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Radiative η' Decays, the Topological Susceptibility and the Witten–Veneziano Mass Formula

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The formulae describing the radiative decays $\eta'(\eta) \to \gamma\gamma$ in QCD beyond the chiral limit are derived. The modifications of the conventional PCAC formulae due to the gluonic contribution to the axial anomaly in the flavour singlet channel are precisely described. The decay constants are found to satisfy a modified Dashen formula which generalises the Witten–Veneziano formula for the mass of the η'. Combining these results, it is shown how the topological susceptibility in QCD with massive, dynamical quarks may be extracted from measurements of $\eta'(\eta) \to \gamma\gamma$.

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1. Introduction and Results

The radiative decay $\eta' \to \gamma\gamma$ is of special interest in particle theory since it involves the gluonic as well as electromagnetic contribution to the axial anomaly. In this paper, we show how this decay may be used to measure the topological susceptibility $\chi(0)$ in QCD with dynamical quarks and thus open a window on the topology of the gluon field.

The flavour non-singlet decay $\pi^0 \to \gamma\gamma$ is of course well understood and has played an important role in the development of the standard model, providing early evidence for the existence of colour. Since the pion is a (pseudo) Goldstone boson for the spontaneously broken chiral symmetry of QCD, the decay is easily calculated from the electromagnetic contribution to the anomaly in the corresponding axial current.

However, the theory underlying the flavour singlet decay is less developed and indeed the special new features arising from the gluon contribution to the divergence of the $U_A(1)$ axial current are usually still ignored in phenomenological analyses. In a previous paper[1], we presented an analysis of the theory of $\eta' \to \gamma\gamma$ decay in the chiral limit of QCD, taking into account the gluonic anomaly and the associated anomalous scaling implied by the renormalisation group. Here, we extend that analysis to QCD with massive quarks, incorporating $\eta - \eta'$ mixing. In particular, we show how a combination of the radiative decay formula and a generalisation of the Witten–Veneziano mass formula[2,3] for the η' can be used, under reasonable assumptions, to measure the gluon topological susceptibility $\chi(0)$ in full QCD with massive quarks.

Our main result is summarised in the following two formulae:

$$f^{a\alpha}\ g_{\eta^\alpha\gamma\gamma}\ +\ 2n_f A\ g_{G\gamma\gamma}\ \delta_{a0}\ =\ a^a_{\rm em}\frac{\alpha}{\pi} \tag{1.1}$$

which describes the radiative decays, and

$$f^{a\alpha}(m^2)_{\alpha\beta}f^{T\beta b} = -2d_{abc}{\rm tr}\ T^c \begin{pmatrix} m_u\langle\bar{u}u\rangle & 0 & 0 \\ 0 & m_d\langle\bar{d}d\rangle & 0 \\ 0 & 0 & m_s\langle\bar{s}s\rangle \end{pmatrix} + (2n_f)^2 A\ \delta_{a0}\delta_{b0} \tag{1.2}$$

which defines the decay constants appearing in (1.1) through a modification of Dashen's formula to include the gluon contribution to the $U_A(1)$ anomaly.

In these formulae, η^α denotes the neutral pseudoscalars π^0, η, η'. The (diagonal) mass matrix is $(m^2)_{\alpha\beta}$ and $g_{\eta^\alpha\gamma\gamma}$ is the appropriate coupling, defined as usual from the decay amplitude by

$$\langle\gamma\gamma|\eta^\alpha\rangle\ =\ -i\ g_{\eta^\alpha\gamma\gamma}\ \epsilon_{\lambda\rho\alpha\beta}p_1^\alpha p_2^\beta\epsilon^\lambda(p_1)\epsilon^\rho(p_2) \tag{1.3}$$

in obvious notation. The constant $a^a_{\rm em}$ is the coefficient of the electromagnetic contribution to the axial current anomaly:

$$\partial^\mu J^a_{\mu5}\ =\ d_{acb}m^c\bar{q}\gamma_5 T^b q\ +\ 2n_f\delta_{a0}\frac{\alpha_s}{8\pi}{\rm tr}G^{\mu\nu}\tilde{G}_{\mu\nu}\ +\ a^a_{\rm em}\frac{\alpha}{8\pi}F^{\mu\nu}\tilde{F}_{\mu\nu} \tag{1.4}$$

where $a = 0, 3, 8$ is the flavour index and the d-symbols are defined from the anticommutation relations of the generators, $\{T^a, T^b\} = d_{abc}T^c$. $G_{\mu\nu}$ and $F_{\mu\nu}$ are the gluon and photon field strengths respectively. (More precise definitions and further notation are given in sect.2.)

The decay constants $f^{a\alpha}$ in (1.1) are defined by the relation (1.2) . Notice immediately that the decay constants which enter the formula (1.1) are *not* defined by the coupling of the pseudoscalar mesons π^0, η, η' to the axial current[1]. In the flavour singlet sector, such a definition would give a RG non-invariant decay constant which would not coincide with the quantities arising in the correct decay formula (1.1) . In practice, since flavour $SU(2)$ symmetry is almost exact, the relations for π^0 decouple and are simply the standard ones with $f^{3\pi}$ identified as f_π, viz.

$$f_\pi g_{\pi\gamma\gamma} = \frac{N_c}{3}\frac{\alpha_{\rm em}}{\pi} \tag{1.5}$$

together with the Dashen formula

$$f_\pi^2 m_\pi^2 = -(m_u\langle\bar{u}u\rangle + m_d\langle\bar{d}d\rangle) \tag{1.6}$$

In the octet-singlet sector, however, there is mixing and the decay constants form a 2×2 matrix:

$$f^{a\alpha} = \begin{pmatrix} f^{0\eta'} & f^{0\eta} \\ f^{8\eta'} & f^{8\eta} \end{pmatrix} \tag{1.7}$$

The four components are independent. In particular, for broken $SU(3)$, there is no reason to express $f^{a\alpha}$ as a diagonal matrix times an orthogonal $\eta - \eta'$ mixing matrix, which would give just three parameters. Several convenient parametrisations may be made, e.g. involving two constants and two mixing angles[4,5,6], but this does not seem to reflect any special dynamics.

The novelty of our results of course lies in the extra terms arising in (1.1) and (1.2) due to the gluonic contribution to the $U_A(1)$ anomaly. The coefficient A is the non-perturbative number which specifies the topological susceptibility in full QCD with massive dynamical quarks. Defining the topological susceptibility as

$$\chi(0) = \int d^4x\ i\langle 0|T\ Q(x)\ Q(0)|0\rangle \tag{1.8}$$

where $Q = \frac{\alpha_s}{8\pi}{\rm tr}G^{\mu\nu}\tilde{G}_{\mu\nu}$ is the gluon topological charge, the anomalous chiral Ward identities determine its dependence on the quark masses and condensates up to an undetermined parameter, viz.[7]

$$\chi(0) = \frac{-A\ m_u m_d m_s\langle\bar{u}u\rangle\langle\bar{d}d\rangle\langle\bar{s}s\rangle}{m_u m_d m_s\langle\bar{u}u\rangle\langle\bar{d}d\rangle\langle\bar{s}s\rangle - A\big(m_u m_d\langle\bar{u}u\rangle\langle\bar{d}d\rangle + m_u m_s\langle\bar{u}u\rangle\langle\bar{s}s\rangle + m_d m_s\langle\bar{d}d\rangle\langle\bar{s}s\rangle\big)} \tag{1.9}$$

or in more compact notation,

$$\chi(0) = -A\left(1 - A\sum_q \frac{1}{m_q\langle\bar{q}q\rangle}\right)^{-1} \tag{1.10}$$

Notice how this satisfies the well-known result that $\chi(0)$ vanishes if any quark mass is set to zero.

