CERN-TH/99-210
hep-th/9907133
revised Oct 1999

1/4 BPS States and Non–Perturbative Couplings in N=4 String Theories

W. Lerche and S. Stieberger

Theory Division, CERN, Geneva, Switzerland

Abstract

We compute certain $(2K+4)$–point, one–loop couplings in the type IIA string compactified on $K3 \times T^2$, which are related to a topological index on this manifold. Their special feature is that they are sensitive to only short and intermediate BPS multiplets. The couplings derive from underlying prepotentials $\mathcal{G}^{[2K+4]}(T,U)$, which can be summed up to a generating function in the form: $\sum_{K=0}^{\infty} V^{2K}/(2K)!\ \mathcal{G}^{[2K+4]}(T,U) = \sum d(4kl-m^2)\ \mathcal{L}i_3[e^{2\pi i(kT+lU)}e^{mV}]$. In the dual heterotic string on T^6, the amplitudes describe non–perturbative gravitational corrections to K-loop amplitudes due to bound states of fivebrane instantons with heterotic world-sheet instantons. We argue, as a consequence, that our results also give information about instanton configurations in six dimensional $Sp(2k)$ gauge theories on T^6.

CERN-TH/99-210
July 1999

1. Introduction

BPS-saturated string loop amplitudes [1–7] play an important rôle since they can give exact non-perturbative answers for appropriate dual formulations of a given theory. The corresponding pieces of the effective action are often given by holomorphic prepotentials, and it is this holomorphicity which underlies their computability. Particularly well-known are the couplings in $N=2$ supersymmetric string compactifications that describe gauge and certain gravitational interactions. They are characterized by holomorphic prepotentials $\mathcal{F}_g$ [8–12], which can be geometrically computed via mirror symmetry [13] on Calabi-Yau threefolds. An analogous holomorphic structure arises also in certain eight dimensional string vacua [5].

However, a comparable systematic treatment for four dimensional string theories with more, notably $N=4$ supersymmetries has been lacking so far. The main novel feature in $N=4$ supersymmetry is the appearance of intermediate ("1/4 BPS") besides the short ("1/2 BPS") supermultiplets. An example for a 1/2 BPS saturated amplitude is given by $\partial_T\langle R^2\rangle$, which is perturbatively exact at one loop order in the type IIA string compactified on $K3\times T^2$. It has been shown in [8,14,15] to be given by the T–derivative of:†

$$\begin{aligned}
\mathcal{F}_1^{(K3\times T^2)}(T,U) &= \int \frac{d^2\tau}{\tau_2} \operatorname*{Tr}_{K3\times T^2}\Big[(-1)^{J_R+J_L} J_R J_L q^{L_0}\overline{q}^{\overline{L}_0} - 24\Big] \\
&= 24\Big[\ln(T_2|\eta(T)|^4) + \ln(U_2|\eta(U)|^4) - \ln\kappa\Big] \\
&\equiv 24\,\mathcal{F}_1^{(T^2)}(T,U) - 24\,\ln\kappa\ ,
\end{aligned} \tag{1.1}$$

where $T\equiv B_{45}+i\sqrt{|G|}$ and $U\equiv(G_{45}+i\sqrt{|G|})/G_{44}$ are the Kähler and complex structure moduli of the two-torus, respectively. Since there are no contributions from the $K3$ apart from the 24 zero modes, the result is proportional to the topological partition function on T^2 [8]. Indeed $\mathcal{F}_1^{(T^2)}(T,U)$ is precisely what counts the 1/2 BPS states in the theory.

In the dual heterotic string on the six-torus T^6, the type IIA modulus T plays the rôle [16,17] of the heterotic dilaton: $T=S_{\rm het}$. Thus (1.1) represents a non-perturbative result from the heterotic point of view, where the $S_{\rm het}$ dependence reflects contributions from 1/2 BPS fivebrane instantons [14].

