



String theory on ADS_3 : some open questions^{*†}

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ABSTRACT: String theory on curved backgrounds has received much attention on account of both its own interest, and of its relation with gauge theories. Despite the progress made in various directions, several quite elementary questions remain unanswered, in particular in the very simple case of three-dimensional anti-de Sitter space. I will very briefly review these problems, discuss in some detail the important issue of constructing a consistent spectrum for a string propagating on ADS_3 plus torsion background, and comment on potential solutions.

1. String theory on ADS_3 : the simplest setting beyond flat space

String theory is certainly the most appropriate setting for studying quantum-gravity phenomena. This includes big-bang cosmology, black-hole physics, and more general gauge/gravitational solitons or other exotic objects. In the absence of a truly non-perturbative approach to string theory, the usual method consists in analysing the propagation of the string on non-trivial backgrounds generated by some sources, which correspond either to perturbative or to non-perturbative string states. Consistency of string propagation imposes severe restrictions on the allowed backgrounds, which must be conformal so as to satisfy the whole set of requirements exactly in α' . Approximations can also be found by solving the relevant equations of motion up to some order in α' .

Three-dimensional anti-de Sitter (or de Sitter) space was recognized long ago as a case of interest with respect to the above motivations [1]–[6]. It is a maximally symmetric solution of Einstein's equations with negative cosmological constant, and time is embedded non-trivially in the

curved geometry. Alternatively, it corresponds to the Freedman–Gibbons electrovac solution of gauged supergravity, which can be shown to leave space-time supersymmetry unbroken [7]. Other peculiar features of ADS_3 are the absence of asymptotically flat regions, the presence of boundaries (when conformally compactified), as well as a rich causal structure, which makes it possible to obtain three-dimensional black holes after modding out some discrete symmetry [8].

As far as string theory is concerned, ADS_3 is an *exact* background, provided an NS–NS two-form is switched on. In fact, three-dimensional anti-de Sitter space is the universal covering of the $SU(1, 1)$ group manifold, and the corresponding two-dimensional conformally invariant sigma model is a Wess–Zumino–Witten model, which naturally accounts for the torsion background.

Several remarks can be made here in order to argue that ADS_3 provides the simplest setting for string theory beyond flat space. General non-compact group manifolds define a natural framework for studying strings on space-times with non-trivial geometry. Restricting ourselves to the case of simple groups, however, only $SU(1, 1)$ possesses a single time direction; ADS_3 is therefore *the only* exact background where string propagation leads to a WZW model. Of course, cosets with one time direction can be constructed out of simple non-compact group manifolds. This is

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the case, for instance, for ADS_n , which appears as $O(2, n - 1)/O(1, n - 1)$. However, these geometries cannot be obtained by the usual GKO construction in the framework of gauged WZW models.

Last but not least, the motivation for understanding string theory on ADS_3 , and more generally on ADS_n , is related to the recent developments on ADS/CFT correspondence [9]. There, supergravity in the bulk of anti-de Sitter space is argued to be in some sense equivalent to a large- N super-Yang–Mills theory on the boundary. Since the supergravity theory under consideration is the low-energy limit of a more fundamental superstring theory, the question arises of the exact structure of the latter on the anti-de Sitter background, and its connexion to the super-Yang–Mills theory on the boundary. Here also ADS_3 plays a particular role. The asymptotic isometry group is infinite-dimensional [10], and the theory on the boundary is a two-dimensional conformal field theory. The latter is different from the two-dimensional sigma model whose target space is the “bulk ADS_3 ” on which the string propagates. Considerable efforts have been made for understanding the relationship between these two conformal theories, in order to both set more precisely the ADS/CFT correspondence, and try to get some feedback on the structure of the string theory on ADS_3 [11]².

Despite those efforts and the apparent simplicity of the model at hand, I will show that several important and elementary issues, such as the determination of the spectrum, consistent with the basic requirements of string theory and conformal field theory, are still beyond our understanding. I will also try to motivate various suggestions for further analysis.

Although we are ultimately interested in understanding how superstrings behave on ADS_3 background, I will concentrate in the sequel on the bosonic case, where the issues I would like to address are already visible. Moreover, this case might have some relevance in the framework of recent attempts at establishing some relationship

²Notice that most of these works deal with “Euclidean ADS_3 ”, H_+^3 , whereas I will present here the ordinary Minkowskian situation. Except for the unitarity properties, the two cases share many features.

between various bosonic theories – including perhaps the celebrated 26-dimensional theory.

2. The $SU(1, 1)$ Wess–Zumino–Witten model

The analysis of string theory on ADS_3 plus torsion background can be performed in two steps. First, we must study the sigma model whose target space has the above geometry; this is a WZW model on the $SU(1, 1)$ group manifold. Then, the latter has to be coupled to two-dimensional gravity. At the level of the Hilbert space, this amounts to the decoupling of a certain subspace, which becomes unphysical.

As a general remark, it should be stressed here that the geometrical interpretation of a conformal field theory as a string propagating in some backgrounds, is sometimes loose. It becomes unambiguous only in some semi-classical limits, or in the presence of a dense spectrum of Kaluza–Klein modes. Hence, one should be aware that very often one is not describing the situation for which the model was designed. Conversely, unexpected geometrical interpretations may arise.

Very little is known about WZW models on non-compact groups, at a sufficiently rigorous and general level. Most of our knowledge is based on a formal extension of the compact case to some specific situations, and in the framework of current-algebra techniques. Target-space boundary conditions, in particular, are treated somehow carelessly, although we know how important they are for selecting various representations when studying quantum mechanics on ADS_3 [12, 13], or in the determination of the asymptotic symmetry algebra acting on that space [10]. This should be kept in mind in any attempt to go beyond our present knowledge of the subject.

We usually assume that the symmetry of the above model, $SO(2, 2) \cong SU(1, 1)_L \times SU(1, 1)_R$, is realized in terms of an affine Lie algebra, the level of which is not quantized because the third homotopy group of $SU(1, 1)$ is trivial or, put differently, because of the absence of any Dirac-like singularity in the torsion background.

