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SYMMETRY BREAKING BOUNDARIES
II. MORE STRUCTURES; EXAMPLES

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Abstract

Various structural properties of the space of symmetry breaking boundary conditions that preserve an orbifold subalgebra are established. To each such boundary condition we associate its automorphism type. We show that correlation functions in the presence of such boundary conditions are expressible in terms of twisted boundary blocks which obey twisted Ward identities. The subset of boundary conditions that share the same automorphism type is controlled by a classifying algebra, whose structure constants are shown to be traces on spaces of chiral blocks. T-duality on boundary conditions is not a one-to-one map in general.

These structures are illustrated in a number of examples. Several applications, including the construction of non-BPS boundary conditions in string theory, are exhibited.

Introduction

The study of conformally invariant boundary conditions in two-dimensional conformal field theory is of considerable interest both for applications in condensed matter physics and in string theory. In such applications typically only the (super-)conformal symmetry needs to be preserved by the boundaries, while the rest of the chiral bulk symmetries $\mathfrak{A}$ may be broken.

In [1] we have studied conformally invariant boundary conditions for an arbitrary conformal field theory that preserve a (consistent) subalgebra $\bar{\mathfrak{A}}$ of $\mathfrak{A}$, such that

$$\bar{\mathfrak{A}} = \mathfrak{A}^G$$

is the subalgebra that is fixed under a finite abelian group G of automorphisms of $\mathfrak{A}$. We have shown that such boundary conditions are governed by a *classifying algebra* $\mathcal{C}(\bar{\mathfrak{A}})$, in the sense that the reflection coefficients [2, 3] – the data that characterize the boundary condition – are precisely the one-dimensional irreducible representations of $\mathcal{C}(\bar{\mathfrak{A}})$.

This paper is a continuation to [1]. Our numbering of sections will be consecutive, i.e. by sections 1 to 6 we refer to those of the preceding paper, while here we will start with section 7; accordingly equation numbers with section label ≤ 6 refer to formulas in [1].[1] A very brief summary of the pertinent results of [1] is as follows. The algebra $\mathcal{C}(\bar{\mathfrak{A}})$ is a commutative associative semisimple algebra. Thus its regular representation is fully reducible, and the structure constants are expressible through the corresponding diagonalizing matrix $\tilde{S}$ by an analogue of the Verlinde formula. This matrix $\tilde{S}$, in turn, can be expressed in terms of various quantities that are already known from the chiral conformal field theory associated to $\bar{\mathfrak{A}}$ (see formula $(5.9)_A$).

The conformal field theory with chiral algebra $\bar{\mathfrak{A}}$ can be obtained from the $\mathfrak{A}$-theory as an *orbifold* by the group G; conversely, the original $\mathfrak{A}$-theory is recovered from the $\bar{\mathfrak{A}}$-theory as an integer spin *simple current extension*, with the group $\mathcal{G}$ of simple currents being the character group of the orbifold group, $\mathcal{G} = G^*$. Denoting the labels for the primary fields of the $\mathfrak{A}$-theory by λ and those for the primary $\bar{\mathfrak{A}}$-fields by $\bar{\lambda}$, a natural basis of $\mathcal{C}(\bar{\mathfrak{A}})$ is labelled by pairs $(\bar{\lambda}, \varphi)$, where $\bar{\lambda}$ refers to a $\bar{\mathfrak{A}}$-primary in the untwisted sector and $\varphi \in \mathcal{S}_\lambda^*$ is a group character, while the boundary conditions are labelled by pairs $[\bar{\rho}, \hat{\psi}_\rho]$ consisting of an arbitrary primary label $\bar{\rho}$ of the $\bar{\mathfrak{A}}$-theory and a character $\hat{\psi}_\rho \in \mathcal{U}_\lambda^*$. Here the stabilizer $\mathcal{S}_\lambda$ and the untwisted stabilizer $\mathcal{U}_\lambda$ are subgroups of the simple current group $\mathcal{G} = G^*$; $\mathcal{S}_\lambda$ consists of all simple currents in $\mathcal{G}$ that leave $\bar{\lambda}$ fixed, and $\mathcal{U}_\lambda$ is the subgroup of $\mathcal{S}_\lambda$ on which a certain alternating bi-homomorphism F_λ, which can be defined through the modular properties of one-point chiral blocks on the torus, is trivial. For the precise meaning of these terms we refer to [1] (in particular appendix A) and to [4]. Quite generally, the number of basis elements of $\mathcal{C}(\bar{\mathfrak{A}})$ – or equivalently, the number of independent *boundary blocks*, i.e. chiral blocks for one-point correlation functions of bulk fields on the disk – and the number of boundary conditions have to be equal. It is a rather non-trivial result of the analysis that this equality indeed holds for the two sets of labels, so that in particular the matrix $\tilde{S} \equiv \tilde{S}_{(\bar{\lambda},\varphi),[\bar{\rho},\hat{\psi}_\rho]}$ is a square matrix.

The results established in [1] clearly demonstrate an unexpectedly nice behavior of the space of conformally invariant boundary conditions. In the present paper we show that indeed

[1] A few of those formulas are, however, reproduced in the appendix of the present paper. This is indicated by an additional subscript 'A' of the equation number.

this space is endowed with even more structure. The following section 7 deals with the implementation of the orbifold group on the representation spaces of the chiral algebra $\mathfrak{A}$; this discussion does not yet involve boundary conditions at all, but the results will be needed in the sequel. Sections 8 – 13 are devoted to the discussion of several additional generic features of symmetry breaking boundary conditions. We start in section 8 by analyzing the notion of *automorphism type*, which in the present setting is a derived concept that arises as a direct consequence of the general structure and does not need be introduced by hand. Next we show that boundary conditions of definite automorphism type can be naturally formulated with the help of *twisted boundary blocks*, which satisfy twisted Ward identities (section 9). Furthermore, to the boundary conditions of fixed automorphism type one can associate their own classifying algebra, which is an invariant subalgebra of the total classifying algebra $\mathcal{C}(\bar{\mathfrak{A}})$ and whose structure constants can be understood in terms of suitable traces on chiral blocks; this is done in section 10. It is also shown that the individual classifying algebra for automorphism type g only depends on the automorphism g, but not on the specific orbifold subalgebra $\bar{\mathfrak{A}}$, i.e. not on the group G containing g.

Afterwards, in section 11, we turn to a detailed study of the dependence of the classifying algebra on the chosen torus partition function, which leads to the concept of T-*duality* of boundary conditions. First we show that only the 'difference' between an automorphism characterizing the torus partition function and the automorphism type of a boundary condition is observable, and then we discuss aspects of T-duality among (families of) boundary conditions for fixed choice of the torus partition function. We emphasize that T-duality on boundary conditions is not a one-to-one map, in general. In section 12 we establish an action of the orbifold group G on the space of boundary conditions, which implies a certain 'homogeneity' among the boundary conditions for fixed $\mathcal{G}$-orbit $[\bar{\rho}]$. Finally we introduce in section 13 the concept of a *universal* classifying algebra, which governs *all* conformally invariant boundary conditions at the same time, and discuss the possibility to obtain this algebra by a suitable projective limit.

In sections 14 and 15 we turn our attention to a specific class of boundary conditions to which we refer as involutary, namely those where the orbifold group is $\mathbb{Z}_2$. We first address some general features and then, in section 15, analyze several classes of examples that are of particular interest. Afterwards, in section 16 we provide several classes of examples with more complicated orbifold groups, in which for instance untwisted stabilizer subgroups occur that are proper subgroups of the full stabilizers. Finally, some of the pertinent formulae from [1] that will be needed in the sequel are collected in appendix A.

7 The action of the orbifold group on $\mathfrak{A}$-modules

The elements g of the orbifold group G are automorphisms of the chiral algebra $\mathfrak{A}$. In the sequel we will have to deal with various subgroups of G and their properties. We first observe that every automorphism g of $\mathfrak{A}$ can be implemented on the physical $\mathfrak{A}$-modules $\mathcal{H}_\lambda$ by maps

$$\Theta_g \equiv \Theta_g^{(\lambda)} : \quad \mathcal{H}_\lambda \to \mathcal{H}_{g^\star\lambda} , \tag{7.1}$$

which obey the g-twisted intertwining property

$$\Theta_g \, Y = g(Y) \, \Theta_g \qquad \text{for all } \; Y \in \mathfrak{A} , \tag{7.2}$$

and the maps Θ_g are defined by this property up to a scalar multiple. (For a concrete realization of these maps in WZW theories, see [5].) In general, such an implementation $\Theta_g^{(\lambda)}$ maps a given space $\mathcal{H}_\lambda$ to some other $\mathfrak{A}$-module $\mathcal{H}_{g^\star\lambda}$, thereby organizing the primary fields of the $\mathfrak{A}$-theory into orbits, much like the simple current group $\mathcal{G}\cong G^*$ organizes [6] the $\bar{\mathfrak{A}}$-primaries into orbits.

To every $\mathfrak{A}$-primary λ we can associate the *stabilizer*

$$S_\lambda := \{g\in G \mid g^\star\lambda=\lambda\}\,, \tag{7.3}$$

which is a subgroup of G whose elements constitute *endo*morphisms $\mathcal{H}_\lambda\to\mathcal{H}_\lambda$. Stabilizers of $\mathfrak{A}$-primaries on the same G-orbit are identical (in the more general case of non-abelian G, they are conjugate subgroups); the vacuum has a maximal stabilizer, $S_\Omega=G$. Also, $\mathfrak{A}$-modules on the same G-orbit are isomorphic as $\bar{\mathfrak{A}}$-modules. The endomorphisms Θ_g for $g\in S_\lambda$ provide us with an action of S_λ on $\mathcal{H}_\lambda$ which is, in general, only projective, and hence determines a two-cocycle $\mathcal{E}_\lambda$ of S_λ or, more precisely (in agreement with the fact that the maps Θ_g are defined only up to normalization), the cohomology class of $\mathcal{E}_\lambda$. We denote by U_λ the subgroup of S_λ that corresponds to the regular elements of the associated twisted group algebra $\mathbb{C}_{\mathcal{E}_\lambda}S_\lambda$, i.e.

$$U_\lambda := \{g\in S_\lambda \mid \mathcal{E}_\lambda(g,g')=\mathcal{E}_\lambda(g',g) \text{ for all } g'\in S_\lambda\}\,. \tag{7.4}$$

The scalar factors in the definition of the implementers Θ_g can be chosen in such a way that the maps Θ_g with $g\in U_\lambda$ provide us with a honest representation of U_λ on $\mathcal{H}_\lambda$. It follows that the $\mathfrak{A}$-modules $\mathcal{H}_\lambda$ can be decomposed as

$$\mathcal{H}_\lambda \cong \bigoplus_{\hat{\jmath}\in U_\lambda^*} V_{\hat{\jmath}}\otimes\bar{\mathcal{H}}_{\bar\lambda,\hat{\jmath}}\,, \tag{7.5}$$

where the spaces $V_{\hat{\jmath}}$ are projective S_λ-modules and the spaces $\bar{\mathcal{H}}_{\bar\lambda,\hat{\jmath}}$ are $\bar{\mathfrak{A}}$-modules. We make the mild technical assumption that all these modules $V_{\hat{\jmath}}$ and $\bar{\mathcal{H}}_{\bar\lambda,\hat{\jmath}}$ are *irreducible*; this holds true in all known examples, and is rigorously proven for the vacuum Ω [7,8] as well as [9] for other $\mathfrak{A}$-modules, including twisted sectors. In the case of the vacuum, no multiplicities appear in this decomposition; thus the action of G is genuine and we have $U_\Omega=S_\Omega=G$.

Since by construction g leaves the subalgebra $\bar{\mathfrak{A}}$ of $\mathfrak{A}$ fixed, the maps Θ_g are *ordinary* intertwiners for $\bar{\mathfrak{A}}$; hence in the decomposition (7.5) they act solely on the degeneracy space $V_{\hat{\jmath}}$. Moreover, by the general properties of twisted group algebras (compare appendix B of [1]), all the spaces $V_{\hat{\jmath}}$ have the same dimension $\sqrt{|S_\lambda|/|U_\lambda|}$, and the basis of the twisted group algebra $\mathbb{C}_{\mathcal{E}_\lambda}S_\lambda$ can be chosen in such a way that every $g\in U_\lambda$ is implemented as a diagonal matrix acting on $V_{\hat{\jmath}}$.

The result (7.5) should be compared to the similar decomposition (3.9) that arises from the simple current point of view, i.e.

$$\mathcal{H}_\lambda \equiv \mathcal{H}_{[\bar\lambda,\hat\psi]} = \bigoplus_{\mathrm{J}\in\mathcal{G}/\mathcal{S}_\lambda} \mathcal{V}_{\hat\psi}\otimes\bar{\mathcal{H}}_{\mathrm{J}\bar\lambda}\,. \tag{7.6}$$

Here the spaces $\mathcal{V}_{\hat\psi}$ are projective $\mathcal{S}_\lambda$-modules, with corresponding cocycle $\mathcal{F}_\lambda$ of $\mathcal{S}_\lambda$, while the spaces $\bar{\mathcal{H}}_{\mathrm{J}\bar\lambda}$ are $\bar{\mathfrak{A}}$-modules; by assumption, the latter modules are irreducible (this assumption is indeed satisfied for all cases we know of). Simple current theory [10, 11, 6, 4] shows that in the decomposition (7.6) isomorphic $\bar{\mathfrak{A}}$-modules appear precisely as a consequence of fixed point resolution; therefore the multiplicity of $\bar{\mathcal{H}}_{\mathrm{J}\bar\lambda}$ in this decomposition is given by $|\mathcal{U}_\lambda^*|$. On the other hand, as seen above, elements of the orbifold group G that are not in the stabilizer S_λ relate isomorphic $\bar{\mathfrak{A}}$-modules. Thus we can identify the groups $\mathcal{U}_\lambda^*$ and G/S_λ. To make this manifest, we dualize the exact sequence $0 \to \mathcal{U}_\lambda \to \mathcal{G}$ and complete it to an exact sequence

$$0 \to S_\lambda \to G = \mathcal{G}^* \to \mathcal{U}_\lambda^* \to 0\,. \tag{7.7}$$

Conversely, in the decomposition of a $\mathfrak{A}$-module $\mathcal{H}_\lambda$ there appear $|U_\lambda^*|$ many irreducible $\bar{\mathfrak{A}}$-modules. In simple current language, the number of irreducibles is just the length of the orbit, and hence again we can dualize $0 \to U_\lambda \to G$ and complete it to

$$0 \to \mathcal{S}_\lambda \to \mathcal{G} = G^* \to U_\lambda^* \to 0\,. \tag{7.8}$$

As a consequence, the cardinalities of the respective subgroups of G and $\mathcal{G}$ are related by

$$|\mathcal{S}_\lambda|\,|U_\lambda| = |\mathcal{G}| = |G| = |S_\lambda|\,|\mathcal{U}_\lambda|\,, \tag{7.9}$$

and hence in particular the dimensions of the degeneracy spaces in the decompositions (7.5) and (7.6) coincide:

$$(\dim \mathcal{V}_{\hat\psi})^2 = |\mathcal{S}_\lambda|\,/\,|\mathcal{U}_\lambda| = |S_\lambda|\,/\,|U_\lambda| = (\dim V_{\hat{\mathrm{J}}})^2\,. \tag{7.10}$$

We abbreviate these dimensions by

$$d_\lambda := \dim \mathcal{V}_{\hat\psi} = \dim V_{\hat{\mathrm{J}}}\,. \tag{7.11}$$

There is also a manifest relationship between the groups $\mathcal{S}_\lambda$ and G/U_λ. First we realize that the implementation of G on the whole G-orbit of λ provides us with a two-cocycle of G with values in U(1) whose restriction to $S_\lambda \times S_\lambda$ coincides with $\mathcal{E}_\lambda$; we again denote this cocycle by the symbol $\mathcal{E}_\lambda$. Given such a cocycle, its *commutator cocycle* E_λ, which is defined by

$$E_\lambda(g,g') := \mathcal{E}_\lambda(g,g')/\mathcal{E}_\lambda(g',g) \tag{7.12}$$

for all $g,g' \in G$, constitutes [1] a *bi-homomorphism* on $G\times G$ which is *alternating* in the sense that $E_\lambda(g',g) = E_\lambda(g,g')^*$. Now let us characterize for every $\mathrm{J} \in \mathcal{S}_\lambda$ an element h_J of G by the property that

$$E_\lambda(h_\mathrm{J},g) = \mathrm{J}(g) \qquad \text{for all } \; g \in G\,; \tag{7.13}$$

such a group element h_J exists because, owing to the exactness of the sequence (7.8), for every $\mathrm{J} \in \mathcal{S}_\lambda$ we have $\mathrm{J}(g) = 1$ for all $g \in U_\lambda$, and (7.13) characterizes h_J uniquely up to an element of U_λ. Furthermore, as a consequence of the character property of E_λ in the first argument, we have

$$E_\lambda(h_{\mathrm{J}\mathrm{J}'},g) = E_\lambda(h_\mathrm{J} h_{\mathrm{J}'},g) \tag{7.14}$$

for all $g \in G$, which tells us that $h_\mathrm{J} h_{\mathrm{J}'} = h_{\mathrm{J}\mathrm{J}'}$ modulo U_λ. It follows that the mapping

$$\mathrm{J} \;\leftrightarrow\; h_\mathrm{J}\, U_\lambda \tag{7.15}$$

constitutes an isomorphism between $\mathcal{S}_\lambda$ and G/U_λ. It is worth noting that this isomorphism is logically independent from the isomorphism between $G^*/\mathcal{S}_\lambda$ and U_λ^* that exists according to the sequence (7.8).

Again this result has an obvious dual analogue. To work this out we recall from appendix A of [1] that the commutator cocycle F_λ of $\mathcal{F}_\lambda$, defined by $F_\lambda(\mathrm{J},\mathrm{L}) = \mathcal{F}_\lambda(\mathrm{J},\mathrm{L})/\mathcal{F}_\lambda(\mathrm{L},\mathrm{J})$ for $\mathrm{J},\mathrm{L} \in \mathcal{S}_\lambda$, possesses a natural extension. F_λ is an alternating bi-homomorphism on $\mathcal{S}_\lambda \times \mathcal{S}_\lambda$, while its extension is a bi-homomorphism on $\mathcal{G} \times \mathcal{S}_\lambda$; by imposing the alternating property it can be further extended to a bi-homomorphism on $\mathcal{G} \times \mathcal{G}$, still to be denoted by F_λ. By the character property of F_λ in the second argument, we can then associate to every $g \in S_\lambda$ an element $\mathrm{K}_g \equiv \mathrm{K}_g^{(\lambda)}$ of $\mathcal{G}$ by stipulating that

$$F_\lambda(\mathrm{K}_g,\mathrm{L}) = g(\mathrm{L}) \qquad \text{for all} \;\; \mathrm{L} \in \mathcal{G}\,, \tag{7.16}$$

which determines K_g uniquely up to elements of $\mathcal{U}_\lambda$. The character property of F_λ in the first argument implies

$$F_\lambda(\mathrm{K}_{gg'},\mathrm{L}) = F_\lambda(\mathrm{K}_g\mathrm{K}_{g'},\mathrm{L})\,, \tag{7.17}$$

so that $\mathrm{K}_g\mathrm{K}_{g'} = \mathrm{K}_{gg'}$ modulo $\mathcal{U}_\lambda$. Thus the map

$$g \;\leftrightarrow\; \mathrm{K}_g\,\mathcal{U}_\lambda \tag{7.18}$$

between S_λ and $\mathcal{G}/\mathcal{U}_\lambda$ is an isomorphism.

For later reference we also mention another isomorphism that is similar to (7.18). Namely, we now consider a character $\hat g$ of $\mathcal{S}_\lambda$ rather than a character g of $\mathcal{G}$. Then the requirement that

$$F_\lambda(\hat{\mathrm{K}}_{\hat g},\mathrm{L}) = g(\mathrm{L}) \qquad \text{for all} \;\; \mathrm{L} \in \mathcal{S}_\lambda \tag{7.19}$$

determines $\hat{\mathrm{K}}_{\hat g} \in \mathcal{S}_\lambda$ uniquely modulo $\mathcal{U}_\lambda$, and

$$\hat g \;\leftrightarrow\; \hat{\mathrm{K}}_{\hat g}\,\mathcal{U}_\lambda \tag{7.20}$$

is an isomorphism between those elements of $\mathcal{S}_\lambda^*$ which are the identity on $\mathcal{U}_\lambda$ (and hence can be regarded as restrictions of elements of S_λ to $\mathcal{S}_\lambda$) and $\mathcal{S}_\lambda/\mathcal{U}_\lambda$. Furthermore, combining the prescriptions above we learn that the restrictions of the relevant bi-homomorphisms to the stabilizer groups are closely related. Indeed, denoting by $\hat g$ the restriction of a given element g of S_λ to $\mathcal{S}_\lambda$, we have

$$E_\lambda(g,g') = \hat{\mathrm{K}}_{\hat g'}(g) = \hat g(\hat{\mathrm{K}}_{\hat g'}) = F_\lambda(\hat{\mathrm{K}}_{\hat g},\hat{\mathrm{K}}_{\hat g'}) \tag{7.21}$$

with $g,g' \in S_\lambda$ and $\hat{\mathrm{K}}_{\hat g},\hat{\mathrm{K}}_{\hat g'} \in \mathcal{S}_\lambda$.

8 Automorphism types

We have demonstrated in [1] that the reflection coefficients $\mathrm{R}^{[\bar\rho,\hat\psi_\rho]}_{(\bar\lambda,\varphi);\bar\Omega}$, i.e. the operator product coefficients in the expansion $(5.4)_A$ of a bulk field approaching the boundary, are equal to one-dimensional irreducible representations $R_{[\bar\rho,\hat\psi_\rho]}$ of the classifying algebra $\mathcal{C}(\bar{\mathfrak{A}})$, evaluated at the basis element $\tilde\Phi_{(\bar\lambda,\varphi)}$ of $\mathcal{C}(\bar{\mathfrak{A}})$. Thus they are given by (see formula (5.48))

$$\mathrm{R}^{[\bar\rho,\hat\psi_\rho]}_{(\bar\lambda,\varphi);\bar\Omega} = R_{[\bar\rho,\hat\psi_\rho]}(\tilde\Phi_{(\bar\lambda,\varphi)}) = \tilde S_{(\bar\lambda,\varphi),[\bar\rho,\hat\psi_\rho]}/\tilde S_{\bar\Omega,[\bar\rho,\hat\psi_\rho]}\,. \tag{8.1}$$

Moreover, it follows from the sum rule $(5.22)_A$ and the fact that $\mathcal{C}(\bar{\mathfrak{A}})$ is semisimple, that the reflection coefficients even provide *all* inequivalent irreducible $\mathcal{C}(\bar{\mathfrak{A}})$-representations. The isomorphism classes of irreducible $\mathcal{C}(\bar{\mathfrak{A}})$-representations are in one-to-one correspondence with the conformally invariant boundary conditions that preserve the orbifold subalgebra $\bar{\mathfrak{A}} = \mathfrak{A}^G$ of the chiral algebra $\mathfrak{A}$. In this section we discuss some implications of this basic result.

Let us associate to each boundary condition $\rho \equiv [\bar\rho, \hat\psi_\rho]$ the collection of all monodromy charges $Q_{\mathrm{J}}(\rho)$, $\mathrm{J} \in \mathcal{G}$, of $\bar\rho$. The monodromy charges do not depend on the choice of a representative of the $\mathcal{G}$-orbit $[\bar\rho]$, and via the prescription

$$g_\rho(\mathrm{J}) := \exp(2\pi\mathrm{i} Q_{\mathrm{J}}(\rho)) \tag{8.2}$$

for all $\mathrm{J} \in \mathcal{G}$, they furnish a character

$$g_\rho \in \mathcal{G}^* \tag{8.3}$$

of the simple current group. (This $\mathcal{G}$-character should not be confused with $\hat\psi_\rho$, which is a character of the subgroup $\mathcal{U}_\rho \subseteq \mathcal{G}$.) The group $\mathcal{G}^* = (G^*)^*$ can be naturally identified with the orbifold group, $\mathcal{G}^* \equiv G$, and hence the quantity g_ρ can be regarded as an element of G.

To proceed, we observe that because of the simple current symmetry

$$\tilde S_{\mathrm{J}(\bar\lambda,\varphi),[\bar\rho,\hat\psi]} = g_\rho(\mathrm{J})\, \tilde S_{(\bar\lambda,\varphi),[\bar\rho,\hat\psi]} \tag{8.4}$$

that was established in formula (5.16), we have

$$R_{[\bar\rho,\hat\psi_\rho]}(\tilde\Phi_{\mathrm{J}(\bar\lambda,\varphi)}) = \frac{\tilde S_{\mathrm{J}(\bar\lambda,\varphi),[\bar\rho,\hat\psi_\rho]}}{\tilde S_{\bar\Omega,[\bar\rho,\hat\psi_\rho]}} = g_\rho(\mathrm{J})\, \frac{\tilde S_{(\bar\lambda,\varphi),[\bar\rho,\hat\psi_\rho]}}{\tilde S_{\bar\Omega,[\bar\rho,\hat\psi_\rho]}} = g_\rho(\mathrm{J}) \cdot R_{[\bar\rho,\hat\psi_\rho]}(\tilde\Phi_{(\bar\lambda,\varphi)}) \tag{8.5}$$

for every simple current $\mathrm{J} \in \mathcal{G}$. Now the reflection coefficients constitute the main ingredient in the relation between the boundary *blocks* $\tilde{\mathrm{B}}_{(\bar\lambda,\varphi)}$ (defined in formula $(4.23)_A$) and the boundary *states* $\mathcal{B}_{[\bar\rho,\hat\psi]}$. When using the notation introduced in (8.2), the precise relationship, established in formula (6.4), reads

$$\mathcal{B}_{[\bar\rho,\hat\psi]} = \bigoplus_{\substack{\bar\lambda \\ g_\lambda \equiv 1}} \bigoplus_{\varphi \in \mathcal{S}^*_{\bar\lambda}} \mathrm{R}^{[\bar\rho,\hat\psi]}_{(\bar\lambda,\varphi);\bar\Omega} \, \langle \Psi^{[\bar\rho,\hat\psi]\,[\bar\rho,\hat\psi]}_{\bar\Omega} \rangle \, \tilde{\mathrm{B}}_{(\bar\lambda,\varphi)} \,. \tag{8.6}$$

Thus the observation (8.5) tells us that the boundary blocks $\tilde{\mathrm{B}}_{(\bar\lambda,\varphi)}$ for primary fields $\bar\lambda$ of the $\bar{\mathfrak{A}}$-theory that lie on one and the same $\mathcal{G}$-orbit contribute to the boundary states $\mathcal{B}_{[\bar\rho,\hat\psi]}$ with a fixed relative phase, which is determined by the element g_ρ of the orbifold group. Put differently, in the presence of the boundary condition $\rho \equiv [\bar\rho, \hat\psi]$ the reflection of a bulk field at the boundary is *twisted* by the action of the group element $g_\rho \in G$.