† The regularization constant is $\kappa\equiv\frac{8\pi}{3\sqrt{3}}e^{1-\gamma_E}$, where γ_E is the Euler constant. In the type IIB string on $K3\times T^2$, it is the U–derivative of this function what becomes relevant.

On the other hand, amplitudes sensitive to the intermediate, 1/4 BPS states have not yet been computed, at least as far as we know.$^\diamond$ It is the purpose of the present paper to compute certain of such couplings at one loop order for type IIA strings on $K3 \times T^2$, and investigate their structure.

More specifically, in the next section we will review some features of 1/4 BPS states in relation to the heterotic-type II duality. In section 3 we will then discuss (similar to refs. [7,15]) some facts about BPS saturated helicity traces and their relation to elliptic genera. In section 4 we will first compute quartic couplings in the moduli fields that are sensitive to the 1/2 BPS states only; not surprisingly, their structure turns out to be essentially the same as for the R^2 coupling in (1.1). Subsequently we will then compute certain sextic couplings, some of which will be sensitive to 1/4 BPS states. We will find that these couplings are characterized by two prepotentials $\mathcal{G}_1(T,U)$ and $\mathcal{G}_2(T,\overline{U})$, which enjoy an intriguing factorization property. In their structure they resemble Borcherds-like sum formulas, with counting functions given by the Eisenstein series $E_2(q)$. In the subsequent sections we investigate these prepotentials by rewriting them in various ways, and also obtain a generalization to an infinite sequence of $(2K+4)$-point amplitudes.

Finally, in the last section we will discuss the non-perturbative significance of these amplitudes when they are mapped by duality to the heterotic string. Specifically we will argue that the prepotentials carry non-trivial information about genus K instantons on heterotic fivebranes, and will also present some more speculative remarks.

2. Short and intermediate BPS multiplets

We will consider $N=4$ supersymmetric compactifications of type IIA superstrings on $K3 \times T^2$, or equivalently, of heterotic strings on T^6. Besides the graviton, the bosonic content of the supergravity multiplet in N=4, $d=4$ supergravity is a

$\diamond$ Some works that deal with different but related issues discuss topological amplitudes for six dimensional compactifications of type IIA strings [18], and the counting 1/8 BPS states of type IIA on T^6 [19]. Moreover, counting 1/4 BPS states in 5d black holes has been considered first in [20].

complex scalar (the dilaton $S_{\rm het}$) and six gravi-photons. In addition we have 22 vector multiplets, which contain each six scalars and one vector. In terms of the heterotic variables, the bosonic part of the action reads (up to two derivatives, see e.g. [21]):

$$\begin{aligned} S_{d=4,N=4} = & \int d^4x\sqrt{-g}\Big[R + 2\frac{\partial^\mu S_{\rm het}\partial_\mu \overline{S}_{\rm het}}{(S_{\rm het}-\overline{S}_{\rm het})^2} \\ & -\frac{1}{4}{\rm Im}(S_{\rm het})F_{\mu\nu}LMLF^{\mu\nu} + \frac{1}{4}{\rm Re}(S_{\rm het})F_{\mu\nu}L\widetilde{F}^{\mu\nu} + \frac{1}{8}{\rm Tr}(\partial_\mu ML\partial^\mu ML)\Big]. \end{aligned} \tag{2.1}$$

This action is manifestly invariant under $SO(22,6,{\rm I\!R})$ [22], while the equations of motion show a further invariance under $SL(2,{\rm I\!R})$ acting on $S_{\rm het}$. Accordingly, the local geometry of the scalar manifold is

$$\mathcal{M} \;=\; \frac{SL(2,{\rm I\!R})}{U(1)}\Big|_{S_{\rm het}} \times \frac{SO(22,6,{\rm I\!R})}{SO(22,{\rm I\!R})\times SO(6,{\rm I\!R})}\ . \tag{2.2}$$

