TRANSVERSE INSTABILITIES IN THE SPS

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Abstract

The beam which the SPS has to be provided for the LHC, combines the inconveniences of a high single bunch intensity together with a high total intensity of closely spaced bunches. Single bunches with the intensity required for the LHC were already accelerated the SPS during the collider period and experience could be gained concerning space charge effects and head tail instabilities. The close spacing of the bunches will inevitably lead to multi-bunch instabilities. For this kind of instabilities the details of the transverse impedance (existence of narrow band oscillators) can become important.

1 WAKE FIELDS

The bunches in an accelerator induce currents and changing charge distributions in the vacuum chamber surrounding them, resulting in an electromagnetic field inside the vacuum chamber, called “wake field”. In symmetrical vacuum chambers (circular, elliptic, rectangular,..) the transverse component of the wake field is zero when the bunch is in the centre. For small deviations from the centre, the transverse component of the wake-field is proportional to the transverse co-ordinate of the beam (1). This fields decay normally rather quickly after the bunch passage. The transverse wake field from a smooth vacuum pipe (resistive wall) decays as $1/D^{5/2}$, D being the distance behind the bunch. The field from cavity like structures behaves typically like a damped sine wave. This damping time can vary from the order of 1nsec (broadband resonator) up to some μseconds (High Q resonators such as RF cavities).

2 COUPLED BUNCH MODES

2.1 System of coupled oscillators

The transverse motion of n coupled bunches can generally be described by n oscillators with betatron frequency (ω_β), and coupled to each other by the linear coupling coefficients k_{ij}, representing the normalised force from the wakefield created by particle j on particle i :

$$X_1''+\omega_\beta{}^2 X_1 = -k_{12}X_2 - k_{13}X_3 ...$$

..

$$X_n''+\omega_\beta{}^2 X_n = -k_{n2}X_2 - k_{n3}X_3 ...$$

Where X is the transverse co-ordinate of the different bunches. The theory of coupled linear differential equations says that the motion of each bunch can be described as the sum of n oscillators,

$$X_k = c_{1k}e^{iw_1 t} + ... + c_{nk}e^{iw_n t}$$

each having a frequency w_k such that $w_k{}^2$ are eigenvalues of the following matrix :

$$\begin{bmatrix} \omega_\beta{}^2 & k_{12} & . & k_{1n} \\ . & . & . & . \\ . & . & . & . \\ k_{n1} & . & k_{n,n-1} & \omega_\beta{}^2 \end{bmatrix}$$

At each eigenfrequency w_i , the motion of the different particles is correlated. This correlation is given by the coefficients c_{ij} which describe the amplitude and phase relationship between the different bunches (j) for each oscillator $\exp(iw_i t)$. They correspond to the eigenvectors of each w_i . This correlated motions at each of the n eigenfrequencies are called “modes”.
In order to find the eigenfrequencies *w* on has to put the following determinant to 0 :

$$\left\| \begin{matrix} \omega_\beta{}^2 - w^2 & k_{12} & . & k_{1n} \\ . & . & . & . \\ . & . & . & . \\ k_{n1} & . & k_{n,n-1} & \omega_\beta{}^2 - w^2 \end{matrix} \right\| = 0$$

$$\left\| \begin{matrix} \Delta\omega.2\omega_\beta & k_{12} & . & k_{1n} \\ . & . & . & . \\ . & . & . & . \\ k_{n1} & . & k_{n,n-1} & \Delta\omega.2\omega_\beta \end{matrix} \right\| = 0$$

The second determinant is obtained by making the approximation that the resulting frequencies are not too different from ω_β so that :

$$\omega_\beta^2 - w^2 = (\omega_\beta - w)(\omega_\beta + w) \sim 2\Delta w.\ \omega_\beta$$

