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PARTICLE SPIN MOTION IN A STORAGE RING WITH AN ARBITRARY FIELD

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Introduction

The study of the behaviour of particle polarization in accelerators is normally confined /5-8/ to the case of an almost uniform magnetic field. The present work sets out the general results of spin motion studies in storage rings (accelerators) with an arbitrary electromagnetic field, on the assumption of the existence of a closed orbit.

The present work consists of two parts.

The first part gives a simple and physically transparent derivation of the equations of the spin motion of relativistic particles without the intermediate use of the 4-vector of polarization. In the second part, an investigation is made of the nature of spin motion, disregarding the particle trajectory deviations from the closed (equilibrium) orbit. It is shown that there is always a periodic spin motion with a fixed polarization along the field in an almost uniform magnetic field.

This being the case, it is in practice possible to create, at a given point of the orbit, an arbitrary spin orientation in relation to the velocity and field. This opens up considerable prospects for controlling the polarization in light and heavy particle storage rings.

The following work is devoted to a detailed study of particle spin behaviour on a real trajectory, and of beam polarization as a whole.

1. Spin motion equations

1. General equations

In non-relativistic theory, the quantum mechanical mean of the spin vector $\langle \vec{S} \rangle = \vec{\zeta}$ of a particle moving along a conventional trajectory satisfies the equation:

$$\frac{d\vec{\zeta}}{dt} = [\vec{\mu}\,\vec{\zeta}] \qquad \vec{\mu} = q\vec{\zeta} \qquad (1.1)$$

where $\vec{\mu}$ magnetic moment of the particle, q gyromagnetic ratio.

The polarization state in the relativistic case may also be characterized conveniently by the spin vector $\vec{\zeta}$ in the rest system.

The equations for the vector $\vec{\zeta}$ can be obtained from the well-known BMT co-variant equations for the 4-vector of polarization /1,2/. In this case the physical simplicity of the equations for $\vec{\zeta}$ is masked. In fact, there is a very simple derivation of the equations without intermediate use of the 4-vector of polarization.

Let us define the proper system as a system obtained by Lorentz transformation from the laboratory system. According to the transformation, the orientation of the velocity $\vec{v}$ in relation to the spatial axis of the proper system at any moment of time coincides with the orientation in the laboratory system. The equation for $\vec{\zeta}$ can be obtained by using directly the non-relativistic equation (1.1).

Let us change over to an inertial system coinciding with the proper system at a moment of time t (u - system). The change in the spin vector $\vec{\zeta}$ in this system over an interval of proper time $d\tau = dt$ will be determined by the equation (1.1):

$$d\vec{\zeta}_u = q\left[\vec{\zeta}\vec{H}_c\right]d\tau$$

where $\vec{H}_C$ the magnetic field in the intrinsic system.

However, $d\vec{\zeta}_u$ will not coincide with the unknown increment $d\vec{\zeta}$, because when the particle velocity direction changes, the spatial axes of the proper system (obtained from the laboratory axes by the Lorentz transformation) are rotated in relation to the u - system. Let us assume that $d\vec{\varphi}$ is the angle by which the proper system is rotated in relation to the u - system. Then, we may write

$$d\vec{\zeta} = d\vec{\zeta}_u - \left[d\vec{\varphi}\,\vec{\zeta}\right] \qquad (1.2)$$

The angle $d\vec{\varphi}$ can be found by simple reasoning.

Let us assume that $d\vec{\alpha}$ is the vector of the velocity rotation angle in the laboratory system:

$$d\vec{\alpha} = \frac{1}{v^2}\left[\vec{v}\,d\vec{v}\right]$$

Then the proper system of coordinates, in accordance with its definition, will rotate in relation to the new velocity direction $\vec{v} + d\vec{v}$ by an angle of $-d\vec{\alpha}$. At the same time the direction $(\vec{v}+d\vec{v})_u$ in the u-system will form an angle of $\gamma d\vec{\alpha}$ with the direction $\vec{v}$. In reality, these directions in the u-system are obtained in accordance with the Lorentz transformation by projecting the corresponding directions of the laboratory system onto a plane τ = const. This procedure does in fact correspond with the definition of the angle in the u-system between two "bars" which are at rest in the laboratory system.

In this way the angle of rotation $d\vec{\varphi}$ is :

$$d\vec{\varphi} = \gamma d\vec{\alpha} - d\vec{\alpha} = \frac{\gamma-1}{v^2}\left[\vec{v}\,d\vec{v}\right]$$

By substituting this expression in (1.2), we obtain the desired equation

$$\dot{\vec{\zeta}} = \frac{d\vec{\zeta}}{dt} = \frac{q}{\gamma}\left[\vec{\zeta}\,\vec{H}_c\right] + \frac{\gamma-1}{v^2}\left[\vec{\zeta}\left[\vec{v}\,\dot{\vec{v}}\right]\right] \qquad (1.3)$$

The first term on the right-hand side of the equation is directly related to the magnetic moment of the particle, and the second term is the consequence of the relativistic kinematics of rotation.

The occurence of the last term can be demonstrated by the following model. Let us assume that the centre-of-mass of a bar having a short length moves around a radius R at a speed v, and that the moment of the forces acting on the bar is equal to 0. For simplicity let us locate the bar in the plane of rotation. In non-relativistic mechanics, in the laboratory system the bar does not rotate, whereas in the rest system of the centre of the bar, which rotates with the velocity (C - system), the bar rotates at an angular speed of $-v/R$. In the relativistic case, rotation of the bar in the C -system will appear exactly as in the non-relativistic case: the angle of rotation $d\varphi_C$ of the bar in the C -system for two consecutive positions of the centre, situated at a distance dl_C (measured in the C - system), will be:

$$d\varphi_c = -\frac{dl_c}{R}$$

During a period of velocity rotation, the bar will rotate through an angle of

$$\varphi_c = -\oint \frac{dl_c}{R} = -\oint \frac{\gamma\, dl}{R} = -2\pi\gamma$$

In this way, in relation to the laboratory system, the bar will rotate through an angle $-2\pi(\gamma - 1)$.

The effect under consideration is phenomenologically related to the well-known "twin paradox". The idea for this derivation is contained in early papers by Thomas /4/.