The modified Dashen formula is in fact a generalisation of the Witten–Veneziano mass formula[2,3] for the η'. Here, however, we do *not* impose the leading order in $1/N_c$ approximation that produces the Witten–Veneziano formula. Recall that, for $n_f = 3$ and non-zero quark masses, this states[3]

$$m_{\eta'}^2 + m_\eta^2 - 2m_K^2 = -\frac{6}{f_\pi^2}\chi(0)\big|_{\rm YM} \tag{1.11}$$

To recover (1.11) from our result (see the first of eqs(1.13)) the condensate $m_s\langle\bar{s}s\rangle$ is replaced by the term proportional to $f_\pi^2 m_K^2$ using a standard Dashen equation, and the singlet decay constants are set to $\sqrt{2n_f}f_\pi$. The identification of the large N_c limit of the coefficient A with the non-zero topological susceptibility of pure Yang-Mills theory may be seen in different ways, either from the large N_c counting rules quoted below or perhaps most simply in the effective Lagrangian analysis explained in section 3.

The final element in (1.1) is the extra 'coupling' $g_{G\gamma\gamma}$ in the flavour singlet decay formula, which arises because even in the chiral limit the η' is not a Goldstone boson because of the gluonic $U_A(1)$ anomaly. A priori, this is *not* a physical coupling, although (suitably normalised) it could be modelled as the coupling of the lightest predominantly glueball state mixing with η'. However, this interpretation would probably stretch the basic dynamical assumptions[1] underlying (1.1) too far, and is not necessary either in deriving or interpreting the formula. In fact, the $g_{G\gamma\gamma}$ term arises simply because in addition to the electromagnetic anomaly the divergence of the axial current contains both the quark bilinear operators $\phi_5^a = \bar{q}\gamma_5 T^a q$ and the gluonic anomaly Q. Diagonalising the propagator

[1] The (standard) dynamical assumptions made in deriving (1.1) are described more completely in the following sections. Essentially, (1.1) is based on the *zero-momentum* anomalous chiral Ward identities. It is then assumed that the decay 'constants' $f^{a\alpha}(k^2)$ and couplings $g_{\eta^\alpha\gamma\gamma}(k^2)$ are approximately constant functions of momentum in the range from zero (where the Ward identities are applied) to the relevant physical mass. Notice that this is only applied to pole-free quantities which depend only implicitly on the quark masses. This is of course simply the standard PCAC or chiral Lagrangian assumption, and corresponds to assuming pole-dominance of the propagators for the appropriate operators by the pseudo Goldstone bosons. It becomes exact in the chiral limit.

matrix for these operators isolates the η and η' poles, whose couplings to $\gamma\gamma$ give the usual terms $g_{\eta\gamma\gamma}$ and $g_{\eta'\gamma\gamma}$. However, the remaining operator (which we call G – see following sections) also couples to $\gamma\gamma$ and therefore also contributes to the decay formula, whether or not we assume that its propagator is dominated by a 'glueball' pole.

Of course, the presence of the in general unmeasurable coupling $g_{G\gamma\gamma}$ in (1.1) appears to remove any predictivity from the $\eta' \to \gamma\gamma$ decay formula. In a strict sense this is true, but we shall argue below that it may nevertheless be a good dynamical approximation to assume $g_{G\gamma\gamma}$ is small compared to $g_{\eta'\gamma\gamma}$. In this case, we can combine eqs.(1.1) and (1.2) to give a measurement of the non-perturbative coefficient A in $\chi(0)$. Assume flavour $SU(2)$ is exact so that eqs.(1.6) and (1.7) for the pion decouple. The remaining parts of (1.1) and (1.2) then together provide five equations (since (1.2) is symmetric), in which we assume the physical quantities $m_\eta, m_{\eta'}, g_{\eta\gamma\gamma}$ and $g_{\eta'\gamma\gamma}$ are all known and we neglect $g_{G\gamma\gamma}$. Four of these equations may be used to determine the four decay constants $f^{a\alpha}$. The final equation is the flavour singlet Dashen formula, which may then be solved for A. This is the generalisation of the Witten–Veneziano formula.

Without neglecting $g_{G\gamma\gamma}$, the five equations give a self-consistent description of the radiative decays, but are non-predictive. It is therefore important to analyse more carefully whether it is really legitimate to neglect $g_{G\gamma\gamma}$.[2] The argument is based on the fact that $g_{G\gamma\gamma}$ is both OZI suppressed *and* renormalisation group (RG) invariant[1]. Since violations of the OZI rule[3] are associated with the $U_A(1)$ anomaly, it is a plausible conjecture that we can identify OZI-violating quantities by their dependence on the anomalous dimension associated with the non-trivial renormalisation of $J^0_{\mu 5}$ due to the anomaly. In this way, RG non-invariance can be used as a flag to indicate those quantities expected to show large OZI violations. If this conjecture is correct, then we would expect the OZI rule to be reasonably good for the RG invariant $g_{G\gamma\gamma}$, which would therefore be suppressed relative to $g_{\eta'\gamma\gamma}$.[4] (An important exception is of course the η' mass itself, which although obviously RG invariant is not zero in the chiral limit as it would be in the OZI limit

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[2] Notice that this approximation is implicitly made in almost all the phenomenological analyses of radiative pseudoscalar decays, which use a formula like (1.1) but omitting the $g_{G\gamma\gamma}$ term and (mistakenly) implying that the decay constants are defined as the couplings of the axial current to η and η'.

[3] The OZI approximation to QCD may be given a precise definition [8] as the truncation of full QCD in which non-planar and quark-loop diagrams are retained, but the diagrams which give purely gluonic intermediate states (those in which the external currents are attached to different quark loops) are omitted. This is a closer approximation to QCD than, for example, the leading large N_c limit.

[4] On the other hand, we would *not* expect the RG non-invariant and anomaly sensitive 'decay constant' $\hat{f}^{0\eta'}$ defined by $\langle 0|J^0_{\mu 5}|\eta'\rangle = ik_\mu \hat{f}^{0\eta'}$ to be well-approximated by its OZI value.

of QCD.) Notice that this conjecture has been applied already with some success to the 'proton spin' problem in polarised deep inelastic scattering[9,10,11].

It is also interesting to look at eqs.(1.1) and (1.2) from the point of view of the large N_c expansion. The large N_c counting for the various quantities involved is as follows: $f^{a\alpha} = O(\sqrt{N_c})$, $g_{\eta^\alpha\gamma\gamma} = O(\sqrt{N_c})$, $g_{G\gamma\gamma} = O(1)$, $m^2_{\eta'} = O(1/N_c)$, $m^2_\eta = O(1)$, $\langle\bar{q}q\rangle = O(N_c)$, $a^a_{\rm em} = O(N_c)$ and $A = O(1)$. Notice first that this implies $\chi(0) \simeq A$ in the large N_c limit, as already used above to derive the Witten–Veneziano formula. It also follows that the term in the decay formula involving $Ag_{G\gamma\gamma}$ is suppressed by $O(1/N_c)$ relative to the $f^{a\alpha}g_{\eta^a\gamma\gamma}$ term. If large N_c is reliable in this case, we would therefore indeed expect this contribution to be suppressed. However, the large N_c limit must be used with great caution in the $U_A(1)$ channel. For example, the same argument would equally imply that the additional term A in the flavour singlet Dashen formula is suppressed by $O(1/N_c)$, yet we know that although formally of $O(1/N_c)$, $m^2_{\eta'}$ is not small phenomenologically.

