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INTERACTION OF AN INTENSE BEAM WITH THE CAVITIES OF THE ACCELERATING STATIONS AT TRANSITION

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An examination is made, in this paper, of the phase oscillations of particles in a circular accelerator with strong focusing, at an energy close to the transition energy, and with a high beam intensity. It has already been shown in papers/1, 2/ that once the critical energy has been passed, modulation of the length and width of the bunches may occur. This is caused by the longitudinal Coulomb field of the beam, which lowers the frequency of synchrotron oscillations in the below-transition region and increases the frequency in the above-transition region. The sharp change in focusing leads to a mismatch of the bunches in relation to the shape of the "bucket", i.e. to a modulation of their dimensions.

The voltage induced by the beam on the accelerating cavities influences the frequency in a similar manner, and consequently may also be a cause of the modulation of the bunch dimensions. In the present work, these effects are considered together. It would not be quite correct to consider them separately, since both effects are substantially non-linear, and may mutually reinforce or attenuate each other.

Computation of the field

In the case under consideration, the longitudinal electric field is composed of three parts: the accelerating voltage $V\cos\omega_0 t$, the voltage $V_1(t)$ induced in the cavities by the beam, and the longitudinal Coulomb field of the bunch $\mathcal{E}(t)$. Subsequently, the phase $\varphi = \omega_0 t$ is used instead of t as the independent variable.

We shall consider that all the buckets are filled and that the bunches are identical. Then, the linear density of the charge $\rho(\varphi)$ and the beam current $\frac{\omega_0 R_0}{q}\rho(\varphi)$ represent periodic functions with a period of 2π (R_0 is the mean radius of the accelerator, q is the harmonic number). The voltage $V_1(\varphi)$ produced by such a beam in the cavities contains only the harmonics of the r.f. We shall disregard the first harmonic, since there are devices which can compensate it; if necessary, we can make allowance for it by varying accordingly the amplitude of the accelerating voltage. We shall also assume that in respect of the highest harmonics the cavity behaves as a gap. As, even at the transition energy, the length of the bunch remains considerably greater than the width of the accelerating gap, the voltage induced on one cavity may be represented in the following form:

$$\tilde{V}_1(\varphi) = -\frac{R_0}{qC}\int_0^{\varphi}\tilde{\rho}(\varphi)\,d\varphi\,, \tag{1}$$

where C is the capacity of the gap, and the tilde denotes that the constant component and the first harmonic are excluded from the corresponding value.

In the transition region, the length and width of the beam are constantly changing, so that, strictly speaking, $\rho(\varphi)$ is not completely a periodic function. This does not, however, destroy the validity of formula (1) since, in comparison with the radio frequency, the indicated dependence is fairly slow.

The strength of the Coulomb field of the bunch will be determined on the assumption that its length considerably exceeds its lateral dimensions, which is almost always the case. In this case

$$\mathcal{E}(\varphi) = \frac{q g}{\gamma^2 R_0^2}\,\rho'(\varphi)\,, \tag{2}$$

where γ is the relativistic factor, g is the dimensionless parameter, which for all practical purposes is 3-5 (see Appendix). Generally speaking, g is a variable value, depending on the width of the beam. However, this dependence is fairly weak (logarithmic), and consequently it may be considered that during the intervals of time with which we are concerned, $g \approx$ const.

So, the overall voltage acting on a particle has the form:

$$\begin{aligned} \mathcal{U}(\varphi) &= V\cos\varphi + M\tilde{V}_1(\varphi) + 2\pi R_0\mathcal{E}(\varphi) = \\ &= V\cos\varphi - \frac{R_0 M}{qC}\int_{\varphi_1}^{\varphi}\tilde{\rho}(\varphi)\,d\varphi + \frac{2\pi q g}{\gamma^2}\,\rho'(\varphi)\,, \end{aligned} \tag{3}$$

where M is the number of cavities, the phase φ_1 being chosen in such a manner that the integral does not have a constant component. Following [1, 2] we will write the charge density as follows:

$$\rho(\varphi) = \frac{3eN}{4\pi R_0 \eta_m^3}\left[\eta_m^2 - (\varphi-\varphi_s)^2\right], \qquad |\varphi - \varphi_s| < \eta_m; \tag{4}$$

where $2\eta_m$ is the phase length of the bunch, φ_s is the co-ordinate of its centre, and N is the total number of particles in the accelerator. In the transition region, the length of the bunch is very small: $\eta_m \ll 1$

$$\begin{aligned} \mathcal{U}(\varphi) \approx V\cos\varphi_s - V(\varphi-\varphi_s)\sin\varphi_s - \frac{3\pi q g e N}{R_0 \gamma^2 \eta_m^3}(\varphi-\varphi_s) - \\ - \frac{3eMN}{4qC}\left[(\varphi-\varphi_s)\left(\frac{1}{\eta_m} - \frac{2}{\pi}\right) - \frac{(\varphi-\varphi_s)^3}{3\eta_m^3}\right]. \end{aligned} \tag{5}$$