Using the equation for $\vec{H}_C$ the electromagnetic field in the laboratory system

$$\vec{H}_c = \gamma\left(\vec{H}_{tr} + [\vec{E}\vec{v}]\right) + \vec{H}_v$$

($\vec{H}_{tr}$, $\vec{H}_v$ = the transverse and longitudinal components of the magnetic field in relation to the velocity) and the equations for particle motion

$$\dot{\vec{v}}_{tr} = \frac{e}{\gamma m}\left\{\vec{E}_{tr} + [\vec{v}\vec{H}]\right\}$$

it is possible to transform (1.3) to its well known form /2/:

$$\dot{\vec{\zeta}} = [\vec{W}_n \vec{\zeta}]$$

$$-\vec{W}_n = \left(\frac{q_0}{\gamma} + q'\right)\vec{H}_{tr} + \frac{q}{\gamma}\vec{H}_v + \left(\frac{q_0}{\gamma+1} + q'\right)[\vec{E}\vec{v}]$$

where $q' = q - \frac{e}{m} \equiv q - q_0$ is the anomalous part of the gyromagnetic ratio (c = 1). The vector $\vec{W}_n$ has the direction of the angular velocity of spin rotation relative to frames fixed in the laboratory system. As can be seen, the normal q_0 and anomalous q' parts of the gyromagnetic relation are included non-additively in the spin motion equation. The immediate explanation of this factor is given by equation (1.3) from which it follows that this fact is due to perturbation of the particle's trajectory.

The angular velocity of spin rotation will be proportional to the total magnetic moment if the particle moves along a straight line (this is possible when $e = 0$, or when $\vec{E}_{t\tau} + [\vec{v}\vec{H}] = 0$).

It is convenient to write $\vec{W}_\Omega$ in the form:

$$\vec{W}_\Omega = \left(1+\gamma\frac{q'}{q_0}\right)\frac{[\vec{v}\dot{\vec{v}}]}{v^2} - \frac{q}{\gamma}\vec{H}_\tau + \frac{q}{\gamma^2 c^2}[\vec{E}\vec{v}] \qquad (1.4)$$

with the condition

$$\gamma^2 q' \gg q \qquad (1.5)$$

the last term in (1.4) can be neglected ($[\dot{\vec{v}}\vec{v}] \neq 0$). $\vec{W}_\Omega$ is expressed directly by the parameters of the trajectory and the longitudinal magnetic field which does not change the particle's acceleration.

For even higher energies ($\gamma q' \gg q_0$), $\vec{W}_\Omega$ is determined by the anomalous moment and does not depend on energy:

$$\vec{W}_\Omega \to \gamma\frac{q'}{q_0}[\vec{v}\dot{\vec{v}}] \simeq -q'(\vec{H}_{t\tau} + [\vec{E}\vec{v}])$$

This is explained by the fact that the normal part (the first term in (1.3)) is reduced by a factor of $\sim 1/\gamma$ compared with the kinematic term. If $q' = 0$, then

$$\vec{W}_\Omega = -\frac{q_0}{\gamma}\vec{H} - \frac{q_0}{\gamma+1}[\vec{v}\vec{E}] .$$

when $\vec{E} = 0$. $\vec{W}_\Omega$ coincides with the Larmor frequency $\frac{e\vec{H}}{\gamma m}$.

Let us examine now the question of the transformation of spin equations during transformation to any other system of unit vectors $\vec{e}_\alpha(t)$ (α = 1.2.3.). Let $\vec{W}_b$ be the angular speed of rotation of the basis in relation to the initial system :

$$\dot{\vec{e}}_\alpha = [\vec{W}_b \vec{e}_\alpha] \tag{1.6}$$

The equations for spin motion in the new system will, obviously, be :

$$\dot{\vec{\zeta}} = [\vec{W}_{\Pi} - \vec{W}_b , \vec{\zeta}] = [\vec{W} \vec{\zeta}] \tag{1.7}$$

The expression for $\vec{W}_b$, follows directly from (1.6) if $\vec{e}_\alpha(t)$ are given :

$$\vec{W}_b = \frac{1}{2} \sum_{\alpha=1}^{3} [\vec{e}_\alpha \dot{\vec{e}}_\alpha] \tag{1.8}$$

The spin equations, as any equations of type (1.7) can be written in canonical form :

$$\dot{\vec{\zeta}} = \{\vec{\zeta} ; \mathcal{H}\}$$

where $\{ ; \}$ are Poisson brackets, either conventional or quantum, and

$$\mathcal{H} = \vec{W} \vec{\zeta} \tag{1.9}$$

is a Hamiltonian.

In many respects it is convenient to write the spin motion in the Hamiltonian variables ζ_z and ψ , where ζ_z is the projection on a selected axis, and ψ is the spin rotation phase around this axis. In these variables, the Hamiltonian

$$\mathcal{H} = W_z \zeta_z + W_\perp \zeta_\perp \cos(\psi - \delta) \tag{1.10}$$

where

$$W_\perp e^{i\delta} = W_x + i W_y$$
$$\zeta_\perp e^{i\psi} = \zeta_x + i \zeta_y$$

The Hamiltonians of the equation are of the form

$$\dot{\zeta}_z = -\frac{\partial \mathcal{H}}{\partial \psi} = W_\perp \zeta_\perp \sin(\psi - \delta)$$
$$\dot{\psi} = \frac{\partial \mathcal{H}}{\partial \zeta_z} = W_z - W_\perp \frac{\zeta_z}{\zeta_\perp} \cos(\psi - \delta) \tag{1.11}$$

2. Equations in the "natural" reference system

For certain applications it is convenient to use a moving system of unit vectors, linked with the particle trajectory:

$$\vec{e}_\tau = -\frac{\dot{\vec{v}}_{\tau\tau}}{|\dot{\vec{v}}_{\tau\tau}|} , \quad \vec{e}_v = \frac{\vec{v}}{v} , \quad \vec{e}_z = [\vec{e}_\tau \vec{e}_v] \tag{1.12}$$

The expression for $\vec{W}_b$ is obtained from (1.8):

$$\vec{W}_D = (\vec{e}_\tau \dot{\vec{e}}_z)\vec{e}_v + \frac{1}{v^2}[\vec{v}\dot{\vec{v}}]$$