This observation suggests that, in the terminology of [12], the orbifold group element g_ρ provides us with the *automorphism type* [2] of the boundary condition ρ. To establish that this is indeed the case, we insert the expression (8.1) for the reflection coefficients and the explicit

[2] The corresponding term in [13] is the *gluing automorphism*. The information contained in a boundary condition that goes beyond the automorphism type was referred to as the *Chan–Paton type* in [12]. Thinking in analogy with the general analysis of modular invariant partition functions on the torus, it may seem to be more suggestive to take the fusion rule automorphism $g^\star$ as a starting point for the description of the automorphism

values of the one-point correlators of the boundary vacuum fields $\Psi_{\bar\Omega}^{[\bar\rho,\hat\psi]\,[\bar\rho,\hat\psi]}$ into formula (8.6), so as to arrive at

$$\mathcal{B}_{[\bar\rho,\hat\psi]} = \bigoplus_{\substack{\bar\lambda\\ g_\lambda\equiv 1}} \bigoplus_{\varphi\in\mathcal{S}_\lambda^*} \tilde S_{(\bar\lambda,\varphi),[\bar\rho,\hat\psi]}\, \tilde{\mathrm{B}}_{(\bar\lambda,\varphi)}\,. \tag{8.7}$$

Next we split the summation over all untwisted $\bar\lambda$ into a summation over $\mathcal{G}$-orbits and one within orbits, and the summation over $\mathcal{S}_\lambda^*$ into one over $\mathcal{U}_\lambda^*$ and one over $\mathcal{S}_\lambda^*/\mathcal{U}_\lambda^*$. To this end we choose (once and for all) arbitrarily a set $\{\bar\lambda_\circ\}$ of representatives of the set of $\mathcal{G}$-orbits and a set $\{\varphi_\circ\}$ of representatives of the classes of $\mathcal{S}_\lambda^*/\mathcal{U}_\lambda^*$; more precisely, the symbol $\bar\lambda_\circ$ will refer to the chosen representative of the orbit $[\bar\lambda]$, and $\varphi_\circ\in\mathcal{S}_\lambda^*$ to the chosen representative of the class in $\mathcal{S}_\lambda^*/\mathcal{U}_\lambda^*$ that restricts to $\hat\varphi\in\mathcal{U}_\lambda^*$, i.e. satisfies $\varphi_{\circ|\mathcal{U}_\lambda}=\hat\varphi$. Then (8.7) becomes

$$\begin{aligned}\mathcal{B}_{[\bar\rho,\hat\psi]} &= \bigoplus_{\substack{[\bar\lambda_\circ]\\ g_\lambda\equiv 1}} \bigoplus_{\hat\varphi\in\mathcal{U}_\lambda^*} \bigoplus_{\mathrm{J}\in\mathcal{G}/\mathcal{U}_\lambda} \tilde S_{\mathrm{J}(\bar\lambda_\circ,\varphi_\circ),[\bar\rho,\hat\psi]}\, \tilde{\mathrm{B}}_{\mathrm{J}(\bar\lambda_\circ,\varphi_\circ)}\\ &= \bigoplus_{\substack{[\bar\lambda_\circ]\\ g_\lambda\equiv 1}} \bigoplus_{\hat\varphi\in\mathcal{U}_\lambda^*} \tilde S_{(\bar\lambda_\circ,\varphi_\circ),[\bar\rho,\hat\psi]} \bigoplus_{\mathrm{J}\in\mathcal{G}/\mathcal{U}_\lambda} g_\rho(\mathrm{J})\, \tilde{\mathrm{B}}_{\mathrm{J}(\bar\lambda_\circ,\varphi_\circ)}\,.\end{aligned} \tag{8.8}$$

Here in the second line we have used the simple current relation (8.4), as well as the fact that by this identity the matrix element $\tilde S_{(\bar\lambda,\varphi),[\bar\rho,\hat\psi]}$ vanishes when $g_\rho(\mathrm{J})\neq 1$ for any $\mathrm{J}\in\mathcal{U}_\lambda$. The latter observation shows that only those boundary blocks $\tilde{\mathrm{B}}_{(\bar\lambda,\varphi)}$ contribute to the boundary state $\mathcal{B}_{[\bar\rho,\hat\psi]}$ for which the character g_ρ is equal to one on the whole untwisted stabilizer $\mathcal{U}_\lambda\subseteq\mathcal{G}$, which in turn implies that $g_\rho\in G$ is actually an element of $S_\lambda\subseteq G$, i.e.

$$g_\rho \in S_\lambda\,. \tag{8.9}$$

Thus as a character of $\mathcal{G}$ the function g_ρ factorizes to a character of $\mathcal{G}/\mathcal{U}_\lambda$; according to the sequence (7.7), the latter group can be identified with S_λ^*. As we will see later, this result is perfectly natural.

We now concentrate on the J-summation for fixed values of $\bar\lambda_\circ$ and $\varphi_\circ$. To proceed, we need a few further tools. First, it turns out to be useful to introduce for every $g\in S_\lambda$ and every $J\in\mathcal{G}$ the endomorphism

$$O_{g,\mathrm{J}} := d_\lambda^{-1/2} \sum_{\mathrm{L}\in\mathcal{S}_\lambda/\mathcal{U}_\lambda} g(\mathrm{L})\, \mathcal{O}_{\mathrm{JL}\varphi_\circ} \tag{8.10}$$

of $\mathcal{V}_{\hat\psi}$, where $\mathcal{O}_\psi$ are the endomorphisms defined by $(4.11)_A$. Inserting that formula for $\mathcal{O}_\psi$ and

type [12]. However, several different automorphism types may give rise to one and the same permutation $g^\star$. For instance, in the case of WZW theories, automorphisms of the underlying finite-dimensional compact simple Lie algebra $\bar{\mathfrak{g}}$ provide us with an automorphism type g, but whenever that automorphism of $\bar{\mathfrak{g}}$ is inner, the associated map $g^\star$ is just the identity. Another example is given by the inner automorphisms of the rational free boson theories whose fixed point algebras correspond to the boson theory compactified at an integral multiple of the original radius; this will be discussed in subsection 16.1.

interchanging the order of summations, the maps $O_{g,\mathrm{J}}$ can also be written in the form

$$\begin{aligned} O_{g,\mathrm{J}} &= d_\lambda^{-2} \sum_{\mathrm{L}\in\mathcal{S}_\lambda/\mathcal{U}_\lambda} {}_{\mathrm{J}}\varphi_\circ(\mathrm{L})^*\, R_{\hat\varphi}(\mathrm{L}) \sum_{\mathrm{L}'\in\mathcal{S}_\lambda/\mathcal{U}_\lambda} g(\mathrm{L}')\, F_\lambda(\mathrm{L},\mathrm{L}')^* \\ &= \sum_{\mathrm{L}\in\mathcal{S}_\lambda/\mathcal{U}_\lambda} {}_{\mathrm{J}}\varphi_\circ(\mathrm{L})^*\, R_{\hat\varphi}(\mathrm{L}) \cdot \delta_{\mathrm{L},\hat{\mathrm{K}}_g} = {}_{\mathrm{J}}\varphi_\circ(\hat{\mathrm{K}}_g)^*\, R_{\hat\varphi}(\hat{\mathrm{K}}_g)\,, \end{aligned} \tag{8.11}$$

where $\hat{\mathrm{K}}_g \in \mathcal{S}_\lambda$ is as defined by formula (7.19), with $\hat g \in \mathcal{S}_\lambda^*$ given by $\hat g = g_{|\mathcal{S}_\lambda}$. (Because of $g\in S_\lambda$, the character $\hat g\in\mathcal{S}_\lambda^*$ is the identity on $\mathcal{U}_\lambda \subseteq \mathcal{S}_\lambda$, as required in (7.19). Also recall that $\hat{\mathrm{K}}_g$ is defined only up to elements of $\mathcal{U}_\lambda$; but the result (8.11) for $O_{g,\mathrm{J}}$ is independent of the choice of representative.)

With the help of formula (8.11) one checks that

$$O_{g,\mathrm{JL}} = g(\mathrm{L})^*\, O_{g,\mathrm{J}} \qquad \text{for all } \ \mathrm{L}\in\mathcal{S}_\lambda\,, \tag{8.12}$$

and that

$$O_{g,\mathrm{J}}\, O_{g',\mathrm{J}} = \mathcal{F}_\lambda(\hat{\mathrm{K}}_g,\hat{\mathrm{K}}_{g'})\, O_{gg',\mathrm{J}} = \mathcal{E}_\lambda(g,g')\, O_{gg',\mathrm{J}} \tag{8.13}$$

for all $g,g'\in S_\lambda$, i.e. the endomorphisms $O_{g,\mathrm{J}}$ with $g\in S_\lambda$ furnish a projective representation of the stabilizer S_λ (in the last equality we have used the identity (7.21)). Let us also see to which extent these results depend on the choice of representative $\varphi_\circ$. Any other representative is of the form $\varphi_\circ\psi$ with $\psi\in\mathcal{S}_\lambda^*$ and $\psi_{|\mathcal{U}_\lambda} = id$; thus upon choosing a different representative the endomorphisms $O_{g,\mathrm{J}}$ get replaced by

$$\tilde O_{g,\mathrm{J}} = \psi(\hat{\mathrm{K}}_g)^*\, O_{g,\mathrm{J}}\,. \tag{8.14}$$

On the other hand, the two-cocycle that characterizes the relevant representation of S_λ does not depend on this choice. Indeed, as a consequence of $\hat{\mathrm{K}}_{gg'} = \hat{\mathrm{K}}_g\hat{\mathrm{K}}_{g'}$ modulo $\mathcal{U}_\lambda$ the maps $\tilde O_{g,\mathrm{J}}$ satisfy

$$\tilde O_{g,\mathrm{J}}\, \tilde O_{g',\mathrm{J}} = \psi(\hat{\mathrm{K}}_g\hat{\mathrm{K}}_{g'})^*\, O_{g,\mathrm{J}} O_{g',\mathrm{J}} = \mathcal{F}_\lambda(\hat{\mathrm{K}}_g,\hat{\mathrm{K}}_{g'})\, \tilde O_{gg',\mathrm{J}}\,. \tag{8.15}$$

Next we also choose a set $\{\mathrm{J}_\circ\}$ of representatives for the classes in $\mathcal{G}/\mathcal{S}_\lambda$ and write, for every $g\in S_\lambda$,

$$\Theta_g \equiv \Theta_g^{(\lambda)} := \bigoplus_{\mathrm{J}_\circ\in\mathcal{G}/\mathcal{S}_\lambda} g(\mathrm{J}_\circ) \big(O_{g,\mathrm{J}_\circ} \otimes id \big) \circ P_{\mathrm{J}_\circ\bar\lambda_\circ}\,, \tag{8.16}$$

where $O_{g,\mathrm{J}_\circ}$ is defined in (8.10) and $P_{\bar\mu}$ is the projector from the $\mathfrak{A}$-module $\mathcal{H}_\lambda$ to its isotypical component of type $\bar{\mathcal{H}}_{\bar\mu}$. Owing to the identity (8.12), Θ_g is in fact independent of the choice of representatives $\mathrm{J}_\circ$ (whereas $O_{g,\mathrm{J}_\circ}$ does again depend on that choice), and from (8.13) it follows that

$$\Theta_g\, \Theta_{g'} = \mathcal{F}_\lambda(\hat{\mathrm{K}}_g,\hat{\mathrm{K}}_{g'})\, \Theta_{gg'} \tag{8.17}$$

for all $g,g'\in S_\lambda$. Let us also note that when specializing to the $\mathfrak{A}$-vacuum sector, where $U_\Omega = S_\Omega = G$, we simply have $\hat{\mathrm{K}}_g = \mathbf{1}$ and hence $O_{g,\mathrm{J}} = id$ for all $\mathrm{J}\in\mathcal{G}$, so that $\Theta_g^{(\Omega)} = \bigoplus_{\mathrm{J}\in\mathcal{G}} g(\mathrm{J}) P_{\mathrm{J}}$.

With these results at hand, we can now address the J-summation that appears in formula (8.8). Inserting the definition $(4.23)_A$ of the boundary blocks $\tilde{\mathrm{B}}$, we find that

$$\begin{aligned}\bigoplus_{\mathrm{J}\in\mathcal{G}/\mathcal{U}_\lambda} g(\mathrm{J})\, \tilde{\mathrm{B}}_{\mathrm{J}(\bar\lambda_\circ,\varphi_\circ)} &= \bigoplus_{\mathrm{J}_\circ\in\mathcal{G}/\mathcal{S}_\lambda} g(\mathrm{J}_\circ) \bigoplus_{\mathrm{L}\in\mathcal{S}_\lambda/\mathcal{U}_\lambda} g(\mathrm{L})\, \tilde{\mathrm{B}}_{(\mathrm{J}_\circ\bar\lambda_\circ,\mathrm{J}_\circ\mathrm{L}\varphi_\circ)} \\ &= \mathcal{N}_\lambda\, d_\lambda^{-2} \bigoplus_{\mathrm{J}_\circ\in\mathcal{G}/\mathcal{S}_\lambda} g(\mathrm{J}_\circ) \bigoplus_{\mathrm{L}\in\mathcal{S}_\lambda/\mathcal{U}_\lambda} g(\mathrm{L}) \\ &\qquad \beta_\circ \circ \Big(\bigoplus_{\mathrm{L}'\in\mathcal{S}_\lambda/\mathcal{U}_\lambda} {}_{\mathrm{J}_\circ\mathrm{L}}\varphi_\circ(\mathrm{L}')^*\, R_{\hat\varphi}(\mathrm{L}') \otimes id \Big) \otimes \bar{\mathrm{B}}_{\mathrm{J}_\circ\bar\lambda_\circ} ,\end{aligned} \tag{8.18}$$

where $\beta_\circ\colon \mathcal{V}_{\hat\psi_\lambda} \otimes \mathcal{V}_{\hat\psi_\lambda^+} \to \mathbb{C}$ is the non-degenerate linear form defined in (4.21). Performing the L-summation, this becomes

$$\bigoplus_{\mathrm{J}\in\mathcal{G}/\mathcal{U}_\lambda} g(\mathrm{J})\, \tilde{\mathrm{B}}_{\mathrm{J}(\bar\lambda_\circ,\varphi_\circ)} = \mathcal{N}_\lambda \bigoplus_{\mathrm{J}_\circ\in\mathcal{G}/\mathcal{S}_\lambda} g(\mathrm{J}_\circ)\, \beta_\circ \circ \big({}_{\mathrm{J}_\circ}\varphi_\circ(\hat{\mathrm{K}}_g)^*\, R_{\hat\varphi}(\hat{\mathrm{K}}_g) \otimes id \big) \otimes \bar{\mathrm{B}}_{\mathrm{J}_\circ\bar\lambda_\circ} = \mathcal{N}_\lambda\, \mathrm{B}_\lambda^{(g)} \tag{8.19}$$

with

$$\mathrm{B}_\lambda^{(g)} := \mathrm{B}_\lambda \circ (\Theta_g \otimes id)\,. \tag{8.20}$$

Here $\mathrm{B}_\lambda = \mathrm{B}_\lambda^{(\mathbf{1})}$ are the boundary blocks of the $\mathfrak{A}$-theory, which are given by the expression (4.25), and we have used that via the identity $(4.17)_A$ they can be written in the form

$$\begin{aligned}\mathrm{B}_\lambda \equiv \mathrm{B}_{[\lambda,\hat\varphi]} &= \mathcal{N}_\lambda^{-1} \bigoplus_{\substack{\varphi\in\mathcal{S}_\lambda^* \\ \varphi\succ\hat\varphi}} \bigoplus_{\mathrm{J}\in\mathcal{G}/\mathcal{S}_\lambda} \tilde{\mathrm{B}}_{(\mathrm{J}\bar\lambda_\circ,\varphi)} = \mathcal{N}_\lambda^{-1} \bigoplus_{\mathrm{J}\in\mathcal{G}/\mathcal{S}_\lambda} \bigoplus_{\mathrm{L}\in\mathcal{S}_\lambda/\mathcal{U}_\lambda} \tilde{\mathrm{B}}_{(\mathrm{J}\bar\lambda_\circ,\mathrm{L}\varphi_\circ)} \\ &= d_\lambda^{-1/2} \bigoplus_{\mathrm{J}\in\mathcal{G}/\mathcal{S}_\lambda} \beta_\circ \circ \Big(\sum_{\mathrm{L}\in\mathcal{S}_\lambda/\mathcal{U}_\lambda} \mathcal{O}_{\mathrm{L}\varphi_\circ} \otimes id \Big) \otimes \bar{\mathrm{B}}_{\mathrm{J}\bar\lambda} = \bigoplus_{\mathrm{J}\in\mathcal{G}/\mathcal{S}_\lambda} \beta_\circ \otimes \bar{\mathrm{B}}_{\mathrm{J}\bar\lambda}\,.\end{aligned} \tag{8.21}$$

At this point it is worth realizing that the twisted intertwining property (7.2) of Θ_g is formulated independently of the subalgebra $\bar{\mathfrak{A}}$, and hence Θ_g only depends on the automorphism g itself, but not on the particular orbifold group G containing g we are considering. By the result (8.20), this independence on the choice of G then holds for the quantities $\mathrm{B}_\lambda^{(g)}$, too.

Inserting the result (8.19) into formula (8.8), we finally see that

$$\mathcal{B}_\rho = \bigoplus_{\substack{[\bar\lambda_\circ] \\ g_\lambda\equiv 1}} \mathcal{N}_\lambda \bigoplus_{\hat\varphi\in\mathcal{U}_\lambda^*} \tilde{S}_{(\bar\lambda_\circ,\varphi_\circ),\rho}\, \mathrm{B}_\lambda^{(g_\rho)}\,. \tag{8.22}$$

Thus, in summary, the boundary state $\mathcal{B}_\rho$ can be entirely constructed from the information contained in the boundary blocks $\mathrm{B}_{[\lambda,\hat\varphi]}$ together with the action of S_λ they carry and in the character g_ρ of G. As we will see, this implies that g_ρ indeed constitutes the *automorphism type* of the boundary condition ρ. We have also seen that only those boundary blocks $\tilde{\mathrm{B}}_{(\bar\lambda,\varphi)}$ contribute to the boundary state $\mathcal{B}_\rho$ for which the stabilizer S_λ contains g_ρ (which, incidentally, shows that the factorization to a character of $\mathcal{G}/\mathcal{U}_\lambda$ is a rather natural property of the elements of S_λ). This should be regarded as a *selection rule* on the possible boundary blocks that show up in the boundary state; the concrete form of this selection rule is completely determined by the automorphism type of the boundary condition.

Our derivation also demonstrates that in the case of our interest one can associate an automorphism of the chiral algebra $\mathfrak{A}$ to *every* boundary condition. This comes as a *result* of our analysis and does not have to be put in as an assumption. In contrast, when the subalgebra $\bar{\mathfrak{A}}$ that is preserved by a boundary condition is not an orbifold subalgebra $\mathfrak{A}^G$, then the boundary condition need not necessarily possess an automorphism type. Indeed, as we will see in section 16.3, most conformally invariant boundary conditions of the $\mathbb{Z}_2$-orbifold of a free boson, compactified at a rational radius squared, do not posses an automorphism type.

9 Twisted blocks and twisted Ward identities

In the discussion of automorphism types above we have introduced, for every primary label $\lambda \equiv [\bar{\lambda}, \hat{\psi}]$ of the $\mathfrak{A}$-theory and for every element $g \in S_\lambda$ of the orbifold group that stabilizes λ, the map Θ_g (8.16) as well as the linear form $\mathrm{B}_\lambda^{(g)} = \mathrm{B}_\lambda \circ (\Theta_g \otimes id)$ (8.20) on the tensor product $\mathfrak{A}$-module $\mathcal{H}_\lambda \otimes \mathcal{H}_{\lambda^+}$. We will refer to the latter linear forms as *g-twisted boundary blocks*. As demonstrated in the previous section, every boundary state $\mathcal{B}_\rho$ is a linear combination of those twisted boundary blocks $\mathrm{B}_\lambda^{(g_\rho)}$ for which the twist g_ρ lies in S_λ.

What still remains to be established is that the notation for the maps Θ_g that was introduced in (8.16) is in agreement with the use of the same notation in section 7, i.e. that these maps satisfy the g-twisted intertwining property (7.2). To address this issue, we need some information about the representation R_λ of $\mathfrak{A}$ on the subspaces in the decomposition (7.6) of a $\mathfrak{A}$-module $\mathcal{H}_\lambda \equiv \mathcal{H}_{[\bar{\lambda}_\circ, \hat{\psi}]}$ into irreducible $\bar{\mathfrak{A}}$-modules. While a complete description of the action of the vertex operator algebra that is obtained by a simple current extension is not yet available, the known results naturally suggest the following structure. On the $\bar{\mathfrak{A}}$-module $\mathcal{V}_{\hat{\psi}} \otimes \bar{\mathcal{H}}_{\mathrm{J}_\circ \bar{\lambda}_\circ} \subseteq \mathcal{H}_\lambda$ an element $Y_\mathrm{J}(z) \equiv Y(\bar{v}_\mathrm{J}; z)$ of $\mathfrak{A}$ with $\bar{v}_\mathrm{J} \in \bar{\mathcal{H}}_\mathrm{J} \subseteq \mathcal{H}_\Omega$ is represented by

$$R_{[\bar{\lambda}_\circ, \hat{\psi}]}(Y_\mathrm{J}) = R_{\hat{\varphi}}(\mathrm{J}') \otimes \bigoplus_{\mathrm{J}_\circ \in \mathcal{G}/\mathcal{S}_\lambda} \bar{\mathcal{R}}_{\mathrm{J}_\circ \bar{\lambda}_\circ}(\bar{Y}_\mathrm{J}) \,. \tag{9.1}$$

Here J' is defined by $\mathrm{J} = \mathrm{J}'\mathrm{J}'_\circ$ with $\mathrm{J}'_\circ$ the representative for a class in $\mathcal{G}/\mathcal{S}_\lambda$ (see before formula (8.16)), and $\bar{\mathcal{R}}_{\mathrm{J}_\circ \bar{\lambda}_\circ}(\bar{Y}_\mathrm{J})$ is a map from $\bar{\mathcal{H}}_{\mathrm{J}_\circ \bar{\lambda}_\circ}$ to $\bar{\mathcal{H}}_{\mathrm{J}'_\circ \mathrm{J}_\circ \bar{\lambda}_\circ}$. To proceed we use the commutation properties

$$R_{\hat{\varphi}}(\mathrm{J}')\, R_{\hat{\varphi}}(\hat{\mathrm{K}}_g) = F_\lambda(\mathrm{J}', \hat{\mathrm{K}}_g)\, R_{\hat{\varphi}}(\hat{\mathrm{K}}_g)\, R_{\hat{\varphi}}(\mathrm{J}') = g(\mathrm{J}')^*\, R_{\hat{\varphi}}(\hat{\mathrm{K}}_g)\, R_{\hat{\varphi}}(\mathrm{J}') \tag{9.2}$$

and

$$\bigoplus_{\mathrm{J}_\circ \in \mathcal{G}/\mathcal{S}_\lambda} \big(id \otimes \bar{\mathcal{R}}_{\mathrm{J}_\circ \bar{\lambda}_\circ}(\bar{Y}_\mathrm{J}) \big) \circ P_{\mathrm{J}''_\circ \bar{\lambda}_\circ} = \bar{\mathcal{R}}_{\mathrm{J}''_\circ \bar{\lambda}_\circ}(\bar{Y}_\mathrm{J}) = P_{\mathrm{J}'_\circ \mathrm{J}''_\circ \bar{\lambda}_\circ} \circ \bigoplus_{\mathrm{J}_\circ \in \mathcal{G}/\mathcal{S}_\lambda} \big(id \otimes \bar{\mathcal{R}}_{\mathrm{J}_\circ \bar{\lambda}_\circ}(\bar{Y}_\mathrm{J}) \big) \tag{9.3}$$

as well as [3]

$$\big(R_{\hat{\varphi}}(\mathrm{J}') \otimes id \big) \circ P_{\mathrm{J}'_\circ \mathrm{J}''_\circ \bar{\lambda}_\circ} = F_\lambda(\mathrm{J}'_\circ, \hat{\mathrm{K}}_g)\, P_{\mathrm{J}'_\circ \mathrm{J}''_\circ \bar{\lambda}_\circ} \circ \big(R_{\hat{\varphi}}(\mathrm{J}') \otimes id \big) \,. \tag{9.4}$$

[3] This relation is needed for compatibility with the fact that while the isomorphism class of the projective $\mathcal{S}_\lambda$-representation $R_{\hat{\varphi}}$ is the same for all values of $\bar{\lambda}$ within the class $[\bar{\lambda}_\circ]$, its explicit realization can depend on $\bar{\lambda}$, and this dependence should precisely be characterized by F_λ.

Then we obtain

$$
\begin{aligned}
R_\lambda(Y_{\rm J}) \circ \Theta_g^{(\lambda)} &= g({\rm J}')^*\, F_\lambda({\rm J}'_\circ, \hat{\rm K}_g) \bigoplus_{{\rm J}''_\circ \in \mathcal{G}/\mathcal{S}_\lambda} g({\rm J}''_\circ)\, {}_{{\rm J}''_\circ}\varphi_\circ(\hat{\rm K}_g)^* \left(R_{\hat\varphi}(\hat{\rm K}_g) \otimes id\right) \circ P_{{\rm J}'_\circ {\rm J}''_\circ \bar\lambda_\circ} \circ R_\lambda(Y_{\rm J}) \\
&= g({\rm J}')^*\, g({\rm J}'_\circ)^* \bigoplus_{{\rm J}''_\circ \in \mathcal{G}/\mathcal{S}_\lambda} g({\rm J}''_\circ)\, {}_{{\rm J}''_\circ}\varphi_\circ(\hat{\rm K}_g)^* \left(R_{\hat\varphi}(\hat{\rm K}_g) \otimes id\right) \circ P_{{\rm J}''_\circ \bar\lambda_\circ} \circ R_\lambda(Y_{\rm J}) \\
&= g({\rm J})^* \cdot \Theta_g^{(\lambda)} \circ R_{\lambda_\circ}(Y_{\rm J}) \,\equiv\, \Theta_g^{(\lambda)} \circ R_\lambda(g^{-1}(Y_{\rm J}))\,.
\end{aligned}
\tag{9.5}
$$

Here in the last expression the element g of the orbifold group is to be regarded as an automorphism of the chiral algebra $\mathfrak{A}$, while in the intermediate steps it is interpreted as a character of the simple current group $\mathcal{G} = G^*$. Formula (9.5) reproduces the twisted intertwiner property (7.2) and hence is the desired result.

We also remark that according to section 7 the twisted intertwiners carry a projective representation that is characterized by the cohomology class of the cocycle $\mathcal{E}_\lambda$ or, equivalently, by the commutator cocycle E_λ. On the other hand, according to relation (8.17) the concrete realization (8.16) of the twisted intertwiners can be characterized by the commutator cocycle F_λ. This is compatible because of the identity (7.21).

