INSTITUTE OF NUCLEAR PHYSICS,
USSR ACADEMY OF SCIENCES, SIBERIAN DIVISION

Report IYaF 44-70

DYNAMICS OF PARTICLE POLARIZATION NEAR THE SPIN RESONANCES

Ya.S. Derbenev, A.M. Kondratenko and A.N. Skrinskij

Novosibirsk 1970

Translated at CERN by R. Luther
(Original: Russian)
Not revised by the Translation Service

(CERN Trans. 74-4)

Geneva
April 1974

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ABSTRACT

The report considers the effect of particle deviations from the equilibrium orbit on spin motion in the storage ring. The methods and results of reports /1-6/ are applied to the case of an arbitrary closed orbit /7/. In addition to first-approximation resonances, resonances of higher orders are also investigated. The case of overlapping resonances is considered.

The paper is mainly devoted to resonance crossing. A complete solution to the single crossing problem is given. On this basis, by making use of the general nature of spin motion in a periodic field, as established in /7/, the problem of periodic resonance crossing is raised and solved.

1. INTRODUCTION

The present report concerns the study of the dynamics of particle polarization near the spin resonances. It is known /1-7/ that the spin motion of a particle in the storage ring becomes unstable when the spin precession frequency on the particle's equilibrium trajectory is close to any combination of the orbital motion frequencies. Owing to the spread of the particle trajectories, the beam's initial degree of polarization may be substantially reduced. This phenomenon may cause problems for experiments with polarized beams; on the other hand, it may be used for intentional depolarization.

The report examines stationary resonances and resonance crossing. There are a considerable number of reports /1-6/ dealing with polarization dynamics near resonances when the unperturbed spin motion takes the form of a precession around a fixed axis (flat equilibrium orbit).

The report discusses a generalized case of an arbitrary closed orbit where the direction of the equilibrium particle's spin precession axis is a periodic function of the azimuth /7/. All the methods and qualitative results of reports /1-6/ may be transposed to the general case without any great changes. Apart from the first-approximation resonances, the higher-order resonances are also examined first.

The case of overlapping resonances of substantially differing strengths is considered in section 5. This problem may be solved by a modified form of averaging (modulation resonances).

The major part of the report is devoted to the study of the important problem of resonance crossing. Single crossing was first examined in /1/ for a constant crossing rate and a special initial condition of polarization in terms of the field far from the resonance. For an arbitrary initial condition, the result is known only for fast crossing /1, 4-6/. The methods we use for dealing with the single crossing problem provide results of a more general nature than those of /1/, and a solution is obtained for an arbitrary initial condition. On this basis, by making use of the general nature of the spin motion in a periodic field /7/, a solution is also found to the problem of periodic resonance crossing. This solution sheds light on the conditions of beam depolarization at periodic crossings.

The case of heavily overlapping resonances in an equidistant perturbation spectrum forms an important systematic application of the problem.

2. Basic equations

The motion of the spin vector is described by the equation /7-9/ :

$$\frac{d\vec{J}}{dt} = [\vec{W}_n \vec{J}]$$

$$\vec{W}_n = \left(1+\gamma\frac{q'}{q_0}\right)\frac{[\vec{v}\dot{\vec{v}}]}{v^2} - \frac{q}{\gamma}\frac{(\vec{H}\vec{v})\vec{v}}{v^2} + \frac{q}{\gamma^2 v^2}[\vec{v}\vec{E}] \qquad (2.1)$$

where $q = q_0 + q' = \frac{e}{m} + q'$ is the gyromagnetic ratio, q' is its anomalous part, $\gamma = (1-v^2)^{-1/2}$ $(c=1)$ and $\vec{v}, \dot{\vec{v}}$ – are the velocity and acceleration of a particle moving in an electromagnetic field $\vec{E}$, $\vec{H}$. A specific property of particle motion in storage rings (accelerators) is the small amount of deviation from the equilibrium trajectory. Let us represent $\vec{W}_n$ in the form of a sum

$$\vec{W}_n = \vec{W}_S(\gamma_S, \theta) + \Delta\vec{W}$$

(θ is the particle's azimuth), where $\vec{W}_S$ is the $\vec{W}_n$ value on the equilibrium (closed) orbit. $\vec{W}_S$ possesses periodicity :

$$\vec{W}_S(\gamma_S, \theta) = \vec{W}_S(\gamma_S, \theta + 2\pi)$$

In our previous report /7/, it was shown that the solution of the equation on an arbitrary equilibrium trajectory

$$\dot{\vec{J}} = \frac{d\vec{J}}{d\theta} = \frac{1}{\omega_s}\left[\vec{W}_s, \vec{J}\right] \tag{2.1a}$$

(W_s is the particle's equilibrium rotation frequency) takes the form

$$\vec{J} = J_{\vec{n}}\vec{n} + \frac{1}{2}\left(c\vec{\eta} + c^*\vec{\eta}^*\right)$$

where $\vec{n}, \vec{\eta}, \vec{\eta}^*$ are the intrinsic orthogonal solutions (2.1a) with the properties: (γ_s = const) :

$$\begin{gathered}
\vec{n}(\theta) = \vec{n}(\theta + 2\pi) \\
\vec{\eta}(\theta + 2\pi) = e^{-2\pi i \nu}\vec{\eta}(\theta) \\
\vec{\eta} = \vec{\eta}_1 + i\vec{\eta}_2 \qquad \vec{\eta}_1^{\,2} = \vec{\eta}_2^{\,2} = \vec{n}^{\,2} = 1 \\
\nu = const
\end{gathered} \tag{2.2}$$

$2\pi\nu$ means the spin flip angle per period of particle motion around the periodic solution $\vec{n}$. In the moving periodic system of unit vectors

$$\begin{gathered}
\vec{e}_1, \ \vec{e}_2, \ \vec{n} \\
\vec{e}_1 + i\vec{e}_2 = \vec{\eta}\, e^{i\nu\theta} \equiv \vec{e}
\end{gathered} \tag{2.3}$$

the spin satisfies the equation

$$\begin{gathered}
\dot{\vec{J}} = \left[\vec{W}\, \vec{J}\right] \\
\vec{W} = \nu\vec{n} + \vec{w} = \nu\vec{n} + \frac{\vec{W}_1'}{\dot{\theta}} - \frac{\vec{W}_s}{\omega_s}
\end{gathered} \tag{2.4}$$

(The particle's azimuth θ serves as the time). Equation (2.4) is equivalent to (2.1).

The spin equations in vector form, being geometrically visualizable, are a system of third-order differential equations. To solve them directly, it is sometimes advisable to reduce the order of the equation, which can be done owing to the existence of the motion integral $\vec{J}^2$ = const. In some cases use is made of the variable projections $J_{\vec{n}} \equiv \vec{J}\vec{n}$ and the spin's rotation phase Ψ about $\vec{n}$. The Hamilton equations for $(J_{\vec{n}}, \Psi)$ are non-linear.

A second-order linear system may be obtained for the two complex variables χ_+, χ_- which we shall write in the form :

$$\chi = \begin{pmatrix} \chi_+ \\ \chi_- \end{pmatrix} . \quad \chi^+ = (\chi_+^*, \chi_-^*) \qquad (2.5)$$

For spin s = ½ the variable χ is a spin wave function which satisfies the Schrödinger equation :

$$\begin{aligned} i\dot{\chi} &= \tfrac{1}{2}\vec{\sigma}\vec{W}\chi \\ \chi^+\chi &= 1 \end{aligned} \qquad (2.6)$$

where $\vec{\sigma}$ are the Pauli matrices :

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$\chi^+\vec{\sigma}\chi = \vec{J} \qquad (\vec{J}^2 = 1) \qquad (2.7)$$

Since the vector equations are linear and do not depend on the value of spin S , transform (2.7) also leads to equation (2.6) for an arbitrary spin. The meaning of variables χ is as follows :

$|\chi_+|^2 - |\chi_-|^2 = J_z$, and the phase difference χ_- and χ_+ is the phase ψ of the spin precession around axis z .

We should point out that rotation matrix $O\ (\theta)$

$$\vec{\zeta}(\theta) = O(\theta)\ \vec{\zeta}(0) \qquad (2.8)$$

not only tells one that $\langle \hat{\vec{S}} \rangle = \vec{\zeta}\ (\theta)$ but also gives one a complete physical description. By using $O\ (\theta)$, the wave function of arbitrary spin $\chi_S\ (\theta)$ may be written in the form :

$$\chi_S(\theta) = e^{-i\hat{\vec{S}}\vec{\alpha}(\theta)}\ \chi_S(0)$$

where $\vec{\alpha}\ (\theta)$ is the rotation angle vector (2.8).