In this way the spin equations in this system are

$$\dot{\vec{\zeta}} = [\vec{W}\vec{\zeta}] , \qquad \vec{W} = (W_\tau , W_v , W_z)$$

$$W_\tau = -\frac{q}{\gamma^2 v} E_z$$
$$W_v = -(\vec{e}_\tau \dot{\vec{e}}_z) - \frac{q}{\gamma} H_v \tag{1.13}$$
$$W_z = \gamma \frac{q'}{q_0} \frac{|\dot{\vec{v}}_{\tau\tau}|}{v} + \frac{q}{\gamma^2 v} E_\tau$$

If the condition (1.5) $\gamma^2 y' \gg q$ is fulfilled, then $\vec{W}$ has a simple form :

$$W_\tau = 0$$
$$W_v = -(\vec{e}_\tau \dot{\vec{e}}_z) - \frac{q}{\gamma} H_v$$
$$W_z = \gamma \frac{q'}{q_0} \frac{|\vec{v}_{t\tau}|}{v}$$

Let us consider some specific cases in which the equations (1.13) allow precise solutions.

When $q' = 0$ and $\vec{E}_{t\tau} = 0$ polarization is maintained in the direction of the velocity /2/. This being the case, we have from (1.11):

$$\zeta_v = const.$$
$$\Psi_v = \int W_v \, dt$$

i.e. the spin moves in a plane perpendicular to the velocity and over a time $\pi / \overline{W_v}$ ζ_τ and ζ_z change their sign. In fact if the average value of $\overline{W_v} = 0$, then the spin will oscillate in this plane around a given average position.

In the case of motion in a magnetic field, $W_\tau = W_v = 0$ (see 1.13), if the particle trajectory remains in the same plane and there is no longitudinal magnetic field on the orbit. The same occurs also in the presence of an electric field, if (1.5) is satisfied. In this case the ζ_z polarization component is constant. The angular velocity of rotation around the direction $\vec{e}_z$

$$\dot{\psi} = W_z = \gamma \frac{q'}{q_0} \frac{|\vec{v}_{t\tau}|}{v}$$

is fully determined by the anomalous magnetic moment.

The spin moves in a similar manner, also in an arbitrary electromagnetic field at the limit of high energies, when the field on the particle trajectory varies adiabatically slowly in comparison with spin rotation:

$$\left|\frac{d}{dt} W_{t,v}\right| \Big/ \left|W_{t,v}\right| \quad << \gamma \frac{q'}{q_c} \left|\dot{\vec{v}}_{tr}\right| \tag{1.14}$$

In the laboratory system, the solution in this case is of the form:

$$\vec{\zeta}(t) = \zeta_z \frac{[\vec{v}\dot{\vec{v}}]}{|\dot{\vec{v}}_{tr}|} + \sqrt{s^2 - \zeta_z^2}\left(\vec{v}\sin\psi - \frac{\dot{\vec{v}}_{tr}}{|\dot{\vec{v}}_{tr}|}\cos\psi\right) \tag{1.15}$$

where $\psi = \frac{q'}{q_c}\int \gamma |\dot{\vec{v}}_{tr}| dt$, $\zeta_z = const$, i.e. the spin has a constant projection in a direction perpendicular to the plane ($\vec{v}$, $\dot{\vec{v}}$).

II. Spin motion in a storage ring

1. Formulation of the problem

Let us now turn to an investigation of the dynamics of spin motion, taking into account the specific nature of particle motion in storage rings and accelerators. The main property of particle orbital motion in these systems is the existence of a closed (equilibrium, periodic) orbit, along which all particles move with small coordinate and momentum deviations from the equilibrium values. Consequently, it is reasonable to represent $\vec{W}_n$ in the form:

$$\vec{W}_n = \vec{W}_s(\gamma_s, \theta) + \vec{w} \tag{II.1}$$

where $\vec{W}_s$ is the value of $\vec{W}_n$ on the equilibrium trajectory, $\theta = \int \omega_s dt$ is the generalized azimuth of the particle, ω_s is the equilibrium frequency of rotation. $\vec{W}_s$ is a periodic function of the azimuth:

$$\vec{W}_s(\gamma_s, \theta) = \vec{W}_s(\gamma_s, \theta + 2\pi)$$

If small deviations in the particle trajectory from the equilibrium value lead to small variations in the electromagnetic field on the trajectory, the effect of $\vec{w}$ may be considered as a small disturbance in relation to the motion in the equilibrium field. Consequently when investigating spin motion the normal procedure may be used: First the spin motion integrals on the equilibrium trajectory are determined, and then a study made of the influence of adding $\vec{w}$ to the ideal spin motion.

It is usually assumed /5-8/ that the real conditions on the closed orbit are such that in one revolution the polarization in a direction transverse to the average plane of the orbit varies only slightly:

$$\left| \zeta_z(\theta + \theta') - \zeta_z(\theta) \right| \ll S \qquad 0 \leq \theta' \leq 2\pi \tag{II.2}$$

This condition denotes, in practice, the closeness of the closed orbit to the flat one and the smallness of the longitudinal magnetic field. A considerable variation in the initial polarization ζ_z in this case may be accumulated only after a sufficiently large number of particle revolutions in the region of spin resonances.

We shall study spin motion without imposing any conditions on the electromagnetic field of the storage ring, apart from the existence of a closed particle trajectory. It is important for the treatment that at a given point of the orbit there is a fixed polarization direction which is constant during the motion. In a magnetic field which is almost uniform, this direction is the direction of the field). In view of this, we may formulate the following approach to the problem. First of all, it is necessary to elucidate whether there exists on this equilibrium trajectory a periodic spin motion. If this motion exists, its stability must be investigated.