To make the phenomenological application of our results (1.1) and (1.2) quite clear, we now write out the five equations in the $\eta - \eta'$ sector explicitly. Set $n_f = 3$ and take $m_u = m_d = 0$ for simplicity. The decay equations are:

$$
\begin{aligned}
f^{0\eta'} g_{\eta'\gamma\gamma} + f^{0\eta} g_{\eta\gamma\gamma} + 6Ag_{G\gamma\gamma} &= a^0_{\rm em}\frac{\alpha}{\pi} \\
f^{8\eta} g_{\eta\gamma\gamma} + f^{8\eta'} g_{\eta'\gamma\gamma} &= a^8_{\rm em}\frac{\alpha}{\pi}
\end{aligned}
\tag{1.12}
$$

where $a^0_{\rm em} = \frac{4}{3}N_c$ and $a^8_{\rm em} = \frac{1}{3\sqrt{3}}N_c$, and the Dashen equations are:

$$
\begin{aligned}
\left(f^{0\eta'}\right)^2 m^2_{\eta'} + \left(f^{0\eta}\right)^2 m^2_\eta &= -4m_s\langle\bar{s}s\rangle + 36A \\
f^{0\eta'} f^{8\eta'} m^2_{\eta'} + f^{0\eta} f^{8\eta} m^2_\eta &= \frac{4}{\sqrt{3}} m_s\langle\bar{s}s\rangle \\
\left(f^{8\eta}\right)^2 m^2_\eta + \left(f^{8\eta'}\right)^2 m^2_{\eta'} &= -\frac{4}{3} m_s\langle\bar{s}s\rangle
\end{aligned}
\tag{1.13}
$$

Clearly, the two purely octet formulae can be used to find $f^{8\eta}$ and $f^{8\eta'}$ if both $g_{\eta\gamma\gamma}$ and $g_{\eta'\gamma\gamma}$ are known. The off-diagonal Dashen formula then expresses $f^{0\eta}$ in terms of $f^{0\eta'}$. This leaves the two purely singlet formulae involving the still-undetermined decay constant $f^{0\eta'}$, the topological susceptibility coefficient A, and the coupling $g_{G\gamma\gamma}$. The advertised result follows immediately. If we neglect $g_{G\gamma\gamma}$, we can find $f^{0\eta'}$ from the singlet decay formula and thus determine A from the remaining, generalised Witten–Veneziano, formula.

On the other hand, we may regard A as a number to be predicted by non-perturbative theoretical calculations. In that case, the Dashen formula determines the decay constant $f^{0\eta'}$ in terms of A, in which case everything is known in the singlet decay formula except the coupling $g_{G\gamma\gamma}$, which is therefore predicted. Unfortunately, it is not at all clear how this could be compared to an experimental measurement without invoking 'glueball dominance' in the $\langle G\ G\rangle$ propagator and assuming that this dominates over other neglected pseudoscalar poles in the propagator matrix (see section 3).

Finally, we should discuss briefly whether it is possible to determine any of the quantities $f^{a\alpha}$ or A by non-perturbative calculations, either using QCD spectral sum rules or lattice gauge theory. The decay constant definition (1.2) has an alternative (prior) form in terms of the propagators for the pseudoscalar quark bilinears ϕ_5^a. In fact (for notation and the derivation see sections 2 and 3), we have:

$$f^{a\alpha}(m^2)_{\alpha\beta}f^{T\beta b} \;=\; d_{ace}\langle\phi^e\rangle\ \langle\phi_5\ \phi_5\rangle^{-1}_{cd}\ d_{dbf}\langle\phi^f\rangle \tag{1.14}$$

where on the r.h.s. $\langle\phi_5\ \phi_5\rangle^{-1}_{cd}$ denotes the (cd) component of the inverse of the matrix of two-point functions of the pseudoscalars, taken at zero momentum, and $\langle\phi^a\rangle$ are the usual quark condensates. Comparing with (1.2) shows that a successful calculation of the r.h.s. would imply a determination of the coefficient A governing the topological susceptibility. Although this looks relatively straightforward, perhaps unsurprisingly it turns out to be a very delicate calculation indeed in the QCD spectral sum rule approach,[5] primarily because the effects of gluons and the anomaly have to make an important contribution to what are in first approximation purely quark bilinear propagators. This may nevertheless be an interesting problem to pursue or to study on the lattice.

The other possibility is to determine A by a direct calculation of the topological susceptibility $\chi(0)$ in full QCD with dynamical, massive quarks[6]. Because of the intricate dependence on all the quark masses (including the light quarks) this does not seem feasible using spectral sum rules. The situation may be better on the lattice, however, if the possibility to change quark masses is exploited. Perhaps A could be extracted from a calculation of $\chi(0)$ in the limit of equal, but not too light, quark masses.

A general point highlighted once again by this is that it would be extremely useful to discover a calculational method that yields 1PI vertices directly rather than deducing them by amputation of the related Green functions. For example, as we show in section 2, the coefficent A in the topological susceptibility is in fact just given by the 'two-point vertex' functional Γ_{QQ}. These '1PI' functionals are smooth and free of all the singularities and

[5] I would like to thank Stephan Narison for collaborating on a preliminary investigation of this problem.

[6] See refs.[12,13] for reviews of progress in calculating the topological susceptibility in full QCD on the lattice. Ref.[14] contains results and references relevant to the spectral sum rule approach. (See also [10,11].)

delicate mass dependence associated with external propogator poles. They are therefore the essential non-perturbative quantities we need to find.

It would also be interesting to make a detailed comparison of the formulae presented here with the corresponding results in the recently constructed chiral Lagrangians in which the η' is incorporated in the framework of the $1/N_c$ expansion[4,15]. Although the formalisms look rather different, mainly because of the non-linear realisations used in the chiral Lagrangian approach and the explicit reliance on the $1/N_c$ expansion, the essential dynamical input and assumptions are the same and the final physical predictions should agree within the limitations of the approximations.

The rest of this paper is devoted to the derivation of the above results by different methods. In section 2 we establish some convenient notation and review the basic anomalous chiral Ward identities which are the basis of all the subsequent work. Section 3 gives a derivation of (1.1) and (1.2) using the method of 1PI vertex functionals used to analyse $\eta' \to \gamma\gamma$ in the chiral limit in refs[1]. Then in section 4 we write an effective Lagrangian (extending refs.[7,16]) which incorporates all the constraints of the zero-momentum chiral Ward identities and includes the coupling to electromagnetism. In section 5 we give a slightly simplified derivation which follows as closely as possible the traditional PCAC methods, generalised as necessary to take account of the gluonic $U_A(1)$ anomaly. This section is intended to be as self-contained as possible, and readers interested simply in the phenomenology may prefer to go directly to section 5.

