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Higgs Mass Textures in Flipped $SU(5)$

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Abstract

We analyze the Higgs doublet-triplet mass splitting problem in the version of flipped $SU(5)$ derived from string theory. Analyzing non-renormalizable terms up to tenth order in the superpotential, we identify a pattern of field vev's that keeps one pair of electroweak Higgs doublets light, while all other Higgs doublets and all Higgs triplets are kept heavy, with the aid of the economical missing-doublet mechanism found in the field-theoretical version of flipped $SU(5)$. The solution predicts that second–generation charge -1/3 quarks and charged leptons are much lighter than those in the third generation.

1. Introduction

One of the most challenging problems in supersymmetric GUT model-building is that of the doublet-triplet mass splitting [1]. The problem is less severe than in non-supersymmetric GUTs, because supersymmetry can be used to stabilize light masses for electroweak Higgs doublets, but how did they get to be light in the first place? And can this be arranged in a natural way that also keeps their GUT triplet partners sufficiently heavy, safeguarding the stability of the proton?

One of the most promising approaches to this problem is the missing-doublet idea [2, 3]. One postulates a GUT with additional colour-triplet fields, that couple to the Higgs triplets and give them GUT-scale masses, but no accompanying electroweak doublet fields, so that the Standard Model Higgs doublets may remain light. The most economical realization [4] of this idea is in the field-theoretical version of the flipped SU(5)×U(1) GUT, where the unwanted Higgs triplets pair with triplet fields in the 10-dimensional representations that develop GUT-scale vacuum expectation values (vev's) [4, 5].

This was one of the vaunted advantages of flipped SU(5), along with its lack of adjoint and higher GUT representations, that motivated its derivation from string theory [6]. String-derived flipped SU(5) models have provided one of the more successful avenues in string phenomenology, but there has never been a definitive answer whether they resolve the doublet-triplet mass-splitting problem. This is complicated by the fact that flipped $SU(5)$ models derived from string contain several $\mathbf{5}$ and $\overline{\mathbf{5}}$ Higgs representations with candidate electroweak Higgs fields, as well as considerable ambiguity in the assignments of $\mathbf{10}$ and $\overline{\mathbf{10}}$ representations. On the other hand, the generalized Yukawa couplings of chiral supermultiplets can be calculated, including (in principle) non-renormalizable couplings [7] up to any desired order (depending on one's computational stamina).

At the level of renormalizable Yukawa couplings, it is known that there are pairs of massless electroweak doublets, and some couplings between Higgs triplets and prospective partners in $\mathbf{10}$ and $\overline{\mathbf{10}}$ representations. Some relevant non-renormalizable couplings have also been identified [7], but a systematic survey is lacking. One expects the contributions of such non-renormalizable interactions to be suppressed by powers of V/M, where V is the generic vev of some singlet or $\mathbf{10}/\overline{\mathbf{10}}$ field, and M is related to the Planck scale. One might expect $M \sim \mathcal{O}(M_P/\sqrt{8\pi}) \sim 10^{18}$ GeV in a weakly-coupled string model, and perhaps $M \sim 10^{16}$ GeV in a strongly-coupled model. One might expect $V/M = \mathcal{O}(1/10)$, and non-renormalizable interactions with such a suppression factor have been shown to provide encouraging textures for quark, charged-lepton and neutrino mass matrices. In this paper, we study whether the string-derived flipped $SU(5)$ model contains non-renormalizable interactions which, with a suitable pattern of large vev's, provide an acceptable texture for Higgs masses, with light doublets and heavy triplets.

We first identify textures of Higgs mass matrices that provide one pair of light Higgs doublets while keeping all Higgs triplets heavy. Then we examine all the relevant renormalizable and non-renormalizable Yukawa interactions up to tenth order in the fields. Starting from constraints on large vev's V that are imposed by earlier phenomenological studies, we explore whether there is a suitable refinement of the pattern of vev's that may solve the doublet-triplet mass-splitting problem in flipped SU(5). We exhibit a pattern of vev's, i.e., a possible choice of string vacuum, in which there is one pair of light Higgs doublets that remain massless through tenth order in the superpotential, another pair of Higgs doublets and a pair of Higgs triplets that acquire masses $\mathcal{O}(10^{10}$ to $10^{12})$ GeV, and the remaining Higgs doublets and triplets have masses close to the GUT scale.

2. The Problem of Light Higgs Doublets in Flipped $SU(5)$

In the minimal field-theoretical version of flipped $SU(5)$, there is a single pair of electroweak Higgs doublets , which are separated from their heavy Higgs triplet partners $D_3, \bar{D}_3$ by an economical realization of the missing-doublet mechanism, in which $D_3, \bar{D}_3$ couple to triplet members of **10** and **10** representations $H, \bar{H}$ and acquire large masses from GUT-scale vev's of electroweak-singlet components of $H, \bar{H}$ [4]. The light electroweak Higgs doublets couple via a singlet field ϕ that develops a vev at the supersymmetry-breaking scale, providing an acceptable value of the Higgs mixing parameter μ.

This simple and elegant picture becomes more complicated in the version of flipped $SU(5)$ derived from string theory in the free-fermion formulation. This model is obtained by introducing a set of five vectors of world-sheet boundary conditions $(1, S, b_{1,2,3})$, which define an $SO(10) \times SO(6) \times E_8$ gauge group with N=1 supersymmetry. Next, adding the vectors $b_{4,5}, \alpha$, the number of generations is reduced to three and the observable-sector gauge group obtained is $SU(5) \times U(1)$ accompanied by additional four $U(1)$ factors and a hidden-sector $SU(4) \times SO(10)$ gauge symmetry. The massless spectrum includes the seventy chiral superfields listed in Table 1, together with their non-Abelian group representation contents and their $U(1)$ charges [6].

As seen explicitly in Table 1, the matter and Higgs fields in this string model carry additional charges under extra $U(1)$ symmetries [6], there are a number of neutral singlet fields, and some hidden-sector matter fields which transform non-trivially under the $SU(4) \times SO(10)$ gauge symmetry: sextets under $SU(4)$, namely $\Delta_{1,2,3,4,5}$, and decuplets under $SO(10)$, namely $T_{1,2,3,4,5}$. There are also fourplets of the $SU(4)$ hidden symmetry which possess fractional charges. However, these are confined and will not concern us here.

We recall that the flavour assignments of the light Standard Model particles in this model are as follows:

$$\bar{f}_1 : \bar{u}, \tau;\ \bar{f}_2 : \bar{c}, e/\mu;\ \bar{f}_5 : \bar{t}, \mu/e;\ F_2 : Q_2, \bar{s};\ F_3 : Q_1, \bar{d};\ F_4 : Q_3, \bar{b};\ \ell_1^c : \bar{\tau};\ \ell_2^c : \bar{e};\ \ell_5^c : \bar{\mu} \quad (1)$$

and the trilinear superpotential relevant to fermion and Higgs doublet or triplet masses is

$$\begin{aligned} W^3 \ =\ & \frac{1}{2}F_1F_1h_1 + \frac{1}{2}F_2F_2h_2 + \frac{1}{2}F_4F_4h_1 + \frac{1}{2}\bar{F}_5\bar{F}_5\bar{h}_2 + F_4\bar{f}_5\bar{h}_{45} + F_3\bar{f}_3\bar{h}_3 \\ & + \bar{f}_1\ell_1^c h_1 + \bar{f}_2\ell_2^c h_2 + \bar{f}_5\ell_5^c h_2 + h_1\bar{h}_2\Phi_{12} + h_2\bar{h}_1\bar{\Phi}_{12} + h_2\bar{h}_3\Phi_{23} + h_3\bar{h}_2\bar{\Phi}_{23} \\ & + h_3\bar{h}_1\Phi_{31} + h_1\bar{h}_3\bar{\Phi}_{31} + h_3\bar{h}_{45}\bar{\phi}_{45} + h_{45}\bar{h}_3\phi_{45} + \frac{1}{2}h_{45}\bar{h}_{45}\Phi_3 \qquad (2) \end{aligned}$$

As also seen in Table 1, there are four candidate pairs of electroweak Higgs doublets, and a corresponding number of partner Higgs triplets in the multiplets h_1, h_2, h_3 and h_{45} and

$\bar{h}_1, \bar{h}_2, \bar{h}_3$ and $\bar{h}_{45}$ shown in Table 1. There are also candidates for the GUT Higgs multiplets among the $F_{1,2,3,4}$ and $\bar{F}_5$ shown in Table 1.

Their tree-level couplings are sufficient to give masses to one pair of Higgs triplet components. The question we address in this paper is whether non-renormalizable higher-order terms in the superpotential can give masses to the remaining Higgs triplets, while leaving light at least one pair of Higgs doublets.

The string-derived flipped $SU(5)$ model contains an anomalous $U(1)_A$ gauge factor, leading to symmetry breaking via vev's at the scale $M_A \sim 10^{-1} M_P$ for GUT-singlet fields that are also listed in Table 1. The superpotential of the model contains, in addition to tree-level trilinear terms, higher-order non-renormalizable terms. If all but three (two) of the fields in such an n^{th}-order interaction acquire large vev's $V \sim 10^{-1} M$, one will be left with a residual trilinear (bilinear) interaction with coefficient of order $10^{3-n} M$. Such interactions may provide interesting and realistic textures for quark, charged-lepton and neutrino mass matrices [8], since they provide entries that are hierarchically suppressed. Models of this type must make choices of the field vev's that are consistent with the D and F flatness of the effective scalar potential.