The commutation relations for the modes of the currents (remember $J^a(z) = \sum_{m \in \mathbf{Z}} z^{-m-1} J_m^a$,

$a = 1, 2, 3$) are

$$[J_m^a, J_n^b] = i f_c^{ab} J_{m+n}^c + \frac{k}{2} m g^{ab} \delta_{m+n},$$

with $g^{ab} = \frac{1}{2} f_c^{bd} f_d^{ca} = \text{diag}(- - +)$ and $f_d^{ab} g^{dc} = -\epsilon^{abc}$ ($\epsilon^{123} = 1$). We expect the anomaly k to be negative so that there is only one negative-metric generator that plays the role of the time coordinate, namely the third direction. We also introduce $J_m^\pm = i J_m^1 \mp J_m^2$.

The world-sheet energy-momentum tensor is given by the affine Sugawara construction:

$$T(z) = \frac{1}{k+2} g_{ab} : J^a(z) J^b(z) : .$$

The modes L_m ($T(z) = \sum_{m \in \mathbf{Z}} z^{-m-2} L_m$) satisfy the Virasoro algebra with central charge $c = \frac{3k}{k+2}$, and

$$[L_m, J_n^a] = -n J_{m+n}^a. \quad (2.1)$$

Finally, the Hilbert space is formally constructed as in the compact case: it is a direct sum of products of representations of the left and right current algebras. Highest-weight representations³ of the $SU(1,1)$ current algebra are labelled by the spin j of the primary fields (states of level zero), which form a representation of the global algebra (generated by the zero modes $J_0^{\pm,3}$), and have conformal weight $j(j+1)/k+2$.

Irreducible representations of the global algebra are essentially of two kinds [16]: discrete $\mathcal{D}^\mp(j)$ or continuous principal $\mathcal{C}_p(b, a)$ and continuous supplementary $\mathcal{C}_s(j, a)$. The discrete ones have highest (\mathcal{D}^-) or lowest (\mathcal{D}^+) weight, whereas the continuous ones do not. The spin j of the discrete representations is real⁴, and their states are labelled by $|jm\rangle$, $m = j, j \mp 1, j \mp 2, \dots$. For the principal continuous ones, $j = -\frac{1}{2} + ib$, $b < 0$, and the magnetic number is $m = a, a \pm 1, a \pm 2, \dots$, $-\frac{1}{2} \leq a < \frac{1}{2}$, $a, b \in \mathbf{R}$; for the supplementary continuous ones, $-\frac{1}{2} \leq j < 0$ and $-\frac{1}{2} \leq a < \frac{1}{2}$, with the constraint $|j + \frac{1}{2}| < \frac{1}{2} - |a|$, $a, j \in \mathbf{R}$. These representations are

³Representations without highest or lowest weight do exist [14]. It is, however, not clear how those could be interpreted within a stable string theory. More representations of the $SU(1,1)$ current algebra can be found in [15].

⁴In order to avoid closed time-like curves, we are considering the universal covering of $SU(1,1)$. Therefore, j is not quantized.

unitary and infinite-dimensional; $\mathcal{D}^\pm(j)$ become finite-dimensional when j is a positive integer or half-integer, and are non-unitary for any positive j . Notice finally that the quadratic Casimir ($j(j+1)$) is negative for both continuous series; for the discrete ones it is negative or positive when $-1 < j < 0$ or $j < -1$, respectively.

Highest-weight representations of the current algebra are obtained by acting with $J_{-1}^{\pm,3}$ on the above level-zero states, which are annihilated by all positive-frequency modes. These representations contain an infinite tower of negative-norm states, due to the indefinite metric g^{ab} . Therefore, in the above setting, it is impossible to write down a unitary conformal theory based on the $SU(1,1)$ WZW model. This is not surprising, and the same conclusion holds anyway for three free bosons with metric $(- + +)$, obtained here when $k \rightarrow -\infty$.

I now come to the following crucial question: How should the above representations be combined to form a consistent, though non-unitary, model? In order to answer this question, we can follow the requirement of *modular invariance*. The genus-one partition function reads, in general:

$$Z(\tau, \bar{\tau}) = \sum_{L,R} \mathcal{N}_{L,R} \chi_L(\tau) \bar{\chi}_R(\bar{\tau}), \quad (2.2)$$

where the summation is performed over all left-right representations present in the spectrum, and $\chi(\tau)$ are the corresponding characters:

$$\chi(\tau) = \text{Tr} q^{L_0 - \frac{c}{24}}, \quad (2.3)$$

$q = \exp 2i\pi\tau$. The multiplicities $\mathcal{N}_{L,R}$ must be chosen in such a way that the partition function is invariant under $\tau \rightarrow \tau + 1$ and $\tau \rightarrow -1/\tau$. Notice that, again, Eq. (2.2) is formal in the non-compact case, and one should prove it, e.g. by using path-integral techniques starting directly from the WZW action, as in Ref. [6].

Already at this level, a major problem appears, which is actually generic to all non-compact groups. The unitary representations of the global algebra being infinite-dimensional, there is an infinite degeneracy level by level in the representations of the current algebra, and consequently

the characters (2.3) are ill-defined⁵. This is the price to pay for using the full *non-compact and non-Abelian symmetry* to classify the states of the theory. In fact, this is not specific to the two-dimensional sigma model we are analysing. Similar problems would occur in the relativistic quantum mechanics of a particle on a two-dimensional plane, if we tried to describe its propagator by using wave-function representations of the full Lorentz group $SO(2,1)$. The reason why free bosons can be analysed without trouble just relies on the Abelian nature of the symmetry used to classify their spectrum.

