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The Prepotential of $\mathcal{N}=2$ SU(2) × SU(2) Supersymmetric Yang–Mills Theory with Bifundamental Matter

Ulrike Feichtinger*

Theory Division, CERN,
Geneva, Switzerland

Abstract

We study the non-perturbative, instanton-corrected effective action of the $\mathcal{N} = 2$ $SU(2) \times SU(2)$ supersymmetric Yang–Mills theory with a massless hypermultiplet in the bifundamental representation. Starting from the appropriate hyperelliptic curve, we determine the periods and the exact holomorphic prepotential in a certain weak coupling expansion. We discuss the dependence of the solution on the parameter $q = \frac{{\Lambda_2}^2}{{\Lambda_1}^2}$ and several other interesting properties.

* Ulrike.Feichtinger@cern.ch

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1 Introduction

Many field theoretical results have been obtained by considering brane configurations in string theory and M-theory. In particular M-theory five-branes provide a very fruitful approach to $\mathcal{N} = 2$ [1, 2] and $\mathcal{N} = 1$ [3, 4, 5] supersymmetric field theories. In this note we study the case of product gauge group $SU(2) \times SU(2)$ with a hypermultiplet in the bifundamental representation. Seiberg–Witten curves of theories with product gauge groups have been obtained from the M-theory [2] and the geometric engineering approach [6]. The precise relation between the moduli of the curve and the moduli of the field theory has been described in [7].

Yet another account of supersymmetric field theories with product gauge groups of the form $G = \prod_i SU(N_i)$ has been obtained in the framework of Calogero–Moser systems, using certain limits of $SU(N)$ supersymmetric Yang–Mills theory with matter in the adjoint representation [8]. In this approach the dynamical scales of the different gauge group factors are all equal to the scale of the underlying $SU(N)$ group. One-instanton predictions for the prepotential of $\mathcal{N} = 2$ supersymmetric field theories with gauge group $SU(N_1) \times SU(N_2)$ and massless matter in the bifundamental representation have been made by using a perturbation expansion of the non-hyperelliptic curve around its hyperelliptic approximation [9].

In the present paper we study the instanton expansion for the case of a massless hypermultiplet in the bifundamental representation of $SU(2) \times SU(2)$. Specifically we will compute the periods as solutions of the Picard–Fuchs equations and obtain in this way the holomorphic prepotential $\mathcal{F}$ that governs the low-energy effective action of the theory. The solution reproduces the existing results in the appropriate limits and has interesting properties in the general region of moduli space.

2 The Setup

By considering the appropriate brane configuration in M-theory [2] one finds that the defining polynomial for the Seiberg–Witten curve of $\mathcal{N} = 2$ supersymmetric Yang–Mills theory with a gauge group of the form $\prod_i^n SU(k_i)$ is given by

$$P(x,t) = t^{n+1} + p_{k_1}(x)\, t^n + p_{k_2}(x)\, t^{n-1} + \cdots + p_{k_n}(x)\, t + c = 0\,. \qquad (2.1)$$

The $p_{k_i}(x)$ are polynomials of order k_i in x, and c is a constant that depends on the dynamical scales Λ_i of the different gauge group factors. In this expression the variables x and t correspond to the combinations $x^4 + ix^5$ and $e^{-(x^6+ix^{10})/R}$ in the notation of [2].

For the case of $SU(2) \times SU(2)$ with a massless hypermultiplet in the bifundamental representation, the explicit expressions for the polynomials $p_{k_i}(x)$ in (2.1) have been derived by using symmetries and classical limits [7]. The curve for the massive case is given by

$$P(x,t) = {\Lambda_1}^2 t^3 + t^2 \left[x^2 - u + \frac{{\Lambda_1}^2}{2} \right] + t \left[(x+m)^2 - v + \frac{{\Lambda_2}^2}{2} \right] + {\Lambda_2}^2 = 0. \qquad (2.2)$$

Here u and v are the moduli of the two gauge group factors, Λ_i are the corresponding dynamical scales, and m is the bare mass of the hypermultiplet. Note that the dimensionless parameter $q = \frac{{\Lambda_2}^2}{{\Lambda_1}^2}$ cannot be eliminated from the curve by rescaling of the moduli, unless $q = 1$.