It is not an easy task to secure the existence of massless electroweak doublets in such a model, while also keeping their triplet partners massive. Consider a generic doublet mass term of the general form

$$g \left(\frac{\Phi}{M} \right)^{n-3} \Phi \, \bar{h} \, h \tag{3}$$

where Φ represents a generic field that obtains a large vev $V \sim \mathcal{O}(M/10)$. Even a term of order $n = 17$ of the type (3) could in principle push the Higgs masses above the electroweak scale. In principle, one should calculate all the relevant non-renormalizable terms up to $17^{\rm th}$ order, in order to ensure that there exist flat directions that preserve at least one massless pair of Higgs doublets. This paper does not go that far, but we do extend the analysis to tenth order, as discussed in following sections.

3. Conditions for Light Higgs Doublets

We now discuss how light Higgs doublets may appear, starting with the tree-level contributions to the superpotential. All phenomenologically viable flat directions require non-zero vev's for the singlet fields $\Phi_{31}, \bar{\Phi}_{31}$, $\Phi_{23}, \bar{\Phi}_{23}$ and $\phi_{45}, \bar{\phi}_{45}$ shown in Table 1, which enter the Higgs mass matrix arising from the $SU(5)$ representations $h_{1,2,3}$, h_{45} and $\bar{h}_{1,2,3}$, $\bar{h}_{45}$ already at the tree level. We recall that the tree-level superpotential terms which may provide the third-generation masses involve the Higgses $h_{1,2}$ and $\bar{h}_{45}$. The tree-level doublet Higgs matrix has two zero eigenvalues, and one pair of the corresponding eigenvectors do indeed have h_{45} and $h_{1,2}$ Higgs fields as components. The doublet Higgs mass matrix at the tree level is:

$$M_2^3 = \begin{pmatrix} 0 & \Phi_{12} & \bar{\Phi}_{31} & 0 \\ \bar{\Phi}_{12} & 0 & \Phi_{23} & 0 \\ \Phi_{31} & \bar{\Phi}_{23} & 0 & \bar{\phi}_{45} \\ 0 & 0 & \phi_{45} & \Phi_3 \end{pmatrix}, \tag{4}$$

but with our choice of singlet vacuum expectation values $\langle\Phi_{12},\bar{\Phi}_{12}\rangle = \langle\bar{\Phi}_3\rangle = 0$, so there are two massless pairs [9].

To ensure the masslessness of suitable Higgs mass eigenstates when higher-order non-renormalizable terms are included, we work as follows. We first consider the most general texture for the 4×4 doublet Higgs mass matrix, including arbitrary contributions to the entries that vanish at the tree level:

$$M_2^{all} = \begin{array}{c} \\ h_1 \\ h_2 \\ h_3 \\ h_{45} \end{array} \begin{array}{c} \begin{array}{cccc} \bar{h}_1 & \bar{h}_2 & \bar{h}_3 & \bar{h}_{45} \end{array} \\ \begin{pmatrix} \varepsilon_1 & \varepsilon_2 & \bar{\Phi}_{31} & \varepsilon_3 \\ \varepsilon_4 & \varepsilon_5 & \Phi_{23} & \varepsilon_6 \\ \Phi_{31} & \bar{\Phi}_{23} & \varepsilon_7 & \bar{\phi}_{45} \\ \varepsilon_8 & \varepsilon_9 & \phi_{45} & \varepsilon_{10} \end{pmatrix} \end{array}, \tag{5}$$

where we may ignore higher-order contributions to the entries which are non-zero at the tree level. The parameters $\varepsilon_{1,\ldots,10}$ correspond to all possible non-renormalizable contributions. Next, we check which of the parameters $\varepsilon_{1,\ldots,10}$ must be zero in order to obtain zero eigenvalues whose eigenvectors have components along the $h_{1,2}$ and $\bar{h}_{45}$ directions. We find 54 solutions. Whether $\varepsilon_7 = 0$ or $\neq 0$ does not affect the presence of such massless eigenstates. Factoring out these two options, there remain 27 options for combinations of the other ε_i that need not vanish, as listed in Tables 2 and 3.

The next task is to analyze the Higgs triplet mass matrix, which takes the form

$$M_3^{all} = \begin{array}{c} \\ h_1 \\ h_2 \\ h_3 \\ h_{45} \\ \bar{F}_5 \end{array} \begin{array}{c} \begin{array}{ccccc} \bar{h}_1 & \bar{h}_2 & \bar{h}_3 & \bar{h}_{45} & F_1 \end{array} \\ \begin{pmatrix} \varepsilon_1 & \varepsilon_2 & \bar{\Phi}_{31} & \varepsilon_3 & \lambda_1 \\ \varepsilon_4 & \varepsilon_5 & \Phi_{23} & \varepsilon_6 & \lambda_2 \\ \Phi_{31} & \bar{\Phi}_{23} & \varepsilon_7 & \bar{\phi}_{45} & \lambda_3 \\ \varepsilon_8 & \varepsilon_9 & \phi_{45} & \varepsilon_{10} & \lambda_4 \\ \bar{\lambda}_1 & \bar{\lambda}_2 & \bar{\lambda}_3 & \bar{\lambda}_4 & 0 \end{pmatrix} \end{array}, \tag{6}$$

where the λ_i and $\bar{\lambda}_j$: $i,j = 1,2,3,4$ are generic Yukawa couplings. Two of these, namely λ_1 and $\bar{\lambda}_2$, are non-vanishing at the tree level. The issue then is: for each of the doublet options listed in Tables 2 and 3, which combinations of the λ_i and $\bar{\lambda}_j$ must be non-vanishing in order that all the Higgs triplets are massive?

The answers are that no such satisfactory combinations exist for the two Higgs doublet textures listed in Table 2, whereas there are possible solutions for each of the other 52 textures, as shown in the last column of Table 3.

4. Non-Renormalizable Contributions to the Higgs Mass Textures

We have enumerated all possible non-renormalizable terms up to tenth order, but do not worry: we shall not list them all here. Instead, we present only the contributions up to seventh order in Table 4, and then we discuss those higher-order terms that have a chance of being non-vanishing. For this purpose, we take into account the choices of non-zero field vev's that have been made in previous phenomenological studies of the string flipped $SU(5)$ model. We therefore assume that just the following minimal set of fields have vanishing vev's, satisfying all flatness conditions [10, 8]

$$\begin{aligned}
&\Phi_{12} = \bar{\Phi}_{12} = \Phi_{I=1,\ldots,5} = \phi_1 = \phi_3 = \bar{\phi}_1 = \bar{\phi}_2 = 0 \\
&\phi^+ = \bar{\phi}^- = \bar{\phi}_3\phi_4 = F_4 = F_3 = F_2 = \Delta_1 = T_1 = 0,
\end{aligned} \tag{7}$$

and that the following set of non-Abelian composite gauge-singlet condensates also vanish [1] :

$$[T_4T_4] = [T_5T_5] = [\Delta_4\Delta_4] = [\Delta_4\Delta_5] = [\Delta_5\Delta_5] = [T_2T_4] = [T_4T_5] = [\Delta_2\Delta_2] + [T_2T_2] = 0, \tag{8}$$

We also assume that the following minimal set of elementary fields have non-vanishing vev's

$$\Phi_{23}, \bar{\Phi}_{23}, \Phi_{31}, \bar{\Phi}_{31}, \phi_{45}, \bar{\phi}_{45}, \phi_2, \bar{\phi}_4, F_1, \bar{F}_5 \neq 0 \tag{9}$$

as well as the following composite fields:

$$[\Delta_2\Delta_3] \neq 0 \text{ and } \quad \left\{ \begin{array}{l} [T_3T_4] \neq 0 \\ \text{or} \\ [T_3T_5] \neq 0 \end{array} \right. . \tag{10}$$

We make no *a priori* assumption about the vev's of the remaining elementary or composite fields.

With the above choice (7,8,9,10) of vev's, the first non–renormalizable terms in the doublet mass matrix appear at seventh order. This means that the ε_i are seventh order or higher, so that the Higgs doublet masses are certainly much smaller than M_{GUT}. In order to have the triplet masses as heavy as possible, we search for solutions involving only the tree–level contributions to the triplet couplings λ_i,$\bar{\lambda}_i$. Since the tree–level superpotential gives $\lambda_1 = \langle F_1 \rangle, \bar{\lambda}_2 = \langle \bar{F}_5 \rangle$, we have $\lambda_{i\neq 1} = \bar{\lambda}_{i\neq 2} = 0$ to this order. Examining the last column of Table 3 for textures leading to non–zero triplet masses, we find the 14 options (3-7,10-15,22-24), for each of which $\varepsilon_7 \neq 0$ is a possible option.