In our formal treatment, the only way out is to lift the degeneracy by switching on a source coupled to J_0^3 :

$$\chi(\tau, v) = \text{Tr} q^{L_0 - \frac{c}{24}} e^{2i\pi v J_0^3}. \quad (2.4)$$

Notice, however, that this definition *does not* allow a regularization of the characters of the representations based on the continuous series⁶. This shows that discrete and continuous representations definitely play different roles, and that the continuous ones do not fit into the present current-algebra approach. Moreover, convergence of the trace in (2.4) demands $\text{Im} v > 0$ for \mathcal{D}^+ and $\text{Im} v < 0$ for \mathcal{D}^- . As a consequence, within the present framework, \mathcal{D}^+ and \mathcal{D}^- *cannot appear simultaneously in the spectrum*.

In computing the characters, the main difficulty is to properly identify the *singular vectors*. These are zero-norm states orthogonal to any other state; their descendents possess the same property and they are thus responsible for the reducibility of the Verma modules. Exhaustive and rigorous results can be found in Refs. [17].

There are some particular sets of representations of the current algebra: the *admissible representations*⁷. These are based on the discrete

⁵We could consider finite-dimensional non-unitary representations of the global algebra, since the Verma module built on any representation is anyway non-unitary. However, for later use in string theory, this choice would not be sensible.

⁶For those, one could replace J_0^3 by $|J_0^3|$ in Eq. (2.4). Such characters have never been studied in the mathematical literature.

⁷In the context of the $SU(2)$ WZW, using the GKO coset construction, these series lead to the minimal BPZ models with $c < 1$. For the integer level ($u = 1$ in

series, and appear at the level

$$k = \frac{t}{u} - 2, \quad t \geq 2, \quad u > 0, \quad t, u \in \mathbf{Z}, \quad (2.5)$$

with spins

$$j = \frac{1}{2} \left(n - s \frac{t}{u} \right), \quad 0 \leq n \leq t - 2, \\ 0 \leq s \leq u - 1, \quad (2.6)$$

$n, s \in \mathbf{Z}$. For these representations, $k \geq -2$, and the spin obeys the following bounds:

$$\frac{1-u}{u} \frac{t}{2} \leq j \leq \frac{t}{2} - 1.$$

The primary states do not necessarily belong to some unitary representation of the global algebra, since j can be positive. Singular vectors appear at various levels, which makes the evaluation of (2.4) quite intricate. Nevertheless, the characters of these series were obtained in [18]. They turn out to form finite representations of the modular group, which now acts as:

$$T : (\tau, v) \rightarrow (\tau+1, v), \quad S : (\tau, v) \rightarrow \left(-\frac{1}{\tau}, \frac{v}{\tau} \right).$$

Rational models with an *ADE* type of classification can be constructed by using the above results [19]. Besides being non-unitary, these models have peculiar properties. Their central charge is given by $c = 3 - 6u/t$, which is negative when $t < 2u$. Moreover, the above models are only defined in the presence of a “magnetic field”, which is not invariant under modular transformations. The interpretation of these features is not clear.

For string-theory purposes, the level of the current algebra should satisfy $k < -2$: this ensures positive central charge as well as a single time-like direction in the target space, which are both necessary conditions for the physical spectrum to be free of negative-norm states. Furthermore, as far as the discrete series are concerned, j should be non-positive in order to avoid unitarity problems already at level zero. This excludes the admissible representations, and therefore the possibility of using their modular-invariant combinations.

Eq. (2.5)), unitarity is guaranteed, and one gets the *ADE* invariants for $SU(2)$ as well as the corresponding unitary series at $c < 1$.

In the regime $k < -2$, very little is known about the characters of the $SU(1, 1)$ current algebra. Those characters can be computed in the case of highest-weight representations based on discrete series (Eq. (2.4)), for generic values of k and j , where *no singular vectors* are present, with the result [20, 21]:

$$\chi_j^k(\tau, v) = \frac{q^{\frac{(2j+1)^2}{4(k+2)}} e^{\frac{i\pi}{2}(2v+1)(2j+1)} e^{-i\pi j}}{\vartheta_1(\tau, v)}, \quad (2.7)$$

where $\vartheta_1(\tau, v)$ is the odd Jacobi function

$$\begin{aligned} \vartheta_1(\tau, v) &= -2 \sin(\pi v) q^{\frac{1}{8}} \prod_{n=1}^{\infty} (1 - q^n) \times \\ &\times (1 - q^n e^{2i\pi v}) (1 - q^n e^{-2i\pi v}). \end{aligned}$$

Expression (2.7) does not hold for some discrete sets of (k, j) 's, such as when $2j - k + n = 1$, n a positive integer, or when $j = m/2$, $m \in \mathbf{Z}$, independently of k . In such cases, the presence of null states will obviously spoil (2.7). In those situations it is probably more of a technical problem than a conceptual one to determine the exact characters. A much more difficult issue is certainly how to combine the various characters for obtaining modular-invariant partition functions. As an example, we can consider the modular transformations of the characters (2.7). We obtain:

$$\begin{aligned} \chi_j^k(\tau + 1, v) &= \chi_j^k(\tau, v) e^{\frac{i\pi}{2} \left(\frac{(2j+1)^2}{(k+2)} - \frac{1}{2} \right)}, \quad (2.8) \\ \chi_j^k \left(-\frac{1}{\tau}, \frac{v}{\tau} \right) &= \int_{-\infty}^{+\infty} d\ell e^{-i\pi \frac{(2j+1)(2\ell+1)}{k+2}} \chi_\ell^k(\tau, v) \\ &\times \sqrt{\frac{2}{k+2}} e^{\frac{i\pi}{2} \left(\frac{v^2}{\tau} k + 1 \right)}. \quad (2.9) \end{aligned}$$

These transformations involve *all* values of j , with zero measure for the discrete sets of representations possessing singular vectors. Constructions involving only these generic characters turn out to be too simple, and do not enable us to obtain interesting modular-invariant combinations. In particular, the naive diagonal combination integrated over all values of j (again, all primary states do not belong to unitary representations of the global algebra) is not, strictly speaking, modular-invariant because of the v -dependent

prefactor appearing in (2.9). Nevertheless, considering this combination, we find:

$$\begin{aligned} Z_{\text{diag}}^k(\tau, v) &= \int_{-\infty}^{+\infty} dj |\chi_j^k(\tau, v)|^2 \\ &= \frac{1}{2} \sqrt{\frac{k+2}{\text{Im } \tau}} \frac{e^{\pi(k+2) \frac{(\text{Im } v)^2}{\text{Im } \tau}}}{|\vartheta_1(\tau, v)|^2} \quad (2.10) \end{aligned}$$