This determinant is an equation of nth degree in Δw giving n roots. One has to bear in mind that these roots can be complex so that they have a non-zero imaginary part. The imaginary part of this frequency result in exponential growth or decay when putting them in the general solution :

$$e^{i(\omega_{\mathrm{Re}} \pm i\omega_{\mathrm{Im}})t} = e^{i\omega_{\mathrm{Re}} t}.e^{\mp\omega_{\mathrm{Im}} t}$$

2.2 *A simple example*

As already mentioned, one of the general features of wake fields is that they decay in function of the distance behind the bunch. The wake field seen by the first following bunch is normally much stronger than the wake field seen by the second following bunch. If we ignore the effect on the second bunch and only keep the coupling between subsequent bunches the equation to find the eigenfrequencies is very simple :

renaming

$$2\omega_\beta \Delta\omega = \Delta$$

gives:

$$\left\| \begin{bmatrix} \Delta & 0 & 0 & k \\ k & \Delta & 0 & . \\ 0 & k & . & 0 \\ 0 & . & k & \Delta \end{bmatrix} \right\| = \Delta^n - (-k)^n = 0$$

$$\Delta^n = \pm k^n$$

Depending whether n is odd or even . The complete set of solutions in the complex plane can be found by :

$$\Delta = \pm k e^{\frac{2\pi m i}{n}}$$

Giving n equidistant points on a circle with radius k in the complex plane (fig 1)

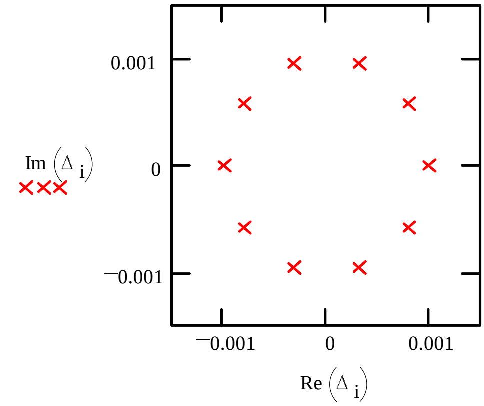

Fig 1: frequency shifts due to coupling between bunches. The imaginary parts give the growth-rate constants.

2.3 *Another extreme case*

Let us now consider a train of bunches, each coupled to next but with no closure at the end, i.e. there is a big gap between the last and the first bunch. The characteristic determinant then looks like this :

$$\left\| \begin{bmatrix} \Delta & 0 & 0 & 0 \\ k & \Delta & 0 & . \\ 0 & k & . & 0 \\ 0 & . & k & \Delta \end{bmatrix} \right\|$$

This gives as solution n equal roots Δ=0 . However, the theory of differential equation says that there have to be n independent solutions. In this case it can be shown that the different solutions look like :

$$x_0 = ae^{i\omega_\beta t}$$

$$x_1 = ae^{i\omega_\beta t} + \frac{ka}{2i\omega_\beta}\ t.e^{i\omega_\beta t}$$

$$x_2 = ae^{i\omega_\beta t} + \frac{ka}{2i\omega_\beta}\ t.e^{i\omega_\beta t} + \frac{1}{2!}\left(\frac{ka}{2i\omega_\beta}\right)^2 t^2 e^{i\omega_\beta t}$$

.....

Each bunch acts on the next one as simple resonant external force. One can see that the amplitude of the second bunch is growing linear with time, the amplitude of the third is growing quadratic etc. This phenomenon is equivalent to the beam break up on can observe in LINAC’s.