All these textures need $\varepsilon_3 = \varepsilon_2 = 0$, so we must check these conditions as far as possible. Imposing $\varepsilon_3 = 0$ up to ninth order yields the conditions

$$\begin{aligned}
&[T_2T_3] = [\Delta_3\Delta_4] = [\Delta_3\Delta_5]\,\bar{\phi}_3 = [T_3T_4T_3T_4]\,[\Delta_3\Delta_3] = 0 \\
&[T_2T_5]\,[\Delta_2\Delta_5] = [T_2T_5]\,[\Delta_2\Delta_4]\,\bar{\phi}_3 = 0 \\
&\bar{\phi}_+\,[\Delta_3\Delta_3\Delta_5\Delta_5] = \bar{\phi}_+\,[\Delta_3\Delta_3\Delta_4\Delta_4] = \bar{\phi}_3\,[\Delta_3\Delta_3\Delta_5\Delta_4] = 0 \\
&\phi_-\left([\Delta_3\Delta_3\Delta_2\Delta_2] + [\Delta_3\Delta_3]\,[T_2T_2]\right) = \bar{\phi}_+\left([\Delta_3\Delta_3\Delta_2\Delta_2] + [\Delta_3\Delta_3]\,[T_2T_2]\right) = 0 \\
&[\Delta_5\Delta_5\Delta_2\Delta_2\Delta_2\Delta_2] + [\Delta_5\Delta_5\Delta_2\Delta_2T_2T_2] = 0
\end{aligned} \tag{11}$$

[1] Here we introduce the notation $[A_1 \ldots A_n]$ to denote a general linear combination of all possible group invariants of the fields $A_1, A_2, \ldots A_n$.

Further imposing $\varepsilon_2 = 0$ up to ninth order yields the additional conditions

$$[T_4T_4T_4T_5T_5T_5] = 0 \tag{12}$$

$$[T_5T_4T_4T_4]\,[\Delta_2\Delta_2] + [T_5T_4T_4T_4T_2T_2] = 0 \tag{13}$$

Both sets of these conditions are simultaneously compatible with our initial choice (7,8,9,10) of vev's.

Next we search for the lowest–order non–vanishing ε_i. We find none at seventh order, and to eighth order only $\varepsilon_1, \varepsilon_4, \varepsilon_6, \varepsilon_{10} \neq 0$. We note also that $\varepsilon_1 \propto \varepsilon_{10}$ and $\varepsilon_4 \propto \varepsilon_6$. Among these options, only $\varepsilon_1 = \varepsilon_{10} = 0$ together with $\varepsilon_6, \varepsilon_4 \neq 0$ give acceptable textures. The condition $\varepsilon_1 = \varepsilon_{10} = 0$ then imposes the extra constraint

$$[\Delta_5\Delta_4\Delta_2\Delta_2] = 0. \tag{14}$$

This selects the textures 15, and possibly 13 if ε_5 is generated at higher order.

All of $\varepsilon_{1,8,9,10}$ have to vanish in order to preserve one massless doublet pair. Examining the ninth-order Higgs mass terms, we find that $\varepsilon_5 = \varepsilon_9 = \varepsilon_{10} = 0$ automatically, whilst in order to keep $\varepsilon_1 = \varepsilon_8 = 0$ we have to impose

$$[\Delta_5\Delta_5\Delta_4\Delta_4\Delta_2\Delta_2] + [\Delta_5\Delta_5\Delta_4\Delta_4]\,[T_2T_2] = 0 \tag{15}$$

$$[\Delta_5\Delta_5\Delta_5\Delta_4\Delta_2\Delta_2] + [\Delta_5\Delta_5\Delta_5\Delta_4]\,[T_2T_2] = 0 \tag{16}$$

$$[\Delta_5\Delta_5\Delta_5\Delta_5\Delta_2\Delta_2] + [\Delta_5\Delta_5\Delta_5\Delta_5]\,[T_2T_2] = 0 \tag{17}$$

Again, these conditions are compatible with the previous choices of vacuum parameters, and amount to a refinement of the string vacuum choice.

With the above vacuum choice, the relevant non–renormalizable contributions to the doublet mass matrix up to the ninth order are

$$\varepsilon_4 \;=\; \mathcal{O}(1)\frac{1}{M^5}\left([\Delta_5\Delta_5\Delta_3\Delta_3]\,\phi_-\phi_{45}\right) \tag{18}$$

$$\varepsilon_6 \;=\; \mathcal{O}(1)\frac{1}{M^5}\left([\Delta_5\Delta_5\Delta_3\Delta_3]\,\phi_-\bar{\Phi}_{31}\right) \tag{19}$$

In order to insure a non–vanishing $\varepsilon_6, \varepsilon_4$, we have to impose

$$[\Delta_5\Delta_5\Delta_3\Delta_3]\,, \phi_- \neq 0 \tag{20}$$

When combined with conditions (11), this implies that the vacuum must also have

$$\bar{\phi}_+ = [\Delta_2\Delta_2\Delta_3\Delta_3] + [\Delta_2\Delta_2]\,[\Delta_3\Delta_3] = 0 \tag{21}$$

which completes our specification of the string vacuum to this order.

We discuss now the Higgs triplet masses. Analyzing texture 15, we first note that there are four pairs which are massive at the tree level, two of them with masses proportional to $\lambda_1\bar{\lambda}_2$. The fifth pair has the lightest mass, which is

$$M_3^{light} \sim \frac{|\phi_{45}|\,\left|\varepsilon_4\bar{\phi}_{45} - \varepsilon_6\Phi_{31}\right|}{(\sqrt{|\Phi_{23}|^2 + |\phi_{45}|^2)(|\Phi_{31}|^2 + |\bar{\phi}_{45}|^2)}} \tag{22}$$

for our vacuum choice. If we assume generic vev's $\mathcal{O}(1/10)$ in natural units, we see that (22) indicates that the lightest triplet mass might be in the range of 10^{10} to 10^{12} GeV, which is perfectly satisfactory in flipped $SU(5)$.

We now turn to the Higgs doublets. By construction, there is a massless pair, and there were two massive pairs *ab initio* at the tree level. The fourth Higgs pair is relatively light, with mass

$$M_2^{light} \sim \frac{\left(|\phi_{45}|^2+|\bar{\Phi}_{31}|^2\right)^{\frac{1}{2}}\left(\left(|\varepsilon_4|^2+|\varepsilon_6|^2\right)|\bar{\Phi}_{23}|^2+\left|\varepsilon_6\Phi_{31}-\varepsilon_4\bar{\phi}_{45}\right|^2\right)^{\frac{1}{2}}}{\left(|\Phi_{23}|^2+|\phi_{45}|^2+|\bar{\Phi}_{31}|^2\right)^{\frac{1}{2}}\left(|\Phi_{31}|^2+|\bar{\Phi}_{23}|^2+|\bar{\phi}_{45}|^2\right)^{\frac{1}{2}}} \tag{23}$$

It is interesting that this mass is comparable in order of magnitude to the lightest triplet mass (22), whereas all the other massive doublets and triplets have masses comparable to M_{GUT}. Thus, below M_{GUT}, we have (effectively) only complete GUT representations, and the standard supersymmetric GUT prediction for the electroweak mixing angle is at least not affected to first order by their appearance in the renormalization-group equations.

According to Table 3, the Higgs doublets that are massless to ninth order are the combinations

$$h_0 \;=\; \frac{1}{\sqrt{\bar{\phi}^{*2}_{45}+\bar{\Phi}^{*2}_{31}}}\left(\bar{\phi}^{*}_{45}h_1-\bar{\Phi}^{*}_{31}h_{45}\right) \tag{24}$$

$$\bar{h}_0 \;=\; \frac{1}{\sqrt{\bar{\Phi}^2_{23}(1+r^2)+(\phi_{45}-\Phi_{31}r)^2}}\left(-\bar{\Phi}_{23}r\bar{h}_1+(\phi_{45}-\Phi_{31}r)\bar{h}_2-\bar{\Phi}_{23}\bar{h}_{45}\right) \tag{25}$$

Where $r=\frac{\varepsilon_6}{\varepsilon_4}=\mathcal{O}(1)\frac{\phi_{45}}{\Phi_{31}}$. Moreover, we have checked that this light Higgs doublet pair remains massless when tenth-order terms in the superpotential involving observable field vev's are taken into account.

We note that h_0 (24) contains components with tree-level couplings only to the third-generation quark and lepton masses. This is welcome since the two lighter generations receive masses from non–renormalizable terms: up to fifth order one finds

$$W_5 = F_2\bar{f}_2\bar{h}_{45}+\bar{F}_2\bar{f}_2\bar{h}_{45}\bar{\phi}_4+F_2F_2h_1\left(\bar{\phi}_4^2+\bar{\phi}_3^2\right)+\bar{f}_2\ell^c_2h_1\left(\bar{\phi}_4^2+\bar{\phi}_3^2\right)+\ldots. \tag{26}$$

Thus, a natural hierarchy between the third- and second-generation quark and lepton masses is a prediction of our doublet-triplet splitting mechanism.

Let us now comment on the uniqueness of the solution presented above. It is the only one that involves tree–level couplings in both the λ_i and the $\bar{\lambda}_i$. If one relaxes this constraint, which will tend to reduce the lightest triplet Higgs mass, many other textures in Table 3 can provide acceptable solutions. For example, a search for FFh and $\bar{F}\bar{F}\bar{h}$ couplings to fifth order yields already

$$\lambda_2 \;\sim\; F_1(\phi_2^2+\phi_4^2) \tag{27}$$

$$\lambda_4 \;\sim\; F_1\phi_{45}\Phi_{31} \tag{28}$$

$$\bar{\lambda}_1 \;\sim\; \bar{F}_5(\phi_2^2+\phi_4^2) \tag{29}$$

$$\bar{\lambda}_4 \;\sim\; F_1\bar{\phi}_{45}\Phi_{23}. \tag{30}$$

As can easily be checked from Table 3, $\lambda_3\bar{\lambda}_3$ is irrelevant and we have not calculated it.