(as usually, in the presence of a time-like coordinate, analytic continuation is needed – here when $k < -2$ in Eqs. (2.9) and (2.10)). Under an S -transformation, an extra factor appears: $|\exp i\pi k v^2 / \tau|$. The latter is irrelevant at $v \rightarrow 0$. Since this limit is singular, however, modular invariance should be demanded for any finite value of v . It can be reached only if, in expression (2.10), the measure dj is replaced with $dj \exp -\pi k \frac{(\text{Im } v)^2}{\text{Im } \tau}$. This formally defines an invariant combination at any $v \neq 0$, because it accounts for the cancellation of the extra v -dependent factor appearing in the transformation (2.9). However, in this way, the diagonal combination no longer depends on k (except for the overall volume factor $\sqrt{k+2}$), which means in particular that *the asymptotic behaviour of the spectrum does not depend on the central charge*. This situation is hardly acceptable (another argument is given at the end of Section 3 for the string theory), and the above results should be interpreted as a sign that, among others, we should consider more carefully the appearance of representations with singular vectors. I will come back to this point when studying the string on ADS_3 . It is also interesting to observe that the result (2.10) was obtained in [6] as the *partition function of Euclidean three-dimensional anti-de Sitter space*, $H_+^3 \equiv SL(2, \mathbf{C})/SU(2)$, by using a rigorous path-integral approach⁸. It is not clear why such a non-modular-invariant partition function would be satisfactory in the case of H_+^3 .

So far, I have been considering the construction of conformal models based on $SU(1, 1)$ WZW at level k . The encountered problems can be summarized as follows. One is the infinite degeneracy at each level in the representations of the

⁸See also [22] for a rigorous treatment of H_+^3 . In Ref. [6], the two-dimensional Euclidean black hole was also analysed. For the latter, the result turns out to be modular-invariant.

current algebra, and in particular the treatment of the representations based on the continuous series for which no character formula has been proposed in the mathematical literature. This problem might be fixed in a path-integral approach, where a zero mode responsible for the corresponding (infinite-volume-like) divergence could be identified and removed. Alternatively, we might also need a deformation of the affine Sugawara construction in order to lift the degeneracy without coupling to an external field. Modification of the current algebra itself has also been advocated [23]. The question then arises whether these deformations still describe the initial WZW theory. For example, in the compact-group WZW models, the natural stress tensor, obtained by differentiating the action with respect to the metric, is precisely the one given by the affine Sugawara construction [24]. Any deformation with respect to the latter, possibly continuous and conformal, will abandon the original WZW theory.

Another problem is related to the construction of various modular-invariant partition functions: What are the sets of representations – including representations based on both discrete and continuous series – which form a well-behaved OPA? Only the sets of admissible representations, based on some discrete series, have been identified.

This question is difficult and we can somehow understand why by comparing our case to the situation of a WZW model on the group manifold of $SU(2)$. The $SU(2)$ theory can be unitary because the affine algebra has unitary highest-weight representations for integer and half-integer spin such that $0 \leq j \leq k/2$ (k is integer here). Modular invariance is therefore expected for combinations of representations falling within this range, and indeed this happens. That is not a miracle: the structure of characters, and thereby their modular transformations, is directly dictated by the presence of singular vectors, which in turn determine the unitarity properties of the representations since they appear as limiting cases of positive-norm states becoming negative-norm. For example, the (sufficient) condition $2j - k + n = 1$ for having a null state at level n has solutions within the range $j \leq k/2$, which em-

beds the unitarity domain. Another instructive example is the case of the free boson. There, all representations of the $U(1)$ algebra are unitary – none if the boson is of time-like signature – and are labelled by a continuous momentum. No null states appear and all representations must be used in a consistent model. They lead to the celebrated $(\sqrt{\text{Im } \tau \eta \bar{\eta}})^{-1}$ partition function. Both for the $SU(2)$ WZW model and for the free boson, unitarity is a guideline for reaching modular-invariance. For $SU(1,1)$ there are *no* unitary highest-weight representations of the current algebra, whereas some have null states and some others do not. Unitarity and presence or absence of singular vectors cannot therefore be successfully advocated for constructing modular-invariant combinations.

3. String theory on $SU(1,1)$

I will now analyse the string propagating over the $SU(1,1)$ group manifold. The coupling of the above conformal model to the two-dimensional gravity creates spurious states that we should eliminate from the spectrum. The most straightforward approach would have been the light-cone-gauge analysis. Unfortunately, this method is hard to implement (despite several attempts [25]) and we have therefore to advocate – without rigorous proof – that going to the conformal gauge and imposing the Virasoro constraints will eliminate all spurious states, provided the conformal anomaly is cancelled.

In this analysis, the natural questions are the following: What are the representations of the $SU(1,1)_L \times SU(1,1)_R$ current algebra that should be kept in order to have a consistent theory (in particular a well-behaved operator algebra)? What are the roles of discrete versus continuous representations? Are all physical states of positive norm? What kind of particles do these states describe, and what are the corresponding vertex operators?

Anomaly cancellation for the bosonic string implies $c = 26$, where c accounts for all matter-field contributions. String theory on $SU(1,1)$ can be critical on its own, since the level of the affine algebra can be freely tuned to reach the critical

central charge: $k = -52/23$. It might be relevant, however, to keep k free, and couple the $SU(1, 1)$ sigma model to some unitary conformal field theory such as d free bosons, a WZW model on $SU(2)$, ... This can help in understanding the theory at large $|k|$, corresponding to the near-flat-space limit.

