.E.T.F. 63, 1198 (1972).

3. P.V. Bliokh. Izv. VUZov, Radiofizika, 2, 63 (1959).

4. M.L. Ter-Mikaelyan. DAN SSSR, 134, 318 (1960).

5. A.Ts. Amatuni, N.A. Korkhmazyan. Izv. AN Arm. SSR, ser. Fiz. Mat. 13, N° 5, 55, (1960).

6. K.F. Casey, C. Yeh, Z.A. Kaprielyan. Phys. Rev. 140. B768 (1965).

7. Ya.B. Fajnberg, N.A. Khizhnyak. Zh.E.T.F. 32, 883 (1957).

8. G.M. Garibyan. Zh.E.T.F. 35, 1435 (1958).

9. V.E. Pafomov, I.M. Frank. Ya. F. 5, 631 (1967).

10. E.M. Laziev, G.G. Oksuzyan, V.L. Serov. Radiotekhnika i Elektronika, 17 , 1335 (1972).

11. A.L. Avakyan, G.M. Garibyan, Yan Shi. Izv. AN Arm. SSR, Fizika, 8, N° 1 (1973).

12. N.V. Mac-Lakhlan. Theory and Applications of Matthieu functions, IL, M., 1953; D.S. Kuznetsov. Special functions, M., 1965.

13. I.S. Gradshtejn, I.M. Ryzhik. Tables of integrals, sums, series and products, M., 1971.

14. G.M. Garibyan. Zh.E.T.F. 60, 39 (1971).

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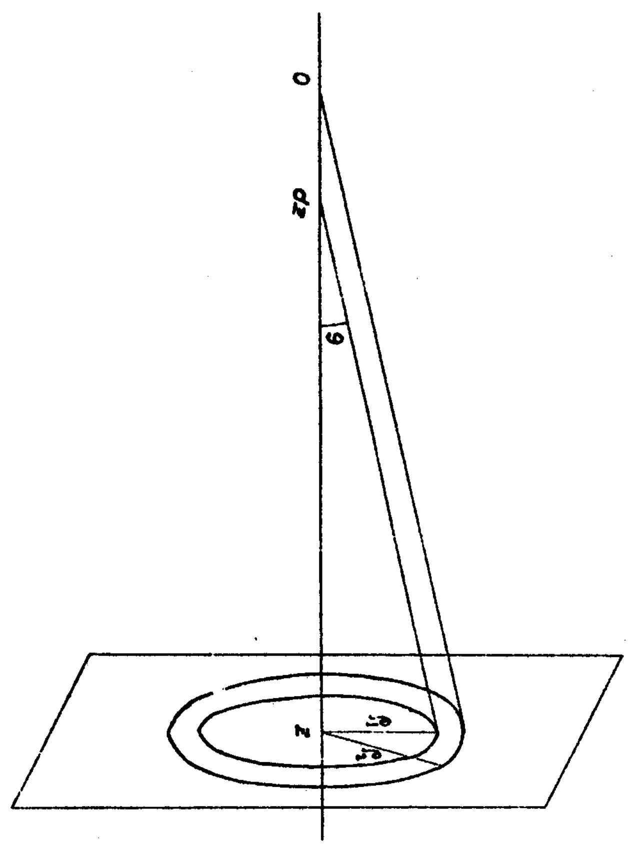