5. Conclusions and Prospects

We have presented in this paper a solution to the triplet–doublet splitting problem in which the light Higgs doublets do not acquire a mass from any superpotential term through ninth order, nor even at tenth order if we discard vev's for higher-order combinations of hidden-sector fields. This solution has the attractive features that there are effectively complete GUT representations of intermediate-mass Higgs doublets and triplets, and predicts that the second-generation charge -1/3 quarks and leptons are much lighter than those in the third generation. As explained earlier, one really needs to check this solution up to 17^{th} order in the superpotential in order to ensure that the light Higgs doublet is sufficiently light. However, the analysis presented here goes to much higher order than had been done previously, and constitutes a promising start. If, eventually, this solution does not survive, there are other possible solutions, as mentioned at the end of the previous section. We are optimistic that the version of flipped $SU(5)$ derived from string can live up to its field-theoretical promise of solving the doublet-triplet mass splitting problem.

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$F_1(\mathbf{10},\frac{1}{2},-\frac{1}{2},0,0,0)$	$\bar{f}_1(\bar{\mathbf{5}}-\frac{3}{2},-\frac{1}{2},0,0,0)$	$\ell_1^c(\mathbf{1},\frac{5}{2},-\frac{1}{2},0,0,0)$
$F_2(\mathbf{10},\frac{1}{2},0,-\frac{1}{2},0,0)$	$\bar{f}_2(\bar{\mathbf{5}}-\frac{3}{2},0,-\frac{1}{2},0,0)$	$\ell_2^c(\mathbf{1},\frac{5}{2},0,-\frac{1}{2},0,0)$
$F_3(\mathbf{10},\frac{1}{2},0,0,\frac{1}{2},-\frac{1}{2})$	$\bar{f}_3(\bar{\mathbf{5}}-\frac{3}{2},0,0,\frac{1}{2},\frac{1}{2})$	$\ell_3^c(\mathbf{1},\frac{5}{2},0,0,\frac{1}{2},\frac{1}{2})$
$F_4(\mathbf{10},\frac{1}{2},-\frac{1}{2},0,0,0)$	$f_4(\mathbf{5},\frac{3}{2},\frac{1}{2},0,0,0)$	$\bar{\ell}_4^c(\mathbf{1},-\frac{5}{2},\frac{1}{2},0,0,0)$
$\bar{F}_5(\overline{\mathbf{10}},-\frac{1}{2},0,\frac{1}{2},0,0)$	$\bar{f}_5(\bar{\mathbf{5}}-\frac{3}{2},0,-\frac{1}{2},0,0)$	$\ell_5^c(\mathbf{1},\frac{5}{2},0,-\frac{1}{2},0,0)$

$h_1(\mathbf{5},-1,1,0,0,0)$	$h_2(\mathbf{5},-1,0,1,0,0)$	$h_3(\mathbf{5},-1,0,0,1,0)$
$h_{45}(\mathbf{5},-1,-\frac{1}{2},-\frac{1}{2},0,0)$		
$\bar{h}_1(\bar{\mathbf{5}},1,-1,0,0,0)$	$\bar{h}_2(\bar{\mathbf{5}},1,0,-1,0,0)$	$\bar{h}_3(\bar{\mathbf{5}},1,0,0,-1,0)$
$\bar{h}_{45}(\bar{\mathbf{5}},1,\frac{1}{2},\frac{1}{2},0,0)$		

$\phi_{45}(\mathbf{1},0,\frac{1}{2},\frac{1}{2},1,0)$	$\phi_+(\mathbf{1},0,\frac{1}{2},-\frac{1}{2},0,1)$	$\phi_-(\mathbf{1},0,\frac{1}{2},-\frac{1}{2},0,-1)$
$\bar{\phi}_{45}(\mathbf{1},0,-\frac{1}{2},-\frac{1}{2},-1,0)$	$\bar{\phi}_+(\mathbf{1},0,-\frac{1}{2},\frac{1}{2},0,-1)$	$\bar{\phi}_-(\mathbf{1},0,-\frac{1}{2},\frac{1}{2},0,1)$
$\Phi_{23}(\mathbf{1},0,0,-1,1,0)$	$\Phi_{31}(\mathbf{1},0,1,0,-1,0)$	$\Phi_{12}(\mathbf{1},0,-1,1,0,0)$
$\phi_i(\mathbf{1},0,\frac{1}{2},-\frac{1}{2},0,0)$	$\bar{\phi}_i(\mathbf{1},0,-\frac{1}{2},+\frac{1}{2},0,0)$,	i=1,2,3,4
$\bar{\Phi}_{23}(\mathbf{1},0,0,1,-1,0)$	$\bar{\Phi}_{31}(\mathbf{1},0,-1,0,1,0)$	$\bar{\Phi}_{12}(\mathbf{1},0,1,-1,0,0)$
$\Phi_i(\mathbf{1},0,0,0,0,0)$,	I=1,2,3,4,5	

$\Delta_1(0,\mathbf{1},\mathbf{6},0,-\frac{1}{2},\frac{1}{2},0)$	$\Delta_2(0,\mathbf{1},\mathbf{6},-\frac{1}{2},0,\frac{1}{2},0)$	$\Delta_3(0,\mathbf{1},\mathbf{6},-\frac{1}{2},-\frac{1}{2},0,\frac{1}{2})$
$\Delta_4(0,\mathbf{1},\mathbf{6},0,-\frac{1}{2},\frac{1}{2},0)$	$\Delta_5(0,\mathbf{1},\mathbf{6},\frac{1}{2},0,-\frac{1}{2},0)$	
$T_1(0,\mathbf{10},\mathbf{1},0,-\frac{1}{2},\frac{1}{2},0)$	$T_2(0,\mathbf{10},\mathbf{1},-\frac{1}{2},0,\frac{1}{2},0)$	$T_3(0,\mathbf{10},\mathbf{1},-\frac{1}{2},-\frac{1}{2},0,\frac{1}{2})$
$T_4(0,\mathbf{10},\mathbf{1},0,\frac{1}{2},-\frac{1}{2},0)$	$T_5(0,\mathbf{10},\mathbf{1},-\frac{1}{2},0,\frac{1}{2},0)$	

Table 1: *The chiral superfields of the version of the flipped $SU(5)$ model derived from string, with their quantum numbers [6]. For F_i, $\bar{f}_i$, ℓ_i^c, h_i, $\bar{h}_i$ and the singlet fields the $SU(5)\times U(1)'\times U(1)^4$ quantum numbers are presented. For Δ_i and T_i we present the $U(1)'\times SO(10)\times SO(6)\times U(1)^4$ quantum numbers.*

	M_2	h_0, h_0'	$\bar{h}_0, \bar{h}_0'$
1	$\begin{pmatrix} 0 & 0 & \bar{\Phi}_{31} & 0 \\ 0 & 0 & \Phi_{23} & 0 \\ \Phi_{31} & \bar{\Phi}_{23} & 0/\varepsilon_7 & \bar{\phi}_{45} \\ 0 & 0 & \phi_{45} & 0 \end{pmatrix}$	$\begin{bmatrix} \bar{\phi}_{45}^* \\ 0 \\ 0 \\ -\bar{\Phi}_{31}^* \end{bmatrix}, \begin{bmatrix} \bar{\Phi}_{23}^* \\ -\bar{\Phi}_{31}^* \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} \bar{\phi}_{45}^* \\ 0 \\ 0 \\ -\bar{\Phi}_{31}^* \end{bmatrix}, \begin{bmatrix} \bar{\Phi}_{23}^* \\ -\bar{\Phi}_{31}^* \\ 0 \\ 0 \end{bmatrix}$

Table 2: Textures leading to two pairs of massless Higgs doublets, with unnormalized indications of the massless eigenstates $(h_0, h_0', \bar{h}_0, \bar{h}_0')$.