As was emphasized in the previous section, very little is known about the WZW model on $SU(1, 1)$. The only guideline is therefore the search for representations of the current algebra leading to a positive-definite physical Hilbert space. In fact, it is straightforward to argue that within the class of highest-weight representations of the $SU(1, 1)$ current algebra we have considered, and in the general framework we have presented so far for analysing the string propagation on a non-compact manifold, *there is no satisfactory selection* of representations that can be performed, which guarantees the absence of negative-norm states in the physical Hilbert space.

The argument goes as follows. I will concentrate on the left-movers, keeping in mind that they should be eventually paired with right-movers. A highest-weight representation of the current algebra is built on a representation of the global algebra, which defines the level-zero states and is annihilated by positive-frequency current modes. Acting on those states with $J_{-1}^{\pm,3}$ will generate the Verma module. At each level, the set of states can be decomposed with respect to the global algebra. Since the Virasoro generators commute with the modes $J_0^{\pm,3}$ (see Eq. (2.1)), Virasoro constraints ($L_m|\text{physical}\rangle = 0 \forall m > 0$) will keep or throw away complete representations of the global algebra. This considerably simplifies the rules for implementing unitarity: (i) the level-zero states should all have positive norm, i.e. be a unitary representation of the global algebra of the type $\mathcal{D}^\pm(j)$, $\mathcal{C}_p(b, a)$ or $\mathcal{C}_s(j, a)$ (see previous section for the allowed values of the parameters j, a, b); (ii) at each level, any physical representation should also have parameters consistent with unitarity; this last statement ensures that all states of the representation at hand are positive-norm, provided the norm of one of them is indeed positive.

A simple computation shows that, irrespectively of the type of unitary level-zero represen-

tation, $\mathcal{D}^\pm(j)$, $\mathcal{C}_p(b, a)$ or $\mathcal{C}_s(j, a)$, at level one there will be generically three representations of the global $SU(1, 1)$ algebra: two Virasoro primaries (i.e. physical up to mass-shell condition) with spin $j + 1$ and $j - 1$, and an unphysical one with spin j . This generalizes at level N , where we meet at least two Virasoro-primary representations, with spin $j \pm N$.

In the case of continuous series $\mathcal{C}_p(b, a)$ or $\mathcal{C}_s(j, a)$, already at level one, the values of the spin are out of the unitarity range: for $\mathcal{C}_p(b, a)$ the quadratic Casimir becomes complex, whereas it becomes positive for $\mathcal{C}_s(j, a)$. On the other hand, at level N , mass-shell condition reads:

$$\frac{j(j+1)}{k+2} + N \leq 1 \quad (3.1)$$

(an internal positive conformal weight is supposed to compensate the difference with respect to 1); this shows that the maximal allowed level for a primary spin- j representation is

$$N_m(j) = \text{integer part} \left(1 - \frac{j(j+1)}{k+2} \right). \quad (3.2)$$

Therefore, as far as the continuous series are concerned, in the regime of interest ($k < -2$), $N_m(j) = 0$. Unitarity is guaranteed, but these representations only describe part of the tachyonic sector of the theory [21].

The case of discrete series goes along the same lines. For $k < -2$, all states have positive norm in the range $-1 < j < 0$, but are all tachyonic (level zero only is allowed). When $j \leq -1$, $N_m(j) \geq 1$, which corresponds to more general massive, massless or tachyonic excitations. However, since $N_m(j)$ grows quadratically with j , for sufficiently large $|j|$, $N_m(j) + j$ becomes positive, and physical non-unitary representations appear [2]. Negative-norm states remain in the spectrum⁹.

The situation described above is not very encouraging. Representations of the current algebra based both on continuous and discrete series seem to be required for generating the complete bosonic spectrum. Virasoro constraints and

⁹As expected, the situation for $k > -2$ is worse and unitarity is definitely lost in that case. For the continuous series, $N_m(j) \geq 1$. Thus, for any spin, non-unitary physical representations appear at several levels. The same conclusion holds for the discrete series with $-1 \leq j < 0$, whereas only tachyons are physical for $j < -1$.

mass-shell condition guarantee unitarity for the continuous series – which describe only tachyons –, but do not succeed in the case of discrete representations. When the spin is of order $j \lesssim k + 2$, the norm of the on-shell states in the current-algebra representation is no longer positive; for these states $M^2 \gtrsim |k + 3|$.

Within this framework, if we insist on having a theory free of negative-norm states, the only possibility left would be to cut the spin j ($j_{\min} \leq j < 0$) in such a way that $N_m(j_{\min}) + j_{\min}$ be non-positive [3]. We thus avoid some physical non-unitary representations that would have been present otherwise, and there is hope that all negative-norm states decouple in this way.

It is important to be aware that *this latter possibility violates the generic structure of the string spectrum itself*. We lose the infinite tower of string modes (the mass is cut off at the scale of the radius of ADS_3), and consequently the hope of constructing a consistent spectrum shrinks. Modular invariance is expected to be spoiled. Moreover, we cannot even keep the unit representation in the spectrum, namely the current-algebra representation with $j = 0$, since $N_m(j = 0) = 1$, and level one contains a representation of the global algebra with spin 1, which is not unitary. Despite these features the above possibility has been worked out because of some appealing properties. Let me briefly summarize the situation.

Starting from a level-zero unitary discrete representation $\mathcal{D}^-(j)$ ($j < 0$), we find a representation $\mathcal{D}^-(j + 1)$ at level one, which is Virasoro-primary. The norm of its highest weight is given by $2j - k$. The mass-shell condition implies that this representation is present as long as $j \leq -1$ (remember $k < -2$). In that range, unitarity thus demands

$$\frac{k}{2} \leq j < 0. \quad (3.3)$$

Condition (3.3) is the key of our analysis [3, 4]. It is similar to the condition appearing in $SU(2)_k$ and has in fact the same origin, although its purpose here is not to guarantee the unitarity of the $SU(1, 1)$ WZW model, but the unitarity of the latter modded out by the Virasoro constraints. The above condition on the spin has drastic consequences over the string spectrum.