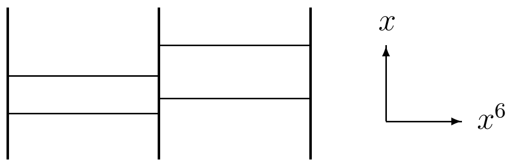

Figure 1: The brane configuration that gives rise to a four-dimensional field theory with gauge group $SU(2) \times SU(2)$ and a hypermultiplet in the bifundamental representation. Vertical lines represent five-branes, horizontal lines are four-branes.

Exchanging the two gauge group factors, i.e. $u \leftrightarrow v$, $\Lambda_1 \leftrightarrow \Lambda_2$, $t \leftrightarrow \frac{1}{t}$, and $x \leftrightarrow -(x+m)$, leaves the curve (2.2) invariant. Pulling the rightmost five-brane to $x^6 = \infty$ gives the brane configuration of a theory with gauge group $SU(2)$ and two fundamental flavours with masses m_1 and m_2. To make this transformation manifest in the curve one has to perform the limit $\Lambda_2 \to 0$, $v \to \frac{1}{4}(m_1 - m_2)^2$ and $m \to \frac{1}{2}(m_1 + m_2)$. The theory with pure gauge group

$SU(2)$ can be obtained by moving the two four-branes of one gauge group factor to $x = \infty$, so that the resulting configuration consists only of two four-branes stretched between two five-branes. To obtain the curve of this theory we have to take the limit $v \to \infty$, $m \to \infty$ while keeping ${\Lambda_1}^2(m^2 - v) = \Lambda_0^4$ fixed.

In the case of gauge group $SU(2) \times SU(2)$ the polynomials $p_1(x)$ and $p_2(x)$ are quadratic in x; we can therefore rewrite (2.2) in hyperelliptic form by redefining[1] $x \to \frac{i\,y}{\sqrt{2}\,t\,(1+t)}$:

$$y^2 = t\,(1+t)\left(2\,{\Lambda_1}^2\,t^3 + t^2\,({\Lambda_1}^2 - 2\,u) + t\,({\Lambda_2}^2 - 2\,v) + 2\,{\Lambda_2}^2\right). \qquad (2.3)$$

The discriminant Δ of this curve consists of two factors, $\Delta = \Delta_1\,\Delta_2$, with

$$\Delta_1 = 4\left(u - v + \frac{1}{2}{\Lambda_1}^2 - \frac{1}{2}{\Lambda_2}^2\right)^2, \qquad (2.4)$$

$$\begin{aligned}\Delta_2 = {} & 8\,{\Lambda_1}^6{\Lambda_2}^2 + 359\,{\Lambda_1}^4{\Lambda_2}^4 + 8\,{\Lambda_1}^2{\Lambda_2}^6 - 48\,{\Lambda_1}^4{\Lambda_2}^2\,u + 148\,{\Lambda_1}^2{\Lambda_2}^4\,u + \\ & 96\,{\Lambda_1}^2\,{\Lambda_2}^2\,u^2 - 4\,{\Lambda_2}^4\,u^2 - 64\,{\Lambda_2}^2\,u^3 + 148\,{\Lambda_1}^4\,{\Lambda_2}^2\,v - 48\,{\Lambda_1}^2\,{\Lambda_2}^4\,v - \\ & 304\,{\Lambda_1}^2\,{\Lambda_2}^2\,u\,v + 16\,{\Lambda_2}^2\,u^2\,v - 4\,{\Lambda_1}^4\,v^2 + 96\,{\Lambda_1}^2\,{\Lambda_2}^2\,v^2 + \\ & 16\,{\Lambda_1}^2\,u\,v^2 - 16\,u^2\,v^2 - 64\,{\Lambda_1}^2\,v^3. \end{aligned} \qquad (2.5)$$

As expected, Δ_1 and Δ_2 are symmetric under the exchange of the two gauge group factors. In general the vanishing of discriminant factors describes points in moduli space where extra massless states appear. Following the analysis of [7], the vanishing of Δ_1 determines the subspace of the moduli space where the Coulomb branch meets the Higgs branch and the gauge group $SU(2) \times SU(2)$ is broken to the diagonal $SU(2)$. Therefore the extra massless states that occur at this point are two components of the bifundamental hypermultiplet. The vanishing of the second discriminant factor describes the singular locus where other massless states appear, in particular monopoles.