Finally, the methods of this paper may equally be applied to other decays involving the η' [17,6,18]. Of special interest are, for example, $\eta' \to V\gamma$ where V is a light 1^- meson such as ρ, which is related to $\eta' \to \gamma\gamma$ by vector meson dominance; $\eta' \to \pi\pi\gamma$, which is determined by the box anomaly for one vector and three axial currents; and $\psi \to \eta'\gamma$ [19]. Some interesting current experimental studies of η' physics include, for example, $B \to \eta' K$ by the CLEO collaboration[20]; $\psi_{2S} \to \eta'\gamma$ by the BES collaboration at BEPC[21]; and η' photoproduction at ELSA[22] and CEBAF[23].

2. Chiral Ward Identities

The anomalous chiral Ward identities for QCD with massive quarks have been written down in the form used here in ref.[11] and reviewed in [24]. We refer to these papers for more complete derivations and in this section simply define our notation and quote the essential identities. In this section, we omit the electromagnetic contributions.

The composite operators involved in the Green functions and 1PI vertices studied here are the currents and pseudoscalar operators $J^a_{\mu 5}$, Q, ϕ^a_5 and the scalar ϕ^a where

$$\begin{aligned} J^a_{\mu 5B} &= \bar{q}\gamma_\mu\gamma_5 T^a q \qquad & Q_B &= \frac{\alpha_s}{8\pi}\mathrm{tr} G_{\mu\nu}\tilde{G}^{\mu\nu} \\ \phi^a_{5B} &= \bar{q}\gamma_5 T^a q \qquad & \phi^a_B &= \bar{q}T^a q \end{aligned} \tag{2.1}$$

$G_{\mu\nu}$ is the gluon field strength. In this notation, $T^i = \frac{1}{2}\lambda^i$ are flavour $SU(n_f)$ generators, and we include the singlet $U_A(1)$ generator $T^0 = \mathbf{1}$ and let the index $a = 0, i$. We only need to consider fields where i corresponds to a generator in the Cartan sub-algebra, so that $a = 0, 3, 8$ for $n_f = 3$ quark flavours. d-symbols are defined by $\{T^a, T^b\} = d_{abc}T^c$. Since this includes the flavour singlet $U_A(1)$ generator, they are only symmetric on the first two indices. For $n_f = 3$, the explicit values are $d_{000} = d_{033} = d_{088} = 2, d_{330} = d_{880} = 1/3, d_{338} = d_{383} = -d_{888} = 1/\sqrt{3}$.

We also use the following compact notation. The quark mass matrix is written as $m^a T^a$, so that for $n_f = 3$,

$$\begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix} = m^0\mathbf{1} + m^3 T^3 + m^8 T^8 \tag{2.2}$$

In the same way, the chiral symmetry breaking condensates may be written as

$$\begin{pmatrix} \langle\bar{u}u\rangle & 0 & 0 \\ 0 & \langle\bar{d}d\rangle & 0 \\ 0 & 0 & \langle\bar{s}s\rangle \end{pmatrix} = \frac{1}{3}\langle\phi^0\rangle\mathbf{1} + 2\langle\phi^3\rangle T^3 + 2\langle\phi^8\rangle T^8 \tag{2.3}$$

where $\langle\phi^c\rangle$ is the VEV $\langle\bar{q}T^c q\rangle$. We then define

$$M_{ab} = d_{acb}m^c \qquad\qquad \Phi_{ab} = d_{abc}\langle\phi^c\rangle \tag{2.4}$$

Eq (2.1) defines the bare operators. The renormalised composite operators are defined as follows:

$$\begin{aligned} J^0_{\mu 5} &= Z J^0_{\mu 5B} \qquad & J^{a\neq 0}_{\mu 5} &= J^{a\neq 0}_{\mu 5B} \\ Q &= Q_B - \frac{1}{2n_f}(1-Z)\partial^\mu J^0_{\mu 5B} & & \\ \phi^a_5 &= Z_\phi \phi^a_{5B} \qquad & \phi^a &= Z_\phi \phi^a_B \end{aligned} \tag{2.5}$$

where Z_ϕ is the inverse of the mass renormalisation, $Z_\phi = Z_m^{-1}$. The non-trivial renormalisation of $J^0_{\mu 5}$ means that its matrix elements scale with an anomalous dimension γ related to Z. This occurs because $J^0_{\mu 5}$ is not a conserved current, due to the anomaly Q. Notice also the mixing of the operator Q with $\partial^\mu J^0_{\mu 5}$ under renormalisation.

The Green functions for these operators are constructed by functional differentiation from the generating functional[24] $W[V_{\mu 5}^a, \theta, S_5^a, S^a]$, where $V_{\mu 5}^a, \theta, S_5^a, S^a$ are the sources for the composite operators $J_{\mu 5}^a, Q, \phi_5^a, \phi^a$ respectively. For example, the Green function $i\langle 0|T\ Q(x)\ Q(y)|0\rangle$ is given by $\frac{\delta^2 W}{\delta\theta(x)\delta\theta(y)}$, which we abbreviate as $W_{\theta\theta}$. In this notation, the anomalous zero-momentum chiral Ward identities are:

$$\begin{aligned} 2n_f\delta_{a0}W_{\theta\theta} + M_{ac}W_{S_5^c\theta} &= 0 \\ 2n_f\delta_{a0}W_{\theta S_5^b} + M_{ac}W_{S_5^c S_5^b} + \Phi_{ab} &= 0 \end{aligned} \tag{2.6}$$

which implies the following identity for the topological susceptibility,

$$(2n_f)^2\chi(0) = M_{0c}W_{S_5^c S_5^d}M_{d0} + (M\Phi)_{00} \tag{2.7}$$

These are derived from the fundamental anomalous Ward identity

$$\partial_\mu W_{V_{\mu 5}^a} - 2n_f\delta_{a0}W_\theta - M_{ac}W_{S_5^c} + d_{adc}S^d W_{S_5^c} - d_{adc}S_5^d W_{S^c} = 0 \tag{2.8}$$

which is the precise expression in the functional formalism of the identity (1.4) .

The 1PI vertices used in section 3 are defined as functional derivatives of a second generating functional (effective action) Γ, constructed from W by a partial Legendre transform with respect to the fields Q, ϕ_5^a and ϕ^a only (*not* the currents $J_{\mu 5}^a$)[24]. The resulting vertices are '1PI' w.r.t. the propagators for these composite operators only. This separates off the particle poles in these propagators, and gives the closest identification of the field theoretic vertices with the physical couplings such as $g_{\eta^\alpha\gamma\gamma}$.

The basic anomalous chiral Ward identity for Γ follows immediately from (2.8) for W:

$$\partial_\mu\Gamma_{V_{\mu 5}^a} - 2n_f\delta_{a0}Q - M_{ac}\phi_5^c + d_{acd}\phi^d\Gamma_{\phi_5^c} - d_{acd}\phi_5^d\Gamma_{\phi^c} = 0 \tag{2.9}$$

and other identities follow simply by functional differentiation. In particular, for the two-point vertices, we find the following zero-momentum identities (analogous to (2.6)):

$$\begin{aligned} \Phi_{ac}\Gamma_{\phi_5^c Q} - 2n_f\delta_{a0} &= 0 \\ \Phi_{ac}\Gamma_{\phi_5^c\phi_5^b} - M_{ab} &= 0 \end{aligned} \tag{2.10}$$

which together imply

$$\Phi_{ac}\Gamma_{\phi_5^c\phi_5^d}\Phi_{db} = -(M\Phi)_{ab} \tag{2.11}$$

These will be useful in section 4.