Now for every field $Y(z) = \sum_{n\in\mathbb{Z}} Y_n z^{-n-\Delta_Y}$ of conformal weight Δ_Y in the chiral algebra $\mathfrak{A}$, the ordinary boundary block ${\rm B}_\lambda^{(\mathbf{1})}$ of the $\mathfrak{A}$-theory obeys the Ward identity appropriate for a two-point block on $\mathbb{P}^1$. That is,

$$
{\rm B}_\lambda^{(\mathbf{1})} \circ \left(R_\lambda(Y_n) \otimes \mathbf{1} + \zeta_Y\, \mathbf{1} \otimes R_\lambda(Y_{-n})\right) = 0 \tag{9.6}
$$

with $\zeta_Y = (-1)^{\Delta_Y - 1}$. When combined with the definition (8.20) of the twisted boundary blocks, the twisted intertwiner property (9.5) therefore allows us to write (suppressing from now on the representation symbol R_λ)

$$
\begin{aligned}
{\rm B}_\lambda^{(g)} \circ (Y_n \otimes \mathbf{1}) &= {\rm B}_\lambda \circ (\Theta_g \otimes id) \circ (Y_n \otimes \mathbf{1}) = {\rm B}_\lambda \circ (g(Y_n) \otimes \mathbf{1}) \circ (\Theta_g \otimes id) \\
&= -\zeta_Y\, {\rm B}_{\bar\lambda} \circ (\mathbf{1} \otimes g(Y_{-n})) \circ (\Theta_g \otimes id) = -\zeta_Y\, {\rm B}_\lambda^{(g)} \circ (\mathbf{1} \otimes g(Y_{-n}))\,.
\end{aligned}
\tag{9.7}
$$

Thus the twisted boundary states satisfy a *twisted Ward identity*

$$
{\rm B}_\lambda^{(g)} \circ \left(Y_n \otimes \mathbf{1} + \zeta_Y\, \mathbf{1} \otimes g(Y_{-n})\right) = 0\,. \tag{9.8}
$$

It is worth noting that we have defined the twisted Ward identities (9.8) through an automorphism of $\mathfrak{A}$ that only acts on the second factor of the tensor product. One could easily generalize this by considering the action of two different automorphisms g_L and g_R on the two factors, according to

$$
{\rm B}_\lambda^{(g)} \circ \left(g_L(Y_n) \otimes \mathbf{1} + \zeta_Y\, \mathbf{1} \otimes g_R(Y_{-n})\right) = 0\,. \tag{9.9}
$$

But obviously what matters is only the combination $g_L^{-1} g_R$. Thus we can describe the space of automorphism types either as the coset space $(G\times G)/G$ or as the group G. In fact, the map $(g_L, g_R) \mapsto g_L^{-1} g_R$ provides a natural bijection between the two sets which, in the case when G is a Lie group, is even an isomorphism of smooth manifolds. (For theories of free bosons, the description in terms of $(G\times G)/G$ has been established in [14].)

10 The classifying algebra $\mathcal{C}^{(g)}$ for automorphism type g

10.1 Individual classifying algebras for fixed automorphism type

We now restrict our attention to the collection of boundary conditions that possess some fixed automorphism type g. According to the results of section 8 the corresponding boundary states can all be written as linear combinations of the twisted boundary blocks (8.20) with fixed g. This suggests to study analogous elements of the classifying algebra; accordingly we introduce for every λ with $g\in S_\lambda$ the linear combination

$$\Phi_\lambda^{(g)} := \frac{|\mathcal{U}_\lambda|}{|\mathcal{G}|} \sum_{\mathrm{J}\in\mathcal{G}/\mathcal{U}_\lambda} g(\mathrm{J})^* \, \tilde\Phi_{\mathrm{J}(\bar\lambda_\circ,\psi_\circ)} \tag{10.1}$$

of basis elements of the classifying algebra $\mathcal{C}(\bar{\mathfrak{A}})$, where by $\bar\lambda_\circ$ and $\psi_\circ$ (which satisfies $\psi_\circ \succ \hat\psi_\circ$ for $\lambda \equiv [\bar\lambda_\circ, \hat\psi_\circ]$) are the representatives introduced in the paragraph before formula (8.8). To proceed, we also note the relation

$$\sum_{\mathrm{J}\in\mathcal{G}/\mathcal{U}_\lambda} g(\mathrm{J})^* \, \tilde S_{\mathrm{J}(\bar\lambda,\psi),\rho} = (|\mathcal{G}|/|\mathcal{U}_\lambda|) \, \delta_{g,g_\rho} \, \tilde S_{(\bar\lambda,\psi),\rho} \tag{10.2}$$

that follows for every $g\in S_\lambda$ with the help of the simple current symmetry (8.4). (When g is not in S_λ, then according to the remarks after (8.8) this expression vanishes.)

Using the Verlinde-like formula that expresses the structure constants of $\mathcal{C}(\bar{\mathfrak{A}})$ in terms of the diagonalizing matrix $\tilde S$, the result (10.2) allows us to compute the product of two elements of $\mathcal{C}(\bar{\mathfrak{A}})$ of the form (10.1) as

$$\Phi_\lambda^{(g)} \star \Phi_{\lambda'}^{(g')} = \delta_{g,g'} \sum_{\lambda''} \mathrm{N}_{\lambda,\lambda'}^{(g)\;\lambda''} \, \Phi_{\lambda''}^{(g)} \tag{10.3}$$

with [4]

$$\mathrm{N}_{\lambda,\lambda'}^{(g)\;\lambda''} \equiv \mathrm{N}_{\lambda,\lambda'}^{(g)\;\lambda''}(\bar{\mathfrak{A}}) := \sum_{\substack{\rho \\ g_\rho = g}} \tilde S_{(\bar\lambda_\circ,\psi_\circ),\rho} \tilde S_{(\bar\lambda'_\circ,\psi'_\circ),\rho} \tilde S^*_{(\bar\lambda''_\circ,\psi''_\circ),\rho} (\tilde S_{\bar\Omega,\rho})^{-1} \, . \tag{10.4}$$

This means that the elements $\Phi_\lambda^{(g)}$ for all λ with $g\in S_\lambda$ span not only a subalgebra, but even an *ideal* of the classifying algebra $\mathcal{C}(\bar{\mathfrak{A}})$. We call this ideal of $\mathcal{C}(\bar{\mathfrak{A}})$ the *individual classifying algebra* for automorphism type g and denote it by $\mathcal{C}^{(g)}(\bar{\mathfrak{A}})$. Also, by construction the $\Phi_\lambda^{(g)}$ are linearly independent, and hence they furnish a basis of $\mathcal{C}^{(g)}(\bar{\mathfrak{A}})$, i.e. for every fixed $\mathfrak{g}\in G$ we have

$$\mathcal{C}^{(g)}(\bar{\mathfrak{A}}) = \mathrm{span}_{\mathbb{C}}\{\Phi_\lambda^{(g)} \,|\, S_\lambda \ni g\} \, . \tag{10.5}$$

Clearly, $\mathcal{C}^{(g)}(\bar{\mathfrak{A}})$ is again semisimple; its one-dimensional irreducible representations are obtained by restriction to $\mathcal{C}^{(g)}(\bar{\mathfrak{A}})$ of those one-dimensional irreducible representations R_ρ of $\mathcal{C}(\bar{\mathfrak{A}})$ which satisfy $g_\rho = g$.

Moreover, the following counting argument shows that together the elements $\Phi_\lambda^{(g)}$ for all λ and all $g\in G$ span all of $\mathcal{C}(\bar{\mathfrak{A}})$. Namely, associated to every $\mathcal{G}$-orbit $[\bar\lambda]$ of $\bar{\mathfrak{A}}$-primaries there are

[4] Note that this expression is only defined when $g\in S_\lambda \cap S_{\lambda'} \cap S_{\lambda''}$.

$|\mathcal{U}_\lambda|$ many $\mathfrak{A}$-primaries $[\bar\lambda,\hat\psi]$, and each of them gives rise to $|S_\lambda|$ many basis elements $\Phi^{(g)}_\lambda$. On the other hand, each such $\mathcal{G}$-orbit contains $|\mathcal{G}|/|\mathcal{S}_\lambda|$ many $\bar{\mathfrak{A}}$-primaries, each of them leading to $|\mathcal{S}_\lambda|$ many basis elements $\tilde\Phi_{(\bar\lambda,\phi)}$ of $\mathcal{C}(\bar{\mathfrak{A}})$. With the help of the identities (7.9) among the sizes of the various subgroups it thus follows that

$$\sum_\lambda |S_\lambda| = \sum_{\substack{\bar\lambda \\ Q_{\mathcal{G}}(\lambda)=0}} |\mathcal{S}_\lambda| \,. \tag{10.6}$$

As a consequence we have indeed – as algebras over $\mathbb{C}$, and with the distinguished bases related by (10.1) – an isomorphism

$$\mathcal{C}(\bar{\mathfrak{A}}) \cong \bigoplus_{g\in G} \mathcal{C}^{(g)}(\bar{\mathfrak{A}}) \,. \tag{10.7}$$

This decomposition may be regarded as expressing the fact that in the situation of our interest every boundary condition has an automorphism type. Put differently, the algebra $\mathcal{C}(\bar{\mathfrak{A}})$ provides a unified description of the boundary conditions for the $|G|$ different automorphism types that correspond to the elements of G.

In the special case of trivial automorphism type, $g=\mathbf{1}$, we can use the result [1] that $\tilde S_{(\bar\lambda,\psi),\rho} = S_{\lambda,\rho}$ for all ρ with $g_\rho=\mathbf{1}$ to see that the ideal $\mathcal{C}^{(\mathbf{1})}(\bar{\mathfrak{A}})$ is nothing but the fusion rule algebra of the $\mathfrak{A}$-theory, so that we recover the known results [15, 3] for boundary conditions that do not break any of the bulk symmetries. It should also be noticed that the precise form of the structure constants of the ideals $\mathcal{C}^{(g)}(\bar{\mathfrak{A}})$ does depend on the choice of representatives $\bar\lambda_\circ$ and $\psi_\circ$ (except when $g=\mathbf{1}$, where independence of this choice follows as a consequence of the simple current relation (8.4)). This is, however, perfectly fine, because the twisted boundary block depends on the choice of representatives as well, and in fact in a manner so as to cancel the overall dependence in all physically meaningful quantities like one-point correlators for bulk fields on the disk.

10.2 Independence of $\mathcal{C}^{(g)}$ on the orbifold group

A remarkable feature of the formula (10.4) for the structure constants of $\mathcal{C}^{(g)}(\bar{\mathfrak{A}})$ is that the automorphism type g just enters by restricting the range of summation. This suggests that the individual classifying algebra for automorphism type g in fact does *not* depend on the specific orbifold subalgebra $\bar{\mathfrak{A}}$, i.e. on the group G that contains the automorphism g, but rather on g alone. Below we will show that this is indeed the case, and actually the statement already applies to the relevant entries of the diagonalizing matrix $\tilde S$. Now recall from section 8 (see the remarks after formula (8.21)) that a similar statement applies to the twisted boundary blocks $\mathrm{B}^{(g)}_\lambda$. When combined with the present result, then according to formula (8.22) this independence property of the *blocks* $\mathrm{B}^{(g)}_\lambda$, which are chiral quantities, extends to the (non-chiral) boundary *states* $\mathcal{B}_\rho$.

To prove the independence of $\mathcal{C}^{(g)}(\bar{\mathfrak{A}})$ on the choice of orbifold group G containing g, it is convenient to study the orbifold with respect to the cyclic group

$$G' := \,<\!g\!>\, \cong \mathbb{Z}_{|G'|} \tag{10.8}$$

that is generated by g. We first note that the diagonalizing matrix $\tilde S'$ for the total classifying algebra $\mathcal{C}(\mathfrak{A}')$ of all boundary conditions preserving $\mathfrak{A}'=\mathfrak{A}^{<g>}\subset\mathfrak{A}$ is given by the expression $(5.9)_A$ which contains contributions involving the various matrices $S^{\rm J}$ (though in this special cyclic case the formula simplifies). However, in the expression for the structure constants of $\mathcal{C}^{(g)}(\mathfrak{A}')$ only those entries $\tilde S'_{(\lambda',\varphi),[\rho',\hat\psi]'}$ appear for which $g\in S'_{\lambda'}$, and these entries turn out to be particularly simple. Indeed, since g generates $<g>$, the latter property means that the stabilizer is maximal, $S'_{\lambda'}=G'$; by the duality relation $\mathcal{U}'_{\lambda'}\cong G'^*/S'^{\,*}_{\lambda'}$ this implies $\mathcal{S}'_{\lambda'}=\mathcal{U}'_{\lambda'}=1$. Similarly, because of $g_{\rho'}=g$ the monodromy charges of ρ' with respect to the extension from $\mathfrak{A}'$ to $\mathfrak{A}$ have denominator $|G'|$, so that ρ' cannot be a fixed point under any simple current in G'^*, which implies that also $\mathcal{S}'_{\rho'}=\mathcal{U}'_{\rho'}=1$. Thus we have to deal with full G'^*-orbits only; in particular the simple current summation in the formula for $\tilde S'$ reduces to the term with ${\rm J}=\mathbf{1}$:

$$\tilde S'_{\lambda',[\rho',\hat\psi]'} = \frac{|G'|}{\sqrt{|\mathcal{S}'_{\lambda'}|\,|\mathcal{U}'_{\lambda'}|\,|\mathcal{S}'_{\rho'}|\,|\mathcal{U}'_{\rho'}|}}\, S'_{\lambda',\rho'} = |G'|\, S'_{\lambda',\rho'}\,. \tag{10.9}$$

To proceed, we note that the algebra $\mathfrak{A}'=\mathfrak{A}^{<g>}$ can be obtained from $\bar{\mathfrak{A}}=\mathfrak{A}^{\rm G}$ as a simple current extension by the subgroup

$$\mathcal{G}'' := \{{\rm J}\in\mathcal{G}\,|\,g({\rm J})=1\} \tag{10.10}$$

of the simple current group $\mathcal{G}=G^*$. (The subgroup $\mathcal{G}''$ has index $|G'|$ in $\mathcal{G}$; in fact, the factor group $\mathcal{G}/\mathcal{G}''$ is cyclic of order $|G'|$.) In particular, the modular S-matrix S' of the $\mathfrak{A}'$-theory can be expressed through quantities of the $\bar{\mathfrak{A}}$-theory as

$$S'_{\lambda',\mu'} = \frac{|\mathcal{G}''|}{\sqrt{|\mathcal{S}''_{\lambda'}|\,|\mathcal{U}''_{\lambda'}|\,|\mathcal{S}''_{\mu'}|\,|\mathcal{U}''_{\mu'}|}} \sum_{{\rm J}\in\mathcal{U}''_{\lambda'}\cap\mathcal{U}''_{\mu'}} \hat\psi_{\lambda'}({\rm J})\, S^{\rm J}_{\bar\lambda,\bar\mu}\, \hat\psi^*_{\mu'}({\rm J}) \tag{10.11}$$

with $\lambda'\equiv[\bar\lambda,\hat\psi_{\lambda'}]''$ and $\mu'\equiv[\bar\mu,\hat\psi_{\mu'}]''$.

The $\mathfrak{A}'$-theory has a simple current of the form

$${\rm J}'_g = {\rm J}_g\cdot\mathcal{G}'' \equiv [{\rm J}_g]''\,, \tag{10.12}$$

where ${\rm J}_g\in\mathcal{G}$ is a simple current of the $\bar{\mathfrak{A}}$-theory that is characterized by the property that $|G'|$ is the smallest positive integer m such that $({\rm J}_g)^m$ lies in $\mathcal{G}''$, so that ${\rm J}'_g$ has order $|G'|$. By this property of ${\rm J}_g$ and the definition (10.10) of $\mathcal{G}''$ it follows that

$$g_\rho({\rm J}_g^m{\rm K}'') = g({\rm J}_g^m) \neq 1 \tag{10.13}$$

for every ${\rm K}''\in\mathcal{G}''$ and every $m=1,2,...\,,|G'|{-}1$. This means that none of the monodromy charges $Q_{{\rm J}_g^m{\rm K}''}(\rho)$ vanishes, so that ρ cannot be a fixed point with respect to any of these simple currents ${\rm J}_g^m{\rm K}''$. It follows that the stabilizer $\mathcal{S}_\rho$ is contained in $\mathcal{G}''$, which in turn implies that $\mathcal{S}''_{\rho'}=\mathcal{S}_\rho$ and hence also $\mathcal{U}''_{\rho'}=\mathcal{U}_\rho$. As a consequence, the boundary labels $[\rho,\hat\psi_\rho]$ of the matrix $\tilde S$ that satisfy $g_\rho=g$ are precisely the same as those appearing in formula (10.11) for S', and the simple current summation in the expression for the corresponding entries of $\tilde S$ only runs over elements of $\mathcal{G}''$. Using also the fact that $\mathcal{S}''_{\lambda'} = \mathcal{S}_\lambda \cap \mathcal{G}''$, we can therefore write

$$\begin{aligned} \tilde S_{(\bar\lambda,\psi_\lambda),[\bar\rho,\hat\psi_\rho]} &= \frac{|G|}{\sqrt{|\mathcal{S}_\lambda|\,|\mathcal{U}_\lambda|\,|\mathcal{S}_\rho|\,|\mathcal{U}_\rho|}} \sum_{\mathrm{J}\in\mathcal{S}_\lambda\cap\mathcal{U}_\rho} \psi_\lambda(\mathrm{J})\, S^{\mathrm{J}}_{\bar\lambda,\bar\rho}\, \hat\psi^*_\rho(\mathrm{J}) \\ &= \frac{|G'|\,|\mathcal{G}''|}{\sqrt{|\mathcal{S}_\lambda|\,|\mathcal{U}_\lambda|\,|\mathcal{S}''_{\rho'}|\,|\mathcal{U}''_{\rho'}|}} \sum_{\mathrm{J}\in\mathcal{S}''_{\lambda'}\cap\mathcal{U}''_{\rho'}} \psi_{\lambda'}(\mathrm{J})\, S^{\mathrm{J}}_{\bar\lambda,\bar\rho}\, \hat\psi^*_{\rho'}(\mathrm{J})\,. \end{aligned} \tag{10.14}$$

Furthermore, because of $g_\rho = g$ the simple current relation (8.4) for $\tilde S$ implies that these entries of $\tilde S$ are identical for all $\bar\lambda$ on one and the same $\mathcal{G}''$-orbit. We may therefore equate the expression (10.14) with its average over $\mathcal{S}''_\lambda$. The simple current relation for the matrices S^{J} then amounts to a restriction of the summation to $\mathcal{U}''_\lambda$, so that

$$\tilde S_{(\bar\lambda,\psi_\lambda),[\bar\rho,\hat\psi_\rho]} = \frac{|G'|\,|\mathcal{G}''|}{\sqrt{|\mathcal{S}_\lambda|\,|\mathcal{U}_\lambda|\,|\mathcal{S}''_{\rho'}|\,|\mathcal{U}''_{\rho'}|}} \sum_{\mathrm{J}\in\mathcal{U}''_{\lambda'}\cap\mathcal{U}''_{\rho'}} \psi_{\lambda'}(\mathrm{J})\, S^{\mathrm{J}}_{\bar\lambda,\bar\rho}\, \hat\psi^*_{\rho'}(\mathrm{J}) = |G'| \sqrt{\frac{|\mathcal{S}''_{\lambda'}|\,|\mathcal{U}''_{\lambda'}|}{|\mathcal{S}_\lambda|\,|\mathcal{U}_\lambda|}}\, S'_{[\bar\lambda,\hat\psi_{\lambda'}]'',[\bar\rho,\hat\psi_{\rho'}]''}\,. \tag{10.15}$$

To analyze the prefactor appearing here, we first remark that the index of the subgroup $\mathcal{S}''_{\lambda'}$ in $\mathcal{S}_\lambda$ is some divisor n_λ of $|G'|$. There is a simple current $\mathrm{J}_\lambda \in \mathcal{S}_\lambda$ which plays an analogous role for the embedding $\mathcal{S}''_{\lambda'} \subseteq \mathcal{S}_\lambda$ as J_g plays for the embedding of $\mathcal{G}''$ in $\mathcal{G}$, i.e. n_λ is the smallest power such that $\mathrm{J}_\lambda^{n_\lambda}$ is in $\mathcal{S}''_{\lambda'}$, and the elements of $\mathcal{S}_\lambda$ are of the form $\mathrm{J}_\lambda^m \mathrm{K}''$ with $\mathrm{K}'' \in \mathcal{S}''_{\lambda'}$ and $m = 1, 2, \dots, n_\lambda{-}1$. Moreover, from the fact that $g \in S_\lambda$ it follows with the help of duality that $\mathrm{J}_\lambda^m \notin \mathcal{U}_\lambda$ for all $m = 1, 2, \dots, n_\lambda{-}1$, and hence we have $\mathcal{U}_\lambda \subseteq \mathcal{S}''_\lambda$. Thus when forming the untwisted stabilizer associated to $\mathcal{S}''_{\lambda'}$ one does not lose any elements of $\mathcal{U}_\lambda$, so that $\mathcal{U}_\lambda \subseteq \mathcal{U}''_{\lambda'}$. Observing that $\mathcal{U}_\lambda$ is precisely the kernel of the group homomorphism from $\mathcal{U}''_{\lambda'}$ to $\mathbb{C}$ that maps $\mathrm{K}'' \in \mathcal{U}''_{\lambda'}$ to the n_λth root of unity $F_\lambda(\mathrm{J}_\lambda, \mathrm{K}'')$, it follows that the index of $\mathcal{U}_\lambda$ in $\mathcal{U}''_{\lambda'}$ is n_λ. Thus we have

$$|\mathcal{S}_\lambda| \,/\, |\mathcal{S}''_{\lambda'}| = n_\lambda = |\mathcal{U}''_{\lambda'}| \,/\, |\mathcal{U}_\lambda|\,, \tag{10.16}$$

so that (10.15) reduces to

$$\tilde S_{(\bar\lambda,\psi_\lambda),[\bar\rho,\hat\psi_\rho]} = |G'|\, S'_{[\bar\lambda,\hat\psi_{\lambda'}]'',[\bar\rho,\hat\psi_{\rho'}]''}\,. \tag{10.17}$$

Comparison with (10.9) then shows that the relevant matrix elements of $\tilde S$ are identical to those of $\tilde S'$. We conclude that

$$\mathcal{C}^{(g)}(\mathfrak{A}^G) = \mathcal{C}^{(g)}(\mathfrak{A}^{<g>}) =: \mathcal{C}^{(g)} \tag{10.18}$$

for every finite abelian orbifold group G with $G \ni g$. It is worth pointing out that the fact that the individual classifying algebra does not depend on the preserved subalgebra constitutes another quite non-trivial check of our ansatz for the diagonalizing matrix $\tilde S$. We also learn that the structure constants of the algebra $\mathcal{C}^{(g)}$ read

$$\mathrm{N}^{(g)\ \nu'}_{\lambda',\mu'} = \mathrm{N}^{(g)\ \nu'}_{\lambda',\mu'}(\mathfrak{A}^{<g>}) = |G'|^2 \sum_{\substack{[\rho']' \\ g_{\rho'}=g}} \frac{S'_{\lambda',\rho'}\, S'_{\mu',\rho'}\, S'^{*}_{\nu',\rho'}}{S'_{\bar\Omega',\rho'}}\,. \tag{10.19}$$

By inserting a suitable projector, they can rewritten in the form

$$\mathrm{N}^{(g)\ \nu'}_{\lambda',\mu'} = \sum_{\mathrm{J}'\in\mathcal{G}'} \sum_{\rho'} g(\mathrm{J}')\, g_{\rho'}(\mathrm{J}')^*\, \frac{S'_{\lambda',\rho'}\, S'_{\mu',\rho'}\, S'^{*}_{\nu',\rho'}}{S'_{\bar\Omega',\rho'}} = \sum_{\mathrm{J}'\in\mathcal{G}'} g(\mathrm{J}')\, \mathrm{N}'^{\ \mathrm{J}'\nu'}_{\lambda',\mu'}\,, \tag{10.20}$$

i.e. as a linear combination of fusion coefficients of the $\mathfrak{A}^{<g>}$-theory.

It is reasonable to expect that similar considerations apply to orbifold subalgebras with respect to non-abelian groups G, too, so that in particular also in that case every boundary condition possesses a definite automorphism type. Assuming this to be true, the classifying algebras for fixed automorphism type studied here should coincide with their analogues in the non-abelian case. In other words, the set of boundary conditions will be exhausted by those boundary conditions that are already known from the cyclic groups $<g>$ for all $g \in G$. (On the other hand, the detailed structure of the classifying algebra, which involves the distinguished basis $\{\tilde\Phi\}$, will still be more involved.)

10.3 Relation with traces on bundles of chiral blocks

The independence on the specific group G finds its natural explanation in the fact that these numbers are interpretable as traces of appropriate maps on bundles of chiral blocks [12]. Namely, since for every λ we are given the twisted intertwiner maps $\Theta_g^{(\lambda)}\colon \mathcal{H}_\lambda \to \mathcal{H}_{g^\star\lambda}$ (7.1), we also have the corresponding tensor product maps

$$\vec\Theta_g \equiv \vec\Theta_{g,g,\dots,g}^{(\lambda_1\,\lambda_2\,\dots\,\lambda_m)} := \Theta_g^{(\lambda_1)} \otimes \Theta_g^{(\lambda_2)} \otimes \cdots \otimes \Theta_g^{(\lambda_m)} \tag{10.21}$$

on tensor products of $\mathfrak{A}$-modules. In view of the definition [16, 17] of chiral blocks $V_{\lambda_1\lambda_2\dots\lambda_m}$ as singlets (with respect to a suitable *block algebra*) in the algebraic dual $(\mathcal{H}_{\lambda_1}\otimes\mathcal{H}_{\lambda_2}\otimes\cdots\otimes\mathcal{H}_{\lambda_m})^*$ of these tensor products, the twisted intertwining property together with the fact that the automorphism g respects the grading of the chiral algebra implies the existence of a linear map

$$\vec\Theta_g^* \equiv \vec\Theta_{g,g,\dots,g}^{*\,(\lambda_1\,\lambda_2\,\dots\,\lambda_m)}:\quad V_{\lambda_1\,\lambda_2\,\dots\,\lambda_m} \to V_{g^\star\lambda_1\,g^\star\lambda_2\,\dots\,g^\star\lambda_m} \tag{10.22}$$

between spaces of chiral blocks. When $g^\star\lambda_i = \lambda_i$ for all $i = 1, 2, \dots, m$, this linear map $\vec\Theta_g^*$ is an *endo*morphism so that one can compute its trace; we will be interested in the traces of three-point blocks. (A concrete description of the block algebras and hence of the maps $\vec\Theta_g^*$ is so far only available for the case of WZW theories, where the situation can be analyzed in terms of the horizontal subalgebra of the relevant affine Lie algebra.) Now in the case of our interest, where all simple current stabilizers in the $\mathfrak{A}'$-theory are trivial, the fusion rules of the $\mathfrak{A}$-theory can be expressed through the modular S-matrix of the $\mathfrak{A}'$-theory as

$$\begin{aligned} \mathrm{N}_{\lambda,\mu}^{\ \ \nu} &= |G'|^2 \sum_{\substack{[\rho']' \\ Q_{\mathcal{G}'}(\rho')=0}} \frac{1}{|\mathcal{S}'_{\rho'}|}\, \frac{S'_{\lambda',\rho'}\, S'_{\mu',\rho'}\, S'^{*}_{\nu',\rho'}}{S'_{\Omega',\rho'}} \\ &= \sum_{\mathrm{J}'\in\mathcal{G}'} \sum_{\rho'} \mathrm{e}^{2\pi\mathrm{i}Q_{\mathrm{J}'}(\rho')}\, \frac{S'_{\lambda',\rho'}\, S'_{\mu',\rho'}\, S'^{*}_{\nu',\rho'}}{S'_{\Omega',\rho'}} = \sum_{\mathrm{J}'\in\mathcal{G}'} \mathrm{N}'^{\ \ \mathrm{J}\nu'}_{\lambda',\mu'} \end{aligned} \tag{10.23}$$

This indicates that the chiral three-point blocks of our interest can be decomposed into the direct sum of spaces of chiral blocks of the $\mathfrak{A}'$-theory. Such a decomposition should in fact be expected on general grounds, and the chiral blocks of the $\mathfrak{A}'$-theory should fit together to sub-bundles of the bundles of chiral blocks of the $\mathfrak{A}$-theory. Now when restricted to irreducible $\mathfrak{A}'$-modules the maps Θ_g are ordinary intertwiners, and as a consequence the map $\vec\Theta_g^*$ acts on the subspaces of chiral blocks as a multiple of the identity. On the subspace of dimension $\mathrm{N}_{\lambda',\mu'}^{(g)\;\mathrm{J}\nu'}$ the map $\vec\Theta_g^*$ should therefore act with eigenvalue $g(\mathrm{J})$. It thus follows that upon choosing representatives of the $\mathcal{G}'$-orbits, the trace of this map is precisely given by the number (10.20):

$$\mathrm{tr}_{V_{\lambda\mu\nu}}\,\vec\Theta_g^{*\,(\lambda\mu\nu)} = \mathrm{N}_{\lambda',\mu'}^{(g)\;\nu'}(\mathfrak{A}^{<g>})\,. \tag{10.24}$$

For $g\,{\ne}\,1$ these numbers do depend on the choice of representatives λ' of the orbits $\lambda\,{\equiv}\,[\lambda']$, in agreement with the fact that the maps $\vec\Theta_g^*$ are defined only up to a phase.