The classical intersection point of the Coulomb branch and the Higgs branch at $u = v$ gets shifted by non-perturbative effects if the two scales do not have equal values, i.e. if $q \neq 1$. This dependence is inferred by the terms $\frac{{\Lambda_i}^2}{2}$ in the curve (2.2) and could formally be avoided by choosing the defining polynomial to be

$$P(x,t) = {\Lambda_1}^2\,t^3 + t^2\left(x^2 - u + {\Lambda_1}^2\right) + t\left(x^2 - v + {\Lambda_2}^2\right) + {\Lambda_2}^2 = 0. \qquad (2.6)$$

[1] In the following we set the bare mass m of the hypermultiplet to zero.

Then the two scales no longer explicitly appear in Δ_1. In order to reduce (2.6) to the curve for $SU(2)$ with two flavours we then have to perform a shift in the modulus u in addition to the above-stated transformation: $u \to \bar{u} = u + \frac{7\Lambda_1{}^2}{8}$. This means that the origins of the two moduli spaces would simply be shifted by $\frac{7\Lambda_1{}^2}{8}$, though classically the moduli still coincide.

The polynomial (2.3) describes a Riemann surface § of genus 2. If we choose a symplectic homology basis of §, i.e. α_i and β_i $(i = 1, 2)$ with intersection pairing $\alpha_i \cap \beta_j = \delta_{ij}$, $\alpha_i \cap \alpha_j = \beta_i \cap \beta_j = 0$, we can define the period integrals in terms of a properly chosen meromorphic one-form λ:

$$a_i = \int_{\alpha_i} \lambda, \qquad\qquad a_{Di} = \int_{\beta_i} \lambda. \tag{2.7}$$

The periods are functions of the moduli u and v, $a_i(u, v)$ is identified with the scalar component of the $\mathcal{N} = 1$ chiral multiplet in the i-th gauge group factor, $a_{Di}(u, v)$ is its dual. Once the periods a_i and a_{Di} are known, the exact quantum-corrected prepotential $\mathcal{F}(a_k)$ of the theory can be computed by integrating the relations

$$a_{Di}(a_k) = \frac{\partial \mathcal{F}(a_k)}{\partial a_i}. \tag{2.8}$$

The second-order derivatives of $\mathcal{F}$ with respect to a_i imply the integrability condition

$$\frac{\partial a_{Di}(a_k)}{\partial a_j} = \frac{\partial a_{Dj}(a_k)}{\partial a_i}. \tag{2.9}$$

The meromorphic one-form λ of the curve (2.2) is given by [1]

$$\lambda \propto \frac{x\, dt}{t}\,, \tag{2.10}$$

or, after the redefinition $x \to \frac{i\, y}{\sqrt{2}\, t\, (1+t)}$, by

$$\lambda \propto \frac{y\, dt}{t^2\, (1+t)}\,, \tag{2.11}$$

where the proportionality factor is determined by the requirements

$$\frac{\partial \lambda}{\partial v} = \frac{dt}{y} \quad \text{and} \quad \frac{\partial \lambda}{\partial u} = \frac{t\, dt}{y}\,. \tag{2.12}$$

It is rather tedious to perform the integrals (2.7) explicitly. However, the periods as functions of the moduli u and v satisfy a system of partial differential equations, the so-called Picard–Fuchs equations. This method to obtain the periods is relatively straightforward and will be the subject of the next sections.

3 The Picard–Fuchs Equations

Starting with the hyperelliptic form (2.3) of the curve[2] for $SU(2) \times SU(2)$, we will derive the system of partial differential equations for the periods $\int_{\alpha_i} \lambda$ and $\int_{\beta_i} \lambda$.

On a Riemann surface § of genus 2 there are two holomorphic differentials and two meromorphic differentials with no residues (abelian differentials of first and second kind) [10]. Therefore we can choose $\{\frac{dx}{y}, \frac{x\,dx}{y}, \frac{x^2\,dx}{y}, \frac{x^3\,dx}{y}\}$ as basis of meromorphic forms on §. When considering derivatives of the meromorphic one-form λ with respect to the moduli, only four of them will be linearly independent up to exact forms. The linear relations between these derivatives define the Picard–Fuchs equations.