Phase Equation

In the presence of an arbitrary (periodic) radio-frequency field, the phase oscillation equation has the following form [3] :

$$\begin{aligned} \frac{d}{dt}\left(\frac{\gamma^3}{\gamma^2-\gamma_K^2}\frac{d\eta}{dt}\right) = \frac{qe}{2\pi m R_0^2 \gamma_K^2}\left[\mathcal{U}(\varphi_s+\eta) - \mathcal{U}(\varphi_s)\right] - \\ - \frac{d}{dt}\left[\frac{\gamma^3}{\gamma^2-\gamma_K^2}\left(\frac{d\varphi_s}{dt} + \omega_0\right)\right], \end{aligned} \tag{6}$$

where γ_K is the transition energy, m is the particle mass, $e\,\mathcal{U}(\varphi_s)$ is the energy gain per turn of the synchronous particle. The charge density (4) is so chosen that the synchronous particle is in the centre of the bunch, and the energy gain is equal to $eV\cos\varphi_s$ independently of the intensity. The last term in (6) disappears if the phase is momentarily switched at the transition energy [3], and this is what is assumed below.

After expanding γ into a series:

$$\gamma \simeq \gamma_K + \dot{\gamma}_K t \;, \tag{7}$$

introducing the dimensionless "time" $\tau = \frac{t}{t_o}$, where

$$t_o = \sqrt[3]{\frac{\pi m \gamma_K^4 R_o^2}{q e V \dot{\gamma}_K |\sin \varphi_s|}} \tag{8}$$

and using the notation:

$$\varkappa_1 = \frac{e M N}{\pi q V C |\sin \varphi_s|} \quad , \tag{9}$$

$$\varkappa_2 = \frac{3 \pi q g e N}{\gamma_K^2 R_o V |\sin \varphi_s|} \quad , \tag{10}$$

we cause equation (6) to acquire the form:

$$\frac{d\eta}{d\tau} = \tau y \;, \tag{11}$$

$$\frac{dy}{d\tau} = \pm \eta - \frac{3\pi}{4} \varkappa_1 \left(\frac{\eta}{\eta_m} - \text{[illegible]}\,\eta - \frac{1}{3} \frac{\eta^3}{\eta_m^3} \right) - \varkappa_2 \frac{\eta}{\eta_m^3} \,, \tag{12}$$

where the plus sign refers to the region below transition $(\tau < 0)$, and the minus sign to the region above transition. The relative deviation of the momentum from its equilibrium value, which determines the radial displacement of the particle, is:

$$\frac{\Delta p}{p} = \frac{e V \dot{\gamma}_K |\sin \varphi_s| t_o}{m c^2 (\gamma_K^2 - 1) T_K} \, y \,, \tag{13}$$

where T_K is the period of revolution at the transition energy.

The system (11) - (12) was solved with the aid of a computer in the interval $-10 < \tau < 5$, where the expansion (7) is fairly accurate. At the beginning of the interval, the particle motion is almost adiabatic, which is confirmed by the case $\varkappa_1 = \varkappa_2 = 0$, which can be solved accurately /3/. To ensure self-consistency of the problem, the following order of computation was used:

1. The constants $\varkappa_1, \varkappa_2$ and a bunch area of $S = \oint y(\eta) d\eta$ were chosen.
2. In the right-hand side of equation (11) the variable $\tau = \tau_0 = -10$ was fixed, after which the boundary of the bunch was determined as a phase trajectory passing through a point $\eta = \eta_m$, $y = 0$. The value η_m was chosen in such a way as to ensure the previously established value for the bunch area.
3. On the boundary curve, 32 equidistant points were selected as the initial conditions for the system (11) - (12). The solution of the system in the interval $\Delta\tau = 0.1$ was determined for the constant η_m, found in the previous paragraph.
4. From the points relating to the moment $\tau = -9.9$, which were obtained by integration, we selected the one whose η co-ordinate was maximum. The value obtained in this way was taken as η_m when solving the system in the following interval $\Delta\tau$ ($\Delta\tau = 0.1$ when $\tau < 0$, $\Delta\tau = 0.005$ when $\tau > 0$). This procedure was repeated until τ reached a value of 5.