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	M_2	h_0	$\bar{h}_0$	$\det M_3$
2	$\begin{pmatrix} 0 & 0 & \bar\Phi_{31} & 0 \\ 0 & 0 & \Phi_{23} & 0 \\ \Phi_{31} & \bar\Phi_{23} & 0/\varepsilon_7 & \bar\phi_{45} \\ 0 & \varepsilon_9 & \phi_{45} & 0 \end{pmatrix}$	$\begin{bmatrix} \bar\Phi^*_{23} \\ -\bar\Phi^*_{31} \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} \bar\phi_{45} \\ 0 \\ 0 \\ -\Phi_{31} \end{bmatrix}$	$\varepsilon_9(\lambda_1-\lambda_2)(\bar\lambda_1-\bar\lambda_4)$
3	$\begin{pmatrix} 0 & 0 & \bar\Phi_{31} & 0 \\ 0 & 0 & \Phi_{23} & 0 \\ \Phi_{31} & \bar\Phi_{23} & 0/\varepsilon_7 & \bar\phi_{45} \\ 0 & \varepsilon_9 & \phi_{45} & \varepsilon_{10} \end{pmatrix}$	$\begin{bmatrix} \bar\Phi^*_{23} \\ -\bar\Phi^*_{31} \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} \bar\Phi_{23}\varepsilon_{10}-\bar\phi_{45}\varepsilon_9 \\ -\Phi_{31}\varepsilon_{10} \\ 0 \\ \Phi_{31}\varepsilon_9 \end{bmatrix}$	$(\Phi_{23}\lambda_1-\bar\Phi_{31}\lambda_2)$ $\times(\bar\phi_{45}\varepsilon_9\bar\lambda_1-\bar\Phi_{23}\varepsilon_{10}\bar\lambda_1$ $+\Phi_{31}\varepsilon_{10}\bar\lambda_2-\Phi_{31}\varepsilon_9\bar\lambda_4)$
4	$\begin{pmatrix} 0 & 0 & \bar\Phi_{31} & 0 \\ 0 & 0 & \Phi_{23} & 0 \\ \Phi_{31} & \bar\Phi_{23} & 0/\varepsilon_7 & \bar\phi_{45} \\ \varepsilon_8 & 0 & \phi_{45} & 0 \end{pmatrix}$	$\begin{bmatrix} \bar\Phi^*_{23} \\ -\bar\Phi^*_{31} \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ \bar\phi_{45} \\ 0 \\ -\bar\Phi_{23} \end{bmatrix}$	$\varepsilon_8(\Phi_{23}\lambda_1-\bar\Phi_{31}\lambda_2)$ $\times(\bar\phi_{45}\bar\lambda_2-\bar\Phi_{23}\bar\lambda_4)$
5	$\begin{pmatrix} 0 & 0 & \bar\Phi_{31} & 0 \\ 0 & 0 & \Phi_{23} & 0 \\ \Phi_{31} & \bar\Phi_{23} & 0/\varepsilon_7 & \bar\phi_{45} \\ \varepsilon_8 & 0 & \phi_{45} & \varepsilon_{10} \end{pmatrix}$	$\begin{bmatrix} \bar\Phi^*_{23} \\ -\bar\Phi^*_{31} \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} \bar\Phi_{23}\varepsilon_{10} \\ \bar\phi_{45}\varepsilon_8-\Phi_{31}\varepsilon_{10} \\ 0 \\ -\bar\Phi_{23}\varepsilon_8 \end{bmatrix}$	$(\Phi_{23}\lambda_1-\bar\Phi_{31}\lambda_2)$ $\times(\bar\phi_{45}\varepsilon_8\bar\lambda_2-\Phi_{31}\varepsilon_{10}\bar\lambda_2$ $+\bar\Phi_{23}\varepsilon_{10}\bar\lambda_1-\bar\Phi_{23}\varepsilon_8\bar\lambda_4)$
6	$\begin{pmatrix} 0 & 0 & \bar\Phi_{31} & 0 \\ 0 & 0 & \Phi_{23} & 0 \\ \Phi_{31} & \bar\Phi_{23} & 0/\varepsilon_7 & \bar\phi_{45} \\ \varepsilon_8 & \varepsilon_9 & \phi_{45} & 0 \end{pmatrix}$	$\begin{bmatrix} \bar\Phi^*_{23} \\ -\bar\Phi^*_{31} \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} \bar\phi_{45}\varepsilon_9 \\ -\bar\phi_{45}\varepsilon_8 \\ 0 \\ \bar\Phi_{23}\varepsilon_8-\Phi_{31}\varepsilon_9 \end{bmatrix}$	$-(\Phi_{23}\lambda_1-\bar\Phi_{31}\lambda_2)$ $\times(\bar\phi_{45}\varepsilon_9\bar\lambda_1-\bar\phi_{45}\varepsilon_8\bar\lambda_2$ $+\bar\Phi_{23}\varepsilon_8\bar\lambda_4-\Phi_{31}\varepsilon_9\bar\lambda_4)$
7	$\begin{pmatrix} 0 & 0 & \bar\Phi_{31} & 0 \\ 0 & 0 & \Phi_{23} & 0 \\ \Phi_{31} & \bar\Phi_{23} & 0/\varepsilon_7 & \bar\phi_{45} \\ \varepsilon_8 & \varepsilon_9 & \phi_{45} & \varepsilon_{10} \end{pmatrix}$	$\begin{bmatrix} \bar\Phi^*_{23} \\ -\bar\Phi^*_{31} \\ 0 \\ 0 \end{bmatrix}$	$\begin{bmatrix} \bar\Phi_{23}\varepsilon_{10}-\bar\phi_{45}\varepsilon_9 \\ \bar\phi_{45}\varepsilon_8-\Phi_{31}\varepsilon_{10} \\ 0 \\ \Phi_{31}\varepsilon_9-\bar\Phi_{23}\varepsilon_8 \end{bmatrix}$	$-(\Phi_{23}\lambda_1-\bar\Phi_{31}\lambda_2)$ $\times(\bar\phi_{45}\varepsilon_9\bar\lambda_1-\bar\Phi_{23}\varepsilon_{10}\bar\lambda_1$ $-\bar\phi_{45}\varepsilon_8\bar\lambda_2+\Phi_{31}\varepsilon_{10}\bar\lambda_2+$ $\bar\Phi_{23}\varepsilon_8\bar\lambda_4-\bar\Phi_{31}\varepsilon_9\bar\lambda_4)$
8	$\begin{pmatrix} 0 & 0 & \bar\Phi_{31} & 0 \\ 0 & \varepsilon_5 & \Phi_{23} & 0 \\ \Phi_{31} & \bar\Phi_{23} & 0/\varepsilon_7 & \bar\phi_{45} \\ 0 & 0 & \phi_{45} & 0 \end{pmatrix}$	$\begin{bmatrix} \phi^*_{45} \\ 0 \\ 0 \\ -\bar\Phi^*_{31} \end{bmatrix}$	$\begin{bmatrix} \bar\phi_{45} \\ 0 \\ 0 \\ -\Phi_{31} \end{bmatrix}$	$\varepsilon_5(\phi_{45}\lambda_1-\bar\Phi_{31}\lambda_4)$ $\times(\bar\phi_{45}\bar\lambda_1-\Phi_{31}\bar\lambda_4)$
9	$\begin{pmatrix} 0 & 0 & \bar\Phi_{31} & 0 \\ 0 & \varepsilon_5 & \Phi_{23} & 0 \\ \Phi_{31} & \bar\Phi_{23} & 0/\varepsilon_7 & \bar\phi_{45} \\ 0 & \varepsilon_9 & \phi_{45} & 0 \end{pmatrix}$	$\begin{bmatrix} \bar\Phi^*_{23}\varepsilon^*_9-\phi^*_{45}\varepsilon^*_5 \\ -\bar\Phi^*_{31}\varepsilon^*_9 \\ 0 \\ \bar\Phi^*_{31}\varepsilon^*_5 \end{bmatrix}$	$\begin{bmatrix} \bar\phi_{45} \\ 0 \\ 0 \\ -\Phi_{31} \end{bmatrix}$	$(\bar\phi_{45}\bar\lambda_1-\Phi_{31}\bar\lambda_4)$ $\times(\phi_{45}\varepsilon_5\lambda_1-\Phi_{23}\varepsilon_9\lambda_1$ $+\bar\Phi_{31}\varepsilon_9\lambda_2-\bar\Phi_{31}\varepsilon_5\lambda_4)$
10	$\begin{pmatrix} 0 & 0 & \bar\Phi_{31} & 0 \\ 0 & \varepsilon_5 & \Phi_{23} & \varepsilon_6 \\ \Phi_{31} & \bar\Phi_{23} & 0/\varepsilon_7 & \bar\phi_{45} \\ 0 & 0 & \phi_{45} & 0 \end{pmatrix}$	$\begin{bmatrix} \phi^*_{45} \\ 0 \\ 0 \\ -\bar\Phi^*_{31} \end{bmatrix}$	$\begin{bmatrix} \bar\Phi_{23}\varepsilon_6-\bar\phi_{45}\varepsilon_5 \\ -\Phi_{31}\varepsilon_6 \\ 0 \\ \Phi_{31}\varepsilon_5 \end{bmatrix}$	$(\phi_{45}\lambda_1-\bar\Phi_{31}\lambda_4)$ $\times(\bar\phi_{45}\varepsilon_5\bar\lambda_1-\bar\Phi_{23}\varepsilon_6\bar\lambda_1$ $+\Phi_{31}\varepsilon_6\bar\lambda_2-\Phi_{31}\varepsilon_5\bar\lambda_4)$

Table 3: *Textures leading to a single pair of massless Higgs doublets, together with unnormalized indications of the associated massless eigenstates* ($h_0, \bar{h}_0$) *and the determinant* $\det(M_3)$ *of the colour–triplet mass matrix.*