Derivatives of λ with respect to u and v involve terms of the form $\frac{\phi(x)\,dx}{y^n}$ with some polynomials $\phi(x)$. In order to express $\frac{\phi(x)\,dx}{y^n}$ in terms of abelian differentials of first and second kind, we need a method to reduce high powers of x in the numerator and high powers of y in the denominator. For the latter we use the fact that the discriminant of a polynomial $p(x)$ of order n can always be written in the form [11]

$$\Delta = a(x)\,p(x) + b(x)\,p'(x), \tag{3.1}$$

where $a(x)$, resp. $b(x)$, is a polynomial of order $(n-2)$, resp. $(n-1)$, in x. With this formula it can easily be shown that the following relation holds up to exact forms:

$$\frac{\phi(x)}{y^n} = \frac{1}{\Delta}\,\frac{a(x)\,\phi(x) + \frac{2}{n-2}\,\frac{d}{dx}\,(b(x)\,\phi(x))}{y^{n-2}}. \tag{3.2}$$

[2]For convenience we replace t by x in the following.

Following [11] we derive a formula to reduce high powers of x that is valid up to addition of total derivatives:

$$\begin{aligned}\frac{x^n}{y} &= -\frac{(2n-7)}{(2n-3)}\frac{{\Lambda_2}^2}{{\Lambda_1}^2}\frac{x^{n-4}}{y} - \frac{(n-3)}{(2n-3)}\frac{(3\,{\Lambda_2}^2 - 2\,v)}{{\Lambda_1}^2}\frac{x^{n-3}}{y} - \\ &\quad \frac{(2n-5)}{(2n-3)}\frac{({\Lambda_1}^2 + {\Lambda_2}^2 - 2\,u - 2\,v)}{2{\Lambda_1}^2}\frac{x^{n-2}}{y} - \\ &\quad \frac{(n-2)}{(2n-3)}\frac{(3\,{\Lambda_1}^2 - 2\,u)}{{\Lambda_1}^2}\frac{x^{n-1}}{y}\,. \end{aligned} \tag{3.3}$$

With these formulae we can re-express the meromorphic one-form (2.11) in terms of abelian differentials of first and second kind

$$\begin{aligned}\lambda &= (2\,v - {\Lambda_2}^2)\,\frac{dx}{y} + (4\,u + 2\,v - 2{\Lambda_1}^2 - {\Lambda_2}^2)\,\frac{x\,dx}{y} + \\ &\quad 4\,(u - 2\,{\Lambda_1}^2)\,\frac{x^2\,dx}{y} - 6\,{\Lambda_1}^2\,\frac{x^3\,dx}{y}\,. \end{aligned} \tag{3.4}$$

When we consider derivatives of λ with respect to u and v up to second order, we find that these derivatives satisfy two linear relations as only four out of six are linearly independent. These two relations give rise to the following Picard–Fuchs operators:

$$\begin{aligned}\mathcal{L}_1 &= (5\,{\Lambda_1}^2 + 2\,u)\,\partial_u^2 + 2\,\left({\Lambda_1}^2 - {\Lambda_2}^2 - 2\,u + 2\,v\right)\partial_u\partial_v - \\ &\quad (5\,{\Lambda_2}^2 + 2\,v)\,\partial_v^2 + \partial_u - \partial_v, \end{aligned} \tag{3.5}$$

$$\begin{aligned}\mathcal{L}_2 &= ({\Lambda_1}^4 + 14\,{\Lambda_1}^2\,{\Lambda_2}^2 - 4\,u^2 - 4\,{\Lambda_1}^2\,v)\,\partial_u^2 - \\ &\quad (14\,{\Lambda_1}^2\,{\Lambda_2}^2 + {\Lambda_2}^4 - 4\,{\Lambda_2}^2\,u - 4\,v^2)\,\partial_v^2 - \\ &\quad 2\,\left(15\,{\Lambda_1}^2\,{\Lambda_2}^2 - 2\,{\Lambda_2}^2\,u - 2\,{\Lambda_1}^2\,v - 4\,u\,v\right)\partial_u\partial_v + 1. \end{aligned} \tag{3.6}$$

Note that the first (second) operator is antisymmetric (symmetric) under the exchange of the two gauge group factors.