Discussion of the Results

Figure 1 shows how the boundary of the bunch varies with time. Four cases were examined: 1 - a beam with a small intensity ($\varkappa_1 = \varkappa_2 = 0$); 2 - consideration of only the induction on the cavities ($\varkappa_1 = 0.05$; $\varkappa_2 = 0$); 3 - consideration of only the Coulomb field of the bunches ($\varkappa_1 = 0$; $\varkappa_2 = 0.004$); 4 - consideration of both factors ($\varkappa_1 = 0.05$; $\varkappa_2 = 0.004$). At the moment $\tau = -10$, all the curves practically coincide, but after transition they diverge sharply from one another.

The effects examined may lead to a limitation in intensity, if the amplitude of the radial phase oscillations which arise is comparable with the width of the vacuum chamber or exceeds it. In accordance with (13), the width of the beam is proportional to y_M.

The dependence of this value on time for the cases 1 to 4 is shown in figure 2. It will be seen that the width of the bunch experiences damped oscillations, and reaches its highest value y_{MM} when $\tau \approx 1.5$. The dependence of this value on the parameters x_1, x_2 and on the value of the phase volume of the bunch, S , is shown in figures 3 to 5.

Altogether, the increase in intensity results in an increase in y_{MM} , but in the region of the very small values of the parameters x_1 , x_2 the dependence is opposite. This is due to the fact that at a sufficiently low intensity the depth of the modulation in the bunch dimensions is so small that the absolute maximum of the width is attained not at $\tau \approx 1.5$, but immediately at the critical energy. This maximum decreases with the rise in x_1 and x_2 on account of the fact that in the below-transition region both the self-field of the beam and the induction on the cavities lower the frequency of the synchronous oscillations, i.e. they increase the length of the bunch and decrease its width.

The lengthening of the bunch in the region below transition also explains the anomalous dependence of y_{MM} on x_1 at large values of x_2 (fig. 3). In such cases, the depth of the modulation of the envelope depends mainly on the Coulomb field of the bunch. An increase in x_1 , accompanied by a lengthening of the bunch, weakens the Coulomb field so much that y_{MM} decreases, in spite of the increase in the partial contribution of the induction on the cavity.

As can be seen from figure 1, the shape of the beam in the phase plane may depart significantly from the initial one. This does not, however, destroy the validity of formula (4), which describes the distribution of the charge density, if the transformation of the phase ellipse is linear. Significant deviations from linearity occur at the moment of maximum lengthening of the bunch, which, as a rule, follows the first maximum of the width (fig. 2). In the interval $\tau \lessapprox 1.5$, the non-linear additions in equation (12) provide an insignificant contribution. Consequently, the data shown in figures 3 to 5 can be used for all intensities which are of practical interest.

As an example, let us consider an accelerator having the following parameters: $N = 5.10^{13}$ ppp; $R_0 = 237$ m; $\gamma_k = 9.46$; $q = 30$; $M = 54$; $V = 380$ kV; $|\varphi_s| = 60°$; $C = 2 \times 10^{-10}$ f; $g = 4.5$. In this case $\varkappa_1 = 0.07$; $\varkappa_2 = 0.013$. By using figures 3 to 5 and formula (13), we will find that the maximum momentum spread will attain values of 1.56% and 1.35% for $S = 0.2$ and 0.3 respectively. For a zero intensity, the momentum spreads for such phase volumes are 0.8% and 1%.

A decrease in the bunch area with constant intensity leads to an increase in the modulation. Thus, when $S = 0.1$, the computed momentum spread amounts to 2.25%, and the width of the beam is about 8 cm, although account must be taken of the fact that in the case of such a substantial variation in the dimensions the initial assumptions may be violated.

In conlusion, we shall touch on the question as to how much the results may be affected if not only the capacity of the cavity is taken into account, but also its inductance and effective resistance. We shall consider the cavity as a simple oscillating circuit with a natural frequency of ω_0 and a Q-factor, Q. Then, use of formula (1) results in an under-estimation of the amplitude of the second harmonic by 25%, and of the third harmonic by roughly 10%. Consequently, the value of the parameter $\varkappa_1$, determined by formula (9) may prove to be under-estimated by 20 to 25%. Furthermore, in view of the losses in the cavity, errors may occur in determining the phases $\Delta\varphi \sim \frac{1}{Q}$; usually, they do not exceed 1°. From the physics standpoint this means that the centre of the bunch shifts in relation to φ_s in such a way as to compensate additional energy losses in the cavity. In principle, this could lead to the occurrence of coherent oscillations of the bunch, since the sign of the displacement changes during passage through the transition energy. This effect, however, may be compensated by an insignificant variation in the phase discontinuity of the accelerating voltage.