2.4 Closer to reality

We now consider a wake field that extends over a longer range. In the example the field is going down quadraticaly with the distance between the bunches. And we consider also a weak coupling through a gap form the last bunch to the first (ε). The result (fig 2) does not differ much from the one calculated in fig 1.

$$\begin{bmatrix} \Delta & 0 & 0 & \varepsilon \\ k & \Delta & 0 & 0 \\ \frac{k}{4} & k & \Delta & 0 \\ \frac{k}{8} & \frac{k}{4} & k & \Delta \end{bmatrix}$$

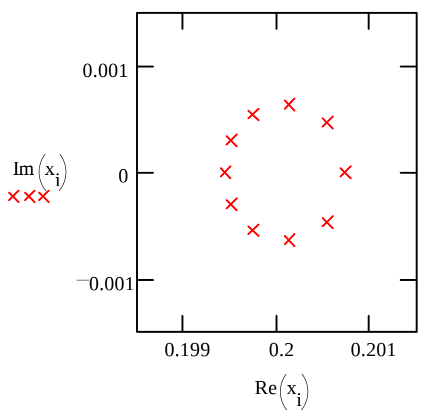

Fig 2 : Eigenfrequencies for a system with longer range and weak closure (i.e. a gap).

It is interesting to see that the results still lie on a circle which means that the maximum growth rates are comparable to the frequency shift. The radius of the circle is proportional to $k.\varepsilon^{1/n}$. It is interesting to look at the eigenvectors which give the amplitude of the oscillation of each bunch (fig 3).

eigenvec (M, x_0) =

	0
0	0.011
1	0.015
2	0.024
3	0.039
4	0.065
5	0.107
6	0.177
7	0.292
8	0.482
9	0.795

Fig 3 : eigenvector for a series of coupled bunches with a big gap between last and first.

As one can see the amplitude is increasing towards the end of the batch. This result shows the same features as the simple example in 2.3. This behaviour could be observed for 2μsec long batches in the SPS.

2.5 The real world

The matrix to calculate the tune shifts in a real machine is given by :

$$\begin{bmatrix} \Delta & . & . & . \\ k_1 & \Delta & . & . \\ k_2 & k_1 & \Delta & . \\ . & . & . & . \end{bmatrix}$$

$$k_m = \frac{Nec^2}{nET} e^{-i\omega_\beta m \frac{T}{n+n_0}} \frac{1}{i} \int_{-\infty}^{\infty} Z(w) e^{iw\frac{T}{n+n_0}m} dw$$

N is the total number protons, n the number of filled bunches, n_0 the number of empty bunches , T is the revolution period and Z the transverse impedance. The factor :

$$e^{-i\omega_\beta m \frac{T}{n+n_0}}$$

comes from the fact that the field seen by subsequent bunches at time t is proportional to the betatron amplitude of the previous bunches at t $-T/(n_0+n)$. This phase shift term makes that the circle in the complex plane is turned so that other modes can become unstable depending on the tune. The result for a single bunch intensity of 10^{11} and 80 bunches gives growth rates of .012 (turns).

3 HEAD TAIL INSTABILITY

In order to understand the instabilities of a single bunch one can imagine the bunch as being composed of smaller bunches, following each other very closely, so coupled together very strongly, and with a negligible coupling to the next bunch (series of micro bunches). The solution looks then like described in 2.3. i.e. the oscillation amplitude of the particles towards the end of bunch increases with an increasing power of time. In linacs this phenomenon leads to so called beam break up. In synchrotrons however, the leading and trailing particles in the bunch exchange their position due to the synchrotron motion and this has normally a stabilising effect. In order to understand this we model the single by only two particles (head and tail) represented by the following two oscillators:

$x_1 = a_1 e^{i\omega_\beta t}$ and $x_2 = a_2 e^{i\omega_\beta t}$. During the half synchrotron period that x_1 is the head, the amplitude of the second particle changes as described in 2.3 :

$$\Delta a_2 = -\frac{ik}{2\omega_\beta} a_1 \frac{T_s}{2} \quad \text{i.e.}$$

$$\frac{da_2}{dt} = -\frac{ik}{2\omega_\beta} a_1$$

During the next half synchrotron period we have :

$$\Delta a_1 = -\frac{ik}{2\omega_\beta} a_2 \frac{T_s}{2} \quad \text{i.e.}$$

$$\frac{da_1}{dt} = -\frac{ik}{2\omega_\beta} a_2$$

Differentiating the second equation and replacing the first in the second we find:

