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TOWARDS THE CLASSIFICATION OF CONFORMAL FIELD THEORIES IN ARBITRARY DIMENSION

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Abstract

I identify the subclass of higher-dimensional conformal field theories that is most similar to two-dimensional conformal field theory. In this subclass the domain of validity of the recently proposed formula for the irreversibility of the renormalization-group flow is suitably enhanced. The trace anomaly is quadratic in the Ricci tensor and contains a unique central charge. This implies, in particular, a relationship between the coefficient in front of the Euler density (charge a) and the stress-tensor two-point function (charge c). I check the prediction in detail in four, six and eight dimensions, and then in arbitrary dimension. In four and six dimensions there is agreement with results from the AdS/CFT correspondence. A by-product is a mathematical algorithm to construct conformal invariants.

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In quantum field theory, scale invariance is broken, in general, by the radiative corrections and measured by a non-vanishing beta function. Scale invariance is recovered in the large- and small-distance limits, where the theory becomes conformal. In the conformal window the UV limit is typically free (asymptotic freedom), while the IR limit is interacting. The motivation for studying the conformal window is that this region separates the perturbative domain from the truly non-perturbative one, where QCD leaves. A better understanding of conformal field theory in four dimensions, hopefully the achievement of a complete classification, is expected to shed light on some of the open problems of quantum field theory.

Four-dimensional conformal field theories have two central charges, c and a, defined by the trace anomaly in a gravitational background. The charge c multiplies the conformal invariant $W^2_{\mu\nu\rho\sigma}$ (square of the Weyl tensor) and is the coefficient of the two-point function of the stress tensor. The quantity a multiplies the Euler density G_4. A third term, $\Box R$, is multiplied by a coefficient a':

$$\Theta = \frac{1}{(4\pi)^2}\left[-cW^2 + \frac{a}{4}\mathrm{G}_4 - \frac{2}{3}a'\Box R\right], \tag{1}$$

where $c = \frac{1}{120}(N_s + 6N_f + 12N_v)$, $a = \frac{1}{360}(N_s + 11N_f + 62N_v)$ for free field theories of $N_{s,f,v}$ real scalars, Dirac fermions and vectors, respectively.

In higher even dimension, the number of independent conformal invariants, and consequently the number of central charges c, increases with the dimension; yet there is a unique term related to the stress-tensor two-point function and it has the form $W_{\mu\nu\rho\sigma}\Box^{n/2-2}W^{\mu\nu\rho\sigma}$+cubic terms proportional to W.

In two dimensions [1], on the contrary, the trace anomaly has a unique term, so that we can say that "$c = a = a'$" there. It is natural to expect that there exists a class of higher-dimensional conformal field theories that is most similar to two-dimensional conformal field theory.

The main purpose of this paper is to identify this special class of conformal theories, collecting present knowledge and offering further evidence in favour of our statement. We use the sum rule of refs. [2, 3] for the irreversibility of the renormalization-group flow to derive a quantitative prediction from this idea. We then proceed to check the prediction. This is first done in detail in four, six and eight dimensions and then extended to the general case. In four and six dimensions, there is agreement with statements coming from the AdS/CFT correspondence [4, 5], which our observations make more precise.

We recall that in four dimensions there are situations where the stress-tensor operator product expansion (OPE) closes with a finite number of operators up to the regular terms [6, 7]. It turns out that when $c = a$ the OPE closure is achieved in a way that is reminiscent of two-dimensional conformal field theory, with the stress tensor and the central extension. Instead, when $c \neq a$ this algebraic structure is enlarged and contains spin-1 and spin-0 operators, yet in finite number.

The subclass of theories we are looking for is therefore identified, in four dimensions, by the equality of c and a and the "closed limit" of [7], which is the limit in which all higher-spin currents decouple from the OPEs. The idea of this limit was suggested by a powerful theorem, due to Ferrara, Gatto and Grillo [8] and to Nachtmann [9], on the spectrum of anomalous dimensions of these higher-spin currents, which follows from very general principles (unitarity) and is therefore expected to hold in arbitrary dimension.

Less straightforward is to formulate the generalization of the identification $c = a$ to arbitrary (even) dimension. It is well known that Θ vanishes on Ricci-flat metrics when $c = a$ in four dimensions. A closer inspection of (1) shows that actually Θ is *quadratic* in the Ricci tensor.