11 T-duality

In all the considerations so far, we have required that the torus partition function, and correspondingly the pairing of the labels λ and $\tilde\lambda$ of the bulk fields $\phi_{\lambda,\tilde\lambda}$, is given by charge conjugation, which we denote by $\sigma^*_{\rm c}$: $\tilde\lambda\,{=}\,\sigma^*_{\rm c}(\lambda)\,{\equiv}\,\lambda^+$. In this section we analyze what happens when a different torus partition function is chosen. [5] To this end we first have to state what we mean by a chiral algebra $\mathfrak{A}$ and its automorphisms in more concrete terms than was done so far. In mathematical terms, a chiral algebra [6] is a *vertex operator algebra* [19, 20]; the relevant data are therefore the vector space $\mathcal{H}_\Omega$, the vacuum vector $\Omega\,{\in}\,\mathcal{H}_\Omega$, the Virasoro element $v_{\rm Vir}\,{\in}\,\mathcal{H}_\Omega$, and a 'vertex operator map' Y. The latter realizes the state-field correspondence, i.e. associates to every $v\,{\in}\,\mathcal{H}_\Omega$ a field operator $Y(v;z)$ (technically, a linear map from $\mathcal{H}_\Omega$ to $\mathrm{End}(\mathcal{H}_\Omega)\otimes\mathbb{C}[[z,z^{-1}]]$ with z a formal variable), e.g. the energy-momentum tensor T to the Virasoro element, $Y(v_{\rm Vir};z)\,{=}\,T(z)$. By an *automorphism* of a vertex operator algebra $\mathfrak{A}$ we mean an invertible linear map

$$\sigma:\quad \mathcal{H}_\Omega\to\mathcal{H}_\Omega \tag{11.1}$$

that is compatible with state-field correspondence, i.e. satisfies

$$\sigma^{-1}\,Y(\sigma v;z)\,\sigma = Y(v;z) \tag{11.2}$$

for all $v\,{\in}\,\mathcal{H}_\Omega$. (Let us stress that – unlike e.g. in [21] – at this point we do *not* require that the map σ leaves the vacuum and the Virasoro element fixed.) As already outlined in section 7, each such map is accompanied by a permutation σ^* of the label set $I\,{:=}\,\{\lambda\}$ of $\mathfrak{A}$-primaries and by twisted intertwiners $\Theta_\sigma\,{\equiv}\,\Theta_\sigma^{(\lambda)}$: $\mathcal{H}_\lambda\,{\to}\,\mathcal{H}_{\sigma^*\lambda}$ between the corresponding irreducible $\mathfrak{A}$-modules.

To proceed, we introduce two particular subgroups of the group Γ of all maps f from I to $\mathrm{Aut}(\mathfrak{A})$. For every $f\,{\in}\,\Gamma$ and every $\lambda\,{\in}\,I$, the image

$$f_\lambda := f(\lambda):\quad \mathcal{H}_\Omega\to\mathcal{H}_\Omega \tag{11.3}$$

[5] It is by no means necessary that the permutation that (for maximally extended chiral algebra) characterizes the torus partition function is equal to the permutation $g^\star$ defined via the automorphism type – if such an automorphism type exists at all – of a boundary condition. While both mappings are associated to the transition from *chiral* conformal field theory (i.e., conformal field theory on a complex curve) to *full* conformal field theory (conformal field theory on a real two-dimensional surface), they refer to such a transition for two different world sheets – the disk and the torus, respectively – which are not related by any factorization rules. As a consequence, they can be chosen independently.

[6] Not to be confused with the global object for which the term chiral algebra has also been used in the recent mathematical literature [18], which we prefer to call a *block algebra*.

is an automorphism in $\mathrm{Aut}(\mathfrak{A})$. We denote by Γ_Z the subgroup of all those elements f of Γ for which the map [7]

$$\begin{aligned} \pi_f^*: \quad & I \to I \\ & \lambda \mapsto f_\lambda^*(\lambda) \end{aligned} \tag{11.4}$$

preserves conformal weights modulo integers, i.e. fulfils $\Delta_{\pi_f^*(\lambda)} = \Delta_\lambda \bmod \mathbb{Z}$ for all $\lambda \in I$, and is an automorphism of the fusion rules, i.e. satisfies

$$\mathrm{N}_{\pi_f^*(\lambda),\pi_f^*(\mu),\pi_f^*(\nu)} \equiv \mathrm{N}_{f_\lambda^*(\lambda),f_\mu^*(\mu),f_\nu^*(\nu)} = \mathrm{N}_{\lambda,\mu,\nu} \tag{11.5}$$

for all $\lambda,\mu,\nu \in I$. (Examples for such automorphisms are those induced by simple currents, see e.g. [6, 22].) Every element of Γ_Z gives rise to a modular invariant torus partition function

$$Z_f(\tau) := \sum_{\lambda\in I} \chi_\lambda(\tau)\, \chi_{\pi_f^*(\lambda^+)}(\tau)^* \,. \tag{11.6}$$

The second subgroup of Γ of our interest consists of *constant* maps f whose image – to be denoted by g_f – is an automorphism of $\mathfrak{A}$ that leaves the Virasoro element fixed, $g_f(v_{\rm Vir}) = v_{\rm Vir}$ (and hence in particular obeys $g_f^*(\Omega) = \Omega$ and preserves conformal weights exactly, not only modulo integers). We denote the subgroup of those maps of this kind that also lie [8] in Γ_Z by Γ_B. Every automorphism g_f of this subgroup of Γ_Z with $f \in \Gamma_B$ can be used to define conformally invariant boundary conditions. Furthermore, for every $f \in \Gamma_B$ and every $f' \in \Gamma_Z$ the modular invariant partition functions $Z_{f'}$ and $Z_{ff'}$ are physically indistinguishable, i.e. upon a suitable relabelling of the fields all correlation functions in the associated conformal field theories coincide. Accordingly, it is appropriate to refer to Γ_B as the T-*duality group* of the theory. For every chiral algebra $\mathfrak{A}$ the T-duality group Γ_B contains in particular the map $f_{\rm c}$ whose image is the charge conjugation automorphism $\sigma_{\rm c}$, i.e. $f_{\rm c}(\lambda) = \sigma_{\rm c}$ for all $\lambda \in I$, and $\pi_{f_{\rm c}}^*(\lambda) = \sigma_{\rm c}(\lambda) = \lambda^+$.

The two theories with partition functions $Z_{f'}$ and $Z_{ff'}$ being indistinguishable, in particular the respective sets of all conformally invariant boundary conditions must be the same. It is worth investigating this correspondence in some detail. Let us denote for any $f \in \Gamma_B$ and any $f' \in \Gamma_Z$ by

$$\mathcal{C}^{(g_f;f')} \equiv \mathcal{C}^{(g_f;f')}(\mathfrak{A}^{<g>}) \tag{11.7}$$

the classifying algebra for boundary conditions of automorphism type g_f for a conformal field theory with torus partition function $Z_{f'}$. Recall from section 10 that this algebra (as well as its distinguished basis) can be constructed by starting with any arbitrary finite abelian group G containing g. As also discussed there, for the case where $f' = f_{\rm c}$ corresponds to charge conjugation, the structure constants of $\mathcal{C}^{(g_f;f')}$ are given by the traces over the linear maps induced by Θ_{g_f} on the three-point chiral blocks:

$$\mathrm{N}_{\lambda,\mu,\nu}^{(g_f;f_{\rm c})} = \mathrm{tr}_{V_{\lambda,\mu,\nu}}\big(\vec{\Theta}^{*\,(\lambda\,\mu\,\nu)}_{g_f,g_f,g_f}\big) \,. \tag{11.8}$$

[7] Note that typically several distinct $f \in \Gamma$ will give rise to one and the same permutation π_f^*.

[8] In fact one should expect that the property of inducing a fusion rule automorphism need not be required independently, but is satisfied automatically as a consequence of the consistency of the relevant orbifold theory. This has been demonstrated in the case of order-two automorphisms in [5]. Indeed, consistency of the orbifold theory requires that the S-matrix of the $\mathfrak{A}$-theory behaves with respect to the permutation $g^\star$ that is induced by the non-trivial element g of $\mathbb{Z}_2$ via the maps Θ_g as $S_{\lambda,g^\star\mu} = S_{g^\star\lambda,\mu}$. When combined with the Verlinde formula, this implies that $g^\star$ furnishes an automorphism of the fusion rules of the $\mathfrak{A}$-theory.

In the general case, what should matter are not the individual maps f and f' by themselves, but rather only the information on how the pairing described by the boundary conditions relates to the pairing in the torus partition function. In other words, a simultaneous action of an element of Γ_B on both labels of the classifying algebra $\mathcal{C}^{(g_f;f')}$ will not change the situation in an observable manner. A well-known example of this effect is seen in the theory of a free boson compactified at radius R, where the T-duality map $R \mapsto 2/R$ (which corresponds to $f \mapsto f f_{\rm c}$) amounts to exchanging Dirichlet and Neumann conditions (compare subsection 15.1). (For orbifolds of free bosons, T-duality has been studied recently e.g. in [23].)

In short, for every $f'' \in \Gamma_B$ there is an isomorphism

$$\mathcal{C}^{(g_{ff''};f'f'')} \cong \mathcal{C}^{(g_f;f')} \tag{11.9}$$

of classifying algebras. With the help of these isomorphisms we deduce that the structure constants of the classifying algebra $\mathcal{C}^{(g_f;f_{\rm c}f')}$ are given by the traces

$$\mathrm{N}_{\lambda,\mu,\nu}^{(g_f;f_{\rm c}f')} = \mathrm{tr}_{V_{\lambda,\mu,\nu}} \big(\vec\Theta^{*\,(\lambda\,\mu\,\nu)}_{g_f g_{f'}^{-1}, g_f g_{f'}^{-1}, g_f g_{f'}^{-1}} \big) \,. \tag{11.10}$$

In this formula it is manifest that simultaneous application of a T-duality transformation $f \in \Gamma_B$ on the bulk modular invariant and on the automorphism type of the boundary conditions does yield isomorphic classifying algebras.

Building on this result, we expect that formula (11.10) extends naturally to arbitrary elements f of Γ_Z, i.e. to the case that f is not necessarily constant on I. We are thus led to conjecture that in this case the relation (11.10) gets generalized to [9]

$$\mathrm{N}_{\lambda,\mu,\nu}^{(g_f;f_{\rm c}f')} = \mathrm{tr}_{V_{\lambda,\mu,\nu}} \big(\vec\Theta^{*\,(\lambda\,\mu\,\nu)}_{g_f f_\lambda'^{-1}, g_f f_\mu'^{-1}, g_f f_\nu'^{-1}} \big) \,. \tag{11.11}$$

In the special case of $D_{\rm odd}$-type modular invariants (which correspond to a simple current automorphism for an order-two simple current of half-integral conformal weight), the relation (11.11) is already known [24, 25] to hold.

Our observations imply in particular that the automorphism type g_ρ of a boundary condition is *not* an *observable* concept. What *is* observable is the product $g_\rho^* \pi_f^{*-1}$, i.e. the 'difference' between an automorphism in the torus partition function and on the boundary. (In other words, one should regard automorphism types as elements of the 'affinum' – see section 12 below – of the group Γ_B rather than as elements of Γ_B itself.) Similarly, also the difference $g(g')^{-1}$ of two automorphism types is observable; e.g. the annulus partition function will be different in the situation where one deals with two boundary conditions of distinct automorphism type as compared to the situation where the automorphism type of both boundary conditions is the same.

While in the considerations above the T-duality transformations had to be applied to the torus partition function and to the boundary conditions simultaneously, there is also a slightly different notion of T-duality for boundary conditions, to which we now turn our attention.

[9] One might have expected that here products of Θ-maps rather than the Θ-map for the product of automorphisms appears. But when the group Γ_Z is realized projectively, this would lead to inconsistencies. When Γ_Z is realized genuinely, then the two descriptions are equivalent.

Namely, we keep the bulk theory fixed, and ask whether boundary conditions of different automorphism type can be associated to each other through a suitable element of the T-duality group. While in many cases this question turns out to have an affirmative answer, the relationship in question is between *families* of boundary conditions rather than between individual boundary conditions. In short, T-duality on boundary conditions is not a one-to-one map, in general. Closer inspection shows that the relevant families of boundary conditions can be understood as orbits of boundary conditions with respect to some suitable symmetry. Here by the term symmetry we mean a bijection γ of the space of boundary conditions of a given automorphism type g such that the annulus amplitudes coincide, i.e.

$$\mathrm{A}_{\gamma(\rho)\,\gamma(\rho')}(t) = \mathrm{A}_{\rho\,\rho'}(t) \tag{11.12}$$

for all ρ, ρ' with automorphism type $g_\rho = g = g_{\rho'}$.

Let us study the presence of such a symmetry first in the example of the critical three-state Potts model. It is known [26] that the duality symmetry of this model maps the free boundary condition to any of the three fixed boundary conditions, and indeed this is a specific case of the general duality [27] between free and 'configurational' boundary conditions of lattice spin models. Similarly, the new boundary condition discovered in [26] gets mapped to any of the three mixed boundary conditions. In this case the symmetry group $\mathrm{H} = \mathbb{Z}_3$ of the fixed or mixed boundary conditions is directly inherited from the lattice realization of the Potts model. We also observe that precisely those boundary conditions for which this $\mathbb{Z}_3$-symmetry is spontaneously broken are grouped in non-trivial orbits. Furthermore, one readily checks that for orbits that are related by the T-duality $\pi_{\mathrm{T}}^* = \sigma_{\mathrm{c}}^*$, the sum rule

$$N_{\pi_{\mathrm{T}}^*(\rho)}^{-1}\, N_{\pi_{\mathrm{T}}^*(\rho')}^{-1} \sum_{\gamma,\gamma'\in\mathrm{H}} \mathrm{A}_{\gamma\pi_{\mathrm{T}}^*(\rho)\,\gamma'\pi_{\mathrm{T}}^*(\rho')} = N_\rho^{-1}\, N_{\rho'}^{-1} \sum_{\gamma,\gamma'\in\mathrm{H}} \mathrm{A}_{\gamma(\rho)\,\gamma'(\rho')} \tag{11.13}$$

for the annulus coefficients holds, where N_ρ is the order of the stabilizer of the H-action on ρ. Roughly speaking, the sum rule (11.13) tells us that T-dual *orbits* give rise to an equal number of open string states on the boundary.

This pattern can be detected in various other examples as well. For instance, in the theory of a single uncompactified free boson, there is a single Neumann boundary condition, whereas the Dirichlet boundary conditions are labelled by a position in $\mathbb{R}$, which should be interpreted as the affine space over the group $\mathrm{H} = \mathbb{R}$ of translations. This group is spontaneously broken for Dirichlet boundary conditions, and it is straightforward to check that relations (11.12) and (11.13) are satisfied in this case as well. Another class of examples is provided by boundary conditions in WZW theories that break the bulk symmetry via an *inner* automorphism of the underlying simple Lie algebra $\bar{\mathfrak{g}}$. In these cases the boundary conditions of each automorphism type are labelled by the same set, namely by the primary labels of the original theory (see [5] for the case of automorphisms of order two). The group H is in this case realized by the action of simple currents of the original theory, which account for the different possible choices of the shift vector that characterizes the inner automorphism of $\bar{\mathfrak{g}}$. Again the validity of relations (11.12) and (11.13) is easily verified.

We would like to emphasize, though, that the existence of such T-duality relations is in fact a quite special feature of an individual model. In general we do *not* expect that boundary conditions of different automorphism type are related in such a manner. For instance, in the case of boundary conditions of WZW theories that break the bulk symmetry via an *outer* automorphism of $\bar{\mathfrak{g}}$, no relations of the form (11.12) or (11.13) are known to us.

12 Boundary homogeneity

In this section we exhibit another general aspect of the space of conformally invariant boundary conditions that preserve only a subalgebra $\bar{\mathfrak{A}}$ of $\mathfrak{A}$. Namely, we show that the orbifold group G is realized as a group of symmetries on the space of boundary conditions. These symmetries permute the boundary conditions within each set $\{[\bar\rho,\hat\psi_\rho]\,|\,\hat\psi_\rho\in\mathcal{U}_\rho^*\}$ with fixed $\mathcal{G}$-orbit $[\bar\rho]$; the permutation is the same for all $\mathcal{G}$-orbits. This behavior, to which we refer as *boundary homogeneity*, is similar to the so-called *fixed point homogeneity* that is present in simple current extensions, and indeed the arguments closely resemble the ones needed in the latter context [4].

We start from the observation that the orbifold group G can be identified with the dual $\mathcal{G}^*$ of the simple current group $\mathcal{G}=G^*$, and consider some character Ψ of $\mathcal{G}$. For every field $\bar\rho$ of the $\bar{\mathfrak{A}}$-theory and every character $\hat\psi\in\mathcal{U}_\rho^*$ we then define a new character ${}^\Psi\hat\psi\in\mathcal{U}_\rho^*$ by

$$ {}^\Psi\hat\psi(\mathrm{J}) := \Psi(\mathrm{J})\,\hat\psi(\mathrm{J}) \tag{12.1} $$

for all $\mathrm{J}\in\mathcal{U}_\rho$. This is indeed again a character of $\mathcal{U}_\rho$. Moreover, manifestly the group law of G is reproduced for different choices of Ψ, so that our construction provides an action of G on the group $\mathcal{U}_\rho^*$. Typically G does not act freely, but it does act transitively.

The next step is to realize that this prescription supplies us with a well defined action on the space of boundary conditions. This is not entirely trivial, because the labels for the boundary conditions are not pairs of orbits, but rather are obtained by the equivalence relation (A.9), i.e. $(\bar\lambda,\hat\psi_\lambda)\sim\mathrm{J}\,(\bar\lambda,\hat\psi_\lambda)\,(\mathrm{J}\star\bar\lambda,{}_{\mathrm{J}}\hat\psi_\lambda)$, which involves a non-trivial manipulation of the characters. However, this complication does not do any harm, because just as for the action of $\Psi\in\mathcal{G}^*$ it consists of a multiplication, so that the two operations commute. We write

$$ {}^\Psi\!\rho := [\bar\rho,{}^\Psi\hat\psi] \quad\text{for}\quad \rho=[\bar\rho,\hat\psi]\,. \tag{12.2} $$

Similarly, when we extend by some smaller group $\mathcal{G}'\subset\mathcal{G}$, we can define the analogous object ${}^\Psi[\bar\rho,\hat\psi']' := [\bar\rho,{}^\Psi\hat\psi']'$.

Let us now explain in which sense the elements of G are to be regarded as symmetries. Using the explicit expression (6.29) for the annulus coefficients, we can establish the identity

$$ \mathrm{A}_{{}^\Psi[\bar\rho_1,\hat\psi_1]\;{}^\Psi[\bar\rho_2,\hat\psi_2]}^{\;{}^\Psi[\bar\sigma,\hat\psi_\sigma]'} = \mathrm{A}_{[\bar\rho_1,\hat\psi_1]\;[\bar\rho_2,\hat\psi_2]}^{\;[\bar\sigma,\hat\psi_\sigma]'} \tag{12.3} $$

between annulus coefficients. Thus when we also act on the corresponding chiral labels of the open string states in the annulus amplitude, we can absorb the transformation into a relabelling of the summation variables, so as to conclude that the annulus partition function is invariant. We expect that this extends indeed to a full-fledged symmetry at the level of correlation functions, where again one has to act on the insertions on the boundary.

As a consequence, the boundary conditions that correspond to one and the same $\mathcal{G}$-orbit $[\bar\rho]$ should in fact better be labelled by the elements of what may be called the *affinum* over the character group $\mathcal{U}_\rho^*$ rather than by $\mathcal{U}_\rho^*$ itself. Here by the term affinum over a group G we refer to the elementary structure of a *set* A_G that carries a free and transitive action of G. [10] Conversely, the group G can be identified with the quotient of $A_G \times A_G$ by the equivalence relation $(p,q) \sim (p^h, q^h)$ for all $h \in \mathrm{G}$. For every $p \in A_G$ we are given an identification $g \leftrightarrow p^g$ between the group G and its affinum A_G, but there does not exist any *canonical* identification. Roughly speaking, in the structure of the affinum one ignores the special role played by the identity element; thus an affinum and its group are related in much the same way as an affine space A_V is related to the corresponding vector space V, which can be identified with its group of translations. Indeed, the group of automorphisms (that is, bijections intertwining the action of G) of an affinum A_G is precisely G. It follows that any two affina over G are isomorphic; but the isomorphism is never canonical; it is always determined only up to the isomorphism group G.

We should admit that, even though we avoided using the term, the concept of an affinum is already implicit at several other places of our discussion of boundary conditions. For instance, the quantities ψ_λ and $\hat{\psi}_\rho$ that appear in the definition $(5.9)_A$ of the matrix $\tilde{S}$ are best regarded as elements of the affina of the respective character groups, because [28] the matrices S^{J} are only defined up to certain changes of basis in the space of one-point blocks on the torus and because such a change amounts to a relabelling of the characters.

We finally add a comment on how this symmetry property of the boundary conditions looks like in the case of the three-state Potts model. In this case it exchanges simultaneously two fixed and two mixed boundary conditions, while it leaves the third fixed and mixed boundary condition invariant. In the Potts model we actually have yet another symmetry on the space of boundary conditions, the action of a $\mathbb{Z}_3$-group. These two symmetries combine to the symmetric group S_3.

13 The universal classifying algebra

The decomposition (10.7) of the classifying algebra $\mathcal{C}(\bar{\mathfrak{A}})$ into individual classifying algebras for fixed automorphism types implies that for every subgroup H of the orbifold group G one has $\mathcal{C}(\mathfrak{A}^H) \subseteq \mathcal{C}(\mathfrak{A}^G)$. Indeed, $\mathcal{C}(\mathfrak{A}^H)$ is just the direct sum of the ideals $\mathcal{C}^{(g)}(\bar{\mathfrak{A}}) \subseteq \mathcal{C}(\mathfrak{A}^G)$ for all $g \in H$. It is also known [8] that the mapping $H \mapsto \mathfrak{A}^H$ provides a bijection between the set of subgroups of G and the set of consistent chiral subalgebras of $\mathfrak{A}$ that contain $\mathfrak{A}^G$. [11] In our situation the orbifold group G is abelian, but the latter result continues to hold for all nilpotent finite groups [9] and is expected to be true for arbitrary finite orbifold groups. [12]

In this section we would like to address the issue of consecutive breakings of bulk symmetries in more generality, which leads us in particular to introduce the notion of a *universal classifying algebra*. While our detailed studies have so far been restricted to cases where the preserved subalgebra $\bar{\mathfrak{A}}$ of the bulk symmetries satisfies $\bar{\mathfrak{A}} = \mathfrak{A}^G$ for some finite abelian group G, it is

[10] Thus there exists a map $A \times \mathrm{G} \to A$ acting as $(p,g) \mapsto p^g$ such that $p^g = p$ if and only if $g = e$, $p^{gh} = (p^g)^h$, and such that for each pair $(p,q) \in A \times A$ there is a unique $g \in \mathrm{G}$ with $p^g = q$. When G is non-abelian, one must distinguish between left and right actions, and hence left and right affina.

[11] A similar Galois correspondence has been established in the context of braided monoidal *-categories in [29].

[12] Assuming that the statement holds for every group within a certain class X, it follows in particular that whenever there exists at least one boundary condition that does not possess an automorphism type, then the Virasoro algebra by itself cannot be the orbifold subalgebra of $\mathfrak{A}$ with respect to any group that belongs to X.

reasonable to expect that several features of our analysis will persist for general $\bar{\mathfrak{A}}$. In particular, it should again be possible to determine a classifying algebra, provided that the following two pieces of information are available:

- the decomposition of $\mathfrak{A}$-modules into direct sums of irreducible $\bar{\mathfrak{A}}$-modules;
- concrete expressions relating the bundles of chiral blocks of the $\mathfrak{A}$-theory with those of the $\bar{\mathfrak{A}}$-theory.

We also expect that the statements about inclusions of classifying algebras valid for the case of finite abelian orbifold groups generalize as follows. To every inclusion $\bar{\mathfrak{A}} \hookrightarrow \mathfrak{A}$ of preserved bulk symmetry algebras there is associated a projection of the corresponding classifying algebras; the classifying algebra for $\mathfrak{A}$ is a suitable quotient of the one for $\bar{\mathfrak{A}}$. More generally, for every chain of inclusions

$$\bar{\mathfrak{A}} \hookrightarrow \mathfrak{A}' \hookrightarrow \mathfrak{A} \tag{13.1}$$

of symmetry algebras, there should exist a corresponding chain of projections

$$\mathcal{C}(\bar{\mathfrak{A}}) \stackrel{\pi}{\twoheadrightarrow} \mathcal{C}(\mathfrak{A}') \stackrel{\pi'}{\twoheadrightarrow} \mathcal{C}(\mathfrak{A}) \tag{13.2}$$

for the associated classifying algebras. As a consequence, every irreducible representation of $\mathcal{C}(\mathfrak{A}')$ gives rise to an irreducible representation of $\mathcal{C}(\bar{\mathfrak{A}})$. This makes sense indeed: an irreducible representation of $\mathcal{C}(\mathfrak{A}')$ corresponds to a boundary condition that preserves $\mathfrak{A}'$ and thus, a fortiori, also preserves the smaller algebra $\bar{\mathfrak{A}}$; it should therefore correspond to some irreducible representation of $\mathcal{C}(\bar{\mathfrak{A}})$. Relation (13.2) clearly holds when both $\bar{\mathfrak{A}} = \mathfrak{A}^G$ and $\mathfrak{A}' = \mathfrak{A}^H$ are orbifold subalgebras for abelian orbifold groups with $H \subseteq G$. (Moreover, the projections are compatible with the distinguished bases of the algebras, compare e.g. the arguments leading to formula (10.18) for the case $H = <g>$). More generally, one can hope to obtain this way also quantitative information on solvable orbifold groups.

This way the following picture emerges. The set $\mathcal{M}$ of all consistent subalgebras of a given chiral algebra $\mathfrak{A}$ that possess the same Virasoro element as $\mathfrak{A}$ is partially ordered by inclusion. It is reasonable to expect that $\mathcal{M}$ is even an inductive system; that is, given any two consistent subalgebras $\bar{\mathfrak{A}}_1$ and $\bar{\mathfrak{A}}_2$ of $\mathfrak{A}$, one can find another consistent subalgebra $\bar{\mathfrak{A}}_3$ that is contained in their intersection,

$$\bar{\mathfrak{A}}_3 \subseteq \bar{\mathfrak{A}}_1 \cap \bar{\mathfrak{A}}_2 \,. \tag{13.3}$$

Note that this implies in particular that we do not have to make the assumption that the intersection of all consistent subalgebras of $\mathfrak{A}$ contains a consistent subalgebra; rather, one only needs to deal with intersections of *finitely many* subalgebras.