In the case of an arbitrary spin S , equation (2.6) may be interpreted as an equation for the wave functions χ_a (a = 1,2 . . . 2) of 2 S independent particles from which a system with spin S may be formally composed /16/.

The possibility of switching from vector equations to "Pauli equations" (2.6) is, in fact, not related to the spin's quantum nature. Generally speaking, transform (2.7) represents a spinor of first rank of a group of rotations in three-dimensional space, and it may be used to describe the rotation of any type of vector.

In the case of complex variables, the formulae obtained in /7/ for vector-type equations are the simplest to use for the periodic solution of spin motion in fields with the periodic dependence $\vec{W}\ (\theta)$. Let us assume that we know matrix Λ , which transforms the solution at $\theta = 0$ χ_0 to time T :

$$\chi_T = \Lambda \chi_0$$

On account of unitarity ($\Lambda^+\Lambda = I$) the matrix Λ takes the form :

$$\Lambda = e^{-i\pi\vec{\sigma}\vec{n}_0 \nu} = \cos\pi\nu - i\vec{\sigma}\vec{n}_0 \sin\pi\nu \qquad (2.9)$$

$\vec{n}_0$ and ν signify the following : $\vec{n}_0$ is the direction about which the spin vector $\vec{\zeta}_0$ must be turned through angle

$2\pi\nu$ in order to obtain a solution in period $\vec{J}_T$. Hence $\vec{n}_o$ obviously coincides with the periodic solution $\vec{n}$ (θ) at instants $\theta = 0, T$ . . . . . . , and $2\pi\nu$ is the angle through which the solution diametrically opposed to $\vec{n}$ turns in a period. Clear expressions for $\vec{n}_o$ and ν may be obtained from (2.9) :

$$\cos\pi\nu = \frac{1}{2}\, Sp\Lambda \quad ; \quad \vec{n}_o = \frac{i}{2\sin\pi\nu}\, Sp\vec{\sigma}\Lambda \qquad (2.10)$$

In the case of integer ν , $\vec{n}_o$ is completely undefined, and any solution is periodic.

This report deals with the study of the effect of $\vec{w}$ perturbation, caused by the deviation of a particle from the equilibrium trajectory, on ideal spin motion (at $\vec{w} = 0$). We would point out that at arbitrary $\vec{n}(\theta)$ this problem is basically the same as the well-studied case of a magnetic field which is almost constant in direction /1-6/ ($\vec{n}$ (θ) = const.). As at $\vec{n}$ = const., $\vec{w}$, owing to its smallness, may lead to a substantial variation of spin motion only near a resonance (the closeness of ν to any frequency in the $\vec{w}$ perturbation spectrum).

3. First-approximation resonances

The part played by spin resonances may be most clearly expressed in the following way.

Let us write solution (2.4) as a first approximation of the conventional perturbation theory,

$$J_{\pi} = J_{\pi}^{\circ} + Re\; i(\vec{J}^{\circ}\vec{e}) \int_0^{\theta} \vec{w}\vec{e}\cdot e^{-i\nu\theta} d\theta \qquad (3.1)$$

Let us express the integral in the form of a sum

$$\begin{aligned} \vec{w}\vec{e}\cdot e^{-i\nu\theta} &= \sum_{K} \omega_k\, e^{i(\nu_{\{K\}}-\nu)\theta} \\ \nu_{\{K\}} &= \sum_i K_i \nu_i \end{aligned} \qquad (3.2)$$

($\cdot\, \nu_i \omega_s$ is the particle's motion frequency and K_i are whole numbers). By substituting into (3.1) and integrating we obtain

$$\begin{aligned} \zeta_{\vec{n}} &= \overline{\zeta_{\vec{n}}} + Re(\vec{\zeta}^{\circ}\vec{e}\,)\sum_{K} \frac{\omega_K}{\nu_{\{K\}}-\nu}\, e^{i(\nu_{\{K\}}-\nu)\theta} \\ \overline{\zeta_{\vec{n}}} &= \zeta^{\circ}_{\vec{n}} - Re(\vec{\zeta}^{\circ}\vec{e}\,)\sum_{K} \frac{\omega_K}{\nu_{\{K\}}-\nu} \end{aligned} \tag{3.3}$$

It is clear from the form of solution (3.3) that it may be used for times $\theta > \frac{1}{|\vec{w}\vec{e}\,|}$ under the condition

$$|\nu - \nu_{\{K\}}| >> |\omega'_K|$$

and $\zeta_{\vec{n}}$ oscillates at low amplitude near $\overline{\zeta_{\vec{n}}}$.

However, in the resonance region

$$|\nu - \nu_{\{K\}}| \lesssim |\omega'_K| \tag{3.4}$$

the conventional perturbation theory is applicable only at

$$\theta \ll \frac{1}{|\omega_K|} \tag{3.5}$$

Thus, the spin behaviour at large time values will be determined by the resonance harmonics which fulfil condition (3.4).

Let us consider the case of an isolated resonance, when (3.4) is fulfilled only for one harmonic $K = K_0$ and the other resonance harmonics are so small that (3.5) is fulfilled. In this case, a solution which is also suitable for times $\theta \gtrsim \frac{1}{|\omega_{K_0}|}$ may be obtained by averaging.

Let us introduce the slow phase

$$\varphi = \psi - \psi_{K_0} \equiv \psi - \nu_{\{K_0\}}\theta \tag{3.6}$$

This signifies a transition to the "resonance" system of co-ordinates rotating about $\vec{n}$ with a frequency $\nu\{K_0\}$ in relation to system (2.3). The new vectors $\vec{\ell}' = \vec{\ell}_1' + i\vec{\ell}_2'$ are connected to unit vectors $\vec{\ell}$ by the relation

$$\vec{\ell} = \vec{\ell} \cdot \ell^{-i\Psi_{K_0}}$$

In this system :

$$\dot{\vec{\zeta}} = [\vec{w}'\vec{\zeta}] \qquad \vec{w}' = (w_1' \; w_2' \; w_3') \tag{3.7}$$

where

$$w_3' = \nu - \nu_{\{K_0\}} + \vec{u}\vec{n} \qquad w_1' + i w_2' = \vec{w}\vec{e}' = \vec{w}\vec{e}\cdot e^{-i\Psi_{K_0}}$$

Equations (3.7) may be solved by an averaging method. In the first approximation, the averaged equations have the form :

$$\dot{\vec{\zeta}} = [\vec{h}\,\vec{\zeta}]$$
$$\vec{h} = \langle \vec{w}' \rangle = const \tag{3.8}$$
$$h_1 + i h_2 = \langle \vec{w}\vec{e}\, e^{-i\Psi_{K_0}} \rangle = \omega_{K_0} \qquad h_3 = \langle w_3' \rangle \equiv \varepsilon$$

Consequently, the averaging method is equivalent to the rejection of all harmonics except the resonance one, and the problem is reduced to motion in a constant "field" $\vec{h}$. The solution is obvious: the spin in the "resonance" system slowly precesses about $\vec{h}$ with a constant angular velocity

$$h = \sqrt{\varepsilon^2 + |\omega_{K_0}|^2} \ll 1$$

The precession axis makes an angle $\vec{n}$ with α which is equal to $arctg \frac{|\omega_{K_0}|}{\varepsilon}$

The spin motion appears as follows from system (2.3) : the spin rotates rapidly with a frequency $\nu\{K_0\}\ \varepsilon^2 - |\omega_{K_0}|/\varepsilon^2 + |\omega_{K_0}|^2$, and slowly alters the $\vec{n}$ component. The rotation taper slowly oscillates with frequency h about a mean value 2α with an amplitude dependent on the initial conditions. When the initial polarization is towards $\vec{n}$, the $J_{\vec{n}}$ component oscillates from 1 to $\varepsilon^2-|\omega_{K_0}|^2/\varepsilon^2+|\omega_{K_0}|^2$. At a precise resonance ($\varepsilon = 0$) after a time π/h, the spin flips. As is to be expected (cf. 3.3), the spin motion differs substantially from the non-resonance case only at $\varepsilon \sim |\omega_{K_0}|$. In this sense, the strength of the resonance $|\omega_{K_0}|$ at the same time defines the width of the resonance.