The fact that the topological susceptibility is zero for vanishing quark mass can be seen immediately from (2.7) . One of the simplest ways to derive the precise form (1.9) or

(1.10) is in fact to use an identity involving Γ. As is well-known, the two-point vertices are simply the inverse of the two-point Green functions (propagators), so in the pseudoscalar sector we have the following matrix inversion formula:

$$\begin{aligned}\Gamma_{QQ} &= -\Big(W_{\theta\theta} - W_{\theta S_5^a}(W_{S_5S_5})^{-1}_{ab}W_{S_5^b\theta}\Big)^{-1} \\ &= -\Big(W_{\theta\theta} - W_{\theta S_5^a}M_{ac}\big(MW_{S_5S_5}M\big)^{-1}_{cd}M_{db}W_{S_5^b\theta}\Big)^{-1}\end{aligned} \tag{2.12}$$

and using the identities (2.6) and (2.7) this implies

$$\Gamma_{QQ}^{-1} = -\chi\Big(1 - (2n_f)^2\chi(M\Phi)^{-1}_{00}\Big)^{-1} \tag{2.13}$$

all at zero momentum. Inverting this relation gives the result for the topological susceptibility:

$$\chi = -\Gamma_{QQ}^{-1}\Big(1 - (2n_f)^2\Gamma_{QQ}^{-1}(M\Phi)^{-1}_{00}\Big)^{-1} \tag{2.14}$$

Substituting the explicit expression for $(M\Phi)^{-1}_{00}$ (which is easily found from the definitions above), viz.

$$(M\Phi)^{-1}_{00} = \frac{1}{(2n_f)^2}\sum_q \frac{1}{m_q\langle\bar{q}q\rangle} \tag{2.15}$$

we see that (2.14) reproduces the general form (1.10) where we can now identify the (mass-independent) non-perturbative coefficient as

$$A = \Gamma_{QQ}^{-1} \tag{2.16}$$

3. $\eta' \to \gamma\gamma$ from 1PI Vertices

In this section, we present the most theoretically complete derivation of the decay formula (1.1) and generalised Dashen formula (1.2) . This follows the derivation previously given in ref.[1] for the chiral limit, extended to include quark masses. The technique relies on the identification of the couplings $g_{\eta^\alpha\gamma\gamma}$ with the zero-momentum limit of certain 1PI vertex functions, precisely defined using the Legendre transform Γ introduced in section 2. These techniques have also been used in our series of papers on the $U_A(1)$ Goldberger–Treiman relation and the 'proton spin' [9,10,11]. See also [25,26] for reviews.

The starting point is the Ward identity (2.9) extended to include the electromagnetic contribution to the anomaly for the axial current:

$$\partial_\mu\Gamma_{V^a_{\mu5}} - 2n_f\delta_{a0}Q - a^a_{\rm em}Q_{\rm em}(A) - M_{ac}\phi^c_5 + d_{acd}\phi^d\Gamma_{\phi^c_5} - d_{acd}\phi^d_5\Gamma_{\phi^c} = 0 \tag{3.1}$$

where $Q_{\rm em}(A)$ is shorthand notation for $\frac{\alpha}{8\pi}F_{\mu\nu}\tilde{F}^{\mu\nu}$, where $F_{\mu\nu}$ is the field strength for the electromagnetic field A_μ. (Since we are working only to leading order in α, it is not necessary to consider $Q_{\rm em}$ as an independent composite operator with non-trivial renormalisation.)

Differentiating twice w.r.t. the field A_μ, evaluating at the VEVs, and taking the Fourier transform, we find

$$ik_\mu \Gamma_{V^a_{\mu 5}A^\lambda A^\rho} + a^a_{\rm em}\frac{\alpha}{\pi}\epsilon_{\lambda\sigma\alpha\beta}p_1^\alpha p_2^\beta + d_{abc}\phi^c\Gamma_{\phi^b_5 A^\lambda A^\rho} = 0 \tag{3.2}$$

where p_1, p_2 are the momenta of the photons. To simplify notation, it will be convenient from now on to define vertices $\hat{\Gamma}$ with the kinematical factors removed, in particular $\Gamma_{\phi^a_5 A^\lambda A^\rho} = -\hat{\Gamma}_{\phi^a_5 A^\lambda A^\rho}\epsilon_{\lambda\sigma\alpha\beta}p_1^\alpha p_2^\beta$. Notice that the mass term in (3.1) does not contribute explicitly to this formula. From its definition as 1PI w.r.t. the pseudoscalar fields, the vertex $\Gamma_{V^a_{\mu 5}A^\lambda A^\rho}$ has no pole at $k^2 = 0$ (even in the chiral limit) so the first term vanishes at zero momentum k, leaving simply

$$\Phi_{ab}\hat{\Gamma}_{\phi^b_5 A^\lambda A^\rho}\Big|_{k=0} = a^a_{\rm em}\frac{\alpha}{\pi} \tag{3.3}$$

The first step in converting (3.3) to the decay formula (1.1) is to identify the physical states η^α. These appear as poles in the propagator matrix for the four pseudoscalar operators Q, ϕ^a_5 $(a = 0, 3, 8)$. To isolate these poles, we diagonalise the propagator matrix in this sector then normalise the three operators coupling to the physical states.

We therefore define the operator

$$G = Q - W_{\theta S^a_5}(W_{S_5 S_5})^{-1}_{ab}\phi^b_5 \tag{3.4}$$

so that by construction the propagators $\langle G\ \phi^a_5\rangle$ all vanish. (Notice that integrations over repeated spacetime arguments are implied in this condensed notation.) Then define operators

$$\eta^\alpha = C^{\alpha b}\phi^b_5 \tag{3.5}$$

such that the propagator matrix

$$\langle\eta^\alpha\ \eta^\beta\rangle \equiv W_{S^\alpha_5 S^\beta_5} = C^{\alpha a}W_{S^a_5 S^b_5}C^{Tb\beta} = \begin{pmatrix} \frac{-1}{k^2-m^2_{\eta'}} & 0 & 0 \\ 0 & \frac{-1}{k^2-m^2_\eta} & 0 \\ 0 & 0 & \frac{-1}{k^2-m^2_\pi} \end{pmatrix} \tag{3.6}$$

where S^α_5 are the sources for the operators η^α.

This change of variable affects the partial functional derivatives in $\hat{\Gamma}_{\phi_5^a A^\lambda A^\rho}$ in (3.3) , which involves $\frac{\delta}{\delta\phi_5^a}$ at fixed Q. In terms of the new variables G, η^α we have

$$\begin{aligned} \frac{\delta}{\delta\phi_5^a}\bigg|_Q &= \frac{\delta\eta^\alpha}{\delta\phi_5^a}\frac{\delta}{\delta\eta^\alpha} + \frac{\delta G}{\delta\phi_5^a}\frac{\delta}{\delta G} \\ &= C^{Ta\alpha}\frac{\delta}{\delta\eta^\alpha} - (W_{S_5S_5})^{-1}_{ab}W_{S_5^b\theta}\frac{\delta}{\delta G} \end{aligned} \tag{3.7}$$

The decay formula therefore becomes

$$\Phi_{ab}C^{Tb\alpha}\ \hat{\Gamma}_{\eta^\alpha A^\lambda A^\rho} - \Phi_{ab}(W_{S_5S_5})^{-1}_{ab}W_{S_5^b\theta}\ \hat{\Gamma}_{GA^\lambda A^\rho} = a^a_{\rm em}\frac{\alpha}{\pi} \tag{3.8}$$

The decay constants are identified as

$$f^{a\alpha} = \Phi_{ab}C^{Tb\alpha} \tag{3.9}$$

In terms of the propagators, we can write (from (3.6))

$$f^{a\alpha}(W_{S_5S_5})^{-1}_{\alpha\beta}f^{T\beta b} = \Phi_{ac}(W_{S_5S_5})^{-1}_{cd}\Phi_{db} \tag{3.10}$$

and so at zero momentum

$$f^{a\alpha}m^2_{\alpha\beta}f^{T\beta b} = \Phi_{ac}(W_{S_5S_5})^{-1}_{cd}\Phi_{db} \tag{3.11}$$

as quoted in (1.14) .