Assuming that also in general the inclusion $\bar{\mathfrak{A}}_1 \subset \bar{\mathfrak{A}}_2$ implies that the classifying algebra for $\bar{\mathfrak{A}}_2$ is a quotient of the one for $\bar{\mathfrak{A}}_1$, one arrives at a projective system $(\bar{\mathfrak{A}}_i)$ of classifying algebras. Taking the *projective limit* over this system, we then arrive at a *universal classifying algebra*

$$\mathcal{C}^\infty := \lim_{\longleftarrow} \bar{\mathfrak{A}}_i \,. \tag{13.4}$$

(The projective limit of the closely related structure of a fusion ring has been studied in [30].) By construction, this algebra $\mathcal{C}^\infty$ governs *all* conformally invariant boundary conditions. In other words, it is the classifying algebra $\mathcal{C}(\mathfrak{Vir})$ for the case where the preserved subalgebra just consists of the Virasoro algebra. The universal classifying algebra can be found explicitly in simple models, e.g. for the free boson compactified on a circle or for the $\mathbb{Z}_2$-orbifold of these theories (see subsection 16.1).

The construction of a surjective homomorphism π: $\mathcal{C}(\bar{\mathfrak{A}}) \to \mathcal{C}(\mathfrak{A}')$ that maps the elements of the distinguished basis $\bar{\mathcal{B}}$ of $\mathcal{C}(\bar{\mathfrak{A}})$ to elements of the distinguished basis $\mathcal{B}'$ of $\mathcal{C}(\mathfrak{A}')$ is in fact straightforward as far as the $\bar{\lambda}$-part of the labels $(\bar{\lambda},\varphi)$ for $\bar{\mathcal{B}}$ are concerned. In contrast, concerning the φ-part one faces complications which stem from the absence of a simple relationship between $\mathcal{U}'_\lambda$ and $\mathcal{U}_\lambda$ (this fact had also to be taken into account in the manipulations that were necessary to establish integrality of the annulus coefficients, see subsection 6.4 of [1]). As a matter of fact, for a complete discussion of this issue even in the case of abelian orbifold subalgebras additional simple current technology is required that goes beyond the results of [4]. In particular it will be necessary to implement the powerful results that have recently been obtained in [31].

We also note that for consistency, along with the projection π there should come an injection ι from the set of boundary conditions that preserve $\mathfrak{A}'$ to those that preserve $\bar{\mathfrak{A}}$, in such a way that the diagonalizing matrices for $\mathcal{C}(\bar{\mathfrak{A}})$ and for $\mathcal{C}(\mathfrak{A}')$ are related as

$$\tilde{S}_{(\bar{\lambda},\varphi),\iota([\bar{\rho},\hat{\psi}]')} \propto \tilde{S}'_{\pi((\bar{\lambda},\varphi)),[\bar{\rho},\hat{\psi}]'} \,. \tag{13.5}$$

Again the explicit construction of ι proves to be difficult; it will not be pursued here.

14 Involutary boundary conditions

In this section we focus our attention to the situation where the symmetries $\bar{\mathfrak{A}} = \mathfrak{A}^G$ that are preserved by the boundary conditions form a subalgebra that is fixed by an involution ϖ. In other words, for such *involutary boundary conditions* the orbifold group G is just the $\mathbb{Z}_2$-group consisting of ϖ and the identity. For the associated $\mathbb{Z}_2$-orbifolds, a lot of information is available (see e.g. [32, 5]. The vacuum sector of the $\mathfrak{A}$-theory decomposes into subspaces as

$$\mathcal{H}_\Omega \cong \bar{\mathcal{H}}_{\bar{\Omega}} \oplus \bar{\mathcal{H}}_{\mathrm{J}} \tag{14.1}$$

with J a simple current of order two, and the automorphism ϖ acts as

$$\varpi|_{\bar{\mathcal{H}}_{\bar{\Omega}}} = id_{\bar{\mathcal{H}}_{\bar{\Omega}}} \,, \qquad \varpi|_{\bar{\mathcal{H}}_{\mathrm{J}}} = -id_{\bar{\mathcal{H}}_{\mathrm{J}}} \,. \tag{14.2}$$

Involutary subalgebras are very special indeed. Several of the structures that required a detailed discussion in the general case are realized rather trivially here. For instance, as cyclic groups have trivial second cohomology, all untwisted stabilizers are equal to the full stabilizers; this already simplifies various equations considerably.

The reason why we nevertheless study this simple situation in quite some detail is that it is realized in various interesting systems. We will soon display several of these examples, but first we summarize some generic features of the $\mathbb{Z}_2$ case; in particular we will study the individual classifying algebra for automorphism type ϖ.

14.1 Even and odd boundary conditions

In the case at hand, the orbifold group G and the simple current group $\mathcal{G} = G^*$ are both isomorphic to $\mathbb{Z}_2$. Thus in particular the exponentiated monodromy charge g takes its value in $\mathbb{Z}_2$, so that there are just two automorphism types of boundary conditions. We refer to those boundary conditions whose automorphism type is given by the identity as *even* boundary conditions while those with automorphism type ϖ will be called *odd*.

The Chan–Paton types for the even boundary conditions are labelled by the primary fields λ of the $\mathfrak{A}$-theory, while those for the odd boundary conditions are labelled by the orbits of primary fields of the orbifold theory (whose chiral algebra is $\bar{\mathfrak{A}} = \mathfrak{A}^{\mathbb{Z}_2}$) with monodromy charge $Q_{\rm J} = 1/2$. The simple current group is $\mathcal{G} = \{\bar\Omega, {\rm J}\}$, so in particular its orbits $[\bar\lambda]$ either have length two (i.e. have stabilizer $\mathcal{S}_\lambda = \{\bar\Omega\}$) or length one (i.e. are fixed points, with $\mathcal{S}_\lambda = \mathbb{Z}_2$). Only fields $\bar\lambda$ with vanishing monodromy charge can be fixed points; therefore fixed points cannot give rise to odd boundary conditions.

The even boundary conditions preserve, of course, all bulk symmetries. The relevant boundary blocks ${}^{\rm e}{\rm B}_\lambda$ can therefore be expressed in terms of the boundary blocks of the $\bar{\mathfrak{A}}$-theory as

$$ {}^{\rm e}{\rm B}_\lambda = \frac{1}{|\mathcal{S}_\lambda|} \left(\bar{\rm B}_{\bar\lambda} \oplus \bar{\rm B}_{{\rm J}\bar\lambda} \right) , \tag{14.3} $$

and the Ward identities that come from a field Y in the chiral algebra $\mathfrak{A}$ look like

$$ {}^{\rm e}{\rm B}_\lambda \circ \left(Y_n \otimes \mathbf{1} + \zeta_Y \, \mathbf{1} \otimes Y_{-n} \right) = 0 \tag{14.4} $$

with $\zeta_Y = (-1)^{\Delta_Y - 1}$. As for any boundary conditions that preserve all of $\mathfrak{A}$, the classifying algebra $\mathcal{C}(\mathfrak{A})$ for the even boundary conditions is the fusion algebra of the $\mathfrak{A}$-theory, with structure constants expressible via the Verlinde formula in terms of the modular transformation matrix S of the $\mathfrak{A}$-theory.

As established in subsection 9, in the case of odd boundary conditions, the Ward identities (14.4) get replaced by those for twisted boundary blocks; here they read

$$ {}^{\rm o}{\rm B}_\lambda \circ \left(Y_n \otimes \mathbf{1} + \zeta_Y \, \mathbf{1} \otimes \varpi(Y_{-n}) \right) = 0 \, . \tag{14.5} $$

The odd boundary blocks ${}^{\rm o}{\rm B}_\lambda$ satisfying these constraints are 'differences'

$$ {}^{\rm o}{\rm B}_\lambda = \bar{\rm B}_{\bar\lambda} \oplus \left(-\bar{\rm B}_{{\rm J}\bar\lambda} \right) \tag{14.6} $$

of the boundary blocks of the $\bar{\mathfrak{A}}$-theory. They are thus related to the ordinary boundary blocks as in formula (8.20), i.e. we have

$$ {}^{\rm o}{\rm B}_\lambda = {}^{\rm e}{\rm B}_\lambda \circ \left(\Theta_\varpi \otimes id \right) , \tag{14.7} $$

where the maps Θ_ϖ satisfy the ϖ-twisted intertwining property $\Theta_\varpi \, Y = \varpi(Y) \, \Theta_\varpi$ (which together with their action on the highest weight vector determines them uniquely).

14.2 The classifying algebra

Let us now display the total classifying algebra $\mathcal{C}(\bar{\mathfrak{A}})$ which governs even and odd boundary conditions simultaneously. We already know what the labels for the basis of $\mathcal{C}(\bar{\mathfrak{A}})$ and for its one-dimensional irreducible representations look like. Moreover, in the formula $(5.9)_A$ for the diagonalizing matrix $\tilde S$, in the $\mathbb{Z}_2$ case only two different matrices appear, namely the modular S-matrix $\bar S \equiv S^{\bar\Omega}$ of the orbifold theory and the matrix $\breve S := S^{\rm J}$ for the simple current J. The length N_λ of an orbit of J can be either one or two; it will be convenient to use different symbols for labels for fixed points and those for length-two orbits; we choose roman letters $f, g, ...$ for the former and greek letters $\alpha, \beta, ...$ from the beginning of the alphabet for the latter. Also, for simplicity we will use one and the same symbol ψ to refer to a $\mathcal{G}$-character and to its value on the non-trivial element $\mathrm{J} \in \mathcal{G}$, which can be either of ± 1. For the subsets of the whole set $I = \{\lambda\}$ of primary labels of the $\mathfrak{A}$-theory that consist of the labels for full orbits and for fixed points we write $I_\circ$ and I_f, respectively, i.e.

$$I_\circ := \{\mu \in I \mid N_\mu = 2\}\,, \qquad I_\mathrm{f} := \{\mu \in I \mid N_\mu = 1\}\,. \tag{14.8}$$

We also introduce the corresponding subsets

$$\bar I_\circ := \{\bar\alpha \in \bar I \mid \alpha = \{\bar\alpha, \mathrm{J}\bar\alpha\} \in I_\circ\}\,, \qquad \bar I_\mathrm{f} := \{\bar f \mid (f, \psi) \in I_\mathrm{f}\} \tag{14.9}$$

of the label set $\bar I$ of the orbifold theory. With these notations the dimension of $\mathcal{C}(\bar{\mathfrak{A}})$ reads $\dim \mathcal{C}(\bar{\mathfrak{A}}) = |\bar I_\circ| + 2\,|\bar I_\mathrm{f}|$. By the sum rule $(5.22)_A$ this must equal the number of boundary conditions, i.e. the number of $\mathbb{Z}_2$-orbits of the $\bar{\mathfrak{A}}$-theory, counted with their stabilizer:

$$\dim \mathcal{C}(\bar{\mathfrak{A}}) = \tfrac{1}{2}\,|\bar I_\circ| + \tfrac{1}{2}\,|\bar I_{1/2}| + 2\,|\bar I_\mathrm{f}|\,. \tag{14.10}$$

This tells us that for every $\mathbb{Z}_2$-orbifold the number $|I_\circ|$ of length-two $Q = 0$ orbits coincides with the total number $|I_{1/2}|$ of $Q = 1/2$ orbits. On the other hand the number $|I_\mathrm{f}|$ of fixed points, which necessarily have $Q = 0$, can be arbitrary. Also note that $\dim \mathcal{C}(\mathfrak{A}) \equiv |I| = |I_\circ| + |I_\mathrm{f}|$, i.e. there are always at least as many even as there are odd boundary conditions. The numbers of odd and even boundary conditions are equal precisely in those cases where there are no fixed points, which happens precisely when the associated automorphism of the fusion rules is the identity.

Further, the entries of the diagonalizing matrix $\tilde S$ read explicitly

$$\begin{aligned} &\tilde S_{\bar\alpha,[\bar\rho]} = 2\,\bar S_{\bar\alpha,\bar\rho}\,, \qquad \tilde S_{(\bar f,\psi),[\bar\rho]} = \bar S_{\bar f,\bar\rho} \qquad \text{for } N_\rho = 2\,, \\ &\tilde S_{\bar\alpha,[\bar g,\psi']} = \bar S_{\bar\alpha,\bar g}\,, \qquad \tilde S_{(\bar f,\psi),[\bar g,\psi']} = \tfrac{1}{2}\,(\bar S_{\bar f,\bar g} + \psi\psi'\,\breve S_{\bar f,\bar g})\,. \end{aligned} \tag{14.11}$$

This leads to the formulæ

$$\begin{aligned} &\tilde{\mathrm{N}}_{\bar\alpha,\bar\beta,\bar\gamma} = 2\,\bar{\mathrm{N}}_{\bar\alpha,\bar\beta,\bar\gamma}\,, \qquad\qquad \tilde{\mathrm{N}}_{\bar\alpha,\bar\beta,(\bar f,\psi)} = \bar{\mathrm{N}}_{\bar\alpha,\bar\beta,\bar f}\,, \\ &\tilde{\mathrm{N}}_{\bar\alpha,(\bar f,\psi),(\bar g,\psi')} = \tfrac{1}{2}\,(\bar{\mathrm{N}}_{\bar\alpha,\bar f,\bar g} + \psi\psi'\,\breve{\mathrm{N}}_{\bar f,\bar\alpha,\bar g}) = {}^{\mathrm{e}}\mathrm{N}_{\alpha,(f,\psi),(g,\psi')}\,, \\ &\tilde{\mathrm{N}}_{(\bar f,\psi),(\bar g,\psi'),(\bar h,\psi'')} = {}^{\mathrm{e}}\mathrm{N}_{(f,\psi),(g,\psi'),(h,\psi'')} \end{aligned} \tag{14.12}$$

for the structure constants with three lower indices. Here $\bar{\mathrm{N}}$ and ${}^{\mathrm{e}}\mathrm{N}$ denote the fusion coefficients of the $\bar{\mathfrak{A}}$-theory and of the $\mathfrak{A}$-theory, respectively. Moreover, we have introduced *twined* fusion coefficients, defined with the help of the twined S-matrix $\breve S$, according to

$$\breve{\mathrm{N}}_{\bar f,\bar\rho,\bar g} := \sum_{\bar h \in \bar I_\mathrm{f}} \frac{\breve S_{\bar f,\bar h}\,\bar S_{\bar\rho,\bar h}\,\breve S^*_{\bar g,\bar h}}{\bar S_{\bar\Omega,\bar h}}\,. \tag{14.13}$$

Recall from subsection 10.3 that these twined fusion coefficients are the traces of the action of the outer automorphisms associated to J on bundles of chiral blocks.

Furthermore, from equation (14.12) we read off that the matrix $C^B = \tilde S \tilde S^{\rm t}$ furnishes a conjugation on the basis labels (not just an involution as in the generic case). We have

$$C^B_{\bar\alpha,\bar\beta} = 2\,\delta_{\bar\alpha,\bar\beta^+}\,, \qquad C^B_{\bar\alpha,(f,\psi)} = 0\,, \qquad C^B_{(f,\psi),(g,\psi')} = \delta_{\bar f,\bar g^+}\, \mathrm{C}^{B\,(\bar f)}_{\eta,\psi'}\,; \tag{14.14}$$

in particular, conjugation on the fixed points is exactly as in the $\mathfrak{A}$-theory. It follows that the structure constants of the classifying algebra read

$$\begin{array}{ll} \tilde{\mathrm N}_{\bar\alpha,\bar\beta}^{\ \ \bar\gamma} = \bar{\mathrm N}_{\bar\alpha,\bar\beta}^{\ \ \bar\gamma}\,, & \tilde{\mathrm N}_{\bar\alpha,\bar\beta}^{\ \ (\bar f,\psi)} = \bar{\mathrm N}_{\bar\alpha,\bar\beta}^{\ \ \bar f}\,, \\ \tilde{\mathrm N}_{\bar\alpha,(\bar f,\psi)}^{\ \ \bar\gamma} = \frac12\, {}^{\rm e}{\mathrm N}_{\alpha,(f,\psi)}^{\ \ \gamma}\,, & \tilde{\mathrm N}_{\bar\alpha,(\bar f,\psi)}^{\ \ (\bar g,\psi')} = {}^{\rm e}{\mathrm N}_{\alpha,(f,\psi)}^{\ \ (g,\psi')}\,, \\ \tilde{\mathrm N}_{(\bar f,\psi),(\bar g,\psi')}^{\ \ \bar\gamma} = \frac12\, {}^{\rm e}{\mathrm N}_{(f,\psi),(g,\psi')}^{\ \ \gamma}\,, & \tilde{\mathrm N}_{(\bar f,\psi),(\bar g,\psi')}^{\ \ (\bar h,\psi'')} = {}^{\rm e}{\mathrm N}_{(f,\psi),(g,\psi')}^{\ \ (h,\psi'')}\,. \end{array} \tag{14.15}$$

14.3 The individual classifying algebra for odd boundary conditions

The basis $\{\tilde\Phi_{(\bar\lambda,\psi)}\,|\,\bar\lambda\in\bar I_\circ,\ \psi\in\mathcal S_\lambda\}$ of the classifying algebra $\mathcal C(\bar{\mathfrak A})$ is mapped by (10.1) to the distinguished basis of the fusion algebra of the $\mathfrak A$-theory. In the present situation we find

$${}^{\rm e}\Phi_\alpha = \tfrac12\,(\tilde\Phi_{\bar\alpha}+\tilde\Phi_{{\rm J}\bar\alpha}) \quad\text{and}\quad {}^{\rm e}\Phi_{(f,\psi)} = \tilde\Phi_{(\bar f,\psi)}\,. \tag{14.16}$$

As we have seen, this provides an algebra homomorphism φ from the classifying algebra $\mathcal C(\bar{\mathfrak A})$ to the fusion algebra of $\mathfrak A$, which is just the classifying algebra $\mathcal C(\mathfrak A)$ for the even boundary conditions, i.e. $\mathcal C(\mathfrak A) = {}^{\rm e}\mathcal C(\bar{\mathfrak A}) = {\rm span}_{\mathbb C}\{{}^{\rm e}\Phi_\lambda\}$. On the other hand the kernel of φ is an ideal of $\mathcal C(\bar{\mathfrak A})$ which provides us with a classifying algebra ${}^{\rm o}\mathcal C(\bar{\mathfrak A}) \equiv \mathcal C^{(\varpi)}(\bar{\mathfrak A})$ for the odd boundary conditions.

Let us analyze this classifying algebra ${}^{\rm o}\mathcal C(\bar{\mathfrak A})$ in more detail. Only $\mathcal G$-orbits of length two contribute. On each such orbit we choose one distinguished representative $\bar\alpha$. We can and will assume that these choices are made in such a manner that, first, for the vacuum orbit the orbifold vacuum $\bar\Omega$ is taken as the representative, and second, on conjugate orbits one chooses conjugate representatives. Then the set of elements

$${}^{\rm o}\Phi_\alpha := \tfrac12\,(\tilde\Phi_{\bar\alpha}-\tilde\Phi_{{\rm J}\bar\alpha})\,, \tag{14.17}$$

with $\bar\alpha$ the distinguished representative, form a basis for ${}^{\rm o}\mathcal C(\bar{\mathfrak A})$. In this basis the structure constants of ${}^{\rm o}\mathcal C(\bar{\mathfrak A})$ are given by the following traces on spaces of chiral blocks (which are integers):

$${}^{\rm o}{\mathrm N}_{\alpha,\beta}^{\ \ \gamma} := \bar{\mathrm N}_{\bar\alpha,\bar\beta}^{\ \ \bar\gamma} - \bar{\mathrm N}_{\bar\alpha,\bar\beta}^{\ \ {\rm J}\bar\gamma}\,. \tag{14.18}$$

As an ideal of $\mathcal C(\bar{\mathfrak A})$, the algebra ${}^{\rm o}\mathcal C(\bar{\mathfrak A})$ inherits several properties of $\mathcal C(\bar{\mathfrak A})$: it is semisimple, commutative and associative; it is unital, the unit element being ${}^{\rm o}\Phi_\Omega = \tilde\Phi_{\bar\Omega} - \tilde\Phi_{\rm J}$; and it has a conjugation which is evaluation at the unit element. The dimension of ${}^{\rm o}\mathcal C(\bar{\mathfrak A})$ is

$$\dim {}^{\rm o}\mathcal C(\bar{\mathfrak A}) \equiv |I_\circ| = |I_{1/2}|\,. \tag{14.19}$$

Since ${}^{\rm o}\!\mathcal{C}(\bar{\mathfrak{A}})$ is semisimple, it must possess a diagonalizing matrix. Indeed, by the Verlinde formula for the orbifold fusion rules $\bar{\rm N}_{\bar\alpha,\bar\beta}^{\ \ \bar\gamma}$ we can write the structure constants (14.18) as

$$ {}^{\rm o}{\rm N}_{\alpha,\beta}^{\ \ \gamma} = \sum_{\bar m\in\bar I} \frac{\bar S_{\bar\alpha,\bar m}\,\bar S_{\bar\beta,\bar m}}{\bar S_{\bar\Omega,\bar m}} \left(\bar S^*_{\bar\gamma,\bar m} - \bar S^*_{{\rm J}\bar\gamma,\bar m}\right). \tag{14.20} $$

Here a priori the summation is over all sectors of the orbifold theory. But only the twisted fields give a non-vanishing contribution, and in that case the two terms with $\bar\gamma$ and ${\rm J}\bar\gamma$ are equal. Labelling these twisted fields by roman letters $\bar a,\bar b,\ldots$ and calling the corresponding index set $\bar I_{1/2}$, we thus obtain ${}^{\rm o}{\rm N}_{\alpha,\beta}^{\ \ \gamma} = 2\sum_{\bar d\in\bar I_{1/2}} \bar S_{\bar\alpha,\bar d}\,\bar S_{\bar\beta,\bar d}\,\bar S^*_{\bar\gamma,\bar d}/\bar S_{\bar\Omega,\bar d}$. Moreover, elements $\bar d$ on the same $\mathbb{Z}_2$-orbit give identical results, hence we may rewrite this formula as a sum over orbits $d:=\{\bar d,{\rm J}\bar d\}$. Denoting the set of these orbits by $I_{1/2}$, so that

$$ \bar I_{1/2} := \{\bar a\in\bar I \,|\, a{=}\{\bar a,{\rm J}\bar a\}\in I_{1/2}\}\,, \tag{14.21} $$

we arrive at

$$ {}^{\rm o}{\rm N}_{\alpha,\beta}^{\ \ \gamma} = 4 \sum_{d\in I_{1/2}} \frac{\bar S_{\bar\alpha,\bar d}\,\bar S_{\bar\beta,\bar d}\,\bar S^*_{\bar\gamma,\bar d}}{\bar S_{\bar\Omega,\bar d}}\,. \tag{14.22} $$

We can interpret this result as stating that the structure constants ${}^{\rm o}{\rm N}$ are governed by the matrix ${}^{\rm o}\!S$ with entries

$$ {}^{\rm o}\!S_{\alpha,b} := 2\,\bar S_{\bar\alpha,\bar b} \qquad \text{with} \ \ \alpha\in I_\circ,\ b\in I_{1/2}\,; \tag{14.23} $$

this is the diagonalizing matrix for ${}^{\rm o}\!\mathcal{C}(\bar{\mathfrak{A}})$. Also note that owing to ${}^{\rm o}\!S_{\Omega,b} = 2\,\bar S_{\bar\Omega,\bar b} > 0$ for all $b\in I_{1/2}$, this matrix shares the positivity property of modular S-matrices (this does *not* already follow from the commutativity and semisimplicity of ${}^{\rm o}\!\mathcal{C}(\bar{\mathfrak{A}})$). Moreover, by combining unitarity of $\bar S$ with the simple current symmetry $\bar S_{{\rm J}\bar m,\bar n} = (-1)^{2Q_{\rm J}(n)}\bar S_{\bar m,\bar n}$ one obtains

$$ \sum_{\bar a\in\bar I_{1/2}} \bar S^*_{\bar a,\bar l}\,\bar S_{\bar a,\bar m} = \tfrac12\,(\delta_{\bar l,\bar m} - \delta_{\bar l,{\rm J}\bar m})\,, \qquad \sum_{\bar n\in\bar I_\circ\cup\bar I_{\rm f}} \bar S^*_{\bar n,\bar l}\,\bar S_{\bar n,\bar m} = \tfrac12\,(\delta_{\bar l,\bar m} + \delta_{\bar l,{\rm J}\bar m})\,, \tag{14.24} $$

with the help of which one can show that the matrix (14.23) is unitary, which in turn tells us once again that ${}^{\rm o}\!S$ is a square matrix.

The structure constants ${}^{\rm o}{\rm N}_{\alpha,\beta}^{\ \ \gamma}$ as defined by (14.18) do depend on the choice of representatives that has been made. Indeed, upon replacing $\bar\gamma$ by ${\rm J}\bar\gamma$, ${}^{\rm o}{\rm N}_{\alpha,\beta}^{\ \ \gamma}$ goes to minus itself. But this does not change the classifying algebra ${}^{\rm o}\!\mathcal{C}(\bar{\mathfrak{A}})$, since the choice of a sign constitutes a one-cocycle on ${}^{\rm o}\!\mathcal{C}(\bar{\mathfrak{A}})$, which in turn can be absorbed by choosing a correlated sign for the boundary blocks. More concretely, for the full $\mathbb{Z}_2$-orbits we have the isomorphism $\mathcal{H}_\alpha \cong \bar{\mathcal{H}}_{\bar\alpha}\oplus\bar{\mathcal{H}}_{{\rm J}\bar\alpha} \cong \mathcal{H}_{\varpi^\star\alpha}$ of $\bar{\mathfrak{A}}$-modules, so that the associated even boundary blocks are precisely ${}^{\rm e}{\rm B}_\alpha = \bar{\rm B}_{\bar\alpha}\oplus\bar{\rm B}_{{\rm J}\bar\alpha}$, as given by formula (14.3), where $\bar{\rm B}_{\bar m}$: $\bar{\mathcal{H}}_{\bar m}\otimes\bar{\mathcal{H}}_{\bar m^+}\to\mathbb{C}$ are the orbifold boundary blocks. In contrast, there are two possible choices of odd boundary blocks, namely ${}^{\rm o}{\rm B}_\alpha = \bar{\rm B}_{\bar\alpha}\oplus(-\bar{\rm B}_{{\rm J}\bar\alpha})$ as in (14.6) as well as

$$ {}^{\rm o}{\rm B}_{\varpi^\star\alpha} = \bar{\rm B}_{{\rm J}\bar\alpha}\oplus(-\bar{\rm B}_{\bar\alpha}) = -\,{}^{\rm o}{\rm B}_\alpha\,. \tag{14.25} $$

By construction both of these linear forms ${}^{\rm o}{\rm B}_\alpha$ and ${}^{\rm o}{\rm B}_{\varpi^\star\alpha}$ on $\mathcal{H}_\alpha\otimes\mathcal{H}_{\alpha^+}$ satisfy the appropriate Ward identities, but of course we must keep just one out of the two. The right prescription is to keep ${}^{\rm o}\mathrm{B}_\beta$ when the label $\bar\beta\in\bar I_\circ$ is the chosen representative for the orbit $\beta\in I_\circ$. (Also, positivity of mixed annulus amplitudes is guaranteed only with this choice, see below.) In short, the label of the orbifold boundary state that appears with a positive sign in the boundary block (14.6) is the one that is to be chosen as the representative of the orbit.