4. Higher-order resonances

As we had expected, in a first order approximation the spin motion may be considerably perturbed only near resonances

$$\nu \approx \nu_{\{K\}}$$

where $\nu_{\{K\}}$ is any frequency from the spectrum $\vec{W}\vec{e}$. If there are no first-approximation resonances, then according to the averaging method it may be stated that the spin component will last until times

$$\theta \ll \frac{1}{|\vec{W}|^2\,\theta_{xap}} = \theta_{max}$$

where $\theta_{xap.}$ is the characteristic "fast" time for the variation of $\vec{W}$. As for the spin behaviour at $\theta \gtrsim \theta_{max.}$, averaged equations allowing for the higher orders must be compiled. Then combination resonances will also be taken into account :

$$\nu = \pm\nu_k \pm \nu_{k'} \pm \dots + \varepsilon \equiv \nu_p + \varepsilon \qquad (4.1)$$

As before, in order to study spin behaviour near resonance ν_p, we shall switch to the resonance system rotating with frequency ν_p about $\vec{n}$ (viz. 3.6) and shall compile averaged equations of the required accuracy.

A matrix description of the spin equations proves convenient for this purpose

$$\dot{J} = W' J$$

where $\mathcal{J}$ is the column from the components of vector $\vec{\mathcal{J}}$ and w' is the matrix

$$w'_{ik} = \varepsilon_{i\alpha k}\, w'_{\alpha}$$

For short times ($T\,|\vec{w}'| \ll 1$) the solution for $\mathcal{J}$ at the instant θ is written in the form of a series :

$$\mathcal{J}_{\theta} = (I + \lambda)\, \mathcal{J}_{0}$$
$$\lambda = \widetilde{w'} + \widetilde{w'\widetilde{w'}} + \dots \tag{4.2}$$

where

$$\widetilde{w}' = \int_0^{\theta} w' d\theta$$

In our case the averaging method indicates the location of a constant effective "field" $\vec{h}$, such that the solution :

$$\mathcal{J}_T = e^{hT}\, \mathcal{J}_0$$

of the averaged equation

$$\dot{\mathcal{J}} = h\, \mathcal{J} \tag{4.3}$$

coincides in time T with (4.2)

$$e^{hT} = I + \lambda$$

hence

$$h = \frac{1}{T} \ln (I + \lambda) = \frac{1}{T}\left(\lambda - \frac{\lambda^2}{2} + \frac{\lambda^3}{3} - \dots\right) \tag{4.4}$$

Here we confine ourselves to plotting $\vec{h}$ accurately up to the fourth order. By using (4.2) we obtain from (4.3)

$$h = \left\langle w' - \frac{1}{2}[\widetilde{w}' w'] - \frac{T}{4}\left[\langle w' \rangle [\widetilde{w}' w']\right] + \frac{1}{3}\left[\widetilde{w}' [\widetilde{w}' w']\right] \right\rangle$$

$[A,B]$ denotes the commutator of matrices A and B. In vector form, the matrix commutator is replaced by the vector product :

$$\vec{h} = \langle \vec{w}' - \frac{1}{2} [\widetilde{\vec{w}}' \vec{w}'] - \\ - \frac{T}{4}\left[\langle \vec{w}' \rangle [\widetilde{\vec{w}}' \vec{w}']\right] + \frac{1}{3}\left[\widetilde{\vec{w}}' [\widetilde{\vec{w}}' \vec{w}']\right] \rangle \tag{4.5}$$

Time T must take into account the relations:

$$\theta_{xap} \ll T \ll \frac{1}{|\vec{w}'|} \qquad (4.6)$$

Under this condition $\vec{h}$ does not depend on T and the brackets $\langle \dots \rangle$ correspond to the definition of an average value.

Solution (4.3) will obviously differ slightly from the accurate solution up to times

$$\theta \ll \frac{1}{|\vec{w}'|} \frac{1}{|\vec{w}'\theta_{xap}|^3} = \theta_{max} \qquad (4.7)$$

If $\vec{w}'(\theta)$ is periodic, then $\vec{h}$ for times $\theta \ll \theta_{max}$ will coincide in direction with the corresponding periodic solution $\vec{m}$ at instants $0, T, 2T \dots$ (in this case T may be identified with the period of variation of $\vec{w}'$). Between these instants solution (4.3) differs from the exact solution by a first-order correction (hence the dependence of direction $\vec{h}$ on the selection of $\theta = 0$: $|\Delta \vec{m}| \sim hT$). Similarly, in the general case, under condition (4.6), T is an approximate period of variation of $\vec{w}'$. The value h defines the frequency of the spin's rotation about the periodic solution (and, therefore, does not depend on when the reading begins).

Let us discuss the types of resonance that are possible in the following approximations. By resonance we mean a situation where the direction of the mean axis of precession $\vec{m}$ differs substantially from $\vec{n}$, i.e. at

$$|h_x + i h_y| \gtrsim |\vec{h}\vec{n}|$$

As we have seen, only the following resonances are possible in the first order approximation:

$$\nu \approx \nu_{\{K\}}$$

where $\nu_{\{K\}}$ is a frequency from the spectrum of the transverse part of $\vec{w}$. It follows from (4.5) that the following resonances are possible in the second order approximation.

$$\nu \approx \nu_e + \nu_K$$

(Here the subscripts "ℓ" and "K" denote frequencies from the spectra $\vec{n}\vec{w}$ and $\vec{w}\vec{\ell}$ respectively). Consequently, second-order resonances occur only as a result of the correlation between the transverse perturbation and oscillations of the precession frequency. At constant $\vec{w}\vec{n}$ ($\nu_e = 0$), the second approximation dose not produce a new resonance but rather forms a correction to the first.

In the third-order approximation, combination resonances are possible :

$$\begin{aligned} \nu &\approx \nu_e + \nu_{e'} + \nu_K \\ \nu &\approx \nu_K + \nu_{K'} - \nu_{K''} \end{aligned} \tag{4.7}$$

There are usually enough of these approximations. To complete the picture, we shall point out a simple rule for the selection of frequency combinations for a resonance of arbitrary order. The general condition for an n-th order resonance is : a given combination of n frequencies of the spectra $\vec{w}\vec{n}$ and $\vec{w}\vec{\ell}$ (some of them may be the same) must be close to ν . In the resonance system this condition may be expressed as :

$$\nu_{e_1} + \nu_{e_2} + \dots + \nu_{e_{n_\parallel}} + S_1(\nu_{K_1} - \nu) + \dots + S_{n_\perp}(\nu_{K_{n_\perp}} - \nu) \approx 0 \quad .$$

$$S_i = \pm 1 \qquad\qquad n_\parallel + n_\perp = n$$

(The choice of sign to precede ν_ℓ may be included in subscript ℓ , since frequencies ν_ℓ form a spectrum of real value $\vec{w}\vec{n}$). Since only linear resonances are possible for the spin system, then we must have :

$$S_1 + \dots + S_{n_\perp} = 1 \tag{4.8}$$

Hence, $n_\perp$ is odd. Thus, the following nth-order resonances are possible

$$\nu \approx \nu_{\ell_1} + \ldots + \nu_{\ell_{n_\parallel}} + S_1 \nu_{K_1} + \ldots + S_{n_\perp} \nu_{K_{n_\perp}} \qquad (4.9)$$

in conformity with condition (4.8). Sometimes, nth-order resonances amount to a correction of lower resonances. It is then necessary to separate out the low-value part from the full combination; the number of frequencies ν_K in the subcombination with positive ($S_i > 0$) signs must be the same as that with negative ($S_i < 0$) signs.

For example, third-order resonances (4.7) may be reduced to corrections of first-order and second-order resonances if $\nu_\ell \gtrsim 0$, $\nu_\ell + \nu_{\ell'} \approx 0$ or $\nu_{K'} - \nu_{K''} \approx 0$.

5. Modulation resonances

The above averaging method may be applied directly to the case of a single resonance, i.e. when the condition

$$|\nu - \nu_p| \lesssim |\vec{w}| \qquad (5.1)$$

is fulfilled only for one combination ν_p. After averaging over the remaining fast oscillations, we obtain the motion in the effective constant field correctly describing the spin motion at large time values. If condition (5.1) satisfies several combinations of ν_p , then after averaging over the fast oscillations, the time dependence of the effective field $\vec{h}$ will have a frequency spread of order of the value h :

$$\vec{h} = \vec{h}_0 + \vec{\Delta}(\theta)$$

$$\vec{\Delta}(\theta) = \sum_{m \neq 0} \Delta_m e^{i\Omega_m \theta} \qquad (\Omega_m \lesssim h) \qquad (5.2)$$

$$\vec{h}_0 = (h_\perp, 0, \varepsilon)$$

where $h_\perp$ and ε are the width and frequency difference of the selected (main resonance, and Δ_m defines the strength of the other resonances. Ω_m is the distance between the main and other resonances. This problem may be solved in certain cases of practical significance.