$$\frac{d^2 a_1}{dt^2} + \left(\frac{k}{2\omega_\beta}\right)^2 a_1 = 0$$

And the same for a_2. This just means that both amplitudes are oscillating with a frequency $\omega_{ht} = k/2\omega_b$. The equation of motion for the two particles is then :

$$x_{1,2} = A_{1,2} e^{i(\omega_\beta \pm \omega_{ht})t}$$

So the first result of this head tail interaction is that there are two different frequencies, one shifted up and the other one shifted down, proportional to the intensity. It is easy to see that one mode corresponds to the two particles moving in phase and the other one coresponding to the particles moving with opposite phase.

In the case of finite chromaticity one has to take into account a phase shift between the particle that moves from head to tail and visa versa. The amplitude change per half synchrotron period is than :

$$\Delta a_2 = -\frac{ik}{2\omega_\beta} e^{i\psi} a_1 \frac{T_s}{2}$$

With ψ proportional to the chromaticity.

The tune shift ω_{ht} is then :

$$\omega \qquad \omega_{ht} = k\,(\cos\psi + i\sin\psi)\,/2\omega_\beta$$

This means, that depending on the chromaticity on can make one or the other mode unstable (imaginary frequency shift). Experience shows that it is better to damp the dipole mode with the chromaticity, the quadrupole mode (particles having opposite phase) being much stronger landau-damped.

4 LASSLETT TUNE SHIFT

Particles travelling in a dense bunch experience a defocusing force from the charges in the bunch. This defocusing force is strongest in the centre of the bunch going to zero for large amplitude particles resulting in a tune spread. The tune shift for the central particles is given by :

$$dQy = \frac{N \cdot rp \cdot 1.5L}{\beta \cdot \sigma s \cdot \gamma^3} \cdot \frac{1}{4\pi} \cdot \frac{\beta x}{\sigma y \cdot (\sigma y + \sigma x)} \qquad dQy = 0.03241$$

$$dQx = \frac{N \cdot rp \cdot 1.5L}{\beta \cdot \sigma s \cdot \gamma^3} \cdot \frac{1}{4\pi} \cdot \frac{\beta x}{\sigma x \cdot (\sigma y + \sigma x)} \qquad dQx = 0.02569$$

The numeric values correspond to the typical LHC bunch at 26 GeV.

In Fig 4 is the summary given of the different tune shifts involved for an LHC bunch ($1\ 10^{11}$) at 26 GeV.

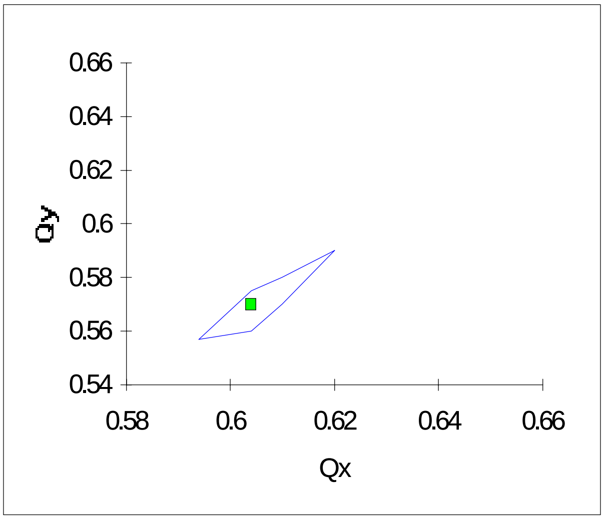

Fig 4 : Tune footprint of an LHC bunch at 26 GeV. The square is the coherent tune i.e. the tune as measured.

5 REFERENCES

(1) L. Palumbo, V.G. Vaccaro, M. Zobov, *Wake fields and impedance*, CERN CAS proceedings Rhodes, 1983.