By using Eq. (3.2), there appears an *absolute upper bound on the string level*,

$$N_{\max} = \text{integer part} \left(1 - \frac{k}{4} \right), \quad (3.4)$$

and similarly for the mass squared. For instance, if the string is a pure WZW model on $SU(1, 1)$, $k = -52/23$, and the physical spectrum is made out of tachyons and massless states only. On the other hand, we can add an internal unitary conformal field theory with positive central charge c_{int} . The bigger c_{int} is, the larger $|k|$ is, and more and more massive are the states that the physical spectrum acquires¹⁰.

As was already stressed, the consistency of a string with a finite number of mass levels is questionable. One can in particular wonder what the issue of modular invariance could be. Following our discussion of Section 2, it appears that modular transformations of characters for generic values of (k, j) (see Eq. (2.9)) violate the bound (3.3). Of course, special values of the spin where singular vectors appear in the Verma module can lead to characters with different modular properties, and modular-invariant combinations could eventually be reached. Unfortunately, interesting situations arise when $2j - k + n = 1$ [17], which is out of the would-be unitarity range. Anyhow, since we do not know the $SU(1, 1)$ characters in the regime $k < -2$, we cannot go any further in the present analysis.

Finally, the question to be answered is still whether the above condition (3.3) can indeed help to restore unitarity.

As already mentioned earlier, at level N , there appears one representation of the global algebra with spin $j + N$, which is Virasoro-primary. Constraint (3.3) combined with mass-shell condition (3.1) is sufficient to guarantee that $j + N$ never becomes positive. Unitarity also requires the highest-weight vector of that representation to be positive-norm. This makes condition (3.3) necessary and sufficient (the norm vanishes at $k = 2j$).

¹⁰Notice that in the flat-space limit, the upper bound on the mass disappears. This limit cannot therefore rule out the above analysis. It can, however, serve as a guideline to check the consistency of the results obtained in ADS_3 .

There also appears at level N a Virasoro-primary representation with spin $j - N$; since $j - N$ is always negative, all we must check is the norm of its highest-weight vector. For $j < -1/2$ and $k < -2$, its norm is strictly positive, at any N (it vanishes at $j = -1/2$ and is negative for $-1/2 < j < 0$, but in this range only level zero is allowed by (3.1)). Condition (3.3) plays no role here.

Although technically involved, it is quite straightforward to prove explicitly the absence of negative-norm states at both level one and level two [3]. The first level is the only one allowed by condition (3.3) for a pure WZW $SU(1, 1)$ model (see Eq. (3.4) with $k = -52/23$), and does not contain, in that case, other Virasoro primaries. Unitarity is therefore proved¹¹.

In order to see what happens at level two, i.e. which representations survive the Virasoro constraints, we must consider some extra unitary conformal field theory. It is simple, and quite instructive as far as counting of states is concerned, to add d free bosons. The total central charge is now $d+3k/(k+2)$, whereas the space-time dimension becomes $D = d + 3$. The critical dimension is $D_{\text{cr}} = 29 - \frac{3k}{k+2}$.

At level one the total number of representations of the global $SU(1, 1)$ algebra is D (1 with spin $j + 1$, $D - 2$ with spin j , and 1 with spin $j - 1$). Among them, $D - 1$ are Virasoro-primary: 1 with spin $j + 1$, $D - 3$ with spin j , and 1 with spin $j - 1$. On shell, $D - 2$ have positive norm, and 1 has zero norm (with spin j).

At level two, the total number of representations is $D(D + 3)/2$; 1 has spin $j + 2$, $D - 1$ have spin $j + 1$, $D(D - 1)/2$ have spin j , $D - 1$ have spin $j - 1$, and 1 has spin $j - 2$. There are $(D + 2)(D - 1)/2$ Virasoro primaries: 1 with spin $j + 2$, $D - 2$ with spin $j + 1$, $(D - 1)(D - 2)/2$ with spin j , $D - 2$ with spin $j - 1$, and 1 with spin $j - 2$. The on-shell positivity properties of these representations are the following:

- (i) For $D < D_{\text{cr}}$: $D(D - 1)/2$ positive-norm; $D - 1$ zero-norm, among which 1 with spin $j + 1$, $D - 3$ with spin j , and 1 with spin $j - 1$.
- (ii) For $D = D_{\text{cr}}$: $(D + 1)(D - 2)/2$ positive-

¹¹By mass-shell condition (3.4), level two would be allowed for $j = -1/2 - \sqrt{47/23}/2 < k/2 = -26/23$. Condition (3.3) is violated, and unitarity is lost.

norm; D zero-norm, among which 1 with spin $j + 1$, $D - 2$ with spin j , and 1 with spin $j - 1$.

- (iii) For $D > D_{\text{cr}}$: $(D + 1)(D - 2)/2$ positive-norm; $D - 1$ zero-norm, among which 1 with spin $j + 1$, $D - 3$ with spin j , and 1 with spin $j - 1$; 1 negative-norm with spin j .

We thus conclude that *all negative-norm states decouple from the physical spectrum, provided $D \leq D_{\text{cr}}$* . Unitarity is lost otherwise.

These level-two unitarity properties *assume condition (3.3)*. If j becomes smaller than $k/2$, not only does the extremal representation with spin $j + 2$ become non-unitary, but also $D - 4$ representations with spin $j + 1$, and 1 with spin j , no matter if we are below, at, or above the critical dimension. This emphasizes the role played by our unitarity condition, and gives some credit to the method we have presented so far, despite the consistency problems that such a bound on the spin creates at the level of the spectrum. It is even more puzzling that a real no-ghost theorem might exist, based on the above observations and more specifically on the constraint (3.3) over the spin of the allowed discrete representations. Various works seem to confirm this viewpoint [26].