4 The Periods and the Prepotential

We are interested in the prepotential of $SU(2) \times SU(2)$ supersymmetric Yang–Mills theory in the weak coupling regime, which is parametrized by large values of the moduli u and v. As usual, if we have two variables that simultaneously become large, we have to specify an appropriate coordinate patch in which the limit is well-defined. By inspection of the first discriminant factor (2.4), we choose the variables to be

$$z_1 = \frac{{\Lambda_1}^2}{u} \quad \text{and} \quad z_2 = \frac{u - v + \frac{1}{2}({\Lambda_1}^2 - {\Lambda_2}^2)}{{\Lambda_1}^2}. \tag{4.1}$$

This choice is not symmetric under the exchange of the two gauge group factors, but of course there exists the analogous patch in the moduli space where the rôle of the two moduli is interchanged. Using Frobenius' method, we find two series solutions and two logarithmic solutions with indices $\left(-\frac{1}{2}, 0\right)$ and $\left(\frac{1}{2}, 1\right)$ in the coordinate patch of the moduli space parametrized by (4.1):

$$w_1(z_1, z_2) = z_1^{-\frac{1}{2}} \sum_{i,j=0}^{\infty} a_{ij}\, z_1^i\, z_2^j, \tag{4.2}$$

$$w_2(z_1, z_2) = z_1^{\frac{1}{2}} z_2 \sum_{i,j=0}^{\infty} b_{ij}\, z_1^i\, z_2^j, \tag{4.3}$$

$$w_3(z_1, z_2) = w_1(z_1, z_2) \ln{(z_1)} + z_1^{-\frac{1}{2}} \sum_{i,j=0}^{\infty} c_{ij}\, z_1^i\, z_2^j, \tag{4.4}$$

$$w_4(z_1, z_2) = w_2(z_1, z_2) \ln{(z_1^3 z_2^2)} + z_1^{-\frac{1}{2}} \sum_{i,j=0}^{\infty} d_{ij}\, z_1^i\, z_2^j. \tag{4.5}$$

The first few coefficients of these series expansions are listed in the appendix.

In order to get the periods we have to consider linear combinations of the solutions that match the infrared asymptotic behaviour. To analyse these leading terms one can either evaluate the lowest order of the integrals (2.7) explicitly or one uses semi-classical relations of the gauge couplings and the integrability condition (2.9).

The gauge couplings τ_{ij} of the theory are related to the prepotential $\mathcal{F}$ by

$$\tau_{ij} = \frac{\partial^2 \mathcal{F}(a_k)}{\partial a_i \partial a_j} = \frac{\partial a_{Di}}{\partial a_j}. \tag{4.6}$$

The one-loop terms of the pure couplings τ_{ii} consist of the logarithmic contributions of the massless spectrum. In each gauge group factor there is a gauge field in the adjoint representation; moreover there is a hypermultiplet in the bifundamental representation. Therefore the one-loop contributions of the pure gauge couplings are given by

$$\begin{aligned} \tau_{11} &\sim -4\ln(a_1) + \ln({a_1}^2 - {a_2}^2), \\ \tau_{22} &\sim -4\ln(a_2) + \ln({a_1}^2 - {a_2}^2). \end{aligned} \tag{4.7}$$

In order to examine the asymptotic behaviour of the periods, we integrate the gauge couplings τ_{ii} in (4.7) to obtain $a_{Di}(a_k) = \int \tau_{ii}(a_k)\, da_i$, introduce a new variable $\varepsilon = a_1 - a_2$, and expand around the point $(a_1, \varepsilon) = (\infty, 0)$. This procedure gives the leading terms of the periods a_{Di}:

$$\begin{aligned} a_{D1} &\sim -2\, a_1\, \ln(a_1) - \varepsilon\, \ln(a_1) + \varepsilon\, \ln(\varepsilon) + \dots\,, \\ a_{D2} &\sim -2\, a_1\, \ln(a_1) + 3\,\varepsilon\, \ln(a_1) - \varepsilon\, \ln(\varepsilon) + \dots\,. \end{aligned} \tag{4.8}$$