11	$\begin{pmatrix} 0 & 0 & \bar{\Phi}_{31} & 0 \\ \varepsilon_4 & 0 & \Phi_{23} & 0 \\ \Phi_{31} & \bar{\Phi}_{23} & 0/\varepsilon_7 & \bar{\phi}_{45} \\ 0 & 0 & \phi_{45} & 0 \end{pmatrix}$	$\begin{bmatrix} \bar{\phi}^*_{45} \\ 0 \\ 0 \\ -\bar{\Phi}^*_{31} \end{bmatrix}$	$\begin{bmatrix} 0 \\ \bar{\phi}_{45} \\ 0 \\ -\bar{\Phi}_{23} \end{bmatrix}$	$\varepsilon_4(\bar{\Phi}_{31}\lambda_4 - \phi_{45}\lambda_1)$ $\times(\bar{\phi}_{45}\bar{\lambda}_2 - \bar{\Phi}_{23}\bar{\lambda}_4)$
12	$\begin{pmatrix} 0 & 0 & \bar{\Phi}_{31} & 0 \\ \varepsilon_4 & 0 & \Phi_{23} & 0 \\ \Phi_{31} & \bar{\Phi}_{23} & 0/\varepsilon_7 & \bar{\phi}_{45} \\ \varepsilon_8 & 0 & \phi_{45} & 0 \end{pmatrix}$	$\begin{bmatrix} \bar{\Phi}^*_{23}\varepsilon^*_8 - \bar{\phi}^*_{45}\varepsilon^*_4 \\ -\bar{\Phi}^*_{31}\varepsilon^*_8 \\ 0 \\ \bar{\Phi}^*_{31}\varepsilon^*_4 \end{bmatrix}$	$\begin{bmatrix} 0 \\ \bar{\phi}_{45} \\ 0 \\ -\bar{\Phi}_{23} \end{bmatrix}$	$-(\bar{\phi}_{45}\bar{\lambda}_2 - \bar{\Phi}_{23}\bar{\lambda}_4)$ $\times(\phi_{45}\varepsilon_4\lambda_1 - \Phi_{23}\varepsilon_8\lambda_1$ $+\bar{\Phi}_{31}\varepsilon_8\lambda_2 - \bar{\Phi}_{31}\varepsilon_4\lambda_4)$
13	$\begin{pmatrix} 0 & 0 & \bar{\Phi}_{31} & 0 \\ \varepsilon_4 & 0 & \Phi_{23} & \varepsilon_6 \\ \Phi_{31} & \bar{\Phi}_{23} & 0/\varepsilon_7 & \bar{\phi}_{45} \\ 0 & 0 & \phi_{45} & 0 \end{pmatrix}$	$\begin{bmatrix} \bar{\phi}^*_{45} \\ 0 \\ 0 \\ -\bar{\Phi}^*_{31} \end{bmatrix}$	$\begin{bmatrix} \bar{\Phi}_{23}\varepsilon_6 \\ \bar{\phi}_{45}\varepsilon_4 - \Phi_{31}\varepsilon_6 \\ 0 \\ -\bar{\Phi}_{23}\varepsilon_4 \end{bmatrix}$	$(\bar{\Phi}_{31}\lambda_4 - \phi_{45}\lambda_1)$ $\times(\bar{\phi}_{45}\varepsilon_4\bar{\lambda}_2 - \Phi_{31}\varepsilon_6\bar{\lambda}_2$ $+\bar{\Phi}_{23}\varepsilon_6\bar{\lambda}_1 - \bar{\Phi}_{23}\varepsilon_4\bar{\lambda}_4)$
14	$\begin{pmatrix} 0 & 0 & \bar{\Phi}_{31} & 0 \\ \varepsilon_4 & \varepsilon_5 & \Phi_{23} & 0 \\ \Phi_{31} & \bar{\Phi}_{23} & 0/\varepsilon_7 & \bar{\phi}_{45} \\ 0 & 0 & \phi_{45} & 0 \end{pmatrix}$	$\begin{bmatrix} \bar{\phi}^*_{45} \\ 0 \\ 0 \\ -\bar{\Phi}^*_{31} \end{bmatrix}$	$\begin{bmatrix} \bar{\phi}_{45}\varepsilon_5 \\ -\bar{\phi}_{45}\varepsilon_4 \\ 0 \\ \bar{\Phi}_{23}\varepsilon_4 - \Phi_{31}\varepsilon_5 \end{bmatrix}$	$(\phi_{45}\lambda_1 - \bar{\Phi}_{31}\lambda_4)$ $\times(\bar{\phi}_{45}\varepsilon_5\bar{\lambda}_1 - \bar{\phi}_{45}\varepsilon_4\bar{\lambda}_2$ $+\bar{\Phi}_{23}\varepsilon_4\bar{\lambda}_4 - \Phi_{31}\varepsilon_5\bar{\lambda}_4)$
15	$\begin{pmatrix} 0 & 0 & \bar{\Phi}_{31} & 0 \\ \varepsilon_4 & \varepsilon_5 & \Phi_{23} & \varepsilon_6 \\ \Phi_{31} & \bar{\Phi}_{23} & 0/\varepsilon_7 & \bar{\phi}_{45} \\ 0 & 0 & \phi_{45} & 0 \end{pmatrix}$	$\begin{bmatrix} \bar{\phi}^*_{45} \\ 0 \\ 0 \\ -\bar{\Phi}^*_{31} \end{bmatrix}$	$\begin{bmatrix} \bar{\phi}_{45}\varepsilon_5 - \bar{\Phi}_{23}\varepsilon_6 \\ \bar{\phi}_{45}\varepsilon_4 - \Phi_{31}\varepsilon_6 \\ 0 \\ \Phi_{31}\varepsilon_5 - \bar{\Phi}_{23}\varepsilon_4 \end{bmatrix}$	$(\phi_{45}\lambda_1 - \bar{\Phi}_{31}\lambda_4)$ $\times(\bar{\phi}_{45}\varepsilon_5\bar{\lambda}_1 - \bar{\Phi}_{23}\varepsilon_6\bar{\lambda}_1$ $-\bar{\phi}_{45}\varepsilon_4\bar{\lambda}_2 + \Phi_{31}\varepsilon_6\bar{\lambda}_2$ $+\bar{\Phi}_{23}\varepsilon_4\bar{\lambda}_4 - \Phi_{31}\varepsilon_5\bar{\lambda}_4)$
16	$\begin{pmatrix} 0 & \varepsilon_2 & \bar{\Phi}_{31} & 0 \\ 0 & 0 & \Phi_{23} & 0 \\ \Phi_{31} & \bar{\Phi}_{23} & 0/\varepsilon_7 & \bar{\phi}_{45} \\ 0 & 0 & \phi_{45} & 0 \end{pmatrix}$	$\begin{bmatrix} 0 \\ \bar{\phi}^*_{45} \\ 0 \\ -\bar{\Phi}^*_{23} \end{bmatrix}$	$\begin{bmatrix} \bar{\phi}_{45} \\ 0 \\ 0 \\ -\Phi_{31} \end{bmatrix}$	$\varepsilon_2(\Phi_{23}\lambda_4 - \phi_{45}\lambda_2)$ $\times(\bar{\phi}_{45}\bar{\lambda}_1 - \Phi_{31}\bar{\lambda}_4)$
17	$\begin{pmatrix} 0 & \varepsilon_2 & \bar{\Phi}_{31} & 0 \\ 0 & 0 & \Phi_{23} & 0 \\ \Phi_{31} & \bar{\Phi}_{23} & 0/\varepsilon_7 & \bar{\phi}_{45} \\ 0 & \varepsilon_9 & \phi_{45} & 0 \end{pmatrix}$	$\begin{bmatrix} \bar{\Phi}^*_{23}\varepsilon^*_9 \\ \bar{\phi}^*_{45}\varepsilon^*_2 - \bar{\Phi}^*_{31}\varepsilon^*_9 \\ 0 \\ -\bar{\Phi}^*_{23}\varepsilon^*_2 \end{bmatrix}$	$\begin{bmatrix} \bar{\phi}_{45} \\ 0 \\ 0 \\ -\Phi_{31} \end{bmatrix}$	$(\Phi_{31}\bar{\lambda}_4 - \bar{\phi}_{45}\bar{\lambda}_1)$ $\times(\phi_{45}\varepsilon_2\lambda_2 - \bar{\Phi}_{31}\varepsilon_9\lambda_2$ $+\Phi_{23}\varepsilon_9\lambda_1 - \Phi_{23}\varepsilon_2\lambda_4)$
18	$\begin{pmatrix} 0 & \varepsilon_2 & \bar{\Phi}_{31} & 0 \\ 0 & \varepsilon_5 & \Phi_{23} & 0 \\ \Phi_{31} & \bar{\Phi}_{23} & 0/\varepsilon_7 & \bar{\phi}_{45} \\ 0 & 0 & \phi_{45} & 0 \end{pmatrix}$	$\begin{bmatrix} \bar{\phi}^*_{45}\varepsilon^*_5 \\ -\bar{\phi}^*_{45}\varepsilon^*_2 \\ 0 \\ \bar{\Phi}^*_{23}\varepsilon^*_2 - \bar{\Phi}^*_{31}\varepsilon^*_5 \end{bmatrix}$	$\begin{bmatrix} \bar{\phi}_{45} \\ 0 \\ 0 \\ -\Phi_{31} \end{bmatrix}$	$(\bar{\phi}_{45}\bar{\lambda}_1 - \Phi_{31}\bar{\lambda}_4)$ $\times(\phi_{45}\varepsilon_5\lambda_1 - \phi_{45}\varepsilon_2\lambda_2$ $+\Phi_{23}\varepsilon_2\lambda_4 - \bar{\Phi}_{31}\varepsilon_5\lambda_4)$
19	$\begin{pmatrix} 0 & \varepsilon_2 & \bar{\Phi}_{31} & 0 \\ 0 & \varepsilon_5 & \Phi_{23} & 0 \\ \Phi_{31} & \bar{\Phi}_{23} & 0/\varepsilon_7 & \bar{\phi}_{45} \\ 0 & \varepsilon_9 & \phi_{45} & 0 \end{pmatrix}$	$\begin{bmatrix} \bar{\phi}^*_{45}\varepsilon^*_5 - \bar{\Phi}^*_{23}\varepsilon^*_9 \\ \bar{\Phi}^*_{31}\varepsilon^*_9 - \bar{\phi}^*_{45}\varepsilon^*_2 \\ 0 \\ \bar{\Phi}^*_{23}\varepsilon^*_2 - \bar{\Phi}^*_{31}\varepsilon_5 \end{bmatrix}$	$\begin{bmatrix} \bar{\phi}_{45} \\ 0 \\ 0 \\ -\Phi_{31} \end{bmatrix}$	$(\bar{\phi}_{45}\bar{\lambda}_1 - \Phi_{31}\bar{\lambda}_4)$ $\times(\phi_{45}\varepsilon_5\lambda_1 - \Phi_{23}\varepsilon_9\lambda_1$ $+\Phi_{23}\varepsilon_2\lambda_4 - \bar{\Phi}_{31}\varepsilon_5\lambda_4$ $-\phi_{45}\varepsilon_2\lambda_2 + \bar{\Phi}_{31}\varepsilon_9\lambda_2)$