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References

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Appendix

Computation of the parameter g

The dimensionless parameter g has the form:

$$g = \iint f(x,y)\,\varphi(x,y)\,dx\,dy , \qquad (A.1)$$

where the normalized function $f(x,y)$ describes the particle distribution in the transverse cross-section of the bunch, $\varphi(x,y)$ is the potential of this distribution with a boundary condition $\varphi = 0$ on the walls of the chamber. Physically, g^{-1} is the capacity of the beam/chamber system, calculated per unit of length.

For a round beam having a radius a in a round chamber having a radius b, and with a particle density falling off from the centre in accordance with the law:

$$f(r) = \frac{s+2}{\pi s a^2}\left[1-\left(\frac{r}{a}\right)^s\right] , \qquad (A.2)$$

formula (A.1) gives:

$$g = 2\left[\ln\frac{b}{a} + \frac{s^2+10s+20}{4(s+2)(s+4)}\right]. \qquad (A.3)$$

In the case of $s \to \infty$ (uniform distribution) the following well-known expression can be obtained from (A.3), (see for example[2]):

$$g = 2\left(\ln\frac{b}{a} + \frac{1}{4}\right). \qquad (A.4)$$

When $s = 2$ (parabolic law) we have:

$$g = 2\left(\ln\frac{b}{a} + \frac{11}{24}\right). \qquad (A.5)$$

If we consider a round beam between parallel plates whose distance apart is $2b$, we will obtain[4]:

$$g = 2\left(\ln\frac{4b}{\pi a} + \frac{1}{2}\right). \qquad (A.6)$$

Formulas (A.4) to (A.6) give similar results, i.e. g depends weakly on the shape of the vacuum chamber and the particle distribution over the transverse cross-section.