To compare the solutions (4.4) and (4.5) with these expansions, we have to replace the moduli u and v by the corresponding classical expressions $u = {a_1}^2$ and $v = {a_2}^2$. We apply the same procedure as above to a linear combination of the logarithmic solutions $A\, w_3 + B\, w_4$ and find, in leading order:

$$-2\, A\, a_1\, \ln(a_1) + (A - 8\, B)\, \varepsilon\, \ln(a_1) + 4\, B\, \varepsilon\, \ln(\varepsilon) + \dots\,. \tag{4.9}$$

Hence the periods a_{Di} are reproduced by the following linear combinations of the solutions

$$\begin{aligned} \frac{a_{D1}}{\Lambda_1} &= A_1 w_1 + B_1 w_2 + w_3 + \frac{1}{4} w_4, \\ \frac{a_{D2}}{\Lambda_1} &= A_2 w_1 + B_2 w_2 + w_3 - \frac{1}{4} w_4, \end{aligned} \tag{4.10}$$

where the coefficients A_1, A_2, B_1, B_2 are still undetermined. We also find that the periods a_i are given by

$$\begin{aligned} \frac{a_1}{\Lambda_1} &= w_1 + \frac{1}{4} w_2, \\ \frac{a_2}{\Lambda_1} &= w_1 - \frac{1}{4} w_2. \end{aligned} \tag{4.11}$$

It can easily be checked that in the limits of $SU(2)$ $N_f = 0$, resp. $N_f = 2$, stated in section 2, a_1 reduces to the corresponding periods known in the literature. By imposing the integrability condition (2.9), one finds that $A_2 = A_1$ and $B_2 = -B_1$.

The prepotential can be computed by integrating the equations (2.8). To this end we have to know the magnetic periods a_{Di} as functions of the electric periods a_k. By inverting the series expansions of a_k in (4.11) we get the variables z_i and in consequence the magnetic periods as functions of a_k:

$$\begin{aligned} a_{D1}(a_1,a_2) &= a_1\left(\frac{1}{2}A_1+2B_1+\ln(2)\right)+ \\ &\quad a_2\left(\frac{1}{2}A_1-2B_1-\ln(2)\right)- \\ &\quad 4a_1\ln(a_1)+(a_1+a_2)\ln(a_1+a_2)+(a_1-a_2)\ln(a_1-a_2)+ \\ &\quad \frac{1}{2}\frac{\Lambda_1{}^2}{a_1}\left(\frac{a_2{}^2}{a_1{}^2}-q\frac{a_1{}^2}{a_2{}^2}\right)+ \\ &\quad \frac{\Lambda_1{}^4}{32}\left[2\frac{1}{a_1{}^7}\left(a_1{}^4-12a_1{}^2a_2{}^2+15a_2{}^4\right)-\right. \\ &\qquad \left. 8q\frac{1}{a_1{}^3}+q^2\frac{a_1}{a_2{}^6}\left(5a_1{}^2-3a_2{}^2\right)\right]+\ldots \end{aligned} \tag{4.12}$$

$$\begin{aligned} a_{D2}(a_1,a_2) &= a_1\left(\frac{1}{2}A_1-2B_1-\ln(2)\right)+ \\ &\quad a_2\left(\frac{1}{2}A_1+2B_1+\ln(2)\right)- \\ &\quad 4a_2\ln(a_2)+(a_1+a_2)\ln(a_1+a_2)-(a_1-a_2)\ln(a_1-a_2)- \\ &\quad \frac{1}{2}\frac{\Lambda_1{}^2}{a_2}\left(\frac{a_2{}^2}{a_1{}^2}-q\frac{a_1{}^2}{a_2{}^2}\right)- \\ &\quad \frac{\Lambda_1{}^4}{32}\left[\frac{a_2}{a_1{}^6}\left(3a_1{}^2-5a_2{}^2\right)+8q\frac{1}{a_2{}^3}-\right. \\ &\qquad \left. 2q^2\frac{1}{a_2{}^7}\left(15a_1{}^4-12a_1{}^2a_2{}^2+a_2{}^4\right)\right]+\ldots \end{aligned} \tag{4.13}$$