We are led to conjecture that the subclass of "$c = a$"-theories are those that have a trace anomaly quadratic in the Ricci tensor.

Therefore, in arbitrary dimension we can distinguish the following important subclasses of conformal field theories:

i) The "closed" theories, when the quantum conformal algebra, i.e. the algebra generated by the singular terms of the stress-tensor OPE, closes with a finite number of operators. They can have $c = a$, but also $c \neq a$.

ii) The $c = a$-theories, whose trace anomaly is quadratic in the Ricci tensor. They can be either closed or open.

iii) The closed $c = a$-theories, which exhibit the highest degree of similarity with two-dimensional conformal field theory.

While the equality $c = a$ is a restriction on the set of conformal field theories, the equality of a and a' is not. In refs. [2, 3] the equality $a = a'$ was studied in arbitrary even dimension n, leading to the sum rule

$$a_n^{\mathrm{UV}} - a_n^{\mathrm{IR}} = \frac{1}{2^{\frac{3n}{2}-1} n\, n!} \int \mathrm{d}^n x\, |x|^n \langle \Theta(x)\, \Theta(0) \rangle, \tag{2}$$

expressing the total renormalization-group (RG) flow of the central charge a_n induced by dimensionless couplings. No restriction or identification of the central charges is required here. The charge a_n is normalized so that the trace anomaly reads

$$\Theta = a_n \mathrm{G}_n = a_n (-1)^{\frac{n}{2}} \varepsilon_{\mu_1 \nu_1 \cdots \mu_{\frac{n}{2}} \nu_{\frac{n}{2}}} \varepsilon^{\alpha_1 \beta_1 \cdots \alpha_{\frac{n}{2}} \beta_{\frac{n}{2}}} \prod_{i=1}^{\frac{n}{2}} R_{\alpha_i \beta_i}^{\mu_i \nu_i}$$

plus conformal invariants and trivial total derivatives.

Direct inspection of the arguments of [2, 3] shows that they do not apply to massive flows, for example free massive scalar fields. (In general, the effect of masses can be included straightforwardly [10].) The sum rule (2) measures the pure effect of the dynamical RG scale μ in lowering the amount of massless degrees of freedom of the theory along the RG flow.

The basic reason why massive flows behave differently is that in a finite theory Duff's identification [11] $a' = c$ is consistent (but not unique), while along a RG flow the only consistent identification is $a' = a$, as shown in [2]. Divergences are crucial in discriminating between the two cases.

Repeating the arguments of [2, 3] in two dimensions, we are led to the same conclusion: that the sum rule (2) works for RG flows and not necessarily for massive ones. The point is, nevertheless, that the two-dimensional version of (2), due to Cardy [12], is universal; in particular, it does work for massive flows. It is therefore compulsory to understand in what cases the domain of validity of our sum rule (2) is similarly enhanced in higher dimensions.

The arguments and explicit checks that we now present show that this enhancement takes place in the subclass of theories with $c = a$ (classes *ii* and *iii* above), because of the higher similarity with the two-dimensional theories.

The two relevant terms of the trace anomaly are

$$\Theta = a_n \mathrm{G}_n - \frac{c_n (n-2)\, (\frac{n}{2})!}{4(4\pi)^{\frac{n}{2}} (n-3)\, (n+1)!} W \Box^{\frac{n}{2}-2} W + \cdots,$$

where

$$c_n = N_s + 2^{\frac{n}{2}-1}(n-1)N_f + \frac{n!}{2\left[(\frac{n}{2}-1)!\right]^2} N_v$$

is the value of the central charge c for free fields, and in arbitrary dimension n, N_v denotes the number of $(n/2-1)$-forms. This calculation is done in ref. [13], section 9, starting from the stress-tensor two-point function.

Massive flows have been considered, among other things, by Cappelli et al. in [14]. An explicit computation for free massive scalar fields and fermions gives [14]

$$\int \mathrm{d}^n x\, |x|^n \, \langle \Theta(x)\, \Theta(0) \rangle = \frac{c_n\, (\frac{n}{2})!}{\pi^{\frac{n}{2}}\, (n+1)}. \tag{3}$$

Repeating the computation for massive vectors, or $(n/2-1)$-forms, is problematic in the UV. However, the relative coefficient between the scalar and fermion contributions is sufficient to show that the result is proportional to c_n and not a_n.