The remaining steps in finding (1.1) and (1.2) are an exercise in manipulating the zero-momentum Ward identities (2.6) . First note that combining the two identities in (2.6) gives

$$M_{ac}W_{S_5^cS_5^d}M_{db} = -(M\Phi)_{ab} + (2n_f)^2\chi(0)\delta_{a0}\delta_{b0} \tag{3.12}$$

whose $a, b = 0$ component is just (2.7) . Note that $(M\Phi)_{ab}$ is symmetric. Also define $\mathbf{1}_{00} = \delta_{a0}\delta_{b0}$. Then we can write

$$\begin{aligned} \Phi_{ab}(W_{S_5S_5})^{-1}_{ab}W_{S_5^b\theta} &= (\Phi M)_{ac}\big(MW_{S_5S_5}M\big)^{-1}_{cd}M_{de}W_{S_5^e\theta} \\ &= -2n_f(M\Phi)_{ac}\Big(-(M\Phi) + (2n_f)^2\chi(0)\mathbf{1}_{00}\Big)^{-1}_{c0}\chi(0) \\ &= 2n_f\chi(0)\Big(1 - (2n_f)^2\chi(0)(M\Phi)^{-1}_{00}\Big)^{-1}\delta_{a0} \\ &= -2n_f\Gamma^{-1}_{QQ}\ \delta_{a0} \end{aligned} \tag{3.13}$$

where in the final step we have used the identification (2.13) . Similarly,

$$
\begin{aligned}
\Phi_{ac}(W_{S_5S_5})^{-1}_{cd}\Phi_{db} &= (\Phi M)_{ac}\big(MW_{S_5S_5}M\big)^{-1}_{cd}(M\Phi)_{db} \\
&= (M\Phi)_{ac}\Big(-(M\Phi)+(2n_f)^2\chi(0)\mathbf{1}_{00}\Big)^{-1}_{cd}(M\Phi)_{db} \\
&= -(M\Phi)_{ab}+(2n_f)^2\Gamma^{-1}_{QQ}\ \delta_{a0}\delta_{b0}
\end{aligned}
\tag{3.14}
$$

This establishes the required results. Substituting (3.13) and (3.14) into (3.8) and (3.11) we find the decay formula

$$
\Phi_{ab}C^{Tb\alpha}\ \hat{\Gamma}_{\eta^\alpha A^\lambda A^\rho} + 2n_f\Gamma^{-1}_{QQ}\ \hat{\Gamma}_{GA^\lambda A^\rho} = a^a_{\rm em}\frac{\alpha}{\pi}
\tag{3.15}
$$

where the decay constants satisfy

$$
f^{a\alpha}m^2_{\alpha\beta}f^{T\beta b} = -(M\Phi)_{ab}+(2n_f)^2\Gamma^{-1}_{QQ}\ \delta_{a0}\delta_{b0}
\tag{3.16}
$$

The final step is to identify the 1PI vertices with the couplings defined in section 1, viz.

$$
\hat{\Gamma}_{\eta^\alpha A^\lambda A^\rho} = g_{\eta^\alpha\gamma\gamma}
\tag{3.17}
$$

It is at this point that the central dynamical assumption is made. In fact, eqs (3.15) and (3.16) are exact identities, following simply from the definitions and the zero-momentum chiral Ward identities. To make contact with the radiative decays of the physical particles, we must assume in particular that the 1PI vertex *evaluated at* $k=0$ accurately approximates the physical coupling, which is defined *on mass-shell.*[7] This requires that $\hat{\Gamma}_{\eta^\alpha A^\lambda A^\rho}$ has only a weak momentum dependence in the range $0 \le k^2 \le m^2_{\eta^\alpha}$. This is reasonable, since it is defined to be a pole-free, amputated dynamical quantity. However, as in standard PCAC, the assumption is expected to be excellent for the π but progressively worse as the mass of the pseudo-Goldstone bosons increases. The hope here, in common with all attempts to include the η' in the framework of PCAC (including chiral Lagrangians with $1/N_c$ effects included[4,15], is that the approximation remains sufficiently good at the mass of the η'.

It is also important to determine the behaviour of all the quantities appearing in these formulae under the renormalisation group. Recall that RG behaviour was a key factor in the conjecture that $g_{G\gamma\gamma}$ may be neglected in first approximation in the decay

[7] The assumption that the 1PI vertices as defined here can be identified with the decay couplings of the physical particles at all rests on the assumption that the dominant particle poles in the pseudoscalar propagator matrix are indeed those of the η^α (see (3.6)).

formula (1.1) . All the required formulae are given in ref.[24]. The result is that *all* the quantities appearing in the final formulae (1.1) , (1.2) (or alternatively (3.15) , (3.16)) are RG invariant. However, notice that this is only true for the 1PI vertices evaluated at $k = 0$ (and in fact also on-shell), not for arbitray momenta. The proof is quite intricate, but since everything can be read off from ref.[24] (see also [11]) we will not give any further explanation here.

4. Effective Action

The results in the previous section can be summarised by writing an explicit form for the effective action $\Gamma[Q, \phi_5^a]$ compatible with all the zero-momentum anomalous chiral Ward identities. Of course this adds no new physics, but allows the essential results to be read off in a perhaps simpler and more systematic way. The resulting effective action is essentially identical to the di Vecchia–Veneziano[7], Rosenzweig–Schechter–Trahern[16] Lagrangian (with $\theta = 0$) though without explicit reference to the $1/N_c$ expansion. Less obviously, it is also very closely related to the (non-linear) chiral Lagrangians which incorporate the η' in the framework of large N_c[4,15].

The simplest effective action $\Gamma[Q, \phi_5^a]$ compatible with the identities (3.9) , (2.11) has been written down in ref.[11]. It is

$$\begin{aligned}\Gamma[Q, \phi_5^a] \;=\; \int dx \, \Bigg[& \frac{1}{2A} Q^2 \;+\; BQQ_{\mathrm{em}} \;+\; 2n_f Q \Phi_{0a}^{-1} \phi_5^a \;+\; a_{\mathrm{em}}^a Q_{\mathrm{em}} \Phi_{ab}^{-1} \phi_5^b \\ & + \frac{1}{2} \phi_5 \Phi^{-1} f \big(-\partial^2 - \mu^2\big) f \Phi^{-1} \phi_5 \Bigg]\end{aligned} \tag{4.1}$$

The final term is written in matrix notation. $f^{a\alpha}$ and $\mu_{\alpha\beta}^2$ are matrices, where $\mu_{\alpha\beta}^2$ is *defined* by the Dashen formula

$$f^{a\alpha} \mu_{\alpha\beta}^2 f^{T\beta b} = -M_{ac} \Phi_{cb} \tag{4.2}$$

The decay constants will subsequently be identified with those in section 3 (as will the constant A). μ^2 is of course the pseudo-Goldstone boson mass matrix in the OZI limit of QCD, i.e. before including the coupling to the gluonic anomaly operator Q.