As a final remark, I would like to use the above on-shell counting of states to infer what the partition function would look like, at least for the contributions originated from the representations of the current algebra based on the discrete series. String partition functions count precisely on-shell states – up to level-matching condition. We thus obtain¹²:

$$\begin{aligned} Z(\tau, \bar{\tau}, v, \bar{v}) &\sim v^{-1} q^{-1} \times \\ &\times \left(1 + q \left(e^{2i\pi v} + D - 4 + e^{-2i\pi v} \right) \right. \\ &+ q^2 \left(e^{4i\pi v} + e^{-4i\pi v} + \left(e^{2i\pi v} + e^{-2i\pi v} \right) (D - 3) \right. \\ &\left. \left. + \frac{1}{2} (D - 2)(D - 3) \right) + O(q^3) \right) \times \text{c. c.} \quad (3.5) \end{aligned}$$

(here $26 > D \geq 3$ is a free integer parameter, and the level $-\infty < k \leq -52/23$ is chosen such

¹²Expression (3.5), except for the infinite degeneracy level by level, is similar to the corresponding one for the free bosonic string. This is due to the structure of $SU(1, 1)$ algebra, and does not hold at higher levels.

that $D = D_{\text{cr}}$). Expression (3.5) can be seen as the expansion of

$$Z(\tau, \bar{\tau}, v, \bar{v}) = \frac{(\text{Im } \tau)^{\frac{5-D}{2}} F(\tau, \bar{\tau}, v, \bar{v})}{\vartheta_1(\tau, v) \bar{\vartheta}_1(\bar{\tau}, \bar{v}) (\eta(\tau) \bar{\eta}(\bar{\tau}))^{D-5}}, \quad (3.6)$$

where $(\sqrt{\text{Im } \tau \eta \bar{\eta}})^{D-5}$ stands for the free-boson-and-ghost contributions and $F(\tau, \bar{\tau}, v, \bar{v})$ behaves like

$$F(\tau, \bar{\tau}, v, \bar{v}) = e^{-\frac{\pi \text{Im } \tau}{k+2}} e^{-2\pi \text{Im } v} \times (1 + \mathcal{O}(q^3)) (1 + \mathcal{O}(\bar{q}^3)) \quad (3.7)$$

In fact, any power of q and \bar{q} (≥ 3) is expected as a consequence of modular covariance:

$$F\left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}, \frac{v}{\tau}, \frac{\bar{v}}{\bar{\tau}}\right) = \left| e^{-2i\pi \frac{v^2}{\tau}} \right| |\tau| F(\tau, \bar{\tau}, v, \bar{v}),$$

whereas $F(\tau, \bar{\tau}, v, \bar{v})$ should be invariant under $\tau \rightarrow \tau + 1$. These constraints can actually be satisfied with expressions that do not fall in the class of (3.7), such as $F = (2 \text{Im } \tau)^{-1/2} \exp 2\pi \frac{(\text{Im } v)^2}{\text{Im } \tau}$, inspired from Eq. (2.10). The latter expression for F is actually what (2.10) would have given if the measure dj had been replaced with $dj \exp -\pi k \frac{(\text{Im } v)^2}{\text{Im } \tau}$. I have already discussed this issue in Section 2 for the pure WZW model. Here, it becomes clear that such a function is not allowed since *it does not exhibit the correct weight shift to fulfil the mass-shell condition*. Furthermore, at large $|k|$ ($D \rightarrow 26$) and small v , matching with the ordinary 26-dimensional string requires the following behaviour¹³: $F \rightarrow \frac{\kappa |v|^2}{(\text{Im } \tau)^{3/2}}$, where κ is a constant expected to diverge like $|k|^{3/2}$. In order to obtain correctly the various behaviours, one should therefore rely on the generic form $F(\tau, \bar{\tau}, v, \bar{v}) = \sum_{j,\ell} \mathcal{N}_{j,\ell} f_j(\tau, v) \bar{f}_\ell(\bar{\tau}, \bar{v})$ (see Eq. (2.2)), where $f_j(\tau, v)$ is the factor in the character of the spin- j representation, which accounts for the null states in the Verma module. The presence of terms of $\mathcal{O}(q^3)$ and $\mathcal{O}(\bar{q}^3)$ precisely traces back the appearance of representations of the $SU(1, 1)$ current algebra in the spectrum, which contain *null states at levels higher*

¹³This is precisely the next-to-leading behaviour of expression (2.10). The leading term diverges like $\sqrt{k}/|v|^2$, and should be avoided in the presence of world-sheet supersymmetry. The various uncertainties related to the meaning of (2.10), however, do not enable us to draw any conclusion.

than two. As was already mentioned before, such representations are expected to appear for $j \leq k/2$; this bound is in contradiction with the unitarity constraint (3.3).

Obviously, using naive generic- (k, j) characters (2.7) *cannot* lead to an expression for F consistent with (3.7). Furthermore, it is hard to believe that F exists, such that expression (3.6) is polynomial in q, \bar{q} of degree N_{max} , as is suggested by the previous study of unitarity. Once more, *compatibility between unitarity and modular invariance* appears as an important issue in understanding the string propagating over ADS_3 .

4. Summary and comments

String theory on three-dimensional anti-de Sitter space-time has been commented for a long time. No satisfactory understanding of its basic features, such as the complete spectrum of perturbative states, has yet been reached. By formally using the standard current-algebra approach of conformal field theory in the framework of a non-compact group manifold, it seems that unitarity would demand an upper bound to the mass spectrum. This conclusion is in disagreement with elementary principles of string theory, leads to serious problems in computing one-loop amplitudes, and has no physical foundation: nothing similar appears in the quantum motion of a free particle on ADS_3 , which is expected to be the $\alpha' \rightarrow 0$ limit of the string, and where *all* possible representations of $SU(1, 1)$ appear in the wavefunction, with appropriate interpretation [13].