14.4 Annulus amplitudes

Using our general results from section 6 of [1] it is also straightforward to calculate all annulus amplitudes. For the case of two even boundary conditions we obtain

$$\begin{aligned}
\mathrm{A}_{\alpha\,\beta} &= \sum_{\bar m\in\bar I_\circ\cup\bar I_{\rm f}} (\bar{\mathrm{N}}_{\bar\beta,\bar m}{}^{\bar\alpha} + \bar{\mathrm{N}}_{\bar\beta,\mathrm{J}\bar m}{}^{\bar\alpha})\, \bar\chi_{\bar m}(\mathrm{i}t/2)\,, \\
\mathrm{A}_{\alpha\,(f,\psi)} &= \sum_{\bar m\in\bar I_\circ\cup\bar I_{\rm f}} \bar{\mathrm{N}}_{\bar f,\bar m}{}^{\bar\alpha}\, \bar\chi_{\bar m}(\mathrm{i}t/2)\,, \\
\mathrm{A}_{(f,\psi)\,(g,\psi')} &= \tfrac12 \sum_{\bar m\in\bar I_\circ\cup\bar I_{\rm f}} (\bar{\mathrm{N}}_{\bar g,\bar m}{}^{\bar f} + \psi\psi'\,\breve{\mathrm{N}}_{\bar g,\bar m}{}^{\bar f})\, \bar\chi_{\bar m}(\mathrm{i}t/2)\,.
\end{aligned} \tag{14.26}$$

Similarly, for two odd boundary conditions the annulus amplitudes are

$$\mathrm{A}_{a\,b}(t) = \tfrac12 \sum_{\bar m\in\bar I_\circ\cup\bar I_{\rm f}} N_m\, (\bar{\mathrm{N}}_{\bar b,\bar m}{}^{\bar a} + \bar{\mathrm{N}}_{\bar b,\mathrm{J}\bar m}{}^{\bar a})\, \bar\chi_{\bar m}(\mathrm{i}t/2)\,. \tag{14.27}$$

Finally, for *mixed* annuli, i.e. annuli with one even and one odd boundary, we find

$$\mathrm{A}_{a\,\mu}(t) = \tfrac12 \sum_{\bar c\in\bar I_{1/2}} N_\mu\, (\bar{\mathrm{N}}_{\bar\mu,\bar c}{}^{\bar a} + \bar{\mathrm{N}}_{\bar\mu,\mathrm{J}\bar c}{}^{\bar a})\, \bar\chi_{\bar c}(\mathrm{i}t/2)\,. \tag{14.28}$$

Thus in particular we correctly establish the $\mathbb{Z}_2$ selection rule

$$\mathrm{A}_{\rho_1\,\rho_2}^{\sigma} = 0 \qquad \text{whenever} \quad Q_\mathrm{J}(\rho_1) + Q_\mathrm{J}(\rho_2) + Q_\mathrm{J}(\sigma) \in \mathbb{Z}{+}1/2\,, \tag{14.29}$$

which is analogous to the selection rule that is valid for the orbifold fusion rules.

For every triple $\rho_1\,\rho_2,\sigma$ the annulus coefficient $\mathrm{A}_{\rho_1\,\rho_2}^{\sigma}$ is manifestly a *non-negative integer*, in agreement with the physical meaning of the annulus amplitude as a partition function. We would like to stress that this coefficient is not only a non-negative integer, but in addition has a natural representation theoretic interpretation, namely as the dimension of a space of chiral blocks. The expressions $\bar{\mathrm{N}}_{\bar g,\bar m}{}^{\bar f} \pm \breve{\mathrm{N}}_{\bar g,\bar m}{}^{\bar f}$, for example, that appear in the last line of (14.26), are equal [33] to the ranks of two invariant subbundles of bundles of chiral blocks. This observation also seems to be relevant for a better understanding of the multiplicities of boundary fields which are counted by these dimensions (that is, there is a separate boundary field $\Psi_\sigma^{\rho_1,\rho_2;\ell}$ for every $\ell=1,2,...\,,\mathrm{A}_{\rho_1\,\rho_2}^{\sigma}$) and for which a satisfactory understanding is unfortunately so far still lacking. Inspecting the calculations more closely, one also observes that the coefficients $\mathrm{A}_{a\,\mu}^{c}$ check the correctness of the sign convention in the definition of the odd boundary blocks that was discussed after formula (14.25); with a different prescription, some of these coefficients would become negative.

With the results above, it is easily verified that the annulus coefficients $\mathrm{A}_{\rho_1\,\rho_2}^{\sigma}$ satisfy the relations that are expected on the basis of factorization arguments, [13] i.e. that they are 'associative' in the sense that

$$\sum_\sigma \mathrm{A}_{\rho_1\,\rho_2}^{\sigma}\mathrm{A}_{\rho_3\,\rho_4}^{\sigma^+} = \sum_\sigma \mathrm{A}_{\rho_1\,\rho_3^+}^{\sigma}\mathrm{A}_{\rho_2^+\,\rho_4}^{\sigma^+} \tag{14.30}$$

for all possible values of the ρ_i, and that they are 'complete' in the sense that

$$\mathrm{A}^{\rho_1}\,\mathrm{A}^{\rho_2} = \sum_{\rho_3} \mathrm{M}_{\rho_1,\rho_2}^{\;\;\rho_3}\,\mathrm{A}^{\rho_3}\,, \tag{14.31}$$

where the A's are regarded as matrices in their two lower indices. The non-vanishing coefficients M are

$$\begin{aligned} \mathrm{M}_{\lambda,\mu}^{\;\;\nu} &= {}^{\mathrm{e}}\mathrm{N}_{\lambda,\mu}^{\;\;\nu}\,, \\ \mathrm{M}_{a,b}^{\;\;\nu} &= \tfrac12\,N_\nu\,(\bar{\mathrm{N}}_{\bar a,\bar b}^{\;\;\bar\nu} + \bar{\mathrm{N}}_{\bar a,\mathrm{J}\bar b}^{\;\;\bar\nu})\,, \\ \mathrm{M}_{a,\mu}^{\;\;c} &= \tfrac14\,(N_\mu)^2\,(\bar{\mathrm{N}}_{\bar a,\bar\mu}^{\;\;\bar c} + \bar{\mathrm{N}}_{\mathrm{J}\bar a,\bar\mu}^{\;\;\bar c})\,. \end{aligned} \tag{14.32}$$

Note that the matrix M_Ω is the identity matrix.

15 Examples for involutory boundary conditions

We now present several classes of examples with involutory boundary conditions. We do not intend to be exhaustive, but concentrate on particularly interesting conformal field theories.

15.1 Dirichlet and Neumann conditions for the free boson

A simple and well-known realization of involutory boundary conditions is encountered for the $c=1$ theory of a single free boson, compactified on a circle of radius $R=\sqrt{2\mathcal{N}}$, where $\mathcal{N}$ is a positive integer – respectively, at the T-dual radius $\mathrm{T}(R)=2/R=\sqrt{2/\mathcal{N}}$. This theory has $2\mathcal{N}$ primary fields, which at the chiral level we label by integers λ mod $2\mathcal{N}$; their $\mathfrak{u}(1)$ charge modulo $\sqrt{2\mathcal{N}}$ is $q_\lambda=\lambda/\sqrt{2\mathcal{N}}$. At radius $\sqrt{2\mathcal{N}}$ the theory has diagonal torus partition function, i.e. the primaries are of the form (λ,λ), while for radius $\sqrt{2/\mathcal{N}}$ one deals with the charge conjugation invariant, i.e. with primaries $(\lambda,-\lambda)$. We will follow our general convention to describe the boundary conditions for the case of the charge conjugation invariant; the corresponding results for the diagonal invariant can be deduced via T-duality, as explained in section 11.

The chiral algebra $\mathfrak{A}$ of the free boson theory consists of operators of the form

$$P(\partial X)\exp(\mathrm{i}n\sqrt{2\mathcal{N}}\,X)\,, \tag{15.1}$$

where $n\in\mathbb{Z}$ and $P(\partial X)$ is a (normally ordered) polynomial in the $\mathfrak{u}(1)$ current $j=\mathrm{i}\partial X/\sqrt{2\mathcal{N}}$. This algebra has an obvious involutory automorphism, which in terms of the Fubini–Veneziano field X is expressed as

$$X\mapsto -X \tag{15.2}$$

[13] As already pointed out in subsection 6.5 of [1] for the case of general orbifold group G, a rigorous derivation of these relations is, however, not yet available.

and has the physical meaning of charge conjugation. It maps the $\mathfrak{u}(1)$ current to minus itself and exchanges the fields $\exp(\pm\mathrm{i}n\sqrt{2\mathcal{N}}\,X)$; thus its fixed point algebra consists of the even polynomials in j combined with the operators $\cos(n\sqrt{2\mathcal{N}}X)$ with $n\in\mathbb{Z}$ and odd polynomials in j combined with $\sin(n\sqrt{2\mathcal{N}}X)$. This algebra $\bar{\mathfrak{A}}$ is just the chiral algebra of the $\mathbb{Z}_2$-orbifold of the free boson.

We recall [32] that this $\mathbb{Z}_2$-orbifold has $\mathcal{N}{+}7$ primary fields. First, one has the vacuum $\bar\Omega$ and a simple current J of conformal weight $\Delta_{\rm J}=1$, which comes from the $\mathfrak{u}(1)$ current of the $\mathfrak{A}$-theory. Besides these two fields there is one other length-two orbit $\{\psi^1,\psi^2\}$ of monodromy charge zero (coming from the self-conjugate field with $\lambda=\mathcal{N}$ of the $\mathfrak{A}$-theory), as well as $\mathcal{N}{-}1$ fixed points $\varphi_{\bar\lambda}$ with $\bar\lambda=1,2,\dots,\mathcal{N}{-}1$, and finally there is one pair of twisted fields for each of the two self-conjugate fields of the $\mathfrak{A}$-theory, which are denoted by $\{\sigma,\sigma'\}$ and $\{\tau,\tau'\}$.

The number of even boundary conditions is equal to the number $2\mathcal{N}$ of primary fields of the boson theory. According to our general prescription they are labelled by the orbits of the orbifold and the characters of their stabilizers, i.e. there is one even boundary condition for each of $\{\bar\Omega,{\rm J}\}$ and $\{\psi^1,\psi^2\}$ and two for each of the fixed points $\varphi_{\bar\lambda}$. (In the language of the circle theory, the latter correspond to the two primary fields of opposite charge $\pm\bar\lambda$.) There are just two odd boundary conditions, corresponding to the two J-orbits of twisted fields. The classifying algebra ${}^{\circ}\mathcal{C}(\bar{\mathfrak{A}})$ for the odd boundary conditions turns out to be just the group algebra of $\mathbb{Z}_2$, and the total classifying algebra $\mathcal{C}(\bar{\mathfrak{A}})$ is isomorphic to the direct sum of this $\mathbb{C}\mathbb{Z}_2$ and the fusion rule algebra $\mathbb{C}\mathbb{Z}_{2\mathcal{N}}$ of the boson theory.

As is already apparent from (15.2), the odd boundary conditions are nothing but Neumann conditions for the free boson X (with definite rational values of the Wilson line), while the odd ones are Dirichlet conditions, with definite rational values of the position of the D0-brane, namely at ξR with ξ any of the $2\mathcal{N}$th roots of unity. (Recall that this formulation refers to the case where the torus partition function is the charge conjugation invariant; for the true diagonal invariant, in agreement with T-duality the role of Dirichlet and Neumann conditions get interchanged.) The fact that the Wilson line (for Neumann conditions) respectively the position of the brane (for Dirichlet conditions) are restricted to a discrete set of values is of course a consequence of the fact that in the situation considered here the preserved bulk symmetries correspond to a rational conformal field theory. By breaking more bulk symmetries one arrives at more general possibilities. In particular, as will be seen in subsection 16.1 below, one may consider orbifolds that correspond to a change of the radius of the circle and thereby arrive at Wilson lines and brane positions at arbitrary points of the circle.

15.2 $\mathfrak{sl}(2)$ WZW theories

Another example is given by $\mathfrak{sl}(2)$ WZW theories at levels $k\in 4\mathbb{Z}$. In this case the full conformal field theory based on the diagonal modular invariant for the orbifold theory can be realized as a sigma model on the group manifold SU(2), while the diagonal modular invariant for the $\mathfrak{A}$-theory corresponds to a sigma model on the non-simply connected group manifold SO(3). In the usual notation [34] these are the theories labelled A_{k+1} and $D_{k/2+2}$, respectively.

The non-trivial simple current $\mathrm{J}\in\mathcal{G}$ has conformal weight $\Delta_{\rm J}=k/4$. Labelling the sectors of the A_{k+1} model by their highest $\mathfrak{sl}(2)$-weights and taking as representatives of $\mathbb{Z}_2$-orbits those with smaller weight, the various label sets look like follows.

$$\begin{array}{lll} \bar I_\circ = \{0,2,4,\dots,\frac k2{-}2,\frac k2{+}2,\frac k2{+}4,\dots,k\}\,, & \bar I_{\rm f} = \{\frac k2\}\,, & \bar I_{1/2} = \{1,3,5,\dots,k{-}1\}\,, \\ I_\circ = \{0,2,4,\dots,\frac k2{-}2\}\,, & I_{\rm f} = \{(\frac k2,\psi)\,|\,\psi{=}\pm1\}\,, & I_{1/2} = \{1,3,5,\dots,\frac k2{-}1\}\,. \end{array} \tag{15.3}$$

Thus $|I_\circ|=|I_{1/2}|=k/4$ and $|I_{\rm f}|=2$.

For the modular matrix $S\equiv{}^{\rm e}\!S$ of the $D_{k/2+2}$-model, a natural ordering of the labels in $I\equiv I_\circ{\cup}I_{\rm f}$ is

$$\mu = 0\,,\,2\,,\,4\,,\,\dots\,,\,\tfrac k2{-}2\,,\,(\tfrac k2{,}{+})\,,\,(\tfrac k2{,}{-})\,, \tag{15.4}$$

both for rows and for columns. With this labelling the general formula for the modular S-matrix of a simple current extension (established in [4] and displayed also in appendix A of [1]) yields

$${}^{\rm e}\!S_{\mu,\nu} = \left\{ \begin{array}{ll} 2\,\bar S_{\bar\mu,\bar\nu} & \text{for }\ \mu,\nu=0,2,\dots,\frac k2{-}2\,, \\ \bar S_{\bar\mu,\bar\nu} & \text{for }\ \mu=0,2,\dots,\frac k2{-}2,\ \ \nu=(\frac k2{,}\psi) \\ & \ \text{or }\ \mu=(\frac k2{,}\psi),\ \ \nu=0,2,\dots,\frac k2{-}2\,, \\ \frac12\,(\frac{1}{\sqrt{k/2{+}1}}+\psi\psi'\,{\rm I}) & \text{for }\ \mu=(\frac k2{,}\psi),\ \ \nu=(\frac k2{,}\psi')\,. \end{array}\right. \tag{15.5}$$

Here $\bar S$ is the S-matrix of the A_{k+1}-model, i.e. $\bar S_{\bar\mu,\bar\nu}=\sqrt{2/(k{+}2)}\,\sin((\bar\mu{+}1)(\bar\nu{+}1)\pi/(k{+}2))$, and we introduced the number

$${\rm I} := \left\{ \begin{array}{ll} 1 & \text{for }\ k\in8\mathbb Z\,, \\ {\rm i}\equiv\sqrt{-1} & \text{for }\ k\in8\mathbb Z+4\,, \end{array}\right. \tag{15.6}$$

which is nothing but the (one-by-one) matrix $S^{\rm J}$. In contrast, for the diagonalizing matrix ${}^\circ\!S$ of ${}^\circ\mathcal C(\bar{\mathfrak A})$ there will typically not exist any preferred ordering of the labels $\alpha\in I_\circ$ (for the rows of ${}^\circ\!S$) and $a\in I_{1/2}$ (for the columns); amusingly, in the present case it is possible to order rows and columns in such a fashion that ${}^\circ\!S$ is symmetric. We have

$${}^\circ\!S_{\alpha,b} = 2\,\bar S_{\bar\alpha,\bar b} = \sqrt{\frac{8}{k{+}2}}\;\sin\big(\frac{(\alpha{+}1)(b{+}1)\,\pi}{k{+}2}\big)\,. \tag{15.7}$$

Now ordering again $I_\circ$ according to the value of the weight, i.e. as $\alpha=0,2,4,\dots,k/2{-}2$, symmetry is achieved when $I_{1/2}$ is ordered by taking first the weights of the form $4j{+}1$ in ascending order and afterwards the weights of the form $4j{+}3$ in descending order, i.e. $a=1,5,9,13,\dots.,\,15,11,7,3$. Doing so, the matrix (15.7) simply becomes the matrix with entries

$${}^\circ\!\widetilde S_{p,q} = \sqrt{\frac{8}{k{+}2}}\;\sin\big(\frac{(2p{-}1)(2q{-}1)\,\pi}{k/2{+}1}\big)\,, \tag{15.8}$$

where the integers p and q run from 1 to $k/4$. In particular, inspection shows that ${}^\circ\mathcal C(\bar{\mathfrak A})$ coincides with the fusion algebra [35] of the $(\frac k2{+}1,2)$ Virasoro minimal models.

Via the simple current symmetries of the matrices $\bar S$ and $S^{\rm J}$, the diagonalizing matrix $\tilde S$ of the classifying algebra is already completely determined by ${}^{\rm e}\!S$ and ${}^\circ\!S$. Concretely, when we choose the ordering of the rows of $\tilde S$ as

$$(\bar\lambda,\varphi) = 0\,,\,2\,,\,4\,,\,\dots\,,\,\tfrac k2{-}2\,,\,(\tfrac k2{,}{+})\,,\,(\tfrac k2{,}{-})\,,\,k\,,\,k{-}2\,,\,\dots\,,\,\tfrac k2{+}2 \tag{15.9}$$

and the ordering of the columns according to

$$(\bar\rho,\psi) = 0\,,\, 2\,,\, 4\,,\, \dots\,,\, \tfrac{k}{2}{-}2\,,\, (\tfrac{k}{2},+)\,,\, (\tfrac{k}{2},-)\,,\, 1\,,\, 3\,,\, \dots\,,\, \tfrac{k}{2}{-}1\,, \tag{15.10}$$

then $\tilde S$ is block diagonal of the form

$$\tilde S = \begin{pmatrix} {}^{\rm e}S & {}^{\rm eo}S \\ {}^{\rm oe}S & {}^{\rm o}S \end{pmatrix}, \tag{15.11}$$

and the off-diagonal blocks are related to the diagonal ones by

$${}^{\rm eo}S|_{\rm full} = -{}^{\rm o}S\,, \quad {}^{\rm eo}S|_{\rm fixed} = 0\,, \quad {}^{\rm oe}S = {}^{\rm e}S|_{\rm full}\,, \tag{15.12}$$

where the symbols $|_{\rm full}$ and $|_{\rm fixed}$ stand for restriction of the rows to those corresponding to full orbits and to fixed points, respectively.

15.3 Relation with incidence matrices of graphs

In the $\mathfrak{sl}(2)$ case under consideration, it is not too difficult to establish that – just like the algebra ${}^{\rm o}\mathcal{C}(\bar{\mathfrak{A}})$ – also the total classifying algebra $\mathcal{C}(\bar{\mathfrak{A}})$ constitutes a structure that has been encountered in conformal field theory before. Indeed, we will construct an isomorphism between $\mathcal{C}(\bar{\mathfrak{A}})$ and an algebra that appears in the work of Pasquier et al. To simplify some of the formulæ below, let us write $k=4\ell$ with $\ell\in\mathbb{Z}$. Then a $2\ell{+}2$-dimensional associative algebra with structure constants

$$\hat{\rm N}_{r,s}^{\ \ t} := \sum_u \hat S_{r,u} \hat S_{s,u} \hat S^*_{t,u} / \hat S_{1,u} \tag{15.13}$$

for $r,s,t\in\{1,2,\dots,2\ell{+}2\}$ has been considered in [36–38] and been called the *Pasquier algebra* associated to the situation of our interest. In formula (15.13), $\hat S$ is the matrix with entries

$$\hat S_{r,s} := \begin{cases} \frac12\,(\frac{1}{\sqrt{2\ell+1}}+{\rm I}) & \text{for } r=2\ell{+}1,\ s=\ell{+}1 \text{ or } r=s=2\ell{+}2\,, \\ \frac12\,(\frac{1}{\sqrt{2\ell+1}}-{\rm I}) & \text{for } r=2\ell{+}1,\ s=2\ell{+}2 \text{ or } r=2\ell{+}2,\ s=\ell{+}1\,, \\ \frac{1}{\sqrt{2\ell+1}}\,(-1)^{(r-1)/2}\,(1-(-1)^r) & \text{for } r=1,2,\dots,2\ell,\ s=\ell{+}1,2\ell{+}2\,, \\ \frac{1}{\sqrt{4\ell+2}} & \text{for } r=2\ell{+}1,2\ell{+}2,\ s\neq\ell{+}1,2\ell{+}2\,, \\ \frac{1}{\sqrt{4\ell+2}}\cdot 2\,\cos\big(\frac{(2\ell-r+1)(2s-1)\,\pi}{4\ell+2}\big) & \text{else}\,, \end{cases} \tag{15.14}$$

with I as in (15.6). Note that the matrix $\hat N_2$ with entries $(\hat N_2)_s^{\ t} = \hat{\rm N}_{2,s}^{\ \ t}$ is nothing but the *incidence matrix* of the graph $D_{2\ell+2}$. Conversely, up to a phase the columns of $\hat S$ are uniquely determined by the two requirements that $\hat S$ is unitary and diagonalizes $\hat N_2$ – with the exception, however, of the columns numbered $\ell{+}1$ and $2\ell{+}2$, which both are eigenvectors to the same eigenvalue zero. For the latter two columns, in formula (15.14) (unlike in table 2 of [36]) we have chosen specific linear combinations that are singled out by the property that all structure constants $\hat{\rm N}_{r,s}^{\ \ t}$ are non-negative integers. We also remark that the matrix (15.14) is unitary, but it is not symmetric, nor can it be made symmetric by any re-ordering of rows and columns,[14] and finally that

$$\hat{S}_{r,2\ell+2-s} = (-1)^{r+1}\, \hat{S}_{r,s} \qquad \text{for} \quad r \neq 2\ell{+}1, 2\ell{+}2\,,\;\; s \neq \ell{+}1, 2\ell{+}2\,. \tag{15.15}$$

By inspecting the formulæ (15.14) and (15.11), we observe the relation

$$\hat{S}_{\mathrm{r}([\bar\rho,\psi]),\mathrm{s}((\bar\lambda,\varphi))} = \frac{\varepsilon_\lambda}{\sqrt{N_\lambda}}\, \tilde{S}_{(\bar\lambda,\varphi),[\bar\rho,\psi]} \tag{15.16}$$

between the matrices $\hat{S}$ and $\tilde{S}$. Here N_λ is the length of the $\mathcal{G}$-orbit through $\bar\lambda$ (i.e. $N_\lambda = 2$ except for $N_{2\ell} = 1$), ε is the sign factor

$$\varepsilon_\lambda := \left\{ \begin{array}{ll} 1 & \text{for } \lambda = 2\ell\,, \\ (-1)^{\lambda/2} & \text{else}\,, \end{array} \right. \tag{15.17}$$

and we introduced bijections r and s between the respective index sets of $\hat{S}$ and $\tilde{S}$ which act as

$$\mathrm{r}([\bar\rho,\psi]) := \left\{ \begin{array}{ll} \bar\rho + 1 & \text{for } \bar\rho \neq 2\ell\,, \\ 2\ell + 1 & \text{for } (\bar\rho,\psi) = (2\ell,+)\,, \\ 2\ell + 2 & \text{for } (\bar\rho,\psi) = (2\ell,-) \end{array} \right. \tag{15.18}$$

and

$$\mathrm{s}((\bar\lambda,\varphi)) := \left\{ \begin{array}{ll} \bar\lambda/2 + 1 & \text{for } \bar\lambda \neq 2\ell\,, \\ \ell + 1 & \text{for } (\bar\lambda,\varphi) = (2\ell,+)\,, \\ 2\ell + 2 & \text{for } (\bar\lambda,\varphi) = (2\ell,-)\,, \end{array} \right. \tag{15.19}$$

respectively. In particular, the diagonalizing matrices (15.5) and (15.7) for the even and odd boundary conditions obey

$${}^{\mathrm{e}}S_{\mu,\nu} = \sqrt{N_\mu}\, (-1)^{\mu/N_\mu}\, \hat{S}_{\mathrm{r}(\nu),\mathrm{s}(\mu)} \tag{15.20}$$

and

$${}^{\mathrm{o}}S_{\alpha,b} = \sqrt{2}\, (-1)^{\alpha/2}\, \hat{S}_{b+1,\mathrm{s}(\alpha)}\,, \tag{15.21}$$

respectively.

Using the fact that S is symmetric, the result (15.16) tells us that up to normalizations of the rows of $\tilde{S}$, transposition, and reordering of the rows and columns, the two matrices $\tilde{S}$ and $\hat{S}$ coincide. As all these manipulations can be absorbed into a basis transformation, it follows that the two algebras that via these matrices are associated to the simple current extension of $\mathfrak{sl}(2)_\ell$, i.e. the classifying algebra $\mathcal{C}(\bar{\mathfrak{A}})$ and Pasquier's algebra are *isomorphic*.

[14] However, as follows from the identifications below, when combining suitable re-orderings with rescalings, certain submatrices of $\hat{S}$ become symmetric.

For concreteness, let us also display explicitly these matrices in the simplest case, i.e. for $\ell = 1$:

$$\hat{S} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & \sqrt{2} & 1 & \sqrt{2} \\ \sqrt{3} & 0 & -\sqrt{3} & 0 \\ 1 & \frac{1}{\sqrt{2}}(1{+}\mathrm{i}\sqrt{3}) & 1 & \frac{1}{\sqrt{2}}(1{-}\mathrm{i}\sqrt{3}) \\ 1 & \frac{1}{\sqrt{2}}(1{-}\mathrm{i}\sqrt{3}) & 1 & \frac{1}{\sqrt{2}}(1{+}\mathrm{i}\sqrt{3}) \end{pmatrix}, \qquad (15.22)$$

$$\tilde{S} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 & -\sqrt{3} \\ 1 & \frac{1}{2}(1{+}\mathrm{i}\sqrt{3}) & \frac{1}{2}(1{-}\mathrm{i}\sqrt{3}) & 0 \\ 1 & \frac{1}{2}(1{-}\mathrm{i}\sqrt{3}) & \frac{1}{2}(1{+}\mathrm{i}\sqrt{3}) & 0 \\ 1 & 1 & 1 & \sqrt{3} \end{pmatrix}.$$

15.4 Virasoro minimal models

The unitary minimal models of the Virasoro algebra are labelled by $m = 2, 3, \ldots$; they have conformal central charge $c = c_m := 1{-}6/(m{+}1)(m{+}2)$. Via their realization as a coset theory $\mathfrak{sl}(2)_{m-1} {\oplus} \mathfrak{sl}(2)1/\mathfrak{sl}(2)_m$, the $\mathfrak{sl}(2)$ WZW situation of the previous subsection gives rise to similar effects in these minimal models. The requirement that the level must be divisible by four translates to the condition $m \in 4\mathbb{Z} \cup (4\mathbb{Z}{+}1)$ on the label m. In these cases the chiral algebra of the $\mathfrak{A}$-theory is obtained from $\bar{\mathfrak{A}}$, which is just the Virasoro algebra, via extension by the field J with label $\bar{\lambda}_{\rm J} = (m, 1)$ (Kac table notation), which has conformal weight $\Delta_J = m(m{-}1)/4$.