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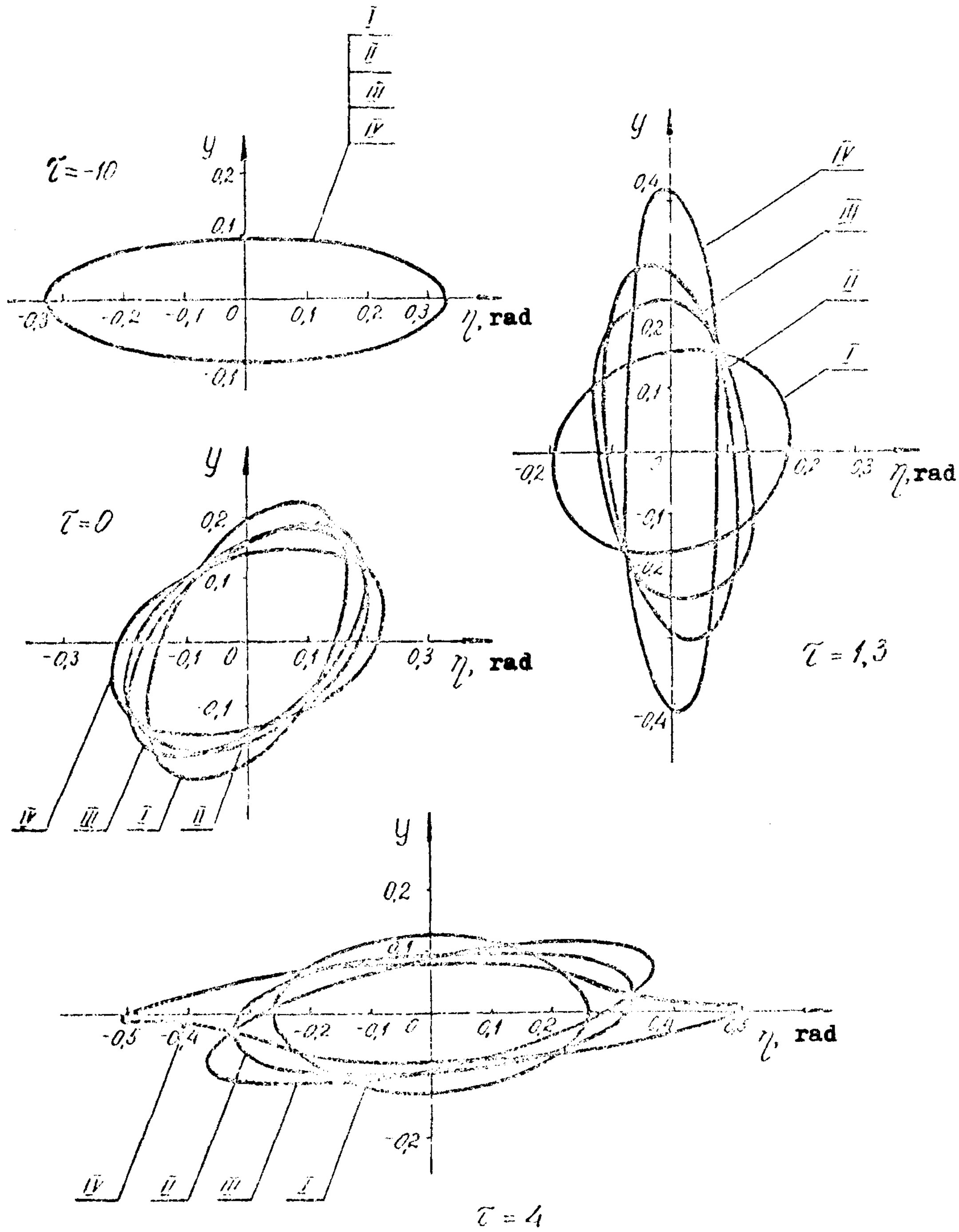
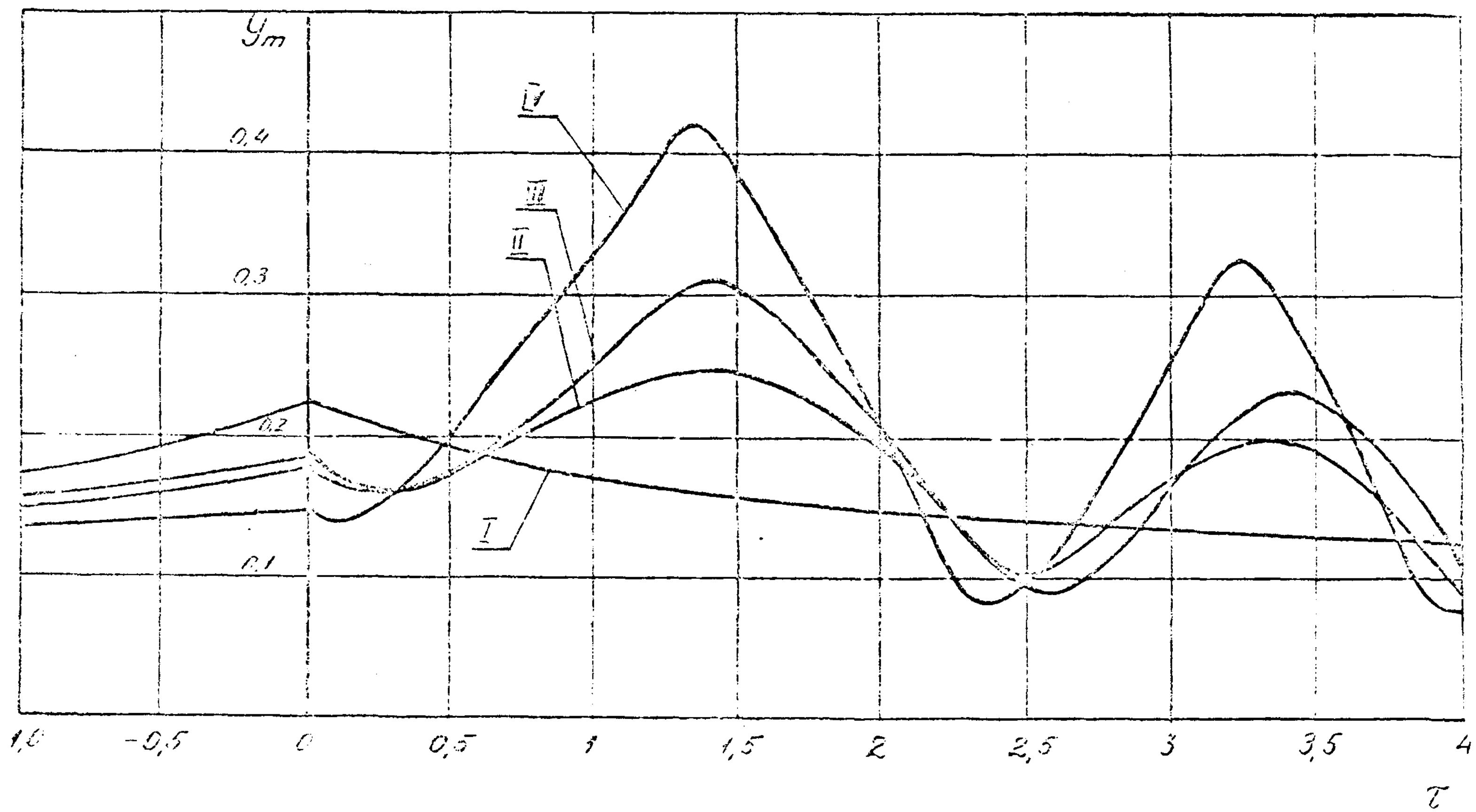