2. Spin motion on an equilibrium trajectory

The spin of an equilibrium particle satisfies the equation

$$\dot{\vec{\zeta}} = [\vec{W}_s \vec{\zeta}] \tag{II.3}$$

We shall consider the equilibrium energy to be constant. Then

$$\theta = \omega_s t \quad , \quad \omega_s = const$$
$$\vec{W}_s(t) = \vec{W}_s\left(t + \frac{2\pi}{\omega_s}\right)$$

In accordance with the objective in mind, let us investigate whether a periodic solution exists

$$\vec{n}(\theta) = \vec{n}(\theta + 2\pi) \qquad \vec{n}^2 = 1$$
$$\dot{\vec{n}} = [\vec{W}_s \vec{n}] \tag{II.4}$$

Let us prove that this solution exists for any $\vec{W}_s(\theta)$ The general solution of equation (II.3) can be represented in the form of a combination of three linearly-independent solutions of $\vec{X}_\alpha(\theta)$ ($\alpha = 1,2,3$). Since the scalar product of any two solutions (II.3) is retained

$$\frac{d}{dt}(\vec{\zeta}_a \vec{\zeta}_b) = 0 \tag{II.5}$$

These three solutions can always be chosen to be orthogonal:

$$\vec{X}_\alpha(\theta)\vec{X}_\beta(\theta) = \delta_{\alpha\beta}$$

We shall seek the periodic equation of the form

$$\vec{n}(\theta) = \sum_{\alpha=1}^{3} n_\alpha \vec{X}_\alpha(\theta)$$

where

$$n_\alpha = \vec{n}\vec{X}_\alpha = const$$

In accordance with the condition of periodicity (II.4)

$$\sum_{\alpha} n_\alpha \vec{X}_\alpha(\theta) = \sum_{\alpha} n_\alpha \vec{X}_\alpha(\theta+2\pi)$$

or

$$\sum_{\beta=1}^{3} (\delta_{\alpha\beta} - \Lambda_{\alpha\beta}) n_\beta = 0$$

where

$$\Lambda_{\alpha\beta} = \vec{X}_\alpha(\theta) \vec{X}_\beta(\theta+2\pi)$$

The matrix Λ does not depend on time, since $\vec{X}_\beta(\theta + 2\pi)$ is again the solution of (II.3) owing to the periodicity of $\vec{W}_S(\theta)$.

The non-trivial solution of n_α exists if the determinant of the system is equal to zero:

$$|I - \Lambda| = 0$$

As the maxtrix Λ is, in fact, the matrix of rotation, it has the following properties:

$$\Lambda\Lambda^T = I \quad ; \quad |\Lambda| = 1$$

Hence:

$$|I-\Lambda| = |I-\Lambda^T| = |\Lambda^T||\Lambda-I| = |\Lambda-I| = (-1)^3|I-\Lambda| = 0$$

In this way for any periodic $\vec{W}_S(\theta)$ there is a periodic solution for $\vec{n}(\theta)$.

This proof does, in fact, contain a possible way of finding the periodic solution.

From the existence of the periodic solution emerges the general character of spin motion in a periodic field. Let $\vec{\zeta}(\theta)$ be the solution of (II.3) for the arbitrary initial condition of $\vec{\zeta}(0) \neq S\vec{n}(0)$: Since

$$\frac{d}{dt}\vec{\zeta}\vec{n} = 0 \qquad \text{(II.5a)}$$

spin motion takes place in the following manner. There is a certain periodic direction $\vec{n}(\theta)$, which has the sense of the direction of polarization of the periodic field, around which the spin rotates, retaining the projection on this direction. At the same time a solution is found also for the problem of the stability of the periodic solution in the case of motion around an equilibrium orbit.

The characteristic of (II.5a) suggests a way of constructing a general solution, if $\vec{n}(\theta)$ is known. Let us introduce the following system of unit vectors:

$$\vec{e}_1(\theta)\ ,\quad \vec{e}_2(\theta)\ ,\quad \vec{n}$$

where $\vec{e}$, $\vec{e}_2$ are any two orthogonal periodic directions in a plane transverse to $\vec{n}(\theta)$. We shall specify the polarization by the projection on to $\vec{n}$ and by the phase ψ of rotation around $\vec{n}$:

$$\vec{\zeta} = \zeta_n \vec{n} + \zeta_\perp Re(\vec{e}\, e^{-i\psi}) \qquad \text{(II.6)}$$

Here $\vec{e} = \vec{e}_1 + i\vec{e}_2$

For the angular velocity $\vec{W}_b$ (see 1.8) we obtain, by taking into account (II.4):

$$\vec{W}_b = [\vec{n}\dot{\vec{n}}] + (\dot{\vec{e}}_1 \vec{e}_2)\vec{n} = (\vec{W}_s)_\perp + \frac{i}{2}(\dot{\vec{e}}\vec{e}^{\,*})\vec{n} \qquad \text{(II.7)}$$

Consequently the Hamiltonian in this system is equal to:

$$\mathcal{H} = (\vec{W}_s - \vec{W}_b, \vec{\zeta}) = (\vec{W}_s\vec{n} - \frac{i}{2}\dot{\vec{e}}\vec{e}^{\,*})\zeta_n$$

As must be the case, $\dot{\zeta}_n = 0$; the spin rotates around $\vec{n}$ at an angular velocity of

$$\dot{\psi} = \frac{\partial \mathcal{H}}{\partial \zeta_n} = \vec{W}_s\vec{n} - \frac{i}{2}\dot{\vec{e}}\vec{e}^{\,*} \qquad \text{(II.8)}$$

The arbitariness in the choice of the transverse unit vectors $\vec{e}_1$ and $\vec{e}_2$ does not, of course, lead to an unique solution of (II.6) in the "laboratory" system. This can easily be proved by checking the invariance of the solution of (II.6) with regard to the substitution

$$\vec{e} \longrightarrow \vec{e}\, e^{-i\alpha(\theta)}$$

The convenience of using precisely the periodic systems $\vec{e}\,(\theta) = \vec{e}\,(\theta + 2\pi)$ is that when the spin is observed at a specific point of the orbit, the variation in polarization during a revolution in relation to the selected periodic system coincides with the variation compared with the stationary system. A special sense is acquired by the fractional part of the mean value, for the period, of the angular speed indicated $\nu = \langle\dot{\Psi}\rangle / \omega_s - K$ (K = nearest whole number), thus specifying the variation in spin orientation during a revolution. This physical value does not depend on the choice of periodic system. In fact, let $\vec{e}$ and $\vec{e}'$ be two periodic systems:

$$\vec{e}\,' = \vec{e} \cdot e^{-i\alpha(\theta)}$$

whilst, in accordance with the condition of periodicity,

$$\langle \dot{\alpha}(\theta) \rangle = K' \omega_s$$

Then from (II.8) we obtain:

$$\langle \dot{\psi}' \rangle = \langle \dot{\psi} \rangle - \langle \dot{\alpha} \rangle = \langle \dot{\psi} \rangle - K' \omega_s$$

i.e. the fractional part is retained.