Nevertheless, it is quite amazing that the cut-off over the spins could really fix the unitarity problem [3, 4, 26], and this may be a good reason to try to keep the above results, and rephrase them within a somewhat less formal approach. This means that we should get a better understanding of the $SU(1, 1)$ WZW model at the *classical level*¹⁴, explore the classical motion of a

¹⁴See [27]. In that spirit, $SU(1, 1)/U(1)$ has been recently revisited [28]. Remember that $SU(1, 1)/U(1)$ was analysed in [29] as an internal conformal field theory for a string compactification. Unitarity was proved, provided $k/2 \leq j < 0$. Later, $SU(1, 1)/U(1)$ was reinterpreted as a two-dimensional black hole [30], and the spectrum was studied in [31]. This could also be a source of inspiration for $SU(1, 1)$ itself, by considering $(SU(1, 1)/U(1)) \times U(1)$.

string, and maybe try to follow a path-integral approach *à la* Gawędzki [6]. This implies in particular a proper treatment of the *target-space boundary conditions* that are hard to implement within the current-algebra method. They might play a role in reconciling unitarity with the appearance of states of the discrete series with spin $j < k/2$, i.e. with mass above the anti-de Sitter radius scale. These states are cut off in our analysis but should be present for physical reasons, and might originate from some *other unitary sector* of the theory, which would have been missed here¹⁵. The role of continuous series could also be clarified. Although they are compatible with unitarity, the corresponding excitations seem to be all tachyonic. Finally, one should wonder whether target-space boundary conditions introduce *ambiguities in the quantization of a string*, similar to those appearing when solving the wave equation for a quantum particle [12]. Such ambiguities may have interesting consequences for the string, as they have when studying e.g. the Unruh effect on $AD\mathcal{S}_3$ [34].

As was pointed out previously, one could try alternatively to avoid the mass/spin cut-off and the purely tachyonic continuous representations in various ways, playing essentially with the current algebra, and/or modifying the affine Sugawara construction. It even seems that contact with some $AD\mathcal{S}/CFT$ -inspired results can be made in that way [35]. However, it is not clear whether such modifications leave unaltered the interpretation of the theory as a string propagating over $AD\mathcal{S}_3$ (in [23], logarithmic cuts and a new zero mode are introduced in the currents, and representations based on discrete series are simply discarded). Furthermore, none of the available attempts treats the *problem of the infinite degeneracy* at each string level, or enlightens the issue of modular invariance. In particular, the role of the “magnetic field” v remains obscure. The possibility of interpreting it as a continu-

There are many ways to define the latter, with various geometrical interpretations – when available [20, 32].

¹⁵A similar viewpoint was somehow taken by the authors of [33]. Their subsequent developments were, however, quite ad-hoc, and the net result for the partition function looks more like an analytic continuation of an $SU(2)_k$ invariant than like a true amplitude computed from first principles in the theory under consideration.

ous twist, similar to those that appear in the parafermionic constructions, has never been exploited. One might, though, relax in this way the constraint of modular invariance: the latter should be recovered only after an appropriate summation over v (like in the case of the two-dimensional Euclidean black hole [6]).

Finally, a somewhat more exotic attitude¹⁶ with respect to the unitarity could be to simply admit the presence of negative-norm states in the physical spectrum, and then try to interpret them or, better, to identify the instability they are related to and its physical origin within the $AD\mathcal{S}_3$ background. Following this line of thought, one could even reconsider the – non-unitary – models based on the admissible representations of the $SU(1,1)$ current algebra; their partition function is known, and one should then try to understand them in the framework of string theory.

Let me emphasize once more that the motivations for studying the string on $AD\mathcal{S}_3$ are wider than expected in the early works. They include the string motion in a three-dimensional black hole [37], which settles the proper framework to address the black-hole evaporation problem. Instead of a NS–NS torsion background, coupling to non-perturbative R–R charges is also a relevant and difficult problem [38]. Finally, the analysis of the $AD\mathcal{S}/CFT$ conjecture, in the framework of the $AD\mathcal{S}_3 \times S^3$ background¹⁷, is probably the issue that has attracted most attention. Ideally, we would like to compute correlators in both sides and compare them. In practice, correlators for $AD\mathcal{S}_3$ string states are out of reach, which makes any rigorous check quite intricate. Therefore, most of the work in that direction has been devoted to trying to express the space-time as well as the asymptotic two-dimensional conformal symmetry, in terms of the fields of the WZW model whose target space is the bulk

¹⁶An even more marginal alternative would be to drop the requirement of modular invariance, and simply face the “stringy exclusion principle” [36]. It is hard to believe that this could be the end of the story, since so many features of string theory, such as the IR/UV duality, strongly rely on modular invariance.

¹⁷Notice that $AD\mathcal{S}_3 \times S^3$ modded out by some discrete symmetry might be more tractable than $AD\mathcal{S}_3 \times S^3$ itself, in particular when we demand supersymmetry.

ADS₃ theory, and to build in that way the boundary conformal field theory. It is fair to say, however, that this approach has not shed any light on the structure of the ADS₃ string itself – at least regarding the questions raised here; as long as one does not handle the ADS₃ side exactly, the achievements are limited both on checking the ADS/CFT correspondence, and on building the boundary theory [11]. Of course, there is still the – weaker – alternative to work with the low-energy supergravity, supplemented with all Kaluza–Klein excitations coming from higher dimensions, thus trying to obtain some feedback for the string on ADS₃. For example, there are signs that all $SU(1, 1)$ representations – discrete and continuous – should appear without bound on the mass. If such a bound were present, assuming the ADS/CFT correspondence, it would be hard to identify states in the bulk ADS₃ supergravity (or string, as a fundamental theory), with states in the boundary conformal field theory.

As a last comment, I would like to stress that string theories on more general ADS_n backgrounds are equally important and more difficult than the three-dimensional case at hand. In odd dimensions, it has been realized very recently that both an antisymmetric tensor and a linear dilaton are needed together with the gravitational background, in order to define an exactly conformal sigma model describing the string [39]. This sigma model *is not* a WZW model, and its spectrum and interactions remain quite unexplored. Furthermore, the exact conformal field theories describing the string propagation on even-dimensional anti-de Sitter spaces have never been investigated. Notice also that some issues, such as the level-by-level infinite degeneracy, or even the unitarity problem, will be generically present.

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material I presented here is not to be considered as a review over the very wide subject that ADS₃ has become on its own. Many interesting aspects are missing or superficially mentioned. I hope that the reader will forgive me for the very incomplete list of references.

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