Our prediction is that in the special $c = a$-theories the sum rule (2) should reproduce (3) for massive flows, which means

$$c_n = a_n \frac{2^{\frac{n}{2}-1} (4\pi)^{\frac{n}{2}}\, n\, (n+1)!}{(\frac{n}{2})!}. \tag{4}$$

The trace anomaly therefore has the form

$$\Theta = a_n \left(\mathrm{G}_n - \frac{2^{\frac{n}{2}-3} n(n-2)}{n-3} W \Box^{n/2-2} W \right) + \cdots \tag{5}$$

Formula (4) is the generalized version of the relation $c = a$. It is uniquely implied by the requirement that Θ be quadratic in the Ricci tensor and Ricci curvature. This condition fixes all the central charges of type c in terms of a_n, not only the constant c_n in front of the stress-tensor two-point function. These further relationships are not important for our purposes.

In four dimensions the combination between the parenthesis in (5) is indeed quadratic in the Ricci tensor:

$$\frac{\mathrm{G}_4}{4} - W^2 = -2R_{\mu\nu}^2 + \frac{2}{3} R^2.$$

For RG flows connecting pairs of conformal field theories with $c = a$ formula (2) holds universally. We regard this observation as a demonstration of the importance of this subclass of theories from the theoretical point of view and as a further evidence in favour of the ideas of refs. [2, 3].

In higher dimensions the check is less straightforward, owing to the high number of invariants. Using the results of Bonora et al. from [15], where the terms occurring in the trace anomaly were classified in six dimensions (see also [16]), we can perform the first non-trivial check of our prediction. The conformal invariants are three:

$$I_1 = W_{\mu\nu\rho\sigma} W^{\mu\alpha\beta\sigma} W^{\nu}{}_{\alpha\beta}{}^{\rho}, \qquad I_2 = W_{\mu\nu\rho\sigma} W^{\mu\nu\alpha\beta} W_{\alpha\beta}{}^{\rho\sigma},$$
$$I_3 = W_{\mu\alpha\beta\gamma}\left(\Box\ \delta^{\mu}_{\nu} + 4R^{\mu}_{\nu} - \frac{6}{5}R\ \delta^{\mu}_{\nu}\right) W^{\nu\alpha\beta\gamma},$$

and the general form of the trace anomaly is

$$\Theta = a_6 \mathrm{G}_6 + \sum_{i=1}^{3} c^{(i)} I_i + \text{t.t.d.},$$

where "t.t.d." means "trivial total derivatives" (as opposed to G_6, which is a non-trivial total derivative). Our notation differs from the one of [15] in the signs of $R_{\mu\nu}$ and R. More importantly, the invariant I_3 differs from the invariant M_3 of [15] and other references [16, 17], the latter containing a spurious contribution proportional to G_6 (see also [3], section 3), as well as a linear combination of I_1 and I_2. Precisely, we find

$$M_3 = \frac{5}{12}\mathrm{G}_6 + \frac{80}{3} I_1 + \frac{40}{3} I_2 - 5 I_3.$$

Finally, our I_3 differs from the expression of ref. [18], formula (19), by the addition of t.t.d.'s, which, however, can be consistently omitted for our purposes.

In [15] it is pointed out that there exists a simple combination of the four invariants G_6 and $I_{1,2,3}$, which reads

$$\begin{aligned} \mathcal{J}_6 = & R_{\mu\nu}\Box R^{\mu\nu} - \frac{3}{10} R\Box R - R R_{\mu\nu} R^{\mu\nu} \\ & -2 R_{\mu\nu} R_{\rho\sigma} R^{\mu\rho\sigma\nu} + \frac{3}{25} R^3 \\ = & -\frac{1}{24}\mathrm{G}_6 - 4 I_1 - I_2 + \frac{1}{3} I_3 + \text{t.t.d.} \end{aligned} \qquad (6)$$

The BPB (Bonora–Pasti–Bregola) term $\mathcal{J}_6$ is precisely the combination we are looking for. A closer inspection of this expression shows that it is uniquely fixed by the requirement that it be quadratic in the Ricci tensor and Ricci curvature. On the other hand, the requirement that $\mathcal{J}_6$ just vanishes on Ricci-flat metrics is not sufficient to fix it uniquely, in particular it does not imply the relation "$c = a$" that we need.