Fig. 1. Passage of the beam through the transition energy:
(1) $\varkappa_1 = \varkappa_2 = 0$; (2) $\varkappa_1 = 0.05$; $\varkappa_2 = 0$;
(3) $\varkappa_1 = 0$; $\varkappa_2 = 0.004$; (4) $\varkappa_1 = 0.05$; $\varkappa_2 = 0.004$; $\delta = 0.1$.



<u>Fig. 2</u>. **Dependence of bunch width on time when δ = 0.1**

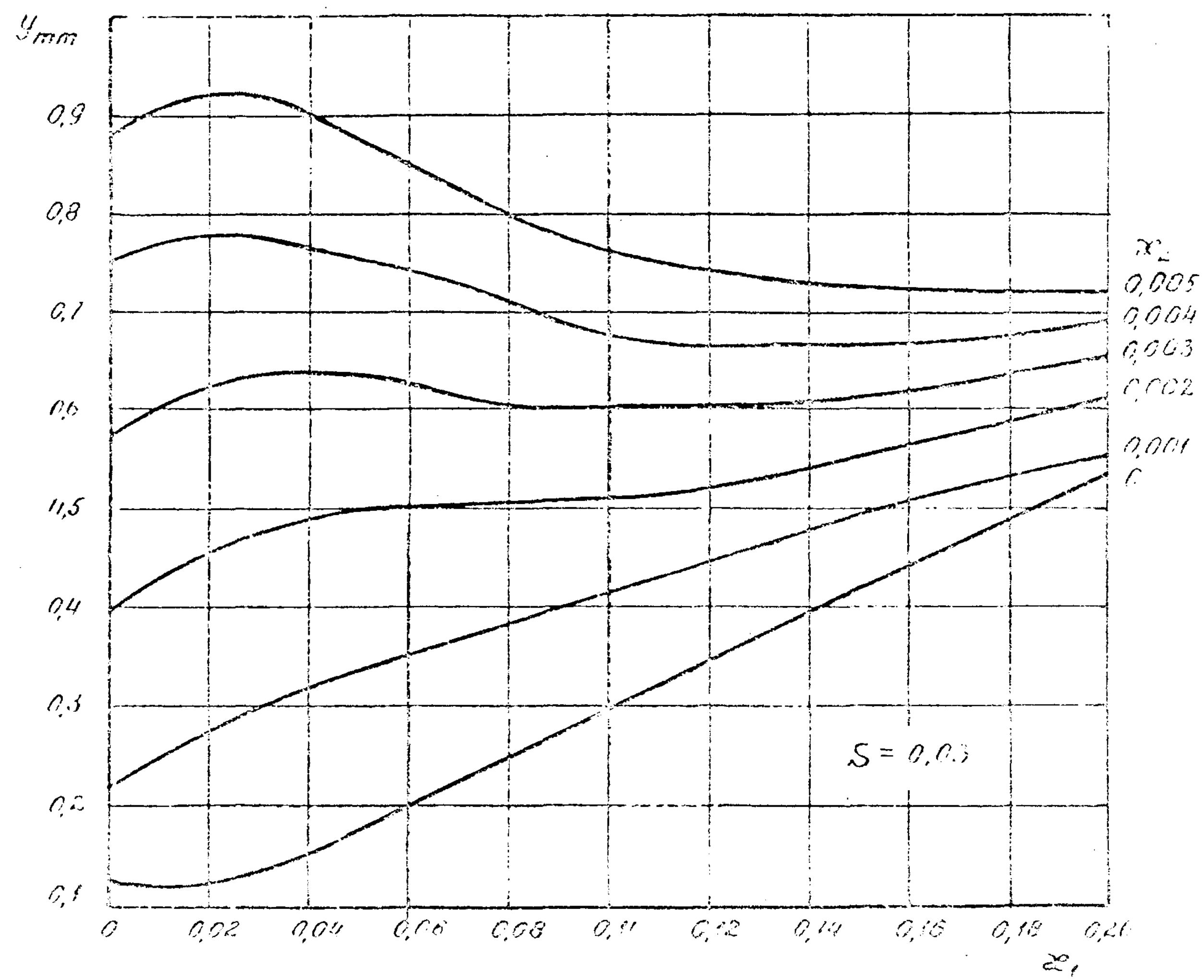


Fig. 3. Dependence of the maximum width of the bunch on $\varkappa_1$ and $\varkappa_2$ when $S = 0.03$.