Fig. 1

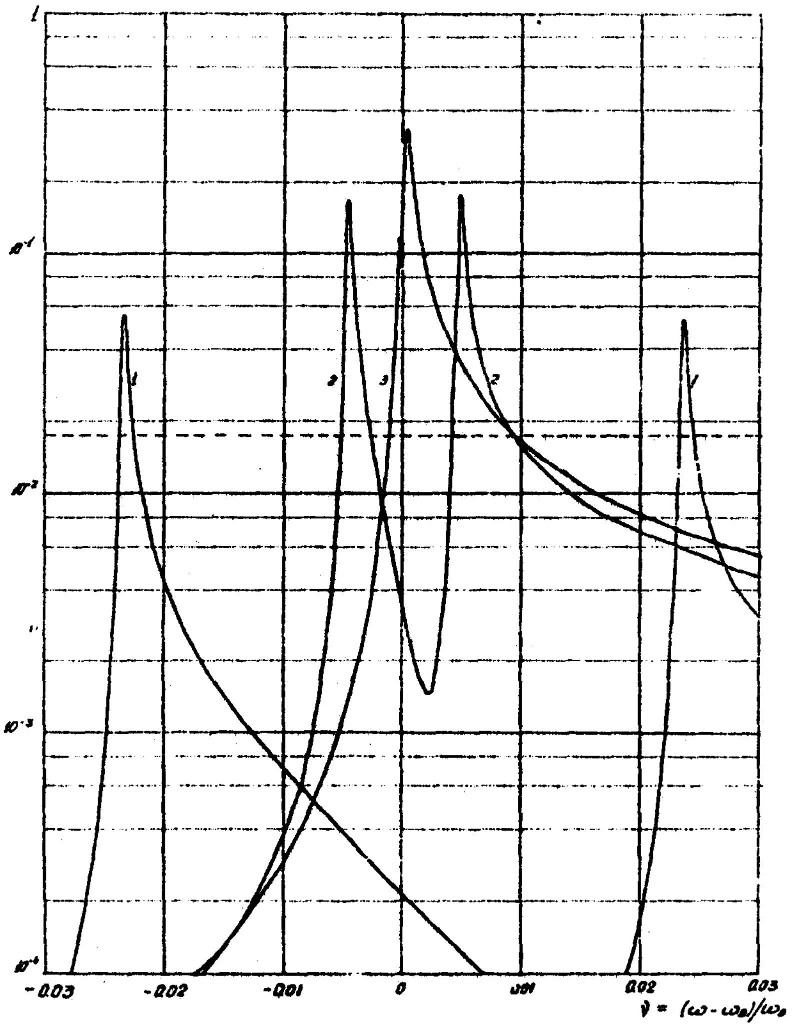
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
1
2
3
2
1
-0.03
-0.02
-0.01
0
0.01
0.02
0.03
$\nu = (\omega - \omega_0)/\omega_0$

Fig. 2

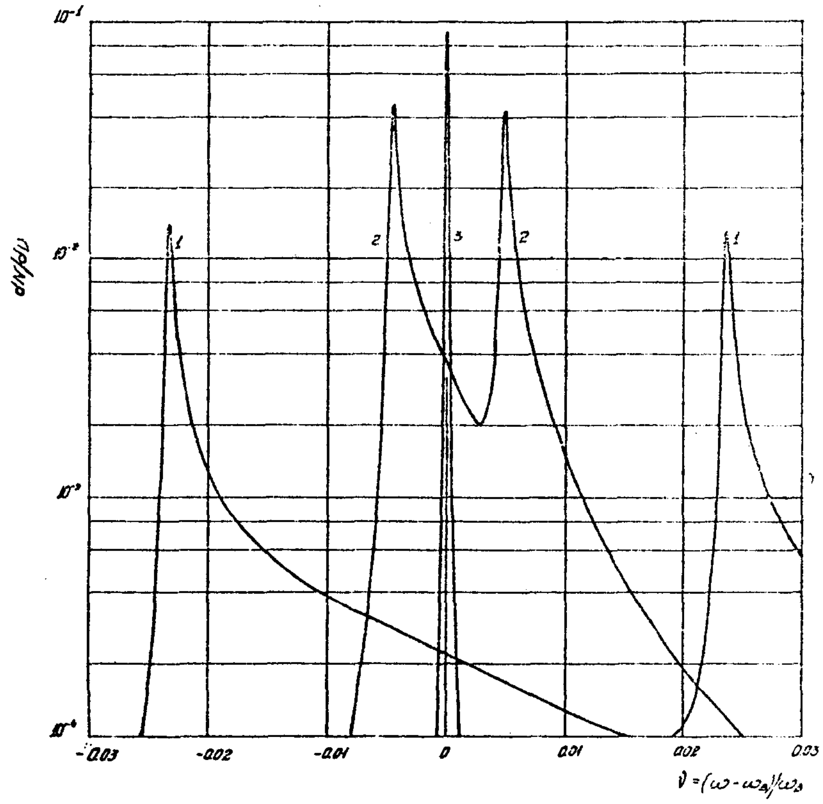

Fig. 3