In (4.1) the simplest choice of kinetic terms for the fields ϕ_5^a has been made, with the $f^{a\alpha}$ chosen to be constants. This is where the dynamical, pole-dominance, assumption of standard PCAC (or chiral Lagrangians) is built in. No kinetic terms are included for the composite operator Q (no glueball poles), and no higher order terms in Q are included (these would be suppressed for large N_C).

We have also included terms in (4.1) involving $Q_{\rm em}(A)$ to satisfy the anomalous chiral Ward identities with the additional electromagnetic contribution. The term coupling $Q_{\rm em}$ to ϕ_5^a is required, whereas the term $QQ_{\rm em}(A)$ is permitted. (Like A, B is taken to be a constant.) Differentiating (4.1) , we immediately obtain

$$\Phi_{ab}\ \hat{\Gamma}_{\phi_5^b A^\lambda A^\rho} = a^a_{\rm em}\frac{\alpha}{\pi} \tag{4.3}$$

as previously found in (3.3) .

The second derivatives of $\Gamma[Q, \phi_5^a]$ are

$$\begin{pmatrix} \Gamma_{QQ} & \Gamma_{Q\phi_5^b} \\ \Gamma_{\phi_5^a Q} & \Gamma_{\phi_5^a \phi_5^b} \end{pmatrix} = \begin{pmatrix} A^{-1} & 2n_f \Phi^{-1}_{0b} \\ 2n_f \Phi^{-1}_{a0} & \Phi^{-1} f\big(k^2-\mu^2\big) f \Phi^{-1} \end{pmatrix} \tag{4.4}$$

which clearly satisfy the identities (3.9) and (2.11) . We confirm the identification $\Gamma^{-1}_{QQ} = A$. The corresponding Green functions are found by inversion:

$$\begin{aligned} W_{\theta\theta} &= -A\ \tilde{\Delta}^{-1} \\ W_{\theta S_5^b} &= 2n_f A\ \Delta^{-1}_{0d}\ \Phi_{db} \\ W_{S_5^a\theta} &= 2n_f A\ \Phi_{ac}\ \Big(f\big(k^2-\mu^2\big)f\Big)^{-1}_{c0}\tilde{\Delta}^{-1} \\ W_{S_5^a S_5^b} &= -\Phi_{ac}\ \Delta^{-1}_{cd}\ \Phi_{db} \end{aligned} \tag{4.5}$$

where

$$\tilde{\Delta} = 1-(2n_f)^2 A\Big(f\big(k^2-\mu^2\big)f\Big)^{-1}_{00} \tag{4.6}$$

and

$$\Delta = f\big(k^2-\mu^2\big)f - (2n_f)^2 A\ \mathbf{1}_{00} \tag{4.7}$$

However, in this form the propagator matrix is clearly not diagonal and the operators are not normalised so as to couple with unit decay constants to the physical states. It is therefore convenient to make a change of variables in Γ so that it is written in terms of operators which are more closely identified with the physical states. Of course this change of variables is precisely that described already in section 3. We therefore define

$$\begin{aligned} G &= Q - W_{\theta S_5^a}(W_{S_5S_5})^{-1}_{ab}\phi_5^b \\ &= Q + 2n_f A\Phi^{-1}_{0b}\phi_5^b \end{aligned} \tag{4.8}$$

using (4.5) for the propagators (c.f. the identity (3.13)), and

$$\eta^\alpha = f^{T\alpha a}\Phi^{-1}_{ab}\phi_5^b \tag{4.9}$$

In terms of these operators, the effective action is

$$\begin{aligned}\Gamma[G,\eta^\alpha] \;=\; \int dx\, \Bigg[&\frac{1}{2A}G^2 \;+\; BGQ_{\rm em} \;+\; a^a_{\rm em} f^{-1}_{a\alpha}\eta^\alpha Q_{\rm em} \;-\; 2n_f ABf^{-1}_{0\alpha}\eta^\alpha Q_{\rm em} \\ &+\frac{1}{2}\eta\Big[(-\partial^2-\mu^2)-(2n_f)^2Af^{T-1}\mathbf{1}_{00}f^{-1}\Big]\eta\;\Bigg]\end{aligned} \tag{4.10}$$

It is then straightforward to read off the propagators

$$\begin{aligned}\langle G\; G\rangle &= -A \\ \langle\eta^\alpha\;\eta^\beta\rangle &= \frac{-1}{k^2-m^2_{\eta^\alpha}}\delta^{\alpha\beta}\end{aligned} \tag{4.11}$$

where the diagonal mass matrix for the physical η^α states satisfies the generalised Dashen fomula

$$f^{a\alpha}m^2_{\alpha\beta}f^{T\beta b} = f^{a\alpha}\mu^2_{\alpha\beta}f^{T\beta b} + (2n_f)^2A\delta_{a0}\delta_{b0} \tag{4.12}$$

Of course this is identical to (1.2) . This confirms the identification of the decay constants in the effective action with those in sections 1 and 3.

Finally, to obtain the decay formula itself, we take functional derivatives of Γ to get

$$\begin{aligned}\hat\Gamma_{GA^\lambda A^\rho} &= B \\ f^{a\alpha}\hat\Gamma_{\eta^\alpha A^\lambda A^\rho} &= a^a_{\rm em}\frac{\alpha}{\pi} - 2n_f AB\delta_{a0}\end{aligned} \tag{4.13}$$

Combining these, we find

$$f^{a\alpha}\;\hat\Gamma_{\eta^\alpha A^\lambda A^\rho} \;+\; 2n_f A\;\hat\Gamma_{GA^\lambda A^\rho} \;=\; a^a_{\rm em}\frac{\alpha}{\pi} \tag{4.14}$$

in agreement with (1.1) . Notice the importance of including the $QQ_{\rm em}$ coupling in (4.1) in obtaining this result.

5. $U_A(1)$ PCAC

For the third of our variations on a theme, we present a derivation of the decay formula following as closely as possible the traditional language of PCAC. This should be reasonably self-contained, though we will use the compact notation defined at the start of section 2 and simply quote the chiral Ward identities without proof.