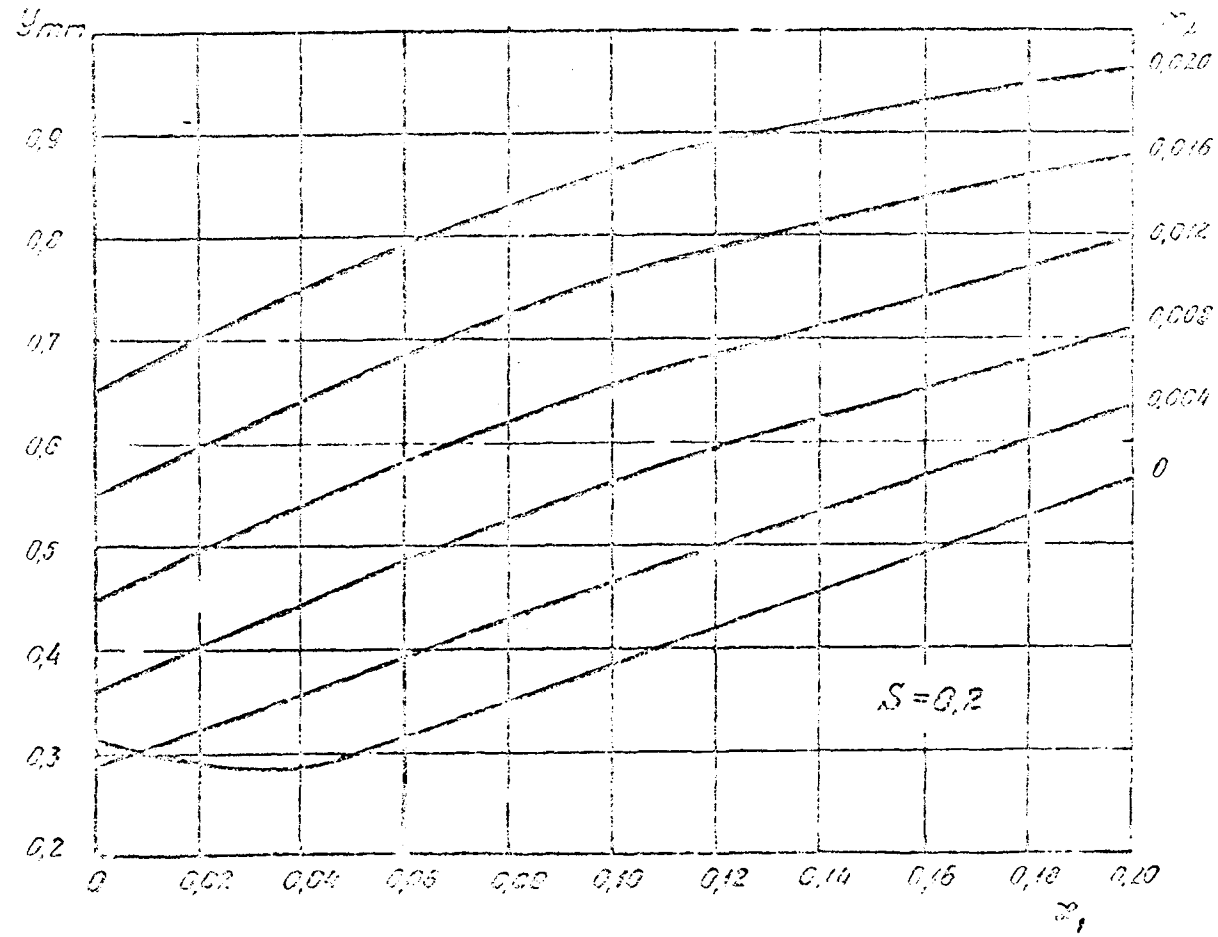
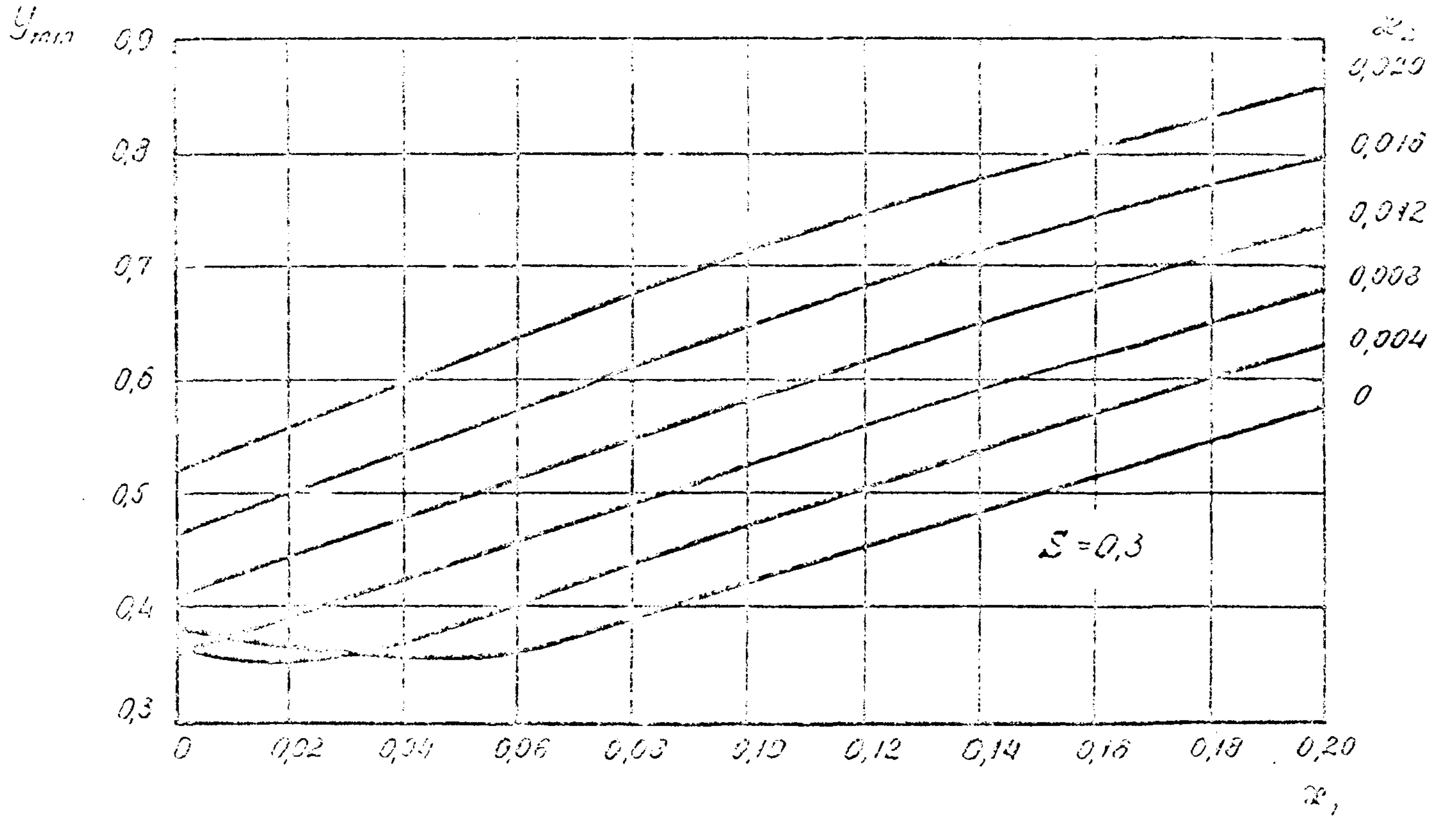


Fig. 4. Dependence of the maximum width of the bunch on $\varkappa_1$ and $\varkappa_2$ when $\mathcal{S}$ = 0.2.



<u>Fig. 5</u>. Dependence of the maximum width of the bunch on x_1 and x_2 when S = 0.3.