Now it is easy to answer the question of the uniqueness of the periodic solution. As all solutions rotate around $\vec{n}$ with one angular frequency $\dot{\Psi}$, then on the condition that

$$\langle \dot{\psi} \rangle = \langle \vec{W}_s \vec{n} - \tfrac{i}{2} \dot{\vec{e}}\, \vec{e}^{\,*} \rangle \neq K \omega_s$$

the periodic solution of $\vec{n}$ is, obviously, unique. In the case of the precise resonance $\langle\dot{\Psi}\rangle = K\omega_s$ any solution is periodic, i.e. $\vec{n}$ is fully undetermined.

The frequency ν can be expressed also through the basic solution of $\vec{X}_\alpha$. By definition, $2\pi\nu$ is the angle through which the solution of equation (II.3) transverse to $\vec{n}$ is rotated during the revolution. In a complex form:

$$\vec{\eta}(\theta+2\pi) = e^{-2\pi i\nu}\vec{\eta}(\theta)$$

By breaking down η by basis (II.5a)

$$\vec{\eta} = \sum_{\alpha=1}^{3} \eta_\alpha \vec{X}_\alpha(\theta)$$

we obtain the equation:

$$\sum_{\beta=1}^{3} \left(e^{-2\pi i\nu}\delta_{\alpha\beta} - \Lambda_{\alpha\beta} \right) \eta_\beta = 0$$

In this manner we have reached the problem of determining the intrinsic values of the matrix Λ :

$$|\lambda I - \Lambda| = 0 \qquad\qquad \lambda \equiv e^{-2\pi i\nu}$$

This equation has three roots λ_1 , λ_2 , λ_3

It has already been proved that $\lambda = 1$ is the intrinsic value corresponding to the periodic solution of $\vec{n}(\Theta)$. Two others can be found by using the following relations:

$$\lambda_1 + \lambda_2 + \lambda_3 = \sum_{\alpha=1}^{3} \Lambda_{\alpha\alpha} = Sp\Lambda$$

$$\lambda_1 \lambda_2 \lambda_3 = |\Lambda| = 1$$

Hence

$$\lambda_2 = \lambda_3^* = e^{-2\pi i\nu}$$

$$\cos 2\pi\nu = \frac{1}{2}\left(Sp\Lambda - 1 \right) \qquad \text{(II.9)}$$

From (II.5) it follows that $\cos 2\pi\nu$, as must be the case, does not depend on the choice of the basis $\vec{X}_\alpha$. The reality of ν follows from the inequalities:

$$-1 \leq Sp\Lambda = \sum_{\alpha=1}^{3} X_\alpha(\theta) X_\alpha(\theta+2\pi) \leq 3$$

which can easily be checked in the system where one of the basic vectors is directed along $\vec{n}$. The eigenvectors of $\vec{n}, \vec{\eta}, \vec{\eta}^*$ which correspond to the eigen values 1, $e^{-2\pi i\nu}$, $e^{2\pi i\nu}$ are orthogonal if $\cos 2\pi\nu \neq 1$:

$$\vec{\eta}^2 = \vec{\eta}\vec{n} = 0 \quad ; \qquad \vec{\eta}\vec{\eta}^* = 2$$

The periodic solution of $\vec{n}$ is unique. In the case of the resonance

$$\cos 2\pi\nu = 1$$

there is a degeneracy

$$\lambda_1 = \lambda_2 = \lambda_3$$

and any solution is periodic.

From the two complex solutions we can construct a pair of real orthogonal solutions $\vec{\eta}_1, \vec{\eta}_2$; $\vec{\eta}_1 + i\vec{\eta}_2 = \vec{\eta}$, which, however, are not eigenvectors.

For future use it is convenient to utilize the periodic basis

$$\{\vec{e}_\alpha\} = \{\vec{e}_1, \vec{e}_2, \vec{n}\} ; \qquad \vec{e}_1 + i\vec{e}_2 = \vec{\eta} e^{i\nu\theta} \equiv \vec{e} \tag{II.10}$$

The general solution of (II.3) is written in the form:

$$\vec{\zeta} = \zeta_n \vec{n} + \frac{1}{2}(c\eta + c^*\eta^*) = \zeta_n \vec{n} + \zeta_\perp Re(\vec{e}\, e^{-i\Psi}) \tag{II.11}$$

$$\dot{\Psi} = \nu\omega_s , \quad \zeta_n = const , \quad c = const, \ |c| = \zeta_\perp = \sqrt{\zeta^2 - \zeta_n^2}$$

The simplest examples of particle spin motion on an equilibrium orbit are given in the Appendix.

Let us examine the question of the stability of the periodic solution of $\vec{n}$ for a slight variation in $\vec{W}_s$. A variation of $\delta\vec{W}_s$ may be linked both with the deviation of the real field and closed orbit from the ideal (calculated) ones, and with the variation in the parameters (for example, energy) which determine $\vec{W}_s$. In the linear approximation $\delta\vec{n}$ satisfies the equation

$$\frac{d}{dt}\delta\vec{n} = [\vec{W}_s\,\delta\vec{n}] + [\delta\vec{W}_s\cdot\vec{n}]$$

In this approximation $\delta\vec{n}$ is transverse to $\vec{n}$, i.e. we can write

$$\delta\vec{n} = Re\, c(\theta)\vec{\eta}$$

For $c(\theta)$ we obtain the equation:

$$\dot{c} = i\,\delta\vec{W}\vec{\eta}^*$$

Hence

$$c = i\int \delta\vec{W}_s\vec{e}^*e^{i\nu\theta}dt + const$$

By expanding $\delta\vec{W}_s\vec{e}^*$ into a Fourier series

$$\delta\vec{W}_s\vec{e}^* = \sum_k(\delta\vec{W}_s\vec{e}^*)_k\,e^{-ik\theta}$$

we obtain

$$\delta\vec{n} = Re\,\vec{e}\sum_k\frac{(\delta\vec{W}_s\vec{e}^*)_k}{(\nu-k)\omega_s}e^{-ik\theta} \qquad \text{(II.12)}$$

(const = 0 from the requirement of periodicty of $\delta\vec{n}$).