In this section we shall consider the case of overlapping resonances with substantially differing strengths. Let us assume that $\Delta_m \ll \Omega_m$. The averaging method may also be used to solve this problem. In the zero approximation ($\vec{\Delta} = 0$), there is an isolated main resonance and the spin precesses around $\vec{h}_o$. By following for $\vec{\Delta}$, the motion is considerably distorted only near (in the first order approximation) the modulation resonances :

$$\Omega_m \approx h_o = \Omega_m + \varepsilon_m$$

In the higher order approximations combination resonances are also possible.

Let us take as an example the case of periodic modulation (for instance, synchrotron oscillations of energy near the spin resonance at fast frequencies). In this case,

$$\vec{\Delta}(\theta) = \vec{\Delta}_\gamma \sin \Omega_\gamma \theta$$
$$\vec{\Delta}_\gamma = const$$

Ω_γ is the synchrotron oscillation frequency. In the first approximation the possible resonances are

$$h_o \approx \Omega_\gamma \qquad\qquad h_o = \Omega_\gamma + \varepsilon_\gamma$$

The strength of this resonance

$$\omega_\gamma = \frac{1}{2h_o} \left| [\vec{\Delta}_\gamma \vec{h}_o] \right| \tag{5.3}$$

is determined by the part $\vec{\Delta}_\gamma$ which is transverse to $\vec{h_o}$. The spin motion takes place as follows. In a system rotating about $\vec{h_o}$ with frequency Ω_γ, the spin slowly precesses with frequency $\sqrt{\varepsilon_\gamma^2 + \omega_\gamma^2}$ about a direction which forms the following angle with $\vec{h_o}$:

$$\alpha = \operatorname{arctg} \frac{\omega_\gamma}{\varepsilon_\gamma}$$

Resonances $h_o \approx K\Omega_\gamma$ occur in the K-th approximation of the averaging method.

This method gives results which differ substantially from the isolated resonance theory only at $|\varepsilon| \lesssim h_\perp$. In the opposite case ($|\varepsilon| \gg h_\perp$) the subsidiary resonances are actually separated from the main resonance and may be considered as independent, isolated resonances.

The case of periodic modulation at $\Delta \gg \Omega$, when many resonances with the same strength overlap, requires a different approach. It is better to tackle this problem by assuming repeated crossings of the "main" resonance rather than an overlapping of separate resonances.

6. Single resonance crossing

Let us now examine the problems of single and multiple (periodic) crossing of the spin resonances. Both these problems are encountered in storage ring technology (for instance, resonance crossing during particle acceleration, modulation of the particle motion frequencies etc.)

Single crossing was first considered in /1/ for a constant crossing rate ($\dot{\varepsilon}$ = const, $h_\perp$ = const.) and a special initial field polarization condition ($\zeta_{z\,\varepsilon\to-\infty} = 1$).

The following result was obtained :

$$\zeta_{z\,\varepsilon\to\infty} = 2e^{-\frac{\pi}{4}\frac{h_\perp^2}{|\dot{\varepsilon}|}} - 1 \qquad (6.1)$$

In the case of the arbitrary initial condition, the result is known only for fast crossing ($|\dot{\varepsilon}| \gg h_\perp^2$) /1, 4-6/.

A complete answer to the problem of single crossing is not only interesting in itself but is also essential for a solution to the case of periodic crossing. The answer to the problem may be expressed in a slightly more general way.

Let us assume that the spin in the resonance system moves in the field

$$\vec{h} = (h_\perp \,,\ 0 \,,\ h_z) \tag{6.2}$$

(Rotation of $\vec{h}_\perp$ may always be excluded by transforming to a system which rotates relative to the resonance system and $\vec{h}_\perp$).

Certain initial conditions are set for spin at $h_z \to -\infty$. It is essential to find the solution of $\vec{J}$ at $h_z \to \infty$. We would point out that, in the region where direction $\vec{h}$ varies adiabatically, the equation in field (6.2) is solved

$$\vec{J} = J_h \frac{\vec{h}}{h} + \sqrt{1-J_h^2}\ Re\ \vec{e}^{\,*}\, e^{i\left(\int_0^\theta h d\theta + \psi_0\right)} \tag{6.3}$$

where $\vec{e}$ is a complex unit vector which is fixed in relation to $\vec{h}$, and $\vec{h}\vec{e} = 0$, i.e. the spin precesses about $\vec{h}$ with a frequency h . The adiabatic condition is easily derived from a system of co-ordinates where the 3 axis lies in direction $\vec{h}$. In this system the field has the following components :

$$(0 \,,\ \dot{\alpha} \,,\ h)$$

where $\dot{\alpha} = \frac{d}{d\theta} \, arctg \frac{h_2}{h_1} = \frac{1}{h^2}(h_2 \dot{h}_1 - h_1 \dot{h}_2)$
is the angular velocity of rotation $\vec{h}$. In order to meet the adiabatic condition (the spin's precession axis lies in the direction of the field). it is essential that $\dot{\alpha}$ is small in comparison with the spin precession frequency h and that the precession frequency itself varies only slightly during the time it takes for the spin to rotate $2\pi/h$ about the field :

Hence it follows that
$$|\dot{\alpha}| \ll h$$
$$|\dot{h}| \ll h^2$$
$$h >> \sqrt[4]{\dot{h}_z^2 + \dot{h}_\perp^2} \qquad (6.4)$$

Condition (6.4) ensures the exponential accuracy of the solution in the adiabatic region when $\vec{h}$ varies monotonically. If $\vec{h}$ oscillates during crossing, then an additional condition for the smallness of the oscillation frequency Ω is essential :

$$\Omega << h \qquad (6.4a)$$

We would point out that if the adiabatic condition (6.4) is met for all values of h_z , ($h_\perp >> (\dot{h}_z^2 + \dot{h}_\perp^2)^{1/4}$), then solution (6.3) is always correct with exponential accuracy in terms of the adiabatic parameter. An exponentially slight inaccuracy builds up in the region where condition (6.4) is the least well satisfied, i.e. in the region $h_z \sim h_\perp$. Therefore, the limit of the <u>effective</u> resonance region will be taken to mean

$$h_z^{эф} \sim max\ (h_\perp\ ,\ (\dot{h}_z^2 + \dot{h}_\perp^2)^{1/4}) \qquad (6.5)$$

Thus, outside the effective region (6.5), the solution takes the form (6.3) with constant parameters J_h and Ψ_o .

Our aim is to connect the components J_h and phases Ψ_o before crossing ($\Theta < O$) and after crossing ($\Theta > O$). This problem may be fully solved for an arbitrary crossing rate $\dot{h}_z$ if the fractional variations of $\dot{h}_z$ and $h_\perp$ are small i.e. in the effective region

$$\left|\frac{\delta \dot{h}_z}{\dot{h}_z}\right| \ll 1 \qquad \left|\frac{\delta h_\perp}{h_\perp}\right| \ll 1 \qquad |h_z| \lesssim h_z^{эф} \qquad (6.6)$$

In this case it is more convenient to use equation (2.6) for the variable

$$i\dot{\chi} = \frac{1}{2}\vec{\sigma}\vec{h}\ \chi \qquad (6.7)$$

In the adiabatic region equivalent to (6.3) in a system where the axis $\mathfrak{z}$ lies along the direction of field $\vec{h}$, solution (6.7) takes the form :

$$\chi = S_{\theta 0}\begin{pmatrix} A \\ B \end{pmatrix} \qquad S_{\theta 0} = \begin{pmatrix} e^{-\frac{i}{2}\int_0^\theta h\,d\theta} & 0 \\ 0 & e^{+\frac{i}{2}\int_0^\theta h\,d\theta} \end{pmatrix} \tag{6.8}$$