In conclusion, the $c = a$-theories have a unique central charge, multiplying the BPB invariant $\mathcal{J}_6$,

$$\Theta = -24\ a_6\ \mathcal{J}_6, \qquad c^{(1)} = 96 a_6, \qquad c^{(2)} = 24 a_6, \qquad c^{(3)} = -8 a_6,$$

so that Θ is of the predicted form (5):

$$\Theta = a_6(\mathrm{G}_6 - 8W\Box W) + \cdots . \tag{7}$$

We now compare these observations with results coming from the AdS/CFT correspondence [4]. Given that the class of closed $c = a$-theories is the one with the highest degree of simplicity, it is conceivable that the conformal field theories admitting the supergravity description envisaged in ref. [4], are precisely those of our class *iii*, at least in four and six dimensions (the AdS/CFT correspondence does not apply to higher dimensions).

Agreement with this picture in four dimensions is well established. Using the results of [5] it is straightforward to show agreement in six dimensions. This can be read from formula (30) of [5], apart from the caveat that in [5] the BPB invariant M_3 is used, which is misleading. The correct decomposition of the anomaly into Euler density and conformal invariants is the last equality of (6), leading directly to (7).

Our prediction, however, is meaningful in arbitrary even dimension. The classification of conformal invariants in arbitrary dimension is involved and the effort that we have done in $n = 6$ suggests that it is preferable to find a clever shortcut leading to the proof of our statement without passing through the detailed analysis of invariants.

A hint for this shortcut comes precisely from the AdS/CFT correspondence, since the arguments of [5] do not appear to be restricted to four and six simensions. The idea is that the correspondence captures a mathematical structure (an algorithm for generating the $\mathcal{J}_n$'s) that is more general than was predicted by the correspondence itself, and that the construction of [5] answers precisely our question.

We begin with $n = 8$. The relevant terms of $\mathcal{J}_8$ are

$$\mathcal{J}_8 = R_{\mu\nu}\Box^2 R^{\mu\nu} - \frac{2}{7}R\Box^2 R + \mathcal{O}(R^3) = \alpha_8\ \mathrm{G}_8 + \text{c.i.t.} + \text{t.t.d.},$$

α_8 being the unknown coefficient and "c.i.t." denoting conformal-invariants. On a sphere, in particular, all terms but $\alpha_8\mathrm{G}_8$ vanish, so that α_8 can be found by evaluating the integral of $\mathcal{J}_8$:

$$\int_{S^8} \sqrt{g}\mathcal{J}_8\ \mathrm{d}^8x = 768\ \alpha_8\ (4\pi)^4.$$

Using

$$W\Box^2 W = \frac{10}{3}\left(R_{\mu\nu}\Box^2 R^{\mu\nu} - \frac{2}{7}R\Box^2 R\right) + \mathcal{O}(R^3) + \text{t.t.d.},$$

our prediction (5) is $\alpha_8 = -1/64$. Indeed, applying the method of [5] on a metric with $R_{\mu\nu} = \Lambda g_{\mu\nu}$, we get, after a non-trivial amount of work,

$$\mathcal{J}_8 = \alpha_8\ \mathrm{G}_8 = -\frac{1440}{343}\Lambda^4,$$

which gives the desired value of α_8.

The check can be generalized for arbitrary n. The invariant $\mathcal{J}_n$ is, up to an overall factor β_n, the coefficient of $\rho^{n/2}$ in the expansion of $\sqrt{\det G}$, where

$$G_{\mu\nu} = g_{\mu\nu} + \sum_{k=1}^{n/2} \rho^k g_{\mu\nu}^{(k)} + \mathcal{O}(\rho^{n/2}\ln\rho, \rho^{n/2+1}, \rho^{n/2+1}\ln\rho, \cdots)$$

and the ρ-dependence is fixed by the equations [5]

$$\begin{aligned} \mathrm{tr}[G^{-1}G''] - \frac{1}{2}\mathrm{tr}[G^{-1}G'G^{-1}G'] &= 0, \\ 2\rho(G'' - G'G^{-1}G') &= (G - \rho G')\mathrm{tr}[G^{-1}G'] \\ &\quad + \mathrm{Ric}(G) + (n-2)G'. \end{aligned} \tag{8}$$