As can be seen, the periodic solution of $\vec{n}$ is very sensitive to a slight change in $\vec{W}_s$ in the region of the $\nu = K$ resonances. This is the physical meaning of the indeterminacy of $\vec{n}$ referred to above in the case of a precise resonance.

3. Spin motion equations for non-equilibrium particles

Let us turn to a study of the dynamics of spin motion in the case of particles moving near to a closed orbit.

We shall examine spin motion in relation to a <u>periodic</u> system of unit vectors (II.10).

For an angular velocity $\vec{W}_b$ (see 1.8) we obtain

$$\vec{W}_b = \vec{W}_s - \nu\omega_s \vec{n}$$

Consequently, the Hamiltonian in this system is

$$\mathcal{H} = (\vec{W}_n - \vec{W}_b, \vec{\zeta}) = (\nu\omega_s \vec{n} + \vec{w}, \vec{\zeta}) = \\ = (\nu\omega_s + \vec{w}\vec{n})\zeta_n + \zeta_\perp w_\perp \cos(\psi - \delta) \tag{II.13}$$

where $w_\perp e^{i\delta} = \vec{w}\vec{e}$ (see II.6; 1.10).

The spin motion equations in vector form and in Hamiltonian variables are of the form:

$$\dot{\vec{\zeta}} = [\nu\omega_s \vec{n} + \vec{w}, \vec{\zeta}] \tag{II.14}$$

$$\dot{\zeta}_n = -\frac{\partial \mathcal{H}}{\partial \psi} = w_\perp \zeta_\perp \sin(\psi - \delta)$$

$$\dot{\psi} = \frac{\partial \mathcal{H}}{\partial \zeta_n} = \nu\omega_s + \vec{w}\vec{n} - w_\perp \frac{\zeta_n}{\zeta_\perp}\cos(\psi - \delta) \tag{II.15}$$

When $\vec{w} = 0$, the equations (II.14, 15) coincide with (II.11) and describe spin motion on the equilibrium trajectory of a particle. The deviation of spin motion from (II.11) is entirely due to the deviation of the particle from the equilibrium orbit.

When investigating the dynamics of spin motion it is useful, both from the mathematical and physical stand-points, to make a comparison of the properties of spin and orbital motion. In many respects, there appears a qualitative and quantitative analogy between the fundamental properties of the dynamics of these degrees of freedom. It is already possible to make such a comparison in general terms.

Tha main characteristic of orbital motion is the equilibrium trajectory. One may compare with it the periodic trajectory of spin $\vec{n}(\theta)$. These trajectories are produced by special initial conditions. In practice, a very important factor is the existence of the periodic trajectory $\vec{n}$ on any closed particle orbit. To the particle oscillations around the equilibrium orbit there corresponds a spin motion along the periodic solution. Corresponding to the non-perturbed spin motion (rotation around $\vec{n}$) there are the "free oscillations" of the particle along the closed orbit, which are obtained in the linear approximation. In this sense $\zeta_\perp$ and the frequency ν play the same part as the amplitudes and frequencies of the normal linear oscillations of orbital motion. In particular, at a resonance $\nu = K$, the periodic solution $\vec{n}$ becomes indeterminate precisely in the same manner as, in the case of the resonance of betatron frequencies with the harmonics of the rotation frequency, the equilibrium orbit loses its determinacy.

If we know the ideal motion, there remains the problem of its stability when the perturbation $\vec{w}$ is included.

The methods of investigating the spin motion of non-equilibrium particles depend substantially on the order of magnitude of $\vec{w}$. As can be seen from (1.4), the relative order of perturbation is almost always determined by the relative scatter of momenta in the beam:

$$|\vec{w}|/|\vec{W}_s| \sim |\Delta\vec{p}|/p_s \qquad \text{(II.16}$$

The general condition for the "slowness" of the perturbation of spin motion is fulfilled during particle deviation from the equilibrium trajectory:

$$\left|\zeta_n(\theta+\theta') - \zeta_n(\theta)\right| \ll \mathcal{S} \quad \text{(see II.2)} \qquad 0 \le \theta' \le 2\pi \qquad \text{(II.17)}$$

and the equations (11.14, 15) can be solved approximately, using in general the same methods as when studying the action of small perturbations on the betatron and synchrotron oscillations of the particle.

As we know, in the solution of the question concerning the stability of the orbital motion a decisive part is played by the resonances between the frequencies of the ideal motion and frequencies of the perturbation. In precisely this way the stability of spin motion should be defined by the strength of the resonances between the frequency of precession ν and the spectral frequencies of perturbation. Basically, the criteria of the stability of spin motion should be the same as for the orbital case. However, it is necessary to bear in mind also the substantial difference in the dynamic properties of orbital and spin motion. As the Hamiltonian of spin motion is linear with respect to spin, resonances of the type

$$\ell \nu = \nu_{\{K\}}$$

($\nu_{\{K\}}$ any frequency from the spectrum $\vec{w}$), with $\ell > 1$ are impossible, i.e. the spectrum of perturbation does not depend on spin (disregarding the effect of spin on orbital motion).

For high energies, the evaluation of (II.16) may be violated at those points where the curvature of the equilibrium orbit is zero (see 1.4), whilst the gradient of the field transverse to the orbit is non-vanishing. In view of the anomalous part of q', the condition (II.17) will not be fulfilled in the limit $\gamma \to \infty$, or for a sufficient length of such a section. This phenomenon is linked with the violation of the similarity of spin trajectories in a magnetic field, which occurs in orbital motion: unlike the case of orbital motion, where the frequencies of motion over a fixed trajectory, when $\gamma \to \infty$, do not vary ($H, E \sim \gamma$), the anomalous part of the spin precession frequency increases in proportion to the energy.