The relations between J_h, Ψ_o and A,B are obvious :

$$J_h = |A|^2 - |B|^2 \qquad \Psi_o = \arg B A^*$$

In order to find the connection between A and B at $\theta < 0$ and $\theta > 0$, we shall use the method for combining the solutions by using a complex time plane /10-12/.
In our case the singular points determined from

$$h = \sqrt{h_z^2 + h_\perp^2} = 0$$

lie in the complex plane θ and the problem is equivalent to that of above-the-barrier reflection in quantum mechanics. If $\dot{h}_z$ and $h_\perp$ are constant, then there are only two singular points - those "of closest approach"

$$\theta_n = \pm i \frac{h_\perp}{\dot{h}_z}$$

Deviations $\dot{h}_z$ and $h_\perp$ lead to a movement of these points and to the appearance of new ones which may be ignored under condition (6.6). As in reports /10-12/, let us avoid the two points of closest approach θ_n, θ_n^* in the complex plane θ (i.e. we skirt around the effective area in which the solution is unknown on a wide circumference where the solution takes the form (6.8)). We obtain the required link between A and B[*)] :

*) In papers /10-12/ the cut in the complex plane θ is made between the points of closest approach. In our determination, h is positive all along the real axis.

$$\begin{pmatrix} A \\ B \end{pmatrix}_{\theta>0} = R \begin{pmatrix} A \\ B \end{pmatrix}_{\theta<0}$$

$$R(h_z>0)=\begin{pmatrix} \sqrt{1-e^{-2\delta}}\,e^{-i\varphi} & -e^{-\delta} \\ e^{-\delta} & \sqrt{1-e^{-2\delta}}\,e^{i\varphi} \end{pmatrix} = R^T(h_z<0) \qquad (6.9)$$

where

$$2\delta = \left| \int_{\theta_n^*}^{\theta_n} h\,d\theta \right| \qquad (6.10)$$

is the value of the integral between the points of closest approach θ_n^*, θ_n. φ is the constant phase which is undefined in /10-12/. The instant $\theta = 0$ is selected so that

$$Re\,\theta_n = 0 \qquad (6.11)$$

From (6.8,9) we obtain the connection

$$\chi_{\theta_2} = S_{\theta_2 0}\, R\, S_{0\theta_1}\, \chi_{\theta_1} \qquad (6.12)$$
$$\theta_2 > 0 \qquad\qquad \theta_1 < 0$$

The phase φ may be found by a comparison with the accurate solution for constant h_z and $h_\perp$ which is set out in Appendix 2. In this case

$$\delta = \frac{\pi}{4}\frac{h_\perp^2}{|h_z|} \equiv \frac{\pi}{4}a^2$$

in accordance with (6.10), and

$$\varphi = -\frac{\pi}{4} + \frac{a^2}{4}\ln\frac{a^2}{4e} - \arg\Gamma\left(i\frac{a^2}{4}\right) \qquad (6.13)$$

A graph of the dependence $\varphi\left(\frac{a^2}{4}\right)$ is shown in Fig. 1 /17/. For small deviations (condition 6.6) of h_z and $h_\perp$ from constant values in the effective area, the difference φ from (6.13) may always be ignored. (We should point out that in the case of slow crossing ($a^2 >> 1$), the absolute variation of δ and the shift $\theta = 0$ may be large.)

Thus, under condition (6.6) matrix R may generally be found from expression (6.9) and φ from (6.13).

The connection between projections on to field J_h and phases ψ may be obtained from (6.12). We give the expression for J_h :

$$J_h(\theta_2) = (1-2e^{-2\delta}) J_h(\theta_1) - -2e^{-\delta}\sqrt{1-e^{-2\delta}}\, J_\perp(\theta_1)\cos\left(\psi(\theta_1) + \int_{\theta_1}^{0} h\, d\theta + \varphi\right) \qquad (6.14)$$
$$\theta_2 > 0 \qquad \theta_1 < 0 .$$

In the case of slow crossing $|\dot{h}_z| << h_\perp^2$, the spin maintains its projection on to $\vec{h}$ with exponential accuracy and flips with $\vec{h}$. In the fast case ($|\dot{h}_z| >> h_\perp^2$), the variation of the projection on $\vec{h}$ is slight ($\sim\sqrt{\delta}$). Formula (6.14) generalizes result /1/ (viz. 6.1).

A practical application of the problem of resonance crossing is given in Appendix 1.

7. Periodic crossing

When a beam is stored in a storage ring for a long time, the problem of periodic resonance crossing may acquire significance. Generally speaking, the problem of periodic resonance crossing is a problem of spin motion in the periodic field

$$\vec{h}(\theta) = \vec{h}\left(\theta + \frac{2\pi}{\Omega}\right) = \left(h_\perp(\theta),\ 0,\ h_z(\theta)\right) \qquad (7.1)$$

where Ω is the crossing frequency.

The results of /7/ illustrate the general nature of spin motion in periodic crossing. There is a certain periodic solution m (θ) for a spin recurring after a period $2\pi/\Omega$. All the other solutions precess with the same frequency μ about $\vec{m}$ ($J_{\vec{m}}$ = const.). Our problem is thus reduced to a search for solution $\vec{m}$ and precession frequency μ.

In section 2 formulae are obtained for $\vec{m}$ and μ in the form of a "wave function" $\mathcal{X}$. Let us assume that the matrix Λ is known ($\Lambda^+\Lambda = 1$). Then

$$\cos\pi\mu = \frac{1}{2} Sp\,\Lambda \qquad \vec{m}(0) = \frac{i}{2\sin\pi\mu} Sp\,\vec{\sigma}\Lambda \tag{7.2}$$

We shall examine a case where condition (6.6) is fulfilled for each separate resonance crossing in the effective area. (It is assumed that the amplitude of the oscillations $\vec{h}_z$ (θ) is large enough). In this case the matrix Λ may be constructed from (6.12). Using the results from the previous section, we obtain :

$$\Lambda = S_{T\theta_2} R_2 S_{\theta_2\theta_1} R_1 S_{\theta_1 0} \qquad \left(T = \frac{2\pi}{\Omega}\right) \tag{7.3}$$

Here θ_2 and θ_1 are the resonance crossing times which meet condition (6.11).

By introducing the notation :

$$x = \varphi_1 + \varphi_2 + \frac{\varkappa}{2} \qquad \varkappa = \varkappa_+ + \varkappa_- = \int_0^T h\,d\theta$$

$$y = \frac{\varkappa_+ - \varkappa_-}{2} \qquad \varkappa_- = \int_0^{\theta_1} h\,d\theta + \int_{\theta_2}^T h\,d\theta \equiv \varkappa_-^{(1)} + \varkappa_-^{(2)}$$

$$\varphi_{1,2} = \varphi\left(\frac{a_{1,2}^2}{4}\right) \tag{7.4}$$

we obtain from (7.3) the matrix elements Λ :

$$\Lambda_{11}=\Lambda_{22}^{*}=\sqrt{(1-e^{-2\delta_1})(1-e^{-2\delta_2})}\,e^{ix}+e^{-\delta_1-\delta_2+iy}$$

$$\Lambda_{12}=-\Lambda_{21}^{*}=e^{\frac{i}{2}(\varphi_1+\varkappa_-^{(1)}-\varphi_2-\varkappa_-^{(2)})}\cdot \tag{7.5}$$

$$\cdot\left\{e^{-\delta_2}\sqrt{1-e^{-2\delta_1}}\,e^{\frac{i}{2}(x+y)}-e^{-\delta_1}\sqrt{1-e^{-2\delta_2}}\,e^{-\frac{i}{2}(x+y)}\right\}$$

Thus, from (7.2) :

$$\cos\pi\mu=\sqrt{(1-e^{-2\delta_1})(1-e^{-2\delta_2})}\cos x+e^{-\delta_1-\delta_2}\cos y \tag{7.6}$$

$$m_z(0)=\frac{\operatorname{Im}\Lambda_{11}}{\sin\pi\mu}\quad,\quad m_x+i\,m_y=-i\,\frac{\Lambda_{12}}{\sin\pi\mu} \tag{7.7}$$

(We remind the reader that $\vec{m}$ is defined in a system connected with field $\vec{h}$.)

These formulae give us all the information we require on spin behaviour during periodic crossings fulfilling condition (6.6). The vector $\vec{m}\,(0)$ defines the direction of the periodic solution at times 0 , $2\pi/\Omega$, ... (The time $\theta=0$ is chosen in an adiabatic region where the spin motion is known). Formula (7.6) defines the angle $2\pi\mu$ at which the spin rotates about $\vec{m}(0)$ over period $2\pi/\Omega$.