In these cases we even know that the Virasoro algebra $\bar{\mathfrak{A}}$ is the only consistent subalgebra of $\mathfrak{A}$, simply because no other unitary conformal field theories exist at the same value of c. Thus our methods supply us [15] with *all* conformally invariant boundary conditions of the $\mathfrak{A}$-theory. In particular there are precisely two automorphism types, the even boundary conditions which preserve the full bulk symmetry $\mathfrak{A}$, and the odd boundary conditions which preserve only the Virasoro subalgebra.

The primary fields of the $\bar{\mathfrak{A}}$-theory with central charge c_m are labelled by $\bar{\lambda} \equiv (r, r')$ with

$$1 \le r \le m\,, \qquad 1 \le r' \le m+1\,, \qquad (15.23)$$

modulo the identification $(r, r') \sim (m{+}1{-}r, m{+}2{-}r')$, so that there is a total of $m(m{+}1)/2$ sectors. We first look at the cases with $m = 4\ell$ for some $\ell \in \mathbb{Z}_{>0}$. Then the $\bar{\mathfrak{A}}$- and $\mathfrak{A}$-theory are commonly [34] denoted by $(A_{4\ell}, A_{4\ell+1})$ and $(A_{4\ell}, D_{2\ell+2})$, respectively. (In the simplest of these, obtained for $\ell = 1$, the $\bar{\mathfrak{A}}$-theory is the tetracritical Ising model (A_4, A_5) while the $\mathfrak{A}$-theory is the three-state Potts model (A_4, D_4).) We have

$$\begin{aligned} \bar{I}_\circ &= \{(r, r') \,|\, r = 1, 3, 5, \ldots, 4\ell{-}1,\ r' = 1, 3, 5, \ldots, 2\ell{-}1, 2\ell{+}3, \ldots, 4\ell{+}1\}\,, \\ \bar{I}_{\rm f} &= \{(r, 2\ell{+}1) \,|\, r = 1, 3, 5, \ldots, 4\ell{-}1\}\,, \\ \bar{I}_{1/2} &= \{(r, r') \,|\, r = 2, 4, 6, \ldots, 4\ell,\ r' = 2, 4, 6, \ldots, 4\ell\}\,, \end{aligned} \qquad (15.24)$$

and hence $|\bar{I}_\circ| = |\bar{I}_{1/2}| = 4\ell^2$, $|\bar{I}_{\rm f}| = 2\ell$. Thus we obtain $2\ell(\ell{+}2)$ even and $2\ell^2$ odd boundary conditions, and hence a total of $4\ell(\ell{+}1)$ conformally invariant boundary conditions.

[15] At least modulo what goes [39] under the name of 'complex charges'.

The similar series with $m = 4\ell{+}1$ for $\ell \in \mathbb{Z}_{>0}$ can be treated analogously. The $\bar{\mathfrak{A}}$- and $\mathfrak{A}$-theory are now known under the names $(A_{4\ell+1}, A_{4\ell+2})$ and $(D_{2\ell+2}, A_{4\ell+2})$, respectively. We have

$$\begin{aligned} \bar{I}_\circ &= \{(r,r') \,|\, r = 1,3,5,\dots,2\ell{-}1, 2\ell{+}3, \dots, 4\ell{+}1\,,\;\; r' = 1,3,5,\dots,4\ell{+}1\,, \\ \bar{I}_{\rm f} &= \{(2\ell{+}1, r') \,|\, r' = 1,3,5,\dots,4\ell{+}1\,, \\ \bar{I}_{1/2} &= \{(r,r') \,|\, r = 2,4,6,\dots,4\ell\,,\;\; r' = 2,4,6,\dots,4\ell{+}2\,, \end{aligned} \tag{15.25}$$

so that $|\bar{I}_\circ| = |\bar{I}_{1/2}| = 2\ell(2\ell{+}1)$, $|\bar{I}_{\rm f}| = 2\ell{+}1$, and the number of even and odd boundary conditions is $(2\ell{+}1)(\ell{+}2)$ and $\ell(2\ell{+}1)$, respectively.

Via the coset construction, it is possible to express all the ingredients in the formula for $\tilde{S}$ through quantities of the underlying $\mathfrak{sl}(2)$ WZW models, so that again the classifying algebra $\mathcal{C}(\bar{\mathfrak{A}})$ can easily be obtained explicitly. We refrain from displaying any details, which are not too illuminating. We would like to mention, however, that these results are in perfect agreement with the findings of [40, 41]. In the latter papers, various statements were encoded in the language of graphs; the following remarks allow to make contact to that point of view.

The total number of conformally invariant boundary conditions is

$$|\bar{I}_\circ| + 2\,|\bar{I}_{\rm f}| = \left\{ \begin{array}{ll} 4\ell(\ell{+}1) = \frac12 \cdot {\rm rank}(A_{4\ell}) \cdot {\rm rank}(D_{2\ell+2}) & \text{for } m = 4\ell\,, \\ 2(\ell{+}1)(2\ell{+}1) = \frac12 \cdot {\rm rank}(A_{4\ell+2}) \cdot {\rm rank}(D_{2\ell+2}) & \text{for } m = 4\ell{+}1\,. \end{array}\right. \tag{15.26}$$

Regarding the graphs $A_{4\ell}$ and $D_{2\ell+2}$ (i.e. the Dynkin diagrams of the respective simple Lie algebras) as bi-colored, starting with (say) a black node, this can be understood as follows. The even boundary conditions are in one-to-one correspondence with pairs of black nodes from the 'product' of the two graphs, while odd boundary conditions are in one-to-one correspondence with pairs of white nodes. Mixed pairs of nodes do not correspond to boundary conditions, which accounts for the factor of 1/2 in (15.26). The latter selection rule may be implemented by a suitable folding of the $A_{4\ell}$ graph. The resulting graph has a loop, hence in particular it is no longer bi-colorable; pairs of nodes from $D_{2\ell+2}$ and the folded graph are then in one-to-one correspondence with *all* conformally invariant boundary conditions, including both even and odd ones.

Further, let us denote by $\mathrm{E}(\tilde{r})$ the $\tilde{r}$th *exponent* of the Lie algebra $D_{2\ell+2}$. For every $\tilde{r} = 1,2,\dots,2\ell{+}2$, the integer $\mathrm{E}(\tilde{r})$ lies in the label set of the $A_{4\ell+1}$ graph; indeed, the exponents correspond precisely to the black nodes of $A_{4\ell+1}$, with the middle node appearing twice. We can therefore define for every $s = 1,2,\dots,4\ell{+}1$ a matrix V_s through

$$(V_r)_{\tilde{s}}^{\ \tilde{t}} := \sum_{\tilde{u}} S^{(A)}_{r,\mathrm{E}(\tilde{u})}\, S^{(D)}_{\tilde{s},\tilde{u}}\, S^{(D)*}_{\tilde{t},\tilde{u}} / S^{(A)}_{1,\mathrm{E}(\tilde{u})}\,, \tag{15.27}$$

where $S^{(A)}$ and $S^{(D)}$ are the unitary diagonalizing matrices for the graphs $A_{4\ell+1}$ and $D_{2\ell+2}$. Thus $S^{(A)}$ is nothing but the modular S-matrix of the $\mathfrak{sl}(2)_{4\ell}$ WZW model. [16] By direct calculation one checks that the matrices (15.27) furnish a representation of the fusion ring of $\mathfrak{sl}(2)_{4\ell}$. Further, it

[16] On the other hand, the matrix $S^{(D)}$ defined this way is by far not unique; in particular, the incidence matrix of $D_{2\ell+2}$ has an eigenvalue with multiplicity 2. For a generic choice of diagonalizing matrix the numbers $\tilde{\mathrm{N}}_{\tilde{r},\tilde{s}}^{\ \ \tilde{t}}$ (9.6) will not be integral. But [40] there is a unique choice such that these numbers *are* integral.

can be shown that V_2 coincides with the incidence matrix of $D_{2\ell+2}$. By the explicit form of the $\mathfrak{sl}(2)_{4\ell}$ fusion rules, this implies that one may equivalently define the matrices V_s inductively via $V_1 := \mathbb{1}$, $V_2 :=$ incidence matrix of $D_{2\ell+2}$ and

$$V_s := V_2 V_{s-1} - V_{s-2} \quad \text{for} \;\; s = 3, 4, \ldots, 4\ell{+}1\,. \tag{15.28}$$

One also has the matrix equation

$$V_q \, \tilde{\mathrm{N}}_{\tilde r} = \sum_{\tilde s} (V_q)_{\tilde r}^{\;\;\tilde s}\, \tilde{\mathrm{N}}_{\tilde s} \tag{15.29}$$

for all $q = 1, 2, \ldots, 4\ell{+}1$ and all $\tilde r = 1, 2, \ldots, 2\ell{+}2$, where $\tilde{\mathrm{N}}_{\tilde s}$ are the 'graph fusion matrices' associated to $S^{(D)}$, i.e. the matrices with entries

$$\tilde{\mathrm{N}}_{\tilde r, \tilde s}^{\;\;\;\tilde t} := \sum_{\tilde u} S^{(D)}_{\tilde r, \tilde u} S^{(D)}_{\tilde s, \tilde u} S^{(D)*}_{\tilde t, \tilde u} / S^{(D)}_{1, \tilde u} \tag{15.30}$$

for $\tilde r, \tilde s, \tilde t = 1, 2, \ldots, 2\ell{+}2$.

In [40,41] also the E-type series of modular invariants for minimal models were discussed in the framework of graphs, and cyclic groups larger than $\mathbb{Z}_2$ have been addressed in [42]. It will certainly be interesting to compare the results obtained there with the ones that can be derived by the methods of the present paper, and in particular to study how the graph oriented approach deals with the case of non-cyclic abelian groups, where non-trivial two-cocycles appear.

16 More examples

In this section we present a few further examples, in which the orbifold group G is larger than $\mathbb{Z}_2$. We first discuss two examples of direct relevance to string theory. Afterwards we turn to some specific examples in which the effects of non-trivial two-cocycles can be analyzed in detail.

16.1 General cyclic groups

Let us study a situation of immediate interest in which the orbifold group is cyclic and hence has trivial second cohomology, so that the untwisted and full stabilizers coincide. We start with the $c = 1$ theory of a free boson, compactified on a circle of radius $R = \sqrt{2\mathcal{N}}$ with $\mathcal{N} \in \mathbb{Z}_{>0}$. The chiral algebra $\mathfrak{A}$ of this theory consists of operators of the form (15.1). Inspection shows that for every $m \in \mathbb{Z}_{>0}$ there is a subalgebra $\bar{\mathfrak{A}}^{(m)}$ of $\mathfrak{A}$ which is obtained by restricting the value of n in (15.1) to be a multiple of m. The algebra $\bar{\mathfrak{A}}^{(m)}$ is nothing else but the chiral algebra of another free boson theory, with the free boson X compactified on a circle of radius $m\sqrt{2\mathcal{N}}$. Therefore it is a consistent subalgebra in the sense that it allows for the construction of chiral blocks which obey factorization rules and have a Knizhnik–Zamolodchikov connection. The orbifold group is $G = \mathbb{Z}_m$; its generator acts on an operator of the form (15.1) by multiplication with the phase $\exp(2\pi i n/m)$; in terms of the Fubini–Veneziano field X this means

$$X \mapsto X + 2\pi/(m\sqrt{2\mathcal{N}})\,. \tag{16.1}$$

This leaves the $\mathfrak{u}(1)$ current j invariant and multiplies $\exp(\mathrm{i}n\sqrt{2\mathcal{N}}\,X)$ by the phase $\exp(2\pi\mathrm{i}n/m)$. (Together with the automorphism $X \mapsto -X$ (15.2), the transformation (16.1) generates the dihedral group D_m.)

In this example no fixed points are present, so that it is most straightforward to write down the classifying algebra. The $\bar{\mathfrak{A}}^{(m)}$-theory has $2m^2\mathcal{N}$ primary fields, which may be labelled by integers $\bar\lambda \bmod 2m^2\mathcal{N}$; their $\mathfrak{u}(1)$ charge is $q_{\bar\lambda} = \bar\lambda/m\sqrt{2\mathcal{N}}$. The twist sector is determined by the value of $\bar\lambda \bmod m$; in particular, those $\bar\lambda = m\bar l$ which are multiples of m are in the untwisted sector and label a basis of the classifying algebra $\mathcal{C}(\bar{\mathfrak{A}}^{(m)})$. The multiplication in $\mathcal{C}(\bar{\mathfrak{A}}^{(m)})$ is just given by the restriction of the fusion product of the $\bar{\mathfrak{A}}^{(m)}$-theory to the untwisted sector, i.e.

$$\tilde{\mathrm{N}}_{\bar l,\bar l'}^{\;\;\bar l''} = \delta_{\bar l+\bar l'+\bar l''\ \mathrm{mod}\ 2m\mathcal{N}}\,. \tag{16.2}$$

Thus $\mathcal{C}(\bar{\mathfrak{A}}^{(m)})$ is the group algebra of the cyclic group $\mathbb{Z}_{m\mathcal{N}}$. The reflection coefficients $\mathrm{R}^a_{\bar\mu;\bar\Omega}$ must therefore obey the relation $\mathrm{R}^a_{m\bar l_1;\bar\Omega}\mathrm{R}^a_{m\bar l_2;\bar\Omega} = \mathrm{R}^a_{m(\bar l_1+\bar l_2);\bar\Omega}$, where the addition is modulo $2m\mathcal{N}$. The solutions to this requirement are

$$\mathrm{R}^a_{m\bar l;\bar\Omega} = \exp(2\pi\mathrm{i}m\bar l a/2m^2\mathcal{N}) = \exp(\pi\mathrm{i}\bar l a/m\mathcal{N}) \tag{16.3}$$

with $a \in \mathbb{Z}$. Moreover, a must be taken only modulo $2m\mathcal{N}$, and hence the possible values of a are in one-to-one correspondence with the $\mathbb{Z}_m$-orbits of the $\bar{\mathfrak{A}}^{(m)}$-theory, in accordance with the general theory.

This result allows for the following geometric interpretation. After performing a suitable T-duality transformation, we can assume that we are dealing with a Dirichlet boundary condition, so that we can characterize the boundary state by the position a of a point-like defect on the circle. Breaking the bulk symmetry to the subalgebra $\bar{\mathfrak{A}}^{(m)}$ correspond to defects located at $2m\mathcal{N}$-th roots of unity on the unit circle. In general, we expect that boundary conditions breaking more chiral symmetries of the bulk correspond to more generic locations in the space of boundary conditions.

In the case at hand we can also study explicitly the projective limit of classifying algebras that was used in the definition of the universal classifying algebra in section 13. Manifestly, when m' divides m, then $\bar{\mathfrak{A}}^{(m')}$ is a subalgebra of $\bar{\mathfrak{A}}^{(m)}$. In more fancy terms this can be expressed as follows. Divisibility introduces a partial ordering on the set I non-negative integers; the subalgebras $\bar{\mathfrak{A}}^{(m)}$ form an inductive system over I. The inductive limit consists just of the subalgebra $\bar{\mathfrak{A}}^{(\infty)}$ of uncharged elements of $\mathfrak{A}$. Moreover, there is a natural projection relating the classifying algebras:

$$\mathcal{C}(\bar{\mathfrak{A}}^{(m)}) \cong \mathbb{C}\mathbb{Z}_{m\mathcal{N}} \;\to\; \mathbb{C}\mathbb{Z}_{m'\mathcal{N}} \cong \mathcal{C}(\bar{\mathfrak{A}}^{(m')})\,. \tag{16.4}$$

There then exists a projective limit

$$\hat{\mathbb{Z}}_{\mathcal{N}} := \lim_{\longleftarrow} \mathbb{Z}_{m\mathcal{N}} \tag{16.5}$$

of the classifying algebras, from which every classifying algebra $\mathcal{C}(\bar{\mathfrak{A}}^{(m)})$ can be obtained as a quotient. The group algebra of the infinite group $\hat{\mathbb{Z}}_{\mathcal{N}}$ is the projective limit of the classifying algebra. (For $\mathcal{N} = 1$ this group is a well-known object; it appears as the absolute Galois group of any finite field, or as the Galois group of the infinite extension $\mathbb{Q}(W)/\mathbb{Q}$, where W is the group of all roots of unity.)

Up to this point we were exclusively considering subalgebras that are rational in the sense that the number of primary fields of the orbifold is finite. When we allow also for boundary conditions that preserve non-rational subalgebras, then the situation simplifies considerably: $\bar{\mathfrak{A}}^{(\infty)}$ is then an allowed subalgebra, and the projective limit becomes isomorphic to the group algebra of $\mathbb{Z}$. [17] In this case the irreducible representations of the classifying algebra, and thus the corresponding boundary conditions, are labelled by the group U(1), the dual group of $\mathbb{Z}$. Recall that depending on whether the trivial or the charge conjugation modular invariant has been chosen in the bulk, the elements of this U(1) group correspond either to values of the Wilson lines or to positions of D0-branes.

16.2 Simple current symmetries in string theory

Our second example concerns the construction of perturbative superstring theories. Such theories contain fermionic degrees of freedom, and one must impose several projections to obtain consistency. All these projections can be formulated in terms of simple currents (see e.g. [6, 43]). First, one has to impose alignment of the fermionic degrees of freedom; the space-time fermions, the fermionic degrees of the internal theory and the superghosts have to be either all in Neveu–Schwarz or all in the Ramond sector. [18] This is accomplished by enlarging the chiral algebra by all bilinears of the supercurrents of the world sheet theory. We assume that the theory has a space-time sector containing D free bosons and D free fermions with a supercurrent J_{st}, which is a simple current of conformal weight $\Delta = 3/2$ and of order two. Similarly, there are supercurrents J_{int} for the inner sector and J_{sgh} for the superghosts. The relevant simple current group $\mathcal{G}$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$, consisting of the identity and the three non-trivial currents

$$(J_{\mathrm{st}}, J_{\mathrm{int}}, \mathbf{1})\,, \quad (J_{\mathrm{st}}, \mathbf{1}, J_{\mathrm{sgh}}) \quad \text{and} \quad (\mathbf{1}, J_{\mathrm{int}}, J_{\mathrm{sgh}})\,. \tag{16.6}$$

This extension ensures supersymmetry of the world sheet theory. Boundary conditions that break this symmetry seem to be unacceptable, since they would spoil the consistency of the theory; they lead to a tachyonic spectrum and other undesired features. These remarks refer to *world sheet supersymmetry* and hence apply to all string theories built from $N = 1$ superconformal field theories.

When we have superconformal theories with extended $N = 2$ superconformal symmetry on the world sheet, another simple current extension allows to build space-time supersymmetric theories. To this end one imposes the GSO projection, which can be achieved by enlarging the chiral algebra by another integer spin simple current J_{GSO}. This simple current J_{GSO} implements the spectral flow in the world sheet theory (see e.g. [6, 43]); it has non-trivial components both in the space-time and in the superghost sector. The order M of J_{GSO} is essentially the common denominator of the $\mathfrak{u}(1)$ charges in the Ramond sector; thus it depends on the specific $N = 2$ model under consideration.

[17] This might look confusing at first sight; but indeed the projective limit does depend on the category in which it is taken, i.e. on the selection of the objects – here boundary conditions – one considers.

[18] For a discussion of boundary states in the (super-)ghost sector, see e.g. [44].

J_{GSO} can be described more explicitly as follows. The $\mathfrak{u}(1)$ current

$$J(z) = J_{\text{st}}(z) + J_{\text{int}}(z) \tag{16.7}$$

of the $N=2$ algebra is the sum of a component J_{st} in the space-time sector and J_{int} in the inner sector; it can be expressed in terms of a standard free boson as $J = \sqrt{c/3}\,\mathrm{i}\partial X \equiv \sqrt{5}\,\mathrm{i}\partial X$. The spectral flow simple current is then realized as the Ramond ground state $\exp(\mathrm{i}(\sqrt{5}/2)X)$ of conformal weight $\Delta = 5/8$. It has to be combined with its counterpart in the superghost sector. Expressing the superghost in terms of a free boson Φ, the spectral flow operator is $\exp(\mathrm{i}\Phi/2)$ which has conformal weight $-5/8$, so that the total simple current

$$J_{\text{GSO}} = \exp(\mathrm{i}\,\tfrac{\sqrt{5}}{2}\,X(z))\,\exp(\mathrm{i}\,\tfrac{1}{2}\,\Phi(z)) \tag{16.8}$$

has integral conformal weight. In total, the projections in the construction of a superstring theory require a simple current extension by the abelian group

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_M\,. \tag{16.9}$$

In this extension typically fixed points do occur, as well as untwisted stabilizers that differ from the full stabilizers.

In contrast to the situation with supersymmetry on the world sheet, it is definitely of interest to study boundary conditions that do not preserve all *space-time* supersymmetries. A particular example is given by BPS conditions, but our formalism allows to describe also boundary conditions in which all space-time supersymmetries are broken. For reviews of non-BPS states and their conformal field theory description, we refer to [45, 46].

Let us mention that the formalism developed in the present work has another application in string theory. Namely, every superconformal field theory with $N=2$ supersymmetry has an automorphism of order two that reverses the sign of the $\mathfrak{u}(1)$ current J of the $N=2$ algebra and exchanges the two supercurrents $G^{\pm}$ of charge ± 1. Accordingly, when studying boundary conditions that correspond to this automorphism we get two automorphism types; they are usually called 'A-type' and 'B-type' (see e.g. [47]). According to the general results in section 11, T-duality interchanges these two automorphism types. Notice that both types of boundary conditions are encompassed by a single classifying algebra.

16.3 The $\mathbb{Z}_2$ orbifold of the free boson and fractional branes

Another illustrative example for our formalism is provided by the $\mathbb{Z}_2$-orbifold of a free boson, compactified at a rational radius squared. For concreteness, we again restrict our attention to the case when $R^2 = 2\mathcal{N}$ with $\mathcal{N} \in \mathbb{Z}_{>0}$, which corresponds to the diagonal modular invariant. The boundary conditions that preserve all bulk symmetries are in one-to-one correspondence to the labels of primary fields, for which we will use the convention of [32]. These boundary conditions can be given the following interpretation.

In the untwisted sector of the orbifold there are $\mathcal{N}-1$ primaries Φ_q, $q = 1, 2, \dots, \mathcal{N}-1$, of conformal weight $\Delta_q = q^2/4\mathcal{N}$, as well as two pairs 1, J (of conformal weight 0 and 1) and ψ^1, ψ^2 (with $\Delta_\psi = \mathcal{N}/4$) which each combine to a single primary field of the underlying circle theory. The boundary conditions labelled by the Φ_q constitute D0-branes sitting at ξR with ξ one of the

$2\mathcal{N}$th roots $e^{\pi i q/\mathcal{N}}$, $0<q<\mathcal{N}$, of unity; those labelled by 1 and J describe D0-branes which are both localized at one orbifold point, and those labelled by ψ^1 and ψ^2 are D0-branes localized at the other orbifold point. The latter boundary conditions deserve particular attention, as in these cases the position in target space is not sufficient to describe uniquely the boundary conditions. Rather, an additional discrete label is needed. Now the primary fields 1, J, ψ^1 and ψ^2 have quantum dimension 1, while the primaries Φ_q all have quantum dimension 2. It is also known that in a string compactification the Ramond–Ramond charge is proportional to certain (generalized) quantum dimensions. Thus we can conclude that the D-branes sitting at orbifold points have a Ramond–Ramond charge that is only half the one of D-branes sitting at smooth points; accordingly [48] they are referred to as *fractional branes*.

In terms of boundary states (or, equivalently, reflection coefficients) this behavior is explained as follows. For boundary conditions that preserve the full bulk symmetry, the reflection coefficients are (ratios of) elements of the modular matrix S. The pairs $1, J$ and ψ^1, ψ^2 form full orbits of the order-two simple current J, while the fields Φ_q are fixed points of J. Since the monodromy charge with respect to J is 0 for fields in the untwisted sector and 1/2 in the twisted sector, the standard simple current relation $S_{J\lambda,\mu} = e^{2\pi i Q_J(\mu)} S_{\lambda,\mu}$ implies that in the boundary conditions labelled by Φ_q boundary blocks of the twisted sector do not appear. Indeed, because of $J\Phi_q = \Phi_q$ we have $S_{q,\mu} \equiv S_{Jq,\mu} = -S_{q,\mu}$ for all μ in the twisted sector. In contrast, orbits of full length do appear, but with opposite sign for the two primaries λ and $J\lambda$ on the orbit. Briefly, the ambiguity for boundary conditions that are localized at the orbifold points reflects the fact that on a disk with such a boundary condition bulk fields in the twisted sector can acquire a non-vanishing one-point function and that two values with opposite sign are possible for this correlation function.

The twisted sector contributes four more primaries $\sigma^{1,2}$ and $\tau^{1,2}$ of conformal weight 1/16 and 9/16, respectively. It turns out that the corresponding boundary conditions are not localized and are thus Neumann-like. It is therefore tempting to identify them with the four different types of $\mathbb{Z}_2$-equivariant line bundles over the circle.

To gain insight in symmetry breaking boundary conditions, we need consistent subalgebras $\bar{\mathfrak{A}}$ of the chiral algebra $\mathfrak{A}$ of the $\mathbb{Z}_2$-orbifold. Examples are easily obtained by observing that the vector space that underlies the chiral algebra $\mathfrak{A}$ can be decomposed according to the absolute value of the $\mathfrak{u}(1)$ charge in the circle theory. This quantity is well-defined because we only have to deal with fields in the untwisted sector and only states with opposite charge are identified. In the decomposition all multiples of $\sqrt{\mathcal{N}}$ appear:

$$\mathcal{H}_\Omega = \bigoplus_{n\in\mathbb{Z}_{\geq 0}} \mathcal{H}^{\mathfrak{u}(1)}_{n\sqrt{\mathcal{N}}} . \tag{16.10}$$

This decomposition does not constitute a grading of $\mathfrak{A}$ over the additive group $\mathbb{Z}_{\geq 0}$, because the fusion structure within the chiral algebra reads $[q_1] \star [q_2] = [q_1{+}q_2] + [|q_1{-}q_2|]$. Still these fusions imply that for every integer $\ell \geq 2$ the subspace

$$\bar{\mathcal{H}}^{(\ell)}_\Omega := \bigoplus_{n\in\mathbb{Z}_{\geq 0}} \mathcal{H}^{\mathfrak{u}(1)}_{n\ell\sqrt{\mathcal{N}}} \tag{16.11}$$

provides a subalgebra of the chiral algebra, and it is in fact a consistent subalgebra because it is nothing but the chiral algebra of the $\mathbb{Z}_2$-orbifold of a free boson at radius $\bar{R}^{(\ell)} = \ell R$.

Moreover, inspection shows that, except for $\ell = 2$, we are *not* dealing with an orbifold subalgebra $\mathfrak{A}^G$ of $\mathfrak{A}$. Indeed, for $\ell \geq 3$ there does not exist any automorphism of $\mathfrak{A}$ for which the fixed point set is $\bar{\mathcal{H}}_\Omega^{(\ell)}$. (Note that for generic $\mathcal{N}$ the orbifold theory also has very few automorphisms of the fusion rules that preserve the conformal weights.) In particular, this situation is not covered by our formalism. However, the specific conformal field theory in question is simple enough to allow for a direct construction of the corresponding boundary states [49]. They are not related to automorphisms of the chiral algebra, and thus it is in general not possible to associate an automorphism type to such boundary conditions. Hence they provide a simple counter example to the common misconception that every boundary condition should possess a definite automorphism type.