If equation (II.17) is not fulfilled, the theory of perturbations cannot be applied to equations (II.14, 15) and they must be solved by other methods. The "fast" spin motion for different beam particles will be very different (see 1.4).

4. Radiation polarization

In accordance with papers /9-12/, the radiation during ultra-relativistic motion in a uniform magnetic field leads to polarization of the electrons and positrons along the magnetic field over a time $T_{c\partial n}$

$$T_{\text{сп}}^{-1} = \frac{15\sqrt{3}}{16}\gamma^2 \frac{\lambda}{R}\delta_{\text{рад}} = \frac{5\sqrt{3}}{8}\frac{\lambda}{R}\frac{r_0}{R}\gamma^5\omega_s \qquad \text{(II.18)}$$

where λ and r_0 are the Compton wave-length and conventional electron radius, R is the radius of the orbit δ rad $= -\frac{1}{\mathcal{E}}\frac{d\mathcal{E}}{dt}$ is the decrement of radiation losses.

The degree of equilibrium polarization

$$\frac{\zeta_H}{\zeta} = 2\zeta_n = \frac{8}{5\sqrt{3}} \qquad \left(\vec{n} = \frac{\vec{H}}{H}\right)$$

It is of interest to investigate how particles will be polarized during motion in an arbitrary periodic field. For this, let us use the equation for the average polarization for the overall system, during motion in an arbitrary external field taking into account the damping obtained in /12/:

$$\frac{d\vec{\zeta}}{dt} = [\vec{w}_s\vec{\zeta}] - \frac{1}{T}\left\{\vec{\zeta} - \frac{2}{9}\vec{v}(\vec{\zeta}\vec{v}) + \frac{4}{5\sqrt{3}}\frac{[\vec{v}\dot{\vec{v}}]}{|\dot{\vec{v}}|}\right\} \qquad \text{(II.19)}$$

where

$$\frac{1}{T} = \frac{5\sqrt{3}}{8}\alpha\frac{\hbar^2}{m^2}\gamma^5|\dot{\vec{v}}|^3 \; ; \quad \alpha = \frac{e^2}{\hbar} = 1/137$$

(the particle deviations from the equilibrium orbit can be disregarded if there is not any spin resonance).

Let us represent $\vec{\zeta}$ in the form (II.11), considering ζ_n and C to be dependent on time. After obtaining the equations for ζ_n and C they may be averaged over time in view of the small size of the radiation term. When $\nu \neq K$ this operation can be reduced to independent averaging over the phase ψ of spin rotation around $\vec{n}$ and over the period of particle motion.

After this we obtain the following equations:

$$\frac{d\zeta_n}{dt} = -\zeta_n \left\langle \frac{1}{T}\left(1-\frac{2}{9}n_v^2\right)\right\rangle + \frac{4}{5\sqrt{3}}\left\langle \frac{n_z}{T}\right\rangle \qquad \text{(II.20)}$$

$$\frac{dC}{dt} = -C\left\langle \frac{1}{9T}\left(8+n_v^2\right)\right\rangle$$

where

$$n_v = \vec{n}\vec{v}\ , \quad n_z = \vec{n}\frac{[\vec{v}\dot{\vec{v}}]}{|\dot{\vec{v}}|}$$

For the equilibrium polarization we obtain:

$$2\vec{\zeta} = 2\zeta_n\vec{n} = \frac{8}{5\sqrt{3}}\frac{\langle |\dot{\vec{v}}|^3 n_z\rangle}{\langle |\dot{\vec{v}}|^3\left(1-\frac{2}{9}n_v^2\right)\rangle}\vec{n}(t) \qquad \text{(II.21)}$$

As was to be expected, the average polarization over the particle beam is directed along the periodic solution of $\vec{n}(\theta)$. The degree of equilibrium polarization is reduced, whilst its actual value depends essentially on $\vec{n}(\theta)$.

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APPENDIX

Let us examine two model examples, where the necessary spin orientation in relation to velocity and field can be produced for a slight deformation of the orbit.

1. For high energies ($\gamma q' \gg q_0$) the following method may be proposed. Let us assume that on a flat equilibrium orbit there are gaps without fields. If we introduce a radial magnetic field H_r in the gap (or a vertical electrical field E_z), the spin may rotate around a radial direction through an angle of ~ 1 for a small deformation of the particle orbit ($\delta p_\perp / p \sim q_0/\gamma q' \ll 1$). Let us change to the system of unit vectors, linked with the undistorted equilibrium orbit ($\vec{e}_r$, $\vec{e}_v$, $\vec{e}_z$). In this system, the angular velocity of spin rotation has the form:

$$\vec{W}(\theta) = -q'\vec{H} = \begin{cases} -q' H_z \vec{e}_z & 0 < \theta < \theta_0 \\ -q' H_r \vec{e}_r & \theta_0 < \theta < 2\pi \end{cases} \qquad (1)$$

The general solution of $\vec{\zeta}(\theta)$ with the initial condition $\vec{\zeta}(0) = \vec{\zeta}^0$ has the form ($0 \leq \theta \leq 2\pi$):

$$\zeta_r = \zeta_r^0 \cos \Phi_\theta - \zeta_v^0 \sin \Phi_\theta$$

$$\zeta_v = (\zeta_r^0 \sin \Phi_\theta + \zeta_v^0 \cos \Phi_\theta)\cos \varphi_\theta - \zeta_z^0 \sin \varphi_\theta \qquad (2)$$

$$\zeta_z = (\zeta_r^0 \sin \Phi_\theta + \zeta_v^0 \cos \Phi_\theta)\sin \varphi_\theta + \zeta_z^0 \cos \varphi_\theta$$

where
$$\Phi_\theta = \int_0^\theta \vec{W}\vec{e}_z \frac{d\theta}{\omega_s} \quad ; \quad \varphi_\theta = \int_0^\theta \vec{W}\vec{e}_r \frac{d\theta}{\omega_s}$$