It is interesting to study how the projection on axis $\vec{h}$ in the adiabatic region varies during periodic crossing. The variation obviously depends to a large extent on the orientation of $\vec{m}$

in relation to $\vec{h}$. For instance, under an initial condition $\zeta_h = 1$, the projection ζ_h varies over the range :

$$1 \div (2m_z^2 - 1) \quad ; \quad m_z = \frac{\vec{m}\vec{h}}{h}$$

Let us see how the orientation of $\vec{m}$ depends on the parameters of the problem.

a) <u>Fast crossing</u> ($\delta_1 \ll 1$, $\delta_2 \ll 1$). It is easy to see that

$$\begin{gathered} \sin \pi\mu \simeq \sqrt{\sin^2 y + 2(\delta_1+\delta_2) - 4\sqrt{\delta_1\delta_2}\cos x \cos y} \\ m_z(0) = \frac{\sin y}{\sin \pi\mu} \end{gathered} \tag{7.8}$$

Clearly $m_z \simeq 1$ almost always, except in the case of narrow bands in terms of y :

$$|y - k\pi| \lesssim \sqrt{\delta_1+\delta_2 - 2\sqrt{\delta_1\delta_2}\,(-1)^k \cos x} \tag{7.9}$$

(7.9) determines the "resonance" region in which the periodic solution depends heavily on the parameters. The polarization $\zeta_{\vec{h}}$ may vary substantially only in this region. The spin then rotates slowly about $\vec{m}$ with a frequency $\sim\sqrt{\delta}$.

For instance, in the case of symmetrical crossing ($\varkappa_+ = \varkappa_-$)

$$\begin{gathered} m_z = 0 \\ \mu = \frac{2}{\pi}\sqrt{2\delta}\,\sin\frac{x}{2} = \sqrt{\frac{2}{\pi}}\,\frac{h_\perp}{\sqrt{|h_z|}}\,\sin\left(\frac{\pi}{4} + \frac{1}{4}\int_0^T h\,d\theta\right) \\ (\delta_1 = \delta_2 = \delta\,) \end{gathered}$$

A slight violation of the crossing symmetry (   ) shifts the resonance (7.9) and then polarization $\mathfrak{J}\vec{n}$ is preserved.

b) <u>Slow crossing</u> ($e^{\delta_1} \gg 1$, $e^{\delta_2} \gg 1$).

In this case

$$\sin \pi \mu \simeq \sqrt{\sin^2 x + e^{-2\delta_1} + e^{-2\delta_2} - 2e^{-\delta_1-\delta_2} \cos x \cos y}$$

$$m_z(0) = -\frac{\sin x}{\sin \pi \mu} \qquad (7.10)$$

As in the previous case, $m_z = 1$ almost always, except in the resonance region :

$$|x - k\pi| \lesssim \sqrt{e^{-2\delta_1} + e^{-2\delta_2} - 2e^{-\delta_1-\delta_2}(-1)^k \cos y} \qquad (7.11)$$

c) <u>Mixed case</u> ($\delta_1 \ll 1$ $e^{\delta_2} \gg 1$)

$$\mu \simeq \frac{1}{2} - \frac{1}{\pi}\sqrt{2\delta_1} \cos x - \frac{1}{\pi} e^{-\delta_2} \cos y$$

$$m_z \simeq e^{-\delta_2} \sin y - \sqrt{2\delta_1} \sin x \qquad |m_z| \ll 1$$

$$m_x + i\, m_y = i\, e^{-\frac{i}{2}(\varkappa_+ + \varkappa_-^{(2)} - \varkappa_-^{(1)})}$$

This solution may be easily explained in the following way. During fast motion "from bottom to top" $\mathfrak{J}\vec{n}$ does not change. Then, during slow crossing "from top to bottom" $\mathfrak{J}\vec{n}$ changes its sign. Consequently, the spin completes half a turn ($\mu \simeq \frac{1}{2}$) about a certain direction diametrically opposed to the axis $\vec{n}$.

Unlike the previous cases, small m_z do not signify a resonance ($\mu \neq K$).

d) <u>Intermediate case</u> ($\delta_1 \sim 1$, $\delta_2 \sim 1$).

Unlike the above cases, the direction of $\vec{m}$ does not depend on (X, y) throughout their range of variation. In this case m_z passes smoothly through all the possible values :

$$0 \leqslant |m_z| \leqslant \sqrt{(1-e^{-2\delta_1})(1-e^{-2\delta_2})} + e^{-\delta_1-\delta_2} \tag{7.12}$$

The greatest sensitivity to the position of point (X , y) is observed near the resonances $\mu = K$, when

$$\delta_1 \simeq \delta_2 = \delta \qquad \cos x \simeq \cos y \simeq \pm 1$$

The approximated formula near the resonance takes the form :

$$\sin \pi \mu \simeq \sqrt{\frac{(\delta_1-\delta_2)^2}{e^{2\delta}-1} + (\Delta x)^2(1-e^{-2\delta}) + (\Delta y)^2 e^{-2\delta}}$$

$$m_z = \frac{e^{-2\delta}\Delta y - (1-e^{-2\delta})\Delta x}{\sin \pi \mu}$$

Here ΔX and Δy are the deviations from the resonance point

$$\cos X = \cos y = \pm 1$$

8. <u>Fast crossings with arbitrary periodic dependence $\vec{h}(\theta)$</u>

Formulae (7.6,7) of the periodic solution are correct for condition (6.6). Normally for fast crossing, the problem may be solved without imposing limitations on the form of the periodic dependence $\vec{h}\ (\theta)$. In order to find the periodic solution, it is sufficient to know the spin motion in the period of variation $\vec{h}\ (\theta)$. In the case of fast crossings, the spin in the effective region does not change a great deal, and the solution in this region may, therefore, be found by using the perturbation theory.

In the case of single crossing, the connection of χ at $\theta > 0$ and $\theta < 0$ in the system rotating about axis $\vec{n}$ at a velocity h_z takes the following form in the first approximation:

$$\chi'_{\theta_2} = \left(1 - \frac{i}{2}\vec{\sigma}\vec{H}_0\right)\chi'_{\theta_1} \qquad H_0 = \int_{-\infty}^{\infty} h_\perp e^{-i\int_{\theta_0}^{\theta} h_z d\theta} d\theta \tag{8.1}$$

$$\theta_2 > 0 \qquad \theta_1 < 0 \qquad h_z(\theta_0) = 0$$

By returning to the original system and matching the solutions in the adiabatic regions, we obtain the matrix Λ (in the system (7.1)):

$$\chi_T = \Lambda \chi_0$$

$$\Lambda = e^{-\frac{i}{2}\sigma_z \varkappa_T}\left(1 - \frac{i}{2}\vec{\sigma}\vec{H}\right) \tag{8.2}$$

$$H_x + i H_y = \int_0^T h_\perp e^{-i\varkappa_\theta} d\theta$$

$$\varkappa_\theta = \int_0^\theta \mathcal{E}_{\text{эф}} d\theta \qquad \mathcal{E}_{\text{эф}} = \begin{cases} h_z & \text{in the effective region} \\ h\dfrac{h_z}{|h_z|} & \text{in the adiabatic region} \end{cases}$$

Hence :

$$\Lambda_{11} = \left(1 - \frac{H^2}{8}\right) e^{-\frac{i}{2}\varkappa_T}$$
$$\Lambda_{12} = -\frac{i}{2}(H_x - iH_y)\, e^{-\frac{i}{2}\varkappa_T} \qquad H = |H_x + iH_y| \ll 1$$

$$\sin \pi\mu = \sqrt{\sin^2\frac{\varkappa_T}{2} + \frac{H^2}{4}}$$
$$m_z = \frac{\sin\frac{1}{2}\varkappa_T}{\sin\pi\mu} \tag{8.3}$$
$$m_x + i\, m_y = \frac{H_x - iH_y}{2\sin\pi\mu}\, e^{-\frac{i}{2}\varkappa_T}$$

Unlike (7.8), this solution may also be used when the adiabatic region is not reached.