Notice that exchanging a_1 and a_2 transforms a_{D1} into a_{D2} and vice versa. Integrating the expression (4.12), resp. (4.13), with respect to a_1, resp. a_2, yields the prepotential

$$\mathcal{F}=\mathcal{F}_{class}+\mathcal{F}_{1-loop}+\sum_{n=1}^{\infty}\mathcal{F}_{n-inst}\,, \tag{4.14}$$

with the following expressions for the first few terms:

$$
\begin{aligned}
\mathcal{F}_{class} &= ({a_1}^2+{a_2}^2)\left(\frac{1}{2}+\frac{1}{4}\,A_1+B_1+\frac{1}{2}\,\ln{(2)}\right)+ \\
&\quad a_1a_2\left(\frac{1}{2}\,A_1-2\,B_1-\ln{(2)}\right)\,, \qquad (4.15)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_{1-loop} &= -2\,{a_1}^2\,\ln{(a_1)}-2\,{a_2}^2\,\ln{(a_2)}+ \\
&\quad \frac{1}{2}(a_1+a_2)^2\ln{(a_1+a_2)}+\frac{1}{2}(a_1-a_2)^2\ln{(a_1-a_2)}\,, \qquad (4.16)
\end{aligned}
$$

$$
\mathcal{F}_{1-inst} = \frac{{\Lambda_1}^2}{4}\left({a_1}^2-{a_2}^2\right)\left(\frac{1}{{a_1}^2}-q\,\frac{1}{{a_2}^2}\right)\,, \qquad (4.17)
$$

$$
\begin{aligned}
\mathcal{F}_{2-inst} &= -\frac{{\Lambda_1}^4}{4}\,q\,\frac{1}{{a_1}^2}-\frac{{\Lambda_1}^4}{128}\left({a_1}^2-{a_2}^2\right)\left[\frac{{a_2}^2}{{a_1}^4}\left(\frac{5}{{a_1}^2}-\frac{1}{{a_2}^2}\right)-\right. \\
&\quad \left. 16\,q\,\frac{1}{{a_1}^2\,{a_2}^2}+q^2\,\frac{{a_1}^2}{{a_2}^4}\left(\frac{1}{{a_1}^2}-\frac{5}{{a_2}^2}\right)\right]\,, \qquad (4.18)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}_{3-inst} &= -\frac{{\Lambda_1}^6}{384}\,({a_1}^2-{a_2}^2)\left[\frac{{a_2}^4}{{a_1}^8}\left(\frac{9}{{a_1}^2}-\frac{5}{{a_2}^2}\right)+3\,q\,\frac{1}{{a_1}^4}\left(\frac{5}{{a_1}^2}-\frac{1}{{a_2}^2}\right)+\right. \\
&\quad \left. 3\,q^2\,\frac{1}{{a_2}^4}\left(\frac{1}{{a_1}^2}-\frac{5}{{a_2}^2}\right)+q^3\,\frac{{a_1}^4}{{a_2}^8}\left(\frac{5}{{a_1}^2}-\frac{9}{{a_2}^2}\right)\right]\,. \qquad (4.19)
\end{aligned}
$$

As expected, all contributions to the prepotential are totally symmetric under the exchange of the two gauge group factors. The constants A_1 and B_1 are determined by the classical coupling constants. The one-loop contribution $\mathcal{F}_{1-loop}$ is in agreement with the corresponding term of the prepotential reported in [8] and [9].

Up to an overall scaling factor of the two scales, $\mathcal{F}_{1-inst}$ coincides with the one-instanton term in the prepotential of [9]. If the bare mass m of the hypermultiplet in the prepotential obtained in [8] is set to zero, the one-instanton term coincides with $\mathcal{F}_{1-inst}$ for $q=1$, but the terms proportional to q in $\mathcal{F}_{2-inst}$ are not present. This is a priori no contradiction, since sending the bare mass to zero is a singular limit.

The instanton terms cannot be written as simple sums of contributions stemming from the two subgroups, but show a non-trivial mixing of a_1 and a_2, as it is expected in the presence of a hypermultiplet in the bifundamental representation. In particular the n-th instanton term is not just a sum of contributions proportional to ${\Lambda_i}^{2n}$, but all possible combinations ${\Lambda_1}^{2i}\,{\Lambda_2}^{2j}$ with $i + j = n$ appear in $\mathcal{F}_{n-inst}$. Terms proportional to odd powers of the scales are forbidden as they would violate the anomaly-free discrete symmetry of the theory [12].