Consider first QCD by itself without the coupling to electromagnetism. The axial anomaly equation is

$$\partial^\mu J^a_{\mu 5} = M_{ab}\phi^a_5 + 2n_f Q\delta_{a0} \tag{5.1}$$

where $J^a_{\mu 5}$ is the axial current, ϕ^a_5 the pseudoscalar quark bilinear operator and Q the topological charge. M_{ab} describes the quark masses and Φ_{ab} the condensates. The anomalous chiral Ward identities, at zero momentum, for the propagators (i.e. two-point Green functions) of these operators are

$$\begin{aligned} &2n_f\langle Q\ Q\rangle\delta_{a0} + M_{ac}\langle\phi^c_5\ Q\rangle = 0 \\ &2n_f\langle Q\ \phi^b_5\rangle\delta_{a0} + M_{ac}\langle\phi^c_5\ \phi^b_5\rangle + \Phi_{ab} = 0 \end{aligned} \tag{5.2}$$

which imply

$$M_{ac}M_{bd}\langle\phi^c_5\ \phi^d_5\rangle = -(M\Phi)_{ab} + (2n_f)^2\langle Q\ Q\rangle\delta_{a0}\delta_{b0} \tag{5.3}$$

We also need the result for the general form of the topological susceptibility:

$$\chi(0) \equiv \langle Q\ Q\rangle = \frac{-A}{1-(2n_f)^2A(M\Phi)^{-1}_{00}} \tag{5.4}$$

Although the pseudoscalar operators ϕ^a_5 and Q indeed couple to the physical states $\eta^\alpha = \eta', \eta, \pi^0$, it is more convenient to redefine linear combinations such that the resulting propagator matrix is diagonal and properly normalised. That is, we define operators η^α and G such that

$$\begin{pmatrix} \langle Q\ Q\rangle & \langle Q\ \phi^b_5\rangle \\ \langle\phi^a_5\ Q\rangle & \langle\phi^a_5\ \phi^b_5\rangle \end{pmatrix} \quad\rightarrow\quad \begin{pmatrix} \langle G\ G\rangle & 0 \\ 0 & \langle\eta^\alpha\ \eta^\beta\rangle \end{pmatrix} \tag{5.5}$$

This is achieved by

$$\begin{aligned} G &= Q - \langle Q\ \phi^a_5\rangle(\langle\phi_5\ \phi_5\rangle)^{-1}_{ab}\phi^b_5 \\ &= Q + 2n_f A\Phi^{-1}_{0b}\phi^b_5 \end{aligned} \tag{5.6}$$

and

$$\eta^\alpha = f^{T\alpha a}\Phi^{-1}_{ab}\phi^b_5 \tag{5.7}$$

With this choice, the $\langle G\ G\rangle$ propagator is

$$\langle G\ G\rangle = -A \tag{5.8}$$

and we impose the normalisation

$$\langle\eta^\alpha\ \eta^\beta\rangle = \frac{-1}{k^2 - m^2_{\eta^\alpha}}\delta^{\alpha\beta} \tag{5.9}$$

This implies that the constants $f^{a\alpha}$ in (5.7) , which we see shortly are simply the decay constants, must satisfy the (Dashen) identity

$$\begin{aligned} f^{a\alpha}m^2_{\alpha\beta}f^{T\beta b} &= \Phi_{ac}(\langle\phi_5\ \phi_5\rangle)^{-1}_{cd}\Phi_{db} \\ &= -(M\Phi)_{ab} + (2n_f)^2A\delta_{a0}\delta_{b0} \end{aligned} \tag{5.10}$$

The last line follows from the Ward identities (5.3) and (5.4) . In terms of these new operators, the anomaly equation (5.1) now reads simply:

$$\partial^\mu J^a_{\mu 5} = f^{a\alpha} m^2_{\alpha\beta} \eta^\beta + 2n_f G \delta_{a0} \tag{5.11}$$

After these preliminaries, we now recall how conventional PCAC is applied to the calculation of $\pi^0 \to \gamma\gamma$. The pion decay constant is defined as the coupling of the pion to the axial current

$$\langle 0|J^3_{\mu 5}|\pi\rangle = ik_\mu f_\pi \qquad \Rightarrow \qquad \langle 0|\partial^\mu J^3_{\mu 5}|\pi\rangle = f_\pi m^2_\pi \tag{5.12}$$

and satisfies the Dashen formula

$$f^2_\pi m^2_\pi = -(m_u + m_d)\langle \bar{q} q\rangle \tag{5.13}$$

The next step is to define a 'phenomenological pion field' π by

$$\partial^\mu J^3_{\mu 5} \to f_\pi m^2_\pi \pi \tag{5.14}$$

This is the step at which the crucial 'pole-dominance' assumption is made. Now include electromagnetism. The full anomaly equation is extended as in (1.4) to include the $F^{\mu\nu}\tilde{F}_{\mu\nu}$ contribution. Using (5.14) we therefore have

$$\begin{aligned} ik^\mu \langle\gamma\gamma|J^3_{\mu 5}|0\rangle &= f_\pi m^2_\pi \langle\gamma\gamma|\pi|0\rangle + a^a_{\rm em}\frac{\alpha}{8\pi}\langle\gamma\gamma|F^{\mu\nu}\tilde{F}_{\mu\nu}|0\rangle \\ &= f_\pi m^2_\pi \langle\pi\ \pi\rangle\langle\gamma\gamma|\pi\rangle + a^a_{\rm em}\frac{\alpha}{8\pi}\langle\gamma\gamma|F^{\mu\nu}\tilde{F}_{\mu\nu}|0\rangle \end{aligned} \tag{5.15}$$

where $\langle\pi\ \pi\rangle$ is the pion propagator $-1/(k^2 - m^2_\pi)$. At zero momentum, the l.h.s. vanishes because of the explicit k_μ factor and the absence of massless poles. We therefore find, defining the couplings as in (1.3) ,

$$f_\pi g_{\pi\gamma\gamma} = a^3_{\rm em}\frac{\alpha}{\pi} \tag{5.16}$$

In the full theory including the flavour singlet sector and the gluonic anomaly, we find a similar result. The 'phenomenological fields' are defined by (5.11) where the decay constants satisfy the generalised Dashen formula (5.10) . Notice, however, that they are *not* simply related to the couplings to the axial current as in (5.12) for the flavour non-singlet. We therefore find:

$$\begin{aligned} ik^\mu\langle\gamma\gamma|J^a_{\mu 5}|0\rangle &= f^{a\alpha} m^2_{\alpha\beta}\langle\gamma\gamma|\eta^\beta|0\rangle + 2n_f\langle\gamma\gamma|G|0\rangle\delta_{a0} + a^a_{\rm em}\frac{\alpha}{8\pi}\langle\gamma\gamma|F^{\mu\nu}\tilde{F}_{\mu\nu}|0\rangle \\ &= f^{a\alpha} m^2_{\alpha\beta}\langle\eta^\beta\ \eta^\gamma\rangle\langle\gamma\gamma|\eta^\gamma\rangle + 2n_f\langle G\ G\rangle\langle\gamma\gamma|G\rangle\delta_{a0} + a^a_{\rm em}\frac{\alpha}{8\pi}\langle\gamma\gamma|F^{\mu\nu}\tilde{F}_{\mu\nu}|0\rangle \end{aligned} \tag{5.17}$$

using the fact that the propagators are diagonal in the basis η^α, G. Using the explicit expressions (5.8) and (5.9) for the propagators, evaluating at zero momentum, and setting the l.h.s. to zero, we find in this case:

$$f^{a\alpha}\ g_{\eta^a\gamma\gamma} + 2n_f A\ g_{G\gamma\gamma}\ \delta_{a0}\ =\ a^a_{\rm em}\frac{\alpha}{\pi} \tag{5.18}$$

where the extra coupling $g_{G\gamma\gamma}$ is defined through (5.17) . This completes the '$U_A(1)$ PCAC' derivation. It is evidently a straightforward generalisation of conventional PCAC with the necessary modification of the usual formulae to take account of the extra gluonic contribution to the axial anomaly in the flavour singlet channel, the key point being the identification of the operators η^α and G in (5.11) .

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