When there are adiabatic regions and condition (6.6) applies, solution (8.3) switches to (7.8). As before, the periodic solution is always directed along the z axis, except in the case of the resonance region :

$$\left|\frac{1}{2}\varkappa_T - K\pi\right| \lesssim H \qquad \text{(c.f. 7.9)} \tag{8.4}$$

We should point out that, if there are no adiabatic regions, then $\varkappa_T = \overline{h}_z T \ll 1$. In this case, as may be seen from (8.3), the spin moves in the mean field :

$$\langle \vec{h} \rangle = (\langle h_1 \rangle,\ 0,\ \langle h_z \rangle)$$

9. Crossing of resonances with rotation frequency $\nu = K$

Up until now we have not studied effects linked with the variation of $\vec{n}(\gamma_s, \theta)$ as a function of energy γ_s (or other parameters) with time. In the periodic system $\vec{n}(\gamma_s, \theta)$, $\vec{\ell}(\gamma_s, \theta)$, owing to the variation of γ_s, equation (2.4) has the additional terms $\sim \frac{\partial \vec{n}}{\partial \gamma_s}\dot{\gamma}_s;\ \frac{\partial \vec{\ell}}{\partial \gamma_s}\dot{\gamma}_s$. Far from the resonance $\nu = K$, they are small and may be ignored like non-resonant terms. Problems arise with those energy values at which the resonance $\nu(\gamma_s) \simeq K$ occurs.

This resonance crossing problem may be solved by the methods described above. Let us assume that the periodic solution $\vec{n}(\gamma_s, \theta)$ and the precession frequency $\nu(\gamma_s)$ are known for every γ_s value. As resonance point $\gamma_s = \gamma_p$ we shall select an energy value at which $\nu(\gamma_s)$ is closest to the whole number

$$\nu(\gamma_p) - K = \Delta\nu_{min}$$

(In particular, $\Delta\nu$ min may equal zero). In the resonance periodic system $\vec{n}_p = \vec{n}(\gamma_p, \theta)$, $\vec{\ell}_p = \vec{\ell}(\gamma_p, \theta)$ the spin motion equations take the form

$$\frac{d\vec{\zeta}}{d\theta} = [\vec{w}'\vec{\zeta}] \quad ; \quad \vec{w}' = (w_x' ; w_y' ; w_z')$$

$$w_x' + i w_y' = \delta\frac{\vec{W}_s}{\omega_s}\vec{e}_p \qquad w_z' = \Delta\nu_{min} + \delta\frac{\vec{W}_s}{\omega_s}\vec{n}_p$$

$$\delta\frac{\vec{W}_s}{\omega_s} = \frac{\vec{W}_s(\gamma_s\,\theta)}{\omega_s(\gamma_s)} - \frac{\vec{W}_s(\gamma_p\,\theta)}{\omega_s(\gamma_p)}$$

(the deviations from the equilibrium orbit are insignificant).

The equations averaged in terms of Θ take the form :

$$\frac{d\vec{\zeta}}{d\theta} = [\vec{h}\vec{\zeta}] \quad ; \qquad \vec{h} = \langle \vec{w}' \rangle$$

$$h_x + i h_y = \frac{1}{\omega_s} \langle \delta \vec{W}_s \vec{e}_p \rangle \quad ; \qquad h_z = \Delta \nu_{min}$$

$$\langle \vec{n}_p \delta \vec{W}_s \rangle = 0$$

$$\frac{\partial \nu}{\partial \gamma}(\gamma_p) = 0$$

, according to the definition of the resonance point

The direction $\vec{h}$ defines the direction of the periodic solution $\vec{n}\,(\gamma_s\ \theta)$:

$$\vec{n}(\gamma_s, \theta) = \frac{\Delta \nu_{min}}{h} \vec{n}_p + Re\, \frac{h_x - i h_y}{h} \vec{e}_p \tag{9.1}$$

The value h defines $\Delta \nu(\gamma_s) = \nu(\gamma_s) - K$

$$\Delta \nu(\gamma_s) = h = \sqrt{(\Delta \nu_{min})^2 + h_x^2 + h_y^2}$$

at $\delta\gamma_s = 0$, as should be the case,

$$\Delta \nu (\gamma_s) = \Delta \nu_{min.}$$

As can be seen from (9.1), the direction of the periodic solution at the resonance point $\vec{n}_p\ (\Theta)$ is transverse to $\vec{n}$ ($\gamma_s\ \Theta$) far from the resonance. This corresponds to the normal behaviour of the precession axis in the region of the spin resonance.

In the case of resonance crossing, according to the condition obtained above (6.4), the variation of $\vec{n}$ and ν may be considered adiabatic if :

$$|\Delta \nu(\gamma_s)| \gg \sqrt{\left| \frac{\partial}{\partial \gamma} \frac{(\vec{W}_s \vec{e}_p)}{\omega_s} \dot{\gamma} \right|} \tag{9.2}$$

In the case of slow crossing

$$|\Delta\nu_{min}| \gg \sqrt{\left|\frac{\partial}{\partial\gamma}\left(\frac{\vec{W}_s\vec{e}_\rho}{\omega_s}\right)\dot{\gamma}\right|}$$

the projection on $\vec{n}$ is maintained and flips together with $\vec{n}$. In the case of fast crossing, the spin cannot follow $\vec{n}(\gamma_s, \theta)$ and the projection $J_{\vec{n}}$ changes sign.

The authors wish to thank V.N. Bajer, S.T. Belyaev, N.S. Dikanskij and Yu. M. Shatunov for studying the report and offering their advice.

Appendix 1.

Let us consider one practical application of the results we have obtained, for instance in a colliding-beam experiment using electrons and positrons with the same polarization. As they move through the storage ring's magnetic field, the beams are polarized in opposite directions under the influence of synchrotron radiation /13,14/. In order to reverse the polarization (of the positrons for instance), we shall separate the energy of the electrons and positrons by introducing a radial electric field E_r . This leads to a distinction between $\nu_{pos.}$ and $\nu_{el.}$.

$$\Delta\nu = \nu_{pos.}\ \nu_{el.} = 2\frac{\partial\nu}{\partial\gamma}\delta\gamma = 2\gamma\frac{\partial\nu}{\partial\gamma}\frac{\langle E_r\rangle}{\langle H\rangle}$$

($\langle H \rangle$ is the storage ring's mean steering magnetic field). Let us construct the resonance by introducing into an orbit section of length θ_o (where $|\vec{n}\,\vec{v}|$ is minimal) a longitudinal (in terms of velocity), variable, magnetic field $H_{\parallel} \sin \nu_{ext.}\theta$ with an external frequency $\nu_{ext.}$.

$$\nu_{pos.} = \nu_{ext.} + K \neq \nu_{el.} \qquad (1)$$

The width of this resonance

$$h_{\perp} = \frac{1}{2}\frac{H_{\parallel}}{\langle H\rangle}\langle g(\theta)\,\vec{v}\vec{e}\cdot e^{-iK\theta}\rangle$$

$$g(\theta) = \begin{cases} 1 & 0 < \theta < \theta_o \\ 0 & \theta_o < \theta < 2\pi \end{cases}$$

must not overlap the next resonance for electrons :

$$|h_{\perp}| \ll |\Delta\nu|$$

by changing $\nu_{ext.}$ it is possible slowly $\sqrt{|\dot{\nu}_{ext.}|} \ll |h_{\perp}|$ to pass resonance (1), so that the polarization of the positrons changes sign.

Appendix 2.

Equations (6.7) may be solved exactly at constant values of $\dot{h}_z$ and $h_\perp$. These equations :

$$\dot{\chi}_+ = -\frac{i}{2}\dot{h}_z \theta \chi_+ - \frac{i}{2} h_\perp \chi_-$$
$$\dot{\chi}_- = \frac{i}{2}\dot{h}_z \theta \chi_- - \frac{i}{2} h_\perp \chi_+$$

are functional relations for functions of the parabolic cylinder /15/ $D_p(z)$. The solution takes the form ($\dot{h}_z > 0$) :

$$\chi(\theta) = \begin{pmatrix} D_p(z) & D_p(-z) \\ \frac{a}{2} e^{i\frac{\pi}{4}} D_{p-1}(z) & -\frac{a}{2} e^{i\frac{\pi}{4}} D_{p-1}(-z) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \equiv M \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad (2)$$

where $a = \frac{h_\perp}{\sqrt{\dot{h}_z}}$; $p = -i\frac{a^2}{4}$; $z = e^{i\frac{\pi}{4}} \sqrt{\dot{h}_z}\, \theta$; α, β

are constants determined by the initial conditions. The determinant $|M|$ ("Wronskian") does not depend on time and may be calculated, for instance, at $\theta \longrightarrow \infty$:

$$|M| = -\frac{a}{2} e^{i\frac{\pi}{4}} \left\{ D_p(z) D_{p-1}(-z) + D_p(-z) D_{p-1}(z) \right\} = \\ = -\frac{a}{2} e^{i\frac{\pi}{4}} \frac{\sqrt{2\pi}}{\Gamma(i\frac{a^2}{4}+1)} \quad (3)$$