It is quite striking that nearly all terms in the instanton expansion of the prepotential are proportional to $({a_1}^2 - {a_2}^2)$ and therefore vanish for $a_1 = \pm\, a_2$. In either of the two cases two components of the hypermultiplet become massless and one expects to recover the point in moduli space where the Coulomb branch meets the Higgs branch and the gauge group is broken to the diagonal $SU(2)$. Continuity of the prepotential at the intersection point ensures that $\mathcal{F}$ collapses to the prepotential of pure $SU(2)$. Indeed, if one sets $a_1 = \pm\, a_2 = a$ in the prepotential (4.14), the non-vanishing terms reproduce the prepotential of $SU(2)$ theory with the dynamical scale $\Lambda^2 = 4\,\Lambda_1\Lambda_2$ to all calculated orders.

In [7] a different argumentation, namely considerations of the brane configuration and the curve, led to the insight that the same point in moduli space is determined by the equation $u + \frac{1}{2}{\Lambda_1}^2 = v + \frac{1}{2}{\Lambda_2}^2$. At first sight it is quite astounding that this condition is equivalent to ${a_1}^2 = {a_2}^2$, but it is clear that these two relations describe the same physical scenario.

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Appendix Coefficients of the Solutions

In table 1 the non-vanishing coefficients of the series solutions (4.2) and (4.3) are listed up to fourth order in the variables $z_1 = \frac{{\Lambda_1}^2}{u}$ and $z_2 = \frac{u-v+\frac{1}{2}({\Lambda_1}^2-{\Lambda_2}^2)}{{\Lambda_1}^2}$. In table 2 the corresponding coefficients for the logarithmic solutions (4.4) and (4.5) are listed.

i	j	a_{ij}	b_{ij}
0	0	1	1
1	0	$\frac{1}{4}$	$-\frac{3}{4}\left(1+\frac{2}{3}q\right)$
2	0	$-\frac{1}{32}(1+8q)$	$\frac{27}{32}\left(1+\frac{4}{3}q+\frac{4}{9}q^2\right)$
3	0	$\frac{1}{128}(1+24q)$	$-\frac{135}{128}\left(1+2q+\frac{4}{3}q^2+\frac{8}{27}q^3\right)$
4	0	$-\frac{5}{2048}(1+48q+96q^2)$	$\frac{2835}{2048}\left(1+\frac{8}{3}q+\frac{8}{3}q^3+\frac{32}{27}q^3+\frac{16}{81}q^4\right)$
1	1	$-\frac{1}{4}$	$\frac{1}{4}$
2	1	$-\frac{1}{16}(1-2q)$	$-\frac{3}{16}(1+4q)$
3	1	$\frac{21}{128}\left(1-\frac{12}{7}q-\frac{4}{7}q^2\right)$	$-\frac{45}{128}\left(1-\frac{16}{3}q-4q^2\right)$
2	2	$-\frac{1}{16}$	$\frac{1}{8}$

Table 1: The non-vanishing coefficients of the series solutions.

i	j	c_{ij}	d_{ij}
0	0	1	1
1	0	$-\frac{1}{4}$	$\frac{9}{4}\left(1-\frac{8}{9}q\right)$
2	0	$-\frac{1}{32}$	$-\frac{25}{32}\left(1-\frac{8}{25}q-\frac{8}{25}q^2\right)$
3	0	$\frac{5}{384}\left(1+\frac{48}{5}q\right)$	$\frac{179}{384}\left(1+\frac{288}{179}q-\frac{360}{179}q^2-\frac{32}{179}q^3\right)$
4	0	$-\frac{31}{6144}\left(1+\frac{768}{31}q+\frac{816}{31}q^2\right)$	$-\frac{2095}{6144}\left(1+\frac{1584}{419}q-\frac{1008}{419}q^2-\frac{512}{419}q^3-\frac{48}{419}q^4\right)$
1	1	$\frac{1}{40}$	$\frac{2}{15}$
2	1	$\frac{37}{160}\left(1+\frac{18}{37}q\right)$	$-\frac{87}{20}\left(1+\frac{214}{261}q\right)$
3	1	$-\frac{193}{1280}\left(1+\frac{564}{193}q+\frac{128}{193}q^2\right)$	$\frac{141}{20}\left(1+\frac{117}{94}q+\frac{289}{564}q^2\right)$
2	2	$-\frac{29}{160}$	$\frac{23}{15}$

Table 2: The non-vanishing coefficients of the logarithmic solutions.