In the particular case $\ell = 2$, our techniques can be applied to obtain still more boundary conditions. We denote primary fields in the orbifold theory at radius $\bar{R} := 2R$ by analogous labels as above, with an additional bar. The boundary conditions then correspond to orbits of primaries with respect to the simple current $\bar{\psi}^1$, which has order two. We only describe the orbits of non-trivial automorphism type. The orbits $\{\bar{\Phi}_q, \bar{\Phi}_{4\mathcal{N}-q}\}$ for $q = 1, 3, \dots, 2\mathcal{N}-1$ give D0-branes localized at R times a $4\mathcal{N}$th root of unity that is not a $2\mathcal{N}$th root of unity. From the other orbits in the untwisted sector of the $\bar{\mathfrak{A}}$-theory we recover the other boundary conditions in the untwisted sector that were described earlier. In the twisted sector of the $\bar{\mathfrak{A}}$-theory we find two orbits that are fixed points, $\bar{\sigma}^1$ and $\bar{\tau}^1$; they give rise to four boundary conditions that preserve all bulk symmetries, corresponding to $\sigma^{1,2}$ and $\tau^{1,2}$ in the $\mathfrak{A}$-theory. Finally, the remaining orbit $\{\bar{\sigma}^2, \bar{\tau}^2\}$ gives rise to a Neumann-like boundary condition which breaks the symmetries down to $\bar{\mathcal{H}}_\Omega^{(2)}$.

We conclude our discussion with the remark that, unlike for the free boson case, the automorphism type of a boundary condition in the $\mathbb{Z}_2$-orbifold of a free boson does not allow any longer to distinguish Dirichlet and Neumann boundary conditions. Still, all Dirichlet boundary conditions come from the untwisted sector while all Neumann boundary conditions come from the twisted sector.

16.4 Examples with genuine untwisted stabilizer

We now consider in detail an example where an untwisted stabilizer occurs that is a proper subgroup of the full stabilizer. It is worth emphasizing that the situation with genuine untwisted stabilizers arises rather naturally in string compactifications. This is a consequence of the following elementary fact that applies to any tensor product of three or more subtheories with three simple currents $\hat{\mathrm{J}}_i$ $(i = 1, 2, 3)$ of half-integral conformal weight in three distinct subtheories. In the Gepner construction of superstring vacua such subtheories can be factors in the inner sector, but also the conformal field theories describing space-time fermions or the (super-)ghosts. Generic examples for simple currents with $\Delta \in \mathbb{Z}+1/2$ are the various components of the supercurrent on the world sheet, but typically in such compactifications other simple currents of this type are present as well.

It has been shown in [4] that in WZW theories the commutator cocycle obeys

$$F_\lambda(\mathrm{J}, \mathrm{K}) = \exp(2\pi \mathrm{i}(\Delta_\mathrm{J} - \Delta_\mathrm{J}^{(\mathrm{K})})) , \tag{16.12}$$

where Δ_J is the conformal weight of J, while $\Delta_\mathrm{J}^{(\mathrm{K})}$ is the conformal weight of the projection of

J in what is called the fixed point theory [6] with respect to K. This formula also applies to those simple currents of coset theories which come from simple currents of the underlying WZW models.[19] In the situation at hand, each of the simple currents $\hat{\mathrm{J}}_i$ is projected to the identity primary field of the fixed point theory, so that (16.12) implies that $F_\lambda(\hat{\mathrm{J}}_i,\hat{\mathrm{J}}_i) = -1$. On the other hand, for $i \neq j$ the two currents are fields in distinct subtheories, so that $F_\lambda(\hat{\mathrm{J}}_i,\hat{\mathrm{J}}_j) = 1$. Now out of the three currents $\hat{\mathrm{J}}_i$ we can form three simple currents J_i of the tensor product theory by setting $\mathrm{J}_1 := \hat{\mathrm{J}}_2\hat{\mathrm{J}}_3$ and cyclic. These have integral conformal weight and hence – unlike the original currents $\hat{\mathrm{J}}_i$ themselves – can be used to extend the chiral algebra, by a $\mathbb{Z}_2{\times}\mathbb{Z}_2$ group. By the bihomomorphism property of F_λ we then see that F_λ is non-trivial on this group:

$$F_\lambda(\mathrm{J}_i,\mathrm{J}_j) = \left\{ \begin{array}{rl} 1 & \text{for } i = j\,, \\ -1 & \text{for } i \neq j\,. \end{array} \right. \tag{16.13}$$

As a consequence, the untwisted stabilizer is a proper subgroup of the full stabilizer.

We will now study the effect of a non-trivial untwisted stabilizer in the situation that the $\mathfrak{A}$-theory is a WZW theory, based on some affine Lie algebra $\mathfrak{g} = \bar{\mathfrak{g}}^{(1)}$ and that the orbifold theory is a WZW theory as well, now based on $\mathfrak{g}' = \bar{\mathfrak{g}}'^{(1)}$ with $\bar{\mathfrak{g}}' \subset \bar{\mathfrak{g}}$. Such WZW orbifolds have been studied in [50, 5]. Here we are interested in conformally invariant boundary conditions. Thus the Virasoro algebras of $\mathfrak{g}$ and $\mathfrak{g}'$ must coincide; this is precisely the case when $\mathfrak{g}' \hookrightarrow \mathfrak{g}$ is a conformal embedding [51, 52].

Many, though not all, conformal embeddings can be understood in terms of simple currents. An example where this is possible is the following. For any three odd positive integers d_1, d_2, d_3 there is a conformal embedding

$$\mathfrak{g}' := \mathrm{so}(d_1)_1 \oplus \mathrm{so}(d_2)_1 \oplus \mathrm{so}(d_3)_1 \;\hookrightarrow\; \mathrm{so}(d_1{+}d_2{+}d_3)_1 =: \mathfrak{g}\,. \tag{16.14}$$

The theory based on $\mathfrak{g}$ can be obtained as an extension of the $\mathfrak{g}'$-theory by a simple current group $\mathcal{G} = \mathbb{Z}_2 \times \mathbb{Z}_2$, consisting of the fields

$$(\mathrm{o,o,o})\,, \quad (\mathrm{o,v,v})\,, \quad (\mathrm{v,o,v}) \quad \text{and} \quad (\mathrm{v,v,o})\,, \tag{16.15}$$

where o and v refer to the singlet and vector representation of $\mathrm{so}(d_i)$, respectively. There is only a single fixed point, namely the tensor product (s,s,s) of three $\mathrm{so}(d_i)$ spinor representations; it has stabilizer $\mathcal{S} = \mathcal{G}$. The conformal weight of the vector simple current v at level 1 is 1/2; according to the general arguments presented at the beginning of this subsection, the untwisted stabilizer is therefore trivial, $\mathcal{U} = \{(\mathrm{o,o,o})\}$. (The value $[\mathcal{S}:\mathcal{U}] = 2^2$ of the index is in agreement with the fact that the ground state degeneracy of the irreducible spinor representation of the $\mathrm{so}(d_1{+}d_2{+}d_3)$ theory is twice as large as the one of (s,s,s), namely $2 \cdot 2^{(d_1-1)/2} 2^{(d_2-1)/2} 2^{(d_3-1)/2}$.)

As a side remark, we mention that these theories can be realized in terms of free fermions. Thus the effect of a genuine untwisted stabilizer can occur even in *free* conformal field theories. This is in fact not too surprising. As we have explained, the presence of untwisted stabilizers is related to the fact that the orbifold group acts only projectively on certain sectors of the theory; this is also known [32] to be true for the action of the three polyhedral groups on the free boson (compactified at the self-dual radius) that gives rise to the exceptional $c = 1$ theories.

[19] In coset theories there can, however, exist additional simple currents which arise from resolving fixed points with integral quantum dimension.

In the case at hand, the relevant automorphisms of the chiral algebra $\mathfrak{A}$ can be understood in terms of the finite-dimensional Lie groups G and G' associated to $\mathfrak{g}$ and $\mathfrak{g}'$. Namely, for every boundary block B_λ: $\mathcal{H}_\lambda {\otimes} \mathcal{H}_{\lambda^+}$ that preserves the full bulk symmetry and every element $\gamma \in \mathsf{G}$ the combination $\mathrm{B}_\lambda^{(\gamma)} := \mathrm{B}_\lambda \circ (\gamma \otimes id)$ provides us with a twisted boundary block. The corresponding automorphism on the affine Lie algebra $\mathfrak{g}$ is the inner automorphism that acts on the modes J^a_n of $\mathfrak{g}$ as $J^a_n \mapsto (\gamma J^a \gamma^{-1})_n$. This automorphism preserves the symmetries in the subalgebra $\mathfrak{g}'$ if and only if γ is in the centralizer $C_\mathsf{G}(\mathsf{G}')$ of G' in G, and it acts trivially if and only if γ is even in the center $Z(\mathsf{G})$ of G. Thus the non-trivial twists are those by elements in the group $C_\mathsf{G}(\mathsf{G}')/Z(\mathsf{G})$.

In the case of our interest, the relevant embedding on the level of Lie groups reads

$$\mathsf{G}' := (\mathrm{Spin}(d_1) \times \mathrm{Spin}(d_2) \times \mathrm{Spin}(d_3))/(\mathbb{Z}_2 \times \mathbb{Z}_2) \;\hookrightarrow\; \mathrm{Spin}(d_1{+}d_2{+}d_3) =: \mathsf{G}\,; \tag{16.16}$$

note that the subgroup G' is not simply connected. For determining $C_\mathsf{G}(\mathsf{G}')/Z(\mathsf{G})$ it is instructive to consider first the embedding

$$\tilde{\mathsf{G}}' := \mathrm{SO}(d_1) \times \mathrm{SO}(d_2) \times \mathrm{SO}(d_3) \;\hookrightarrow\; \mathrm{SO}(d_1{+}d_2{+}d_3) =: \tilde{\mathsf{G}} \tag{16.17}$$

that is obtained from (16.16) by dividing out the center of $\mathrm{Spin}(d_1{+}d_2{+}d_3)$. Then the center of the $\tilde{\mathsf{G}}$ is trivial; moreover, using the matrix realization of these groups, one shows that the centralizer $C_{\tilde{\mathsf{G}}}(\tilde{\mathsf{G}}')$ is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ group consisting of the unit matrix $\mathbb{1}_{d_1+d_2+d_3}$ and the diagonal matrices

$$\tilde{M}_{23} := \begin{pmatrix} \mathbb{1}_{d_1} & 0 & 0 \\ 0 & -\mathbb{1}_{d_2} & 0 \\ 0 & 0 & -\mathbb{1}_{d_3} \end{pmatrix}, \quad \tilde{M}_{13} := \begin{pmatrix} -\mathbb{1}_{d_1} & 0 & 0 \\ 0 & \mathbb{1}_{d_2} & 0 \\ 0 & 0 & -\mathbb{1}_{d_3} \end{pmatrix}, \quad \tilde{M}_{12} := \begin{pmatrix} -\mathbb{1}_{d_1} & 0 & 0 \\ 0 & -\mathbb{1}_{d_2} & 0 \\ 0 & 0 & \mathbb{1}_{d_3} \end{pmatrix}. \tag{16.18}$$

Already at this stage we can conclude that the centralizer $C_{\mathsf{G}'}(\mathsf{G})$ is an extension of this $\mathbb{Z}_2 \times \mathbb{Z}_2$ group by the center $\mathbb{Z}_2$ of $\mathrm{Spin}(d_1{+}d_2{+}d_3)$. To decide which extension we are dealing with, we introduce gamma matrices γ^i that satisfy the Clifford relations $\{\gamma^i, \gamma^j\} = 2\,\delta^{ij}$. We can then determine the two lifts $M^\pm_{ij}$ of the matrices $\tilde{M}_{ij}$ to $\mathrm{Spin}(d_1{+}d_2{+}d_3)$ from the requirement that $M_{12}\gamma^i (M_{12})^{-1} = -\gamma^i$ for $1 \le i \le d_1{+}d_2$ and $M_{12}\gamma^i (M_{12})^{-1} = \gamma^i$ for $i > d_1{+}d_2$. It is easy to verify that

$$\begin{aligned} &M^\pm_{12} = \pm\,\gamma^1\gamma^2\cdots\gamma^{d_1+d_2}\,, \qquad M^\pm_{23} = \pm\,\gamma^{d_1+1}\gamma^{d_1+2}\cdots\gamma^{d_1+d_2+d_3}\,, \\ &M^\pm_{13} = \pm\,\gamma^1\gamma^2\cdots\gamma^{d_1}\,\gamma^{d_1+d_2+1}\cdots\gamma^{d_1+d_2+d_3}\,. \end{aligned} \tag{16.19}$$

It can be checked that these matrices commute with all elements in G'. They form a group of order 8; the structure of this group depends on the values of d_1, d_2 and d_3. First, when all d_i leave the same rest modulo $4\mathbb{Z}$, then the group is isomorphic to the eight-element generalized *quaternion* group; this group has one two-dimensional and four one-dimensional irreducible representations. Otherwise, i.e. when only two of the d_i leave the same rest modulo $4\mathbb{Z}$, the group is isomorphic to the *dihedral* group D_4 of 8 elements. This group has four one-dimensional and one two-dimensional irreducible representation, too. In both cases the centralizer is non-abelian.

Let us present some more details. The boundary blocks are labelled by the fields of monodromy charge zero and characters of their (full) stabilizers; thus there are 8 boundary blocks coming from full orbits; we label them lexicographically,

$$\begin{aligned} &\mathrm{B}_1 := \mathrm{B}_{\mathrm{ooo}}\,, \quad &&\mathrm{B}_2 := \mathrm{B}_{\mathrm{oov}}\,, \quad &&\mathrm{B}_3 := \mathrm{B}_{\mathrm{ovo}}\,, \quad &&\mathrm{B}_4 := \mathrm{B}_{\mathrm{ovv}}\,, \\ &\mathrm{B}_5 := \mathrm{B}_{\mathrm{voo}}\,, &&\mathrm{B}_6 := \mathrm{B}_{\mathrm{vov}}\,, &&\mathrm{B}_7 := \mathrm{B}_{\mathrm{vvo}}\,, &&\mathrm{B}_8 := \mathrm{B}_{\mathrm{vvv}}\,. \end{aligned} \tag{16.20}$$

Note that the blocks numbered as $1,4,6,7$ come from the vacuum of the $\mathfrak{A}$-theory, while the others come from the field that carries the vector representation of $\mathrm{so}(d_1{+}d_2{+}d_3)$. In addition we have 4 boundary blocks coming from the fixed point (s,s,s). They correspond to the four irreducible characters ψ of $\mathbb{Z}_2\times\mathbb{Z}_2$; we label them as

$$\mathrm{B}_9 := \mathrm{B}_{++++}\,, \qquad \mathrm{B}_{10} := \mathrm{B}_{++--}\,, \qquad \mathrm{B}_{11} := \mathrm{B}_{+-+-}\,, \qquad \mathrm{B}_{12} := \mathrm{B}_{+--+}\,, \tag{16.21}$$

where the $\pm$ labels indicate the values ±1 of ψ on the four elements of $\mathbb{Z}_2\times\mathbb{Z}_2$, in the lexicographic order chosen in formula (16.15).

The boundary conditions are labelled by the orbits and characters of their untwisted stabilizers. Thus in addition to the three boundary conditions that preserve all of $\mathfrak{A}$, there are three conditions from the length-4 orbits and two conditions for each of the three length-2 orbits which have stabilizer $\mathbb{Z}_2$. We label them according to

$$\begin{array}{llll} \mathcal{B}_1 \,\hat{=} & \{(\mathrm{ooo})\,,(\mathrm{ovv})\,,(\mathrm{vov})\,,(\mathrm{vvo})\}\,, & \mathcal{B}_7 \,\hat{=} & \{(\mathrm{oss})\,,(\mathrm{vss})\}\ \text{with}\ \psi=1\,, \\ \mathcal{B}_2 \,\hat{=} & \{(\mathrm{oov})\,,(\mathrm{ovo})\,,(\mathrm{voo})\,,(\mathrm{vvv})\}\,, & \mathcal{B}_8 \,\hat{=} & \{(\mathrm{oss})\,,(\mathrm{vss})\}\ \text{with}\ \psi=-1\,, \\ \mathcal{B}_3 \,\hat{=} & \{(\mathrm{sss})\}\,, & \mathcal{B}_9 \,\hat{=} & \{(\mathrm{sos})\,,(\mathrm{svs})\}\ \text{with}\ \psi=1\,, \\ \mathcal{B}_4 \,\hat{=} & \{(\mathrm{soo})\,,(\mathrm{sov})\,,(\mathrm{svo})\,,(\mathrm{svv})\}\,, & \mathcal{B}_{10} \,\hat{=} & \{(\mathrm{sos})\,,(\mathrm{svs})\}\ \text{with}\ \psi=-1\,, \\ \mathcal{B}_5 \,\hat{=} & \{(\mathrm{oso})\,,(\mathrm{osv})\,,(\mathrm{vso})\,,(\mathrm{vsv})\}\,, & \mathcal{B}_{11} \,\hat{=} & \{(\mathrm{sso})\,,(\mathrm{ssv})\}\ \text{with}\ \psi=1\,, \\ \mathcal{B}_6 \,\hat{=} & \{(\mathrm{oos})\,,(\mathrm{ovs})\,,(\mathrm{vos})\,,(\mathrm{vvs})\}\,, & \mathcal{B}_{12} \,\hat{=} & \{(\mathrm{sso})\,,(\mathrm{ssv})\}\ \text{with}\ \psi=-1\,. \end{array} \tag{16.22}$$

With this numbering, the diagonalizing matrix $\tilde S$ looks as follows:

$$\tilde S = \frac12 \begin{pmatrix} 1 & 1 & \sqrt2 & \sqrt2 & \sqrt2 & \sqrt2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -\sqrt2 & \sqrt2 & \sqrt2 & -\sqrt2 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -\sqrt2 & \sqrt2 & -\sqrt2 & \sqrt2 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & \sqrt2 & \sqrt2 & -\sqrt2 & -\sqrt2 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -\sqrt2 & -\sqrt2 & \sqrt2 & \sqrt2 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & \sqrt2 & -\sqrt2 & \sqrt2 & -\sqrt2 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & \sqrt2 & -\sqrt2 & -\sqrt2 & \sqrt2 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -\sqrt2 & -\sqrt2 & -\sqrt2 & -\sqrt2 & 1 & 1 & 1 & 1 & 1 & 1 \\ \sqrt2 & -\sqrt2 & 0 & 0 & 0 & 0 & b_{23} & -b_{23} & b_{13} & -b_{13} & b_{12} & -b_{12} \\ \sqrt2 & -\sqrt2 & 0 & 0 & 0 & 0 & b_{23} & -b_{23} & -b_{13} & b_{13} & -b_{12} & b_{12} \\ \sqrt2 & -\sqrt2 & 0 & 0 & 0 & 0 & -b_{23} & b_{23} & b_{13} & -b_{13} & -b_{12} & b_{12} \\ \sqrt2 & -\sqrt2 & 0 & 0 & 0 & 0 & -b_{23} & b_{23} & -b_{13} & b_{13} & b_{12} & -b_{12} \end{pmatrix} . \tag{16.23}$$

Here we have put

$$b_{ij} := \sqrt2\,\mathrm{i}^{-(d_i+d_j)/2} \tag{16.24}$$

for $i,j\in\{1,2,3\}$. We note that

$$\tilde S\tilde S^\dagger = 4\cdot\mathbb{1} = \tilde S^\dagger\tilde S \quad \text{and} \quad (\tilde S\tilde S^{\mathrm t})^2 = 16\cdot\mathbb{1}\,, \tag{16.25}$$

but $\tilde{S}\tilde{S}^{\rm t}$ is a permutation if and only if precisely one out of the expressions

$$2+b_{23}^2+b_{13}^2+b_{12}^2\,, \quad 2+b_{23}^2-b_{13}^2-b_{12}^2\,, \quad 2-b_{23}^2+b_{13}^2-b_{12}^2\,, \quad 2-b_{23}^2-b_{13}^2+b_{12}^2 \qquad (16.26)$$

is non-vanishing, which are precisely those cases where after division by 2 these numbers furnish a character of $\mathbb{Z}_2{\times}\mathbb{Z}_2$.

Let us now add some general remarks. Every G-module can be decomposed into irreducible modules of the product $C_{\mathsf{G}}(\mathsf{G}')\times\mathsf{G}'$. The fact that the group $C_{\mathsf{G}}(\mathsf{G}')$ is non-abelian implies that it has higher-dimensional irreducible representations, which in turn means that higher-dimensional degeneracy spaces appear; this way we recover a generic feature of genuine untwisted stabilizers. Conversely, we are led to the following conjecture. Let $\mathsf{G}'\hookrightarrow\mathsf{G}$ be an embedding of reductive compact Lie groups such that the associated embedding of affine Lie algebras is a simple current extension. Then the centralizer $C_{\mathsf{G}}(\mathsf{G}')$ is non-abelian if and only if at some value of the level a genuine untwisted stabilizer appears. For instance, there is a conformal embedding $(D_4)_2\hookrightarrow(E_7)_1$ which again is a simple current extension by a $\mathbb{Z}_2{\times}\mathbb{Z}_2$ group. As E_7 is an exceptional Lie algebra, the relevant centralizer, namely the one of $\mathrm{SO}(8)/\mathbb{Z}_2$ in E_7, is now difficult to compute, but in any case the analysis of the boundary conditions indicates that this centralizer is non-abelian. For completeness, we mention that in the E_7 case there are eight boundary conditions preserving the affine D_4 subalgebra, and the matrix $\tilde{S}$ reads

$$\tilde{S}=\frac{1}{\sqrt{2}}\begin{pmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&1&-1&-1&-1&-1\\ 1&1&-1&-1&1&1&-1&-1\\ 1&1&-1&-1&-1&-1&1&1\\ 1&-1&\mathrm{i}&-\mathrm{i}&\mathrm{i}&-\mathrm{i}&\mathrm{i}&-\mathrm{i}\\ 1&-1&\mathrm{i}&-\mathrm{i}&-\mathrm{i}&\mathrm{i}&-\mathrm{i}&\mathrm{i}\\ 1&-1&-\mathrm{i}&\mathrm{i}&\mathrm{i}&-\mathrm{i}&-\mathrm{i}&\mathrm{i}\\ 1&-1&-\mathrm{i}&\mathrm{i}&-\mathrm{i}&\mathrm{i}&\mathrm{i}&-\mathrm{i} \end{pmatrix}. \qquad (16.27)$$

On the other hand, the automorphisms themselves are classified by the group $C_{\mathsf{G}}(\mathsf{G}')/Z(\mathsf{G})$, which in both cases considered above is $\mathbb{Z}_2\times\mathbb{Z}_2$. Thus they precisely correspond to the automorphism types that are predicted by the general analysis.

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A Collection of formulae

Here we collect a few basic formulae from [1] that are used in this paper. The equation numbers are the same as in [1].

▪ The boundary blocks $\tilde{\rm B}_{(\bar\lambda,\varphi)}$ are the linear forms

$$\tilde{\rm B}_{(\bar\lambda,\psi)} := \mathcal{N}_\mu\, d_\lambda^{-1/2}\, {\rm b}_\psi \otimes \bar{\rm B}_{\bar\lambda} \tag{4.23}$$

on

$$\mathcal{H}_\lambda \otimes \mathcal{H}_{\lambda^+} \equiv \bigoplus_{{\rm J}\in\mathcal{G}/\mathcal{S}_\lambda} \mathcal{V}_{\hat\psi} \otimes \bar{\mathcal{H}}_{{\rm J}\bar\lambda_\circ} \otimes \bigoplus_{{\rm J}\in\mathcal{G}/\mathcal{S}_\lambda} \mathcal{V}_{\hat\psi'} \otimes \bar{\mathcal{H}}_{{\rm J}\bar\lambda_\circ^+} \,.$$

Here ${\rm b}_\psi$: $\mathcal{V}_{\hat\psi_\lambda} \otimes \mathcal{V}_{\hat\psi^+_\lambda} \to \mathbb{C}$ is a linear form on the degeneracy spaces, $\bar{\rm B}_{\bar\lambda}$: $\mathcal{H}_{\bar\lambda} \otimes \mathcal{H}_{\bar\lambda} \to \mathbb{C}$ is an ordinary boundary block of the $\bar{\mathfrak{A}}$-theory, $d_\lambda = \sqrt{|\mathcal{S}_\lambda|/|\mathcal{U}_\lambda|}$, and $\mathcal{N}_\lambda$ is a phase that is left undetermined (in the first place, this normalization is introduced as $\mathcal{N}_{\bar\lambda,\psi}$, but as shown in [1] it depends only on the primary label $\lambda = [\bar\lambda, \hat\psi]$ of the $\mathfrak{A}$-theory). The linear form ${\rm b}_\psi$ can be written as

$${\rm b}_\psi = \beta_\circ \circ (\mathcal{O}_\psi \otimes id) \tag{4.20}$$

with $\beta_\circ$ defined by $\beta_\circ(v \otimes w) = {\rm B}_\lambda(v \otimes p_\circ \otimes w \otimes q_\circ) / \bar{\rm B}_{\bar\lambda}(p_\circ \otimes q_\circ)$ (where $p_\circ \in \bar{\mathcal{H}}_{\bar\lambda}$ and $q_\circ \in \bar{\mathcal{H}}_{\bar\lambda^+}$ are any vectors such that $\bar{\rm B}_{\bar\lambda}(p_\circ \otimes q_\circ) \neq 0$) and

$$\mathcal{O}_\psi := d_\lambda^{-3/2} \sum_{{\rm J}\in\mathcal{S}_\lambda/\mathcal{U}_\lambda} \psi({\rm J})^* R_{\hat\psi}({\rm J})\,, \tag{4.11}$$

where $R_{\hat\psi}$ denotes the irreducible representation of the twisted group algebra that is labelled by $\hat\psi \prec \psi$. The $\mathcal{O}_\psi$ with $\psi \succ \hat\psi$ (that is, $\psi_{|\mathcal{U}_\lambda} = \hat\psi$) form a partition of unity:

$$\sum_{\substack{\psi\in\mathcal{S}_\lambda^* \\ \psi\succ\hat\psi}} \mathcal{O}_\psi = d_\lambda^{1/2}\, \mathbb{1}_{d_\lambda} \,. \tag{4.17}$$

▪ The operator product expansion that describes the excitation on the boundary caused by a bulk field approaching it reads

$$\phi_{(\bar\lambda,\psi),(\bar\lambda^+,\psi^+)}(r{\rm e}^{{\rm i}\sigma}) = \sum_{\bar\mu} (1{-}r^2)^{-2\Delta_{\bar\lambda}+\Delta_{\bar\mu}}\, {\rm R}^a_{(\bar\lambda,\psi);\bar\mu}\, \Psi^{aa}_{\bar\mu}({\rm e}^{{\rm i}\sigma}) + \text{descendants} \quad \text{for } r \to 1\,. \tag{5.4}$$

▪ The diagonalizing matrix $\tilde S$ of $\mathcal{C}(\bar{\mathfrak{A}})$ can be written as

$$\tilde S_{(\bar\lambda,\psi_\lambda),[\bar\rho,\hat\psi_\rho]} := \frac{|G|}{[\mathcal{S}_\lambda \mathcal{U}_\lambda \mathcal{S}_\rho \mathcal{U}_\rho]^{1/2}} \sum_{{\rm J}\in\mathcal{S}_\lambda\cap\mathcal{U}_\rho} \psi_\lambda({\rm J})\, \hat\psi_\rho({\rm J})^*\, S^{\rm J}_{\bar\lambda,\bar\rho}\,, \tag{5.9}$$

where the matrices $S^{\rm J}$ represent the modular S-transformation on the one-point chiral blocks with insertion J on the torus. The result that $\tilde S$ is a square matrix is equivalent to the sum rule

$$\sum_{\substack{\bar\lambda \\ Q_\mathcal{G}(\lambda)=0}} |\mathcal{S}_\lambda| = \sum_{[\bar\rho]} |\mathcal{U}_\rho|\,. \tag{5.22}$$

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