Thus we find the elements of the matrix Λ_o :

$$\chi_{\theta_2} = \Lambda_o \chi_{\theta_1}$$

$$(\Lambda_o)_{11} = (\Lambda_o)^*_{22} = \frac{\Gamma(i\frac{a^2}{4}+1)}{\sqrt{2\pi}}\left[D_p(z_2)D_{p-1}(-z_1) + D_p(z_1)D_{p-1}(-z_2)\right]$$

$$(\Lambda_o)_{12} = -(\Lambda_o)^*_{21} = \sqrt{\frac{2}{\pi}}\, e^{-i\frac{\pi}{4}}\, \Gamma(i\frac{a^2}{4}+1)\cdot \tag{4}$$

$$\cdot\left[D_p(z_2)D_p(-z_1) - D_p(-z_2)D_p(z_1)\right]$$

$$z_1 = z(\theta_1) \qquad z_2 = z(\theta_2)$$

To find the matrix Λ_o linking the adiabatic regions

$$\sqrt{h_z^2 + h_\perp^2} \gg \sqrt{\dot{h}_z} \tag{5}$$

it is essential to know the asymptotics $D_p(z), D_{p-1}(z)$. Their extreme expressions are known /15/ in the case of condition :

$$|z| \gg max\,(1,\ |p|)$$
$$|h_z| \gg max\left(\sqrt{\dot{h}_z}\ ,\ \frac{h_\perp^2}{\sqrt{\dot{h}_z}}\right) \tag{6}$$

By using the method of steepest descent and by the constancy of the solutions' "Wronksian", the asymptotics can also be found under condition (5).

It is sufficient for our purposes to know the limiting expressions of D_p (z), since the asymptotic form of Λ (viz 6.12) (and of Λ_0) is known under condition (5). In order to define Λ fully, it is essential to determine the constant (under condition (5) !) phase φ which obviously may also be found in the limit $|\theta_1|$, $|\theta_2| \to \infty$. In order to link Λ and Λ_0 , the matrix of Λ_0 must be converted into a system related to direction $\vec{h}$, which can be done by rotating about axis "y" through an angle

$$\alpha = \operatorname{arctg} \frac{h_\perp}{h_z} \quad :$$

$$\Lambda = e^{\frac{i}{2}\sigma_y \alpha(\theta_1)} \Lambda_0 \, e^{-\frac{i}{2}\sigma_y \alpha(\theta_1)} \tag{7}$$

At the limit $\quad |\theta_1|,\ |\theta_2| \to \infty$

$$\Lambda_{11} = \Lambda_{12}^{c} = \sqrt{1-e^{-2\delta}}\; e^{-i\left(\varphi + \frac{1}{2}\int_{\theta_1}^{\theta_2} h\, d\theta\right)} \tag{8}$$

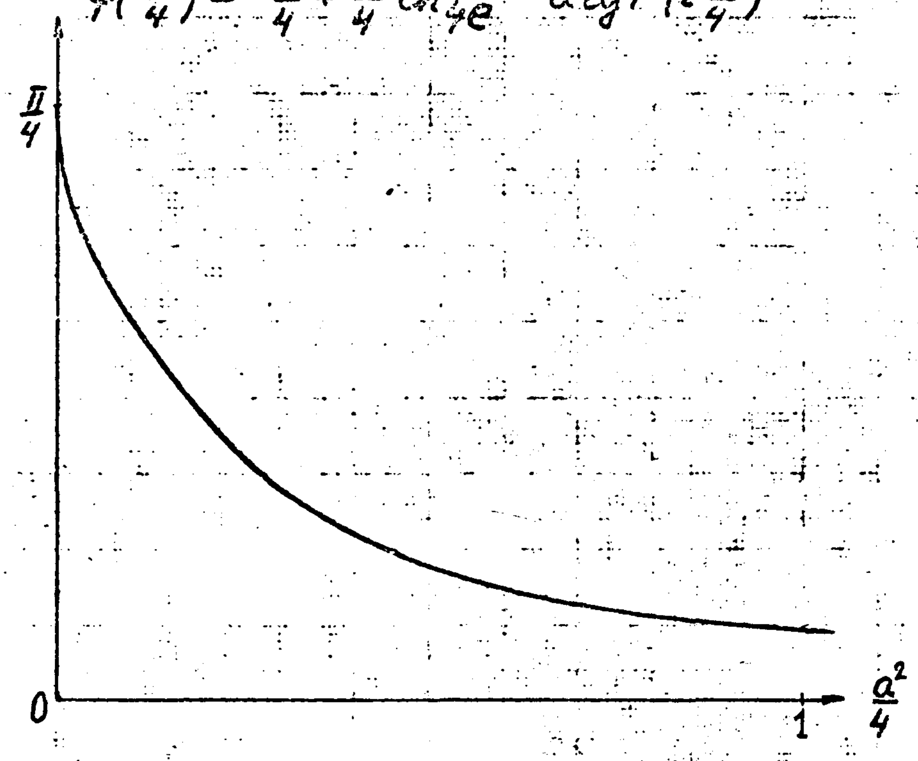
By comparing (8) and (4) at the limit $|\theta_1|, |\theta_2| \to \infty$, we obtain the following expression for phase φ

$$\varphi = -\frac{\pi}{4} + \frac{a^2}{4} \ln \frac{a^2}{4e} - \arg \Gamma\left(i\frac{a^2}{4}\right) \tag{9}$$

where $\Gamma(z)$ is the gamma function.

In paper /1/ condition (6) was used to obtain result (6.1). From what has gone before, it follows that, if matrix Λ is to take the form (6.12) with φ from (9), $\dot{h}_z$ and $h_\perp$ must be constant only in the effective region (6.5).

$$\varphi\left(\frac{a^2}{4}\right) = -\frac{\pi}{4} + \frac{a^2}{4}\ln\frac{a^2}{4e} - \arg\Gamma\left(i\frac{a^2}{4}\right)$$



$$\varphi = \frac{1}{3a^2} + \frac{8}{45a^6} + \cdots \qquad a^2 > 4$$

$$\varphi = \frac{\pi}{4} + \frac{a^2}{4}\left(\ln\frac{a^2}{4e} + 0{,}577\right) + \cdots \qquad a^2 \ll 4$$

Figure 1.

Bibliography

1. Froissart M., StoraR. Nucl. Instr. and Methods 7, 297 (1960)
2. Kh. A. Simonyan, Yu. F. Orlov. Zh.E.T.F. 45, N° 2, 173 (1963)
3. Yu. F. Orlov, S.A. Khejfets. Izv. A.N. Arm. SSR XIII N°1, 169 (1960)
4. Kh. A. Simonyan. Proceedings of the First International Conference on Accelerators, Dubna (1963)
5. Yu. A. Plis, L.M. Soroko. Proceedings of the First International Conference on Accelerators, Dubna (1963)
6. P.R. Zenkevich. Proceedings of the First International Conference on Accelerators, Dubna (1963)
7. Ya. S. Derbenev, A.M. Kondratenko, A.N. Skrinskij. Preprint I.Ya.F. SO AN SSR 2-70; DAN SSR (1970) (in press)
8. V. Bargmann, L. Michel, V. Telegdi. Phys. Rev. Lett. 2, 435 (1959)
9. V.N. Baier, V.M. Katkov, V.M. Strakhovenko. Phys. Lett. 31A N4 (1970)
10. Stuckelberg E.G. Helv. Phys. Acta, 5, 370 (1932)
11. G.M. Zaslavskij. Lektsii po primeneniyu metoda VKB v fizike, NGU, Novosibirsk (1965).
12. Kheding D. Vvedenie v metog fazovykh integralov. M., Mir (1965)
13. A.A. Sokolov, I.M. Ternov. DAN SSSR 153, 1052 (1963)
14. V.N. Bajer, V.M. Katkov. Zh. E.T.F. 52, 1422 (1967)
15. I.S. Gradshtejn, I.M. Ryzhik. "Tablitsy integralov summ, ryadov i proizvedenij". F.M., M (1962)
16. L.D. Landau, E.M. Lifshits. Kvantovaga mekhanika, OGIZ (1948)
17. A.A. Abramov. Tablitsy $\ln \Gamma(z)$ v kompleksnoj oblasti, Izd-vo A.M. SSR (1953)