Table 3 *(continued)*

20	$\begin{pmatrix} 0 & \varepsilon_2 & \bar{\Phi}_{31} & \varepsilon_3 \\ 0 & 0 & \Phi_{23} & 0 \\ \Phi_{31} & \bar{\Phi}_{23} & 0/\varepsilon_7 & \bar{\phi}_{45} \\ 0 & 0 & \phi_{45} & 0 \end{pmatrix}$	$\begin{bmatrix} 0 \\ \bar{\phi}^*_{45} \\ 0 \\ -\bar{\Phi}^*_{23} \end{bmatrix}$	$\begin{bmatrix} \bar{\Phi}_{23}\varepsilon_3 - \bar{\phi}_{45}\varepsilon_2 \\ -\Phi_{31}\varepsilon_3 \\ 0 \\ \Phi_{31}\varepsilon_2 \end{bmatrix}$	$(\Phi_{23}\lambda_4 - \phi_{45}\lambda_2)$ $\times(\bar{\phi}_{45}\varepsilon_2\bar{\lambda}_1 - \bar{\Phi}_{23}\varepsilon_3\bar{\lambda}_1$ $+\Phi_{31}\varepsilon_3\bar{\lambda}_2 - \Phi_{31}\varepsilon_2\bar{\lambda}_4)$
21	$\begin{pmatrix} \varepsilon_1 & 0 & \bar{\Phi}_{31} & 0 \\ 0 & 0 & \Phi_{23} & 0 \\ \Phi_{31} & \bar{\Phi}_{23} & 0/\varepsilon_7 & \bar{\phi}_{45} \\ 0 & 0 & \phi_{45} & 0 \end{pmatrix}$	$\begin{bmatrix} 0 \\ \bar{\phi}^*_{45} \\ 0 \\ -\bar{\Phi}^*_{23} \end{bmatrix}$	$\begin{bmatrix} 0 \\ \bar{\phi}_{45} \\ 0 \\ -\bar{\Phi}_{23} \end{bmatrix}$	$\varepsilon_1(\phi_{45}\lambda_2 - \Phi_{23}\lambda_4)$ $\times(\bar{\phi}_{45}\bar{\lambda}_2 - \bar{\Phi}_{23}\bar{\lambda}_4)$
22	$\begin{pmatrix} \varepsilon_1 & & \bar{\Phi}_{31} & 0 \\ 0 & 0 & \Phi_{23} & 0 \\ \Phi_{31} & \bar{\Phi}_{23} & 0/\varepsilon_7 & \bar{\phi}_{45} \\ \varepsilon_8 & 0 & \phi_{45} & 0 \end{pmatrix}$	$\begin{bmatrix} \bar{\Phi}^*_{23}\varepsilon^*_8 \\ \bar{\phi}^*_{45}\varepsilon^*_1 - \bar{\Phi}^*_{31}\varepsilon^*_8 \\ 0 \\ -\bar{\Phi}^*_{23}\varepsilon^*_1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ \bar{\phi}_{45} \\ 0 \\ -\bar{\Phi}_{23} \end{bmatrix}$	$(\bar{\phi}_{45}\bar{\lambda}_2 - \bar{\Phi}_{23}\bar{\lambda}_4)$ $\times(\phi_{45}\varepsilon_1\lambda_2 - \bar{\Phi}_{31}\varepsilon_8\lambda_2$ $+\Phi_{23}\varepsilon_8\lambda_1 - \Phi_{23}\varepsilon_1\lambda_4)$
23	$\begin{pmatrix} \varepsilon_1 & 0 & \bar{\Phi}_{31} & 0 \\ \varepsilon_4 & 0 & \Phi_{23} & 0 \\ \Phi_{31} & \bar{\Phi}_{23} & 0/\varepsilon_7 & \bar{\phi}_{45} \\ 0 & 0 & \phi_{45} & 0 \end{pmatrix}$	$\begin{bmatrix} \bar{\phi}^*_{45}\varepsilon^*_4 \\ -\bar{\phi}^*_{45}\varepsilon^*_1 \\ 0 \\ \bar{\Phi}^*_{23}\varepsilon^*_1 - \bar{\Phi}^*_{31}\varepsilon^*_4 \end{bmatrix}$	$\begin{bmatrix} 0 \\ \bar{\phi}_{45} \\ 0 \\ -\bar{\Phi}_{23} \end{bmatrix}$	$(-\bar{\Phi}_{23}\bar{\lambda}_4 + \bar{\phi}_{45}\bar{\lambda}_2)$ $\times(\phi_{45}\varepsilon_4\lambda_1 - \phi_{45}\varepsilon_1\lambda_2$ $+\Phi_{23}\varepsilon_1\lambda_4 - \bar{\Phi}_{31}\varepsilon_4\lambda_4)$
24	$\begin{pmatrix} \varepsilon_1 & 0 & \bar{\Phi}_{31} & 0 \\ \varepsilon_4 & 0 & \Phi_{23} & 0 \\ \Phi_{31} & \bar{\Phi}_{23} & 0/\varepsilon_7 & \bar{\phi}_{45} \\ \varepsilon_8 & 0 & \phi_{45} & 0 \end{pmatrix}$	$\begin{bmatrix} \bar{\phi}^*_{45}\varepsilon^*_4 - \bar{\Phi}^*_{23}\varepsilon^*_8 \\ \bar{\Phi}^*_{31}\varepsilon^*_8 - \bar{\phi}^*_{45}\varepsilon^*_1 \\ 0 \\ \bar{\Phi}^*_{23}\varepsilon^*_1 - \bar{\Phi}^*_{31}\varepsilon^*_4 \end{bmatrix}$	$\begin{bmatrix} 0 \\ \bar{\phi}_{45} \\ 0 \\ -\bar{\Phi}_{23} \end{bmatrix}$	$(\bar{\Phi}_{23}\bar{\lambda}_4 - \bar{\phi}_{45}\bar{\lambda}_2)$ $\times(\phi_{45}\varepsilon_4\lambda_1 - \Phi_{23}\varepsilon_8\lambda_1$ $-\phi_{45}\varepsilon_1\lambda_2 + \bar{\Phi}_{31}\varepsilon_8\lambda_2$ $+\Phi_{23}\varepsilon_1\lambda_4 - \bar{\Phi}_{31}\varepsilon_4\lambda_4)$
25	$\begin{pmatrix} \varepsilon_1 & 0 & \bar{\Phi}_{31} & \varepsilon_3 \\ 0 & 0 & \Phi_{23} & 0 \\ \Phi_{31} & \bar{\Phi}_{23} & 0/\varepsilon_7 & \bar{\phi}_{45} \\ 0 & 0 & \phi_{45} & 0 \end{pmatrix}$	$\begin{bmatrix} 0 \\ \bar{\phi}^*_{45} \\ 0 \\ -\bar{\Phi}^*_{23} \end{bmatrix}$	$\begin{bmatrix} \bar{\Phi}_{23}\varepsilon_3 \\ \bar{\phi}_{45}\varepsilon_1 - \Phi_{31}\varepsilon_3 \\ 0 \\ -\bar{\Phi}_{23}\varepsilon_1 \end{bmatrix}$	$(\phi_{45}\lambda_2 - \Phi_{23}\lambda_4)$ $\times(\bar{\phi}_{45}\varepsilon_1\bar{\lambda}_2 - \Phi_{31}\varepsilon_3\bar{\lambda}_2$ $+\bar{\Phi}_{23}\varepsilon_3\bar{\lambda}_1 - \bar{\Phi}_{23}\varepsilon_1\bar{\lambda}_4)$
26	$\begin{pmatrix} \varepsilon_1 & \varepsilon_2 & \bar{\Phi}_{31} & 0 \\ 0 & 0 & \Phi_{23} & 0 \\ \Phi_{31} & \bar{\Phi}_{23} & 0/\varepsilon_7 & \bar{\phi}_{45} \\ 0 & 0 & \phi_{45} & 0 \end{pmatrix}$	$\begin{bmatrix} 0 \\ \bar{\phi}^*_{45} \\ 0 \\ -\bar{\Phi}^*_{23} \end{bmatrix}$	$\begin{bmatrix} \bar{\phi}_{45}\varepsilon_2 \\ -\bar{\phi}_{45}\varepsilon_1 \\ 0 \\ \bar{\Phi}_{23}\varepsilon_1 - \Phi_{31}\varepsilon_2 \end{bmatrix}$	$(\Phi_{23}\lambda_4 - \phi_{45}\lambda_2)$ $\times(\bar{\phi}_{45}\varepsilon_2\bar{\lambda}_1 - \bar{\phi}_{45}\varepsilon_1\bar{\lambda}_2$ $+\bar{\Phi}_{23}\varepsilon_1\bar{\lambda}_4 - \Phi_{31}\varepsilon_2\bar{\lambda}_4)$
27	$\begin{pmatrix} \varepsilon_1 & \varepsilon_2 & \bar{\Phi}_{31} & \varepsilon_3 \\ 0 & 0 & \Phi_{23} & 0 \\ \Phi_{31} & \bar{\Phi}_{23} & 0/\varepsilon_7 & \bar{\phi}_{45} \\ 0 & 0 & \phi_{45} & 0 \end{pmatrix}$	$\begin{bmatrix} 0 \\ \bar{\phi}^*_{45} \\ 0 \\ -\bar{\Phi}^*_{23} \end{bmatrix}$	$\begin{bmatrix} \bar{\phi}_{45}\varepsilon_2 - \bar{\Phi}_{23}\varepsilon_3 \\ \Phi_{31}\varepsilon_3 - \bar{\phi}_{45}\varepsilon_1 \\ 0 \\ \Phi_{23}\varepsilon_1 - \Phi_{31}\varepsilon_2 \end{bmatrix}$	$(\Phi_{23}\lambda_4 - \phi_{45}\lambda_2)$ $\times(\bar{\phi}_{45}\varepsilon_2\bar{\lambda}_1 - \bar{\Phi}_{23}\varepsilon_3\bar{\lambda}_1$ $-\bar{\phi}_{45}\varepsilon_1\bar{\lambda}_2 + \bar{\Phi}_{23}\varepsilon_1\bar{\lambda}_4$ $+\Phi_{31}\varepsilon_3\bar{\lambda}_2 - \Phi_{31}\varepsilon_2\bar{\lambda}_4)$