From the requirement of periodicity $\vec{\zeta}(2\pi) = \vec{\zeta}^0$, we obtain the initial conditions for the periodic solution of $\vec{n}(\theta)$:

$$\frac{\zeta_z^c}{\zeta_r^c} = -\,ctg\,\frac{\Phi}{2} \quad , \quad \frac{\zeta_z^c}{\zeta_r^c} = -\,ctg\,\frac{\varphi}{2}$$

where

$$\Phi \equiv \Phi_{2\pi} \quad , \quad \varphi \equiv \varphi_{2\pi}$$

In this way the periodic solution of $\vec{n}$:

$$\vec{n}_{0,\theta_0} = A\left\{\left[\vec{e}_z\cos(\Phi_\theta - \frac{\Phi}{2}) + \vec{e}_z\sin(\Phi_\theta - \frac{\Phi}{2})\right]\sin\frac{\varphi}{2} + \vec{e}_z\sin\frac{\Phi}{2}\cos\frac{\varphi}{2}\right\}$$

$$\vec{n}_{\theta_0,2\pi} = A\left\{\left[\vec{e}_z\cos(\varphi_\theta - \frac{\varphi}{2}) - \vec{e}_c\sin(\varphi_\theta - \frac{\varphi}{2})\right]\sin\frac{\Phi}{2} + \vec{e}_z\cos\frac{\Phi}{2}\sin\frac{\varphi}{2}\right\} \qquad (3)$$

$$A = \left(1 - \cos^2\frac{\Phi}{2}\cos^2\frac{\varphi}{2}\right)^{-\frac{1}{2}}$$

In order to determine the frequency ν , let us construct a normalized solution of $\vec{n}$ transverse to $\vec{\eta}$:

$$\vec{\eta}_{0,\theta_0} = \frac{\dot{\vec{n}}}{|\dot{n}|} = \vec{e}_z\sin(\Phi_\theta - \frac{\Phi}{2}) - \vec{e}_r\cos(\Phi_\theta - \frac{\Phi}{2})$$

$$\vec{\eta}_{\theta_0,2\pi} = \alpha\dot{\vec{n}} + \beta[\vec{n}\dot{\vec{n}}] = \vec{e}_z\sin\frac{\Phi}{2} - (\vec{e}_r\cos\varphi_\theta + \vec{e}_z\sin\varphi_\theta)\cos\frac{\Phi}{2}$$

The constants α and β are found from the condition of continuity of $\vec{\eta}$ when $\theta = \theta_0$.

By definition:

$$\cos 2\pi\nu = \vec{\eta}(0)\vec{\eta}(2\pi) = 2\cos^2\frac{\Phi}{2}\cos^2\frac{\varphi}{2} - 1 \qquad (4)$$

From this it can be seen that the resonances $\nu = K$ are possible only when

$$\Phi = 2K_1\pi \quad , \quad \varphi = 2K_2\pi$$

i.e. subject to the periodicity of spin motion independently in each of the two sections.

From (3) it can be seen that by varying ϕ and φ it is possible to produce any polarization direction at the necessary point of the orbit.

The simplest example is the case when, on the section of the leading field $0 < \theta < \theta_0$ an equilibrium polarization is directed along the field. As can be seen from (3), this necessitates $\varphi = 0$, i.e. the average radial field in the gap is zero. Then, by changing the value of $H_z(\theta)$ it is possible in this case to produce an equilibrium polarization at any angle in relation to the velocity (including longitudinal). The degree of equilibrium radiation polarization of electrons and positrons is reduced insignificantly if $2\pi - \theta_0 \ll 2\pi$:

Let us examine the case when, in the gap, the magnetic field is directed along $\vec{e}_v$:

$$\vec{W}(\theta) = \begin{cases} -q' H_z \vec{e}_z & 0 < \theta < \theta_0 \\ -\dfrac{q}{\gamma} H_v \vec{e}_v & \theta_0 < \theta < 2\pi \end{cases} \tag{5}$$

The equilibrium orbit is not distorted.

The periodic solution in this case is obtained from (3) by substituting $\vec{e}_r \longrightarrow \vec{e}_v$, $\vec{e}_v \longrightarrow -\vec{e}_r$:

$$\begin{aligned} \vec{n}_{0,\theta_0} &= A\left\{\left[\vec{e}_v \cos\left(\phi_\theta - \frac{\phi}{2}\right) - \vec{e}_z \sin\left(\phi_\theta - \frac{\phi}{2}\right)\right]\sin\frac{\varphi}{2} + \vec{e}_z \sin\frac{\phi}{2}\cos\frac{\varphi}{2}\right\} \\ \vec{n}_{\theta_0,2\pi} &= A\left\{\left[\vec{e}_z \cos\left(\varphi_\theta - \frac{\varphi}{2}\right) + \vec{e}_z \sin\left(\varphi_\theta - \frac{\varphi}{2}\right)\right]\sin\frac{\phi}{2} + \vec{e}_v \cos\frac{\phi}{2}\sin\frac{\varphi}{2}\right\} \end{aligned} \tag{6}$$

here $$\phi_\theta = \int_0^\theta \vec{W}\vec{e}_v \frac{d\theta}{\omega_s} \quad ; \qquad \phi = \varphi_{2\pi} = -\frac{q}{\gamma}\int_{\theta_0}^{2\pi} H_v \frac{d\theta}{\omega_s}$$

A special feature of this example is that when

$$\phi = 2K\pi \tag{7}$$

the velocity polarization (for the periodic solution) is effected over the whole length of the gap. For energies $\gamma \lesssim q_c / q'$, the condition (7) can, in practice, be satisfied only for specific energy values. For large values, under the condition (7) it is always possible to "get by" by introducing an additional magnetic field along $\vec{e}_z$ in the basic section. We note that without a longitudinal field in the gap, (7) would imply a resonance, and such spin motion would be unstable. The inclusion of the longitudinal field displaces the resonance, as can be seen from (4). For stability, we require only:

$$|\omega_s \varphi| = \left| \frac{q}{\gamma} \int_{\theta_0}^{2\pi} H_v \, d\theta \right| \gg |\vec{w}_K|$$

where $|\vec{w}_K|$ is the value of the resonance Fourier-harmonic of perturbation $\vec{w}_\perp$ (see II.12 - II.15).

As can be seen from (II.20) in the case of electrons and positrons, the radiation in this case leads to the disappearance of the initial polarization over a time $\sim T$ (II.18).

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