Precisely,

$$\begin{aligned} \frac{1}{(\frac{n}{2})!}\frac{\mathrm{d}^{\frac{n}{2}}}{\mathrm{d}\rho^{\frac{n}{2}}}\frac{\sqrt{\det G}}{\sqrt{\det g}}\bigg|_{\rho=0} &= \beta_n \mathcal{J}_n \\ &= \beta_n(R_{\mu\nu}\Box^{\frac{n}{2}-2}R^{\mu\nu} + \alpha_n \mathrm{G}_n + \mathrm{rest}). \end{aligned}$$

First, we consider metrics with $R_{\mu\nu} = \Lambda g_{\mu\nu}$. The form of the solution and the first equation of (8) read

$$G_{\mu\nu} = u(\rho\Lambda)g_{\mu\nu}, \qquad \frac{u''}{u} = \frac{1}{2}\left(\frac{u'}{u}\right)^2.$$

The second equation of (8) is used to fix the integration constants, with the result

$$u(\rho\Lambda) = \left(1 - \frac{\rho\Lambda}{4(n-1)}\right)^2, \qquad \beta_n\mathcal{J}_n \to \frac{(-1)^{\frac{n}{2}}\, n!\, \Lambda^{\frac{n}{2}}}{2^n(n-1)^{\frac{n}{2}}\left[(\frac{n}{2})!\right]^2}.$$

Then, we fix the normalization β_n by looking for the term $R_{\mu\nu}\Box^{\frac{n}{2}-2}R^{\mu\nu}$ (we can set the Ricci curvature R to zero for simplicity). We write

$$G_{\mu\nu} = g_{\mu\nu} + \frac{1}{\Box}v(\rho\Box)R_{\mu\nu} + R_{\mu\alpha}\frac{1}{\Box^2}y(\rho\Box)R^{\alpha}_{\nu} + O(R^3),$$

with $v(0) = y(0) = y'(0) = 0$. We have

$$\beta_n = \frac{1}{2\,(\frac{n}{2})!}\frac{\mathrm{d}^{\frac{n}{2}}x}{\mathrm{d}t^{\frac{n}{2}}}\bigg|_{t=0}$$

where $t = \rho\Box$ and $x = y - v^2/2$. Integrating $\mathcal{J}_n$ over a sphere, we can convert our prediction (5) to a prediction for β_n or

$$\frac{\mathrm{d}^{\frac{n}{2}}x}{\mathrm{d}t^{\frac{n}{2}}}\bigg|_{t=0} = -\frac{1}{2^{n-1}\Gamma\left(\frac{n}{2}\right)}. \tag{9}$$

Equations (8) relate y, and therefore x, to v and imply that v is a Bessel function of the second type:

$$x'' = -\frac{(v')^2}{2}, \qquad 2tv'' - 1 + \frac{v}{2} - (n-2)v' = 0. \tag{10}$$

β_n is a coefficient in the series expansion of the square of a Bessel function of the second type, and is not usually in the mathematical tables. Solving (10) recursively with the help of a calculator, we have checked agreement between (9) and (10) up to dimension 1000.

Our picture and the quantitative agreement with prediction (5) explain, among other things, the physical meaning of the mathematical construction of Henningson and Skenderis [5] and its extension to arbitrary dimension. Furthermore, the mathematical properties of the invariant $\mathcal{J}_n$, and therefore the identification of c and a (in the subclasses of theories *ii* and *iii* where it applies), are a nice counterpart of the notion of extended (*pondered*) Euler density introduced in [3], which explained the identification $a = a'$.

The results presented in this paper are a further, unexpected, check of the ideas of [2, 3] and of the picture offered there. These are, we believe, the first steps towards the classification of all conformal field theories. In our investigation, higher-dimensional conformal field theory has mostly been used as a laboratory to better establish ideas on four-dimensional quantum field theory. Above four dimensions, treatable conformal field theories are the higher-derivative ones, which admit RG deformations and should presumably be formulated in the spirit of ref. [19], where it seems that unitarity problems do not arise.

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