Table 3 *(continued)*

$$
\begin{aligned}
&h_1\overline{h}_1D_5D_5\overline{\phi}_3\overline{\phi}_3\Phi_{23} + h_1\overline{h}_1D_5D_5\overline{\phi}_4\overline{\phi}_4\Phi_{23} + h_1\overline{h}_1D_5T_4D_2T_2\phi_2 + h_1\overline{h}_1T_5T_5\phi_2\phi_2\overline{\Phi}_{23} + \\
&h_1\overline{h}_1T_5T_5\phi_4\phi_4\overline{\Phi}_{23} + h_1\overline{h}_1D_2D_2\phi_2\phi_2\overline{\Phi}_{23} + h_1\overline{h}_1D_2D_2\phi_4\phi_4\overline{\Phi}_{23} + h_1\overline{h}_1T_2T_2\phi_2\phi_2\overline{\Phi}_{23} + \\
&h_1\overline{h}_1T_2T_2\phi_4\phi_4\overline{\Phi}_{23} + h_1\overline{h}_2D_5D_4\overline{\phi}_3\overline{\phi}_3\overline{\phi}_3 + h_1\overline{h}_2D_5D_4\overline{\phi}_3\overline{\phi}_4\overline{\phi}_4 + h_1\overline{h}_2T_5T_4\overline{\phi}_3\overline{\phi}_3\phi_2 + \\
&h_1\overline{h}_2T_5T_4\phi_2\overline{\phi}_4\overline{\phi}_4 + h_1\overline{h}_{45}T_5T_5\overline{\phi}_{45} + h_1\overline{h}_{45}D_2D_2\overline{\phi}_{45} + h_1\overline{h}_{45}T_2T_2\overline{\phi}_{45} + h_1\overline{h}_{45}D_5T_5D_3T_3\overline{\phi}_3 + \\
&h_1\overline{h}_{45}D_5T_5D_2T_2\overline{\phi}_{45} + h_1\overline{h}_{45}D_5D_4\overline{\phi}_3\overline{\phi}_{45}\overline{\Phi}_{31} + h_1\overline{h}_{45}T_5T_5\overline{\phi}_{45}\Phi_{23}\overline{\Phi}_{23} + h_1\overline{h}_{45}T_5D_4D_3T_3\overline{\Phi}_{23} + \\
&h_1\overline{h}_{45}T_5T_4\phi_2\overline{\phi}_{45}\overline{\Phi}_{31} + h_1\overline{h}_{45}D_4D_4\overline{\phi}_3\overline{\phi}_3\overline{\phi}_{45} + h_1\overline{h}_{45}D_4D_4\overline{\phi}_4\overline{\phi}_4\overline{\phi}_{45} + h_1\overline{h}_{45}D_2D_2\overline{\phi}_{45}\Phi_{23}\overline{\Phi}_{23} + \\
&h_1\overline{h}_{45}T_2T_2\overline{\phi}_{45}\Phi_{23}\overline{\Phi}_{23} + h_2\overline{h}_1D_5D_4\overline{\phi}_3\phi_2\phi_2 + h_2\overline{h}_1T_5T_4\phi_2\phi_2\phi_2 + \\
&h_2\overline{h}_1T_5T_4\phi_2\phi_4\phi_4 + h_2\overline{h}_2D_4D_4\overline{\phi}_3\overline{\phi}_3\Phi_{31} + h_2\overline{h}_2D_4D_4\overline{\phi}_4\overline{\phi}_4\Phi_{31} + h_2\overline{h}_2T_4T_4\phi_2\phi_2\overline{\Phi}_{31} + \\
&h_2\overline{h}_2T_4T_4\phi_4\phi_4\overline{\Phi}_{31} + h_2\overline{h}_{45}D_4D_4\overline{\phi}_{45} + h_2\overline{h}_{45}D_5D_4\overline{\phi}_3\overline{\phi}_{45}\Phi_{23} + h_2\overline{h}_{45}T_5T_5\phi_2\phi_2\overline{\phi}_{45} + \\
&h_2\overline{h}_{45}T_5T_5\phi_4\phi_4\overline{\phi}_{45} + h_2\overline{h}_{45}T_5D_4D_3T_3\Phi_{31} + h_2\overline{h}_{45}T_5T_4\phi_2\overline{\phi}_{45}\Phi_{23} + h_2\overline{h}_{45}D_4D_4\overline{\phi}_{45}\overline{\Phi}_{31}\Phi_{31} + \\
&h_2\overline{h}_{45}D_4T_4D_3T_3\phi_2 + h_2\overline{h}_{45}D_2D_2\phi_2\phi_2\overline{\phi}_{45} + h_2\overline{h}_{45}D_2D_2\phi_4\phi_4\overline{\phi}_{45} + h_2\overline{h}_{45}T_2T_2\phi_2\phi_2\overline{\phi}_{45} + \\
&h_2\overline{h}_{45}T_2T_2\phi_4\phi_4\overline{\phi}_{45} + h_{45}\overline{h}_1D_5D_5\phi_{45} + h_{45}\overline{h}_1D_5D_5\phi_{45}\Phi_{23}\overline{\Phi}_{23} + h_{45}\overline{h}_1D_5D_4\overline{\phi}_3\phi_{45}\Phi_{31} + \\
&h_{45}\overline{h}_1T_5T_4\phi_2\phi_{45}\Phi_{31} + h_{45}\overline{h}_1T_4T_4\phi_2\phi_2\phi_{45} + h_{45}\overline{h}_1T_4T_4\phi_4\phi_4\phi_{45} + h_{45}\overline{h}_2T_4T_4\phi_{45} + \\
&h_{45}\overline{h}_2D_5D_5\overline{\phi}_3\overline{\phi}_3\phi_{45} + h_{45}\overline{h}_2D_5D_5\overline{\phi}_4\overline{\phi}_4\phi_{45} + h_{45}\overline{h}_2D_5D_4\overline{\phi}_3\phi_{45}\overline{\Phi}_{23} + h_{45}\overline{h}_2T_5T_4\phi_2\phi_{45}\overline{\Phi}_{23} + \\
&h_{45}\overline{h}_2T_4T_4\phi_{45}\overline{\Phi}_{31}\Phi_{31} + h_{45}\overline{h}_{45}D_5D_5\phi_4\overline{\phi}_4\overline{\Phi}_{31} + h_{45}\overline{h}_{45}D_5D_5\phi_{45}\overline{\phi}_{45}\overline{\Phi}_{31} + \\
&h_{45}\overline{h}_{45}D_5D_4\overline{\phi}_3\phi_{45}\overline{\phi}_{45} + h_{45}\overline{h}_{45}T_5T_5\phi_4\overline{\phi}_4\Phi_{31} + h_{45}\overline{h}_{45}T_5T_5\phi_{45}\overline{\phi}_{45}\Phi_{31} + h_{45}\overline{h}_{45}T_5T_4\phi_2\phi_4\overline{\phi}_4 + \\
&h_{45}\overline{h}_{45}T_5T_4\phi_2\phi_{45}\overline{\phi}_{45} + h_{45}\overline{h}_{45}D_4D_4\phi_4\overline{\phi}_4\overline{\Phi}_{23} + h_{45}\overline{h}_{45}D_4D_4\phi_{45}\overline{\phi}_{45}\overline{\Phi}_{23} + h_{45}\overline{h}_{45}T_4T_4\phi_4\overline{\phi}_4\Phi_{23} + \\
&h_{45}\overline{h}_{45}T_4T_4\phi_{45}\overline{\phi}_{45}\Phi_{23} + h_{45}\overline{h}_{45}D_2D_2\phi_4\overline{\phi}_4\Phi_{31} + h_{45}\overline{h}_{45}D_2D_2\phi_{45}\overline{\phi}_{45}\Phi_{31} + h_{45}\overline{h}_{45}T_2T_2\phi_4\overline{\phi}_4\Phi_{31} + \\
&h_{45}\overline{h}_{45}T_2T_2\phi_{45}\overline{\phi}_{45}\Phi_{31}
\end{aligned}
\tag{31}
$$

Table 4: *Non–renormalizable contributions to the Higgs doublet mass matrix that are non–zero up to seventh order for the vacuum choice (7).*