The mass formula for 1/4 BPS states on this space is [17,23]:

$$\begin{aligned} M^2_{BPS} = & \frac{1}{S_{\rm het}-\overline{S}_{\rm het}}\Big[(\mathbf{m}+S_{\rm het}\mathbf{n})^t(M+L)(\mathbf{m}+\overline{S}_{\rm het}\mathbf{n}) \\ & \pm\frac{1}{2}\sqrt{[\mathbf{m}^t(M+L)\mathbf{m}][\mathbf{n}^t(M+L)\mathbf{n}]-[\mathbf{m}^t(M+L)\mathbf{n}]^2}\Big]\ , \end{aligned} \tag{2.3}$$

which involves the electric ($\mathbf{m}$) and magnetic ($\mathbf{n}$) charge vectors. The sign is always meant to be chosen such that M_{BPS} is maximized. The degenerate case, in which the square root vanishes, corresponds to the 1/2 BPS states [21]. Hence these may be viewed as specializations of the more generic 1/4 BPS states that "accidentally" leave more supersymmetries unbroken.

We will consider in the following only the subspace spanned by $S_{\rm het}$, $T_{\rm het}$ and $U_{\rm het}$, so that the relevant moduli sub-space is $\big(\frac{SL(2,{\rm I\!R})}{U(1)}\big)^3$. On this subspace we have $M^2_{BPS}\sim\frac{1}{S_{2\,\rm het}T_{2\,\rm het}U_{2\,\rm het}}|Z|^2$, where $Z=$max$\{|Z^+|,|Z^-|\}$ with [24,23]:

$$\begin{aligned} Z^+ &= m_1+m_2U_{\rm het}+k_1T_{\rm het}+k_2T_{\rm het}U_{\rm het} \\ &\qquad + S_{\rm het}(n_1+n_2U_{\rm het}+p_1T_{\rm het}+p_2T_{\rm het}U_{\rm het}) \\ Z^- &= m_1+m_2U_{\rm het}+k_1T_{\rm het}+k_2T_{\rm het}U_{\rm het} \\ &\qquad + \overline{S}_{\rm het}(n_1+n_2U_{\rm het}+p_1T_{\rm het}+p_2T_{\rm het}U_{\rm het})\ . \end{aligned} \tag{2.4}$$

One can check that in the degenerate case, $|Z^+|=|Z^-|$, these central charges reduce to the 1/2 BPS mass formula [21]:

$$Z = (q_1+q_2S_{\rm het})(m_1+m_2U_{\rm het}+k_1T_{\rm het}+k_2T_{\rm het}U_{\rm het})\ . \tag{2.5}$$

Note that the $S_{\rm het}$-independent terms of (2.4) and (2.5) coincide, which implies that the perturbative states are at least 1/2 BPS and thus that the 1/4 BPS states are intrinsically non-perturbative from the heterotic point of view. That is, there is no perturbative calculation in the heterotic string that could possibly see the 1/4 BPS states.

However, as noted above, the heterotic and type IIA compactifications are dual to each other provided [16,25,17] we exchange: $T_{\rm het} = S$, $S_{\rm het} = T$ and $U_{\rm het} = U$. Inserting this into (2.4) and (2.5), we see that at least some of the 1/4 BPS states do have a perturbative description on the type IIA side, though certainly not all of them.

Indeed we cannot expect to exactly compute the simultaneous dependence on all three moduli S, T, U in perturbation theory, in whatever framework. But what we can do is to simply focus on the T, U subspace on the type IIA side, while going to weak coupling, $S \equiv T_{\rm het} \to i\infty$. In this limit, further non-perturbative corrections on the type IIA side are suppressed. Even though this will not capture the full story, it will capture at least some of the non-perturbative physics in the heterotic string that is related to 1/4 BPS states.

The relevant physical states we thus consider are tensor products of the states on the $K3$ together with the momentum and winding modes on T^2, characterized by

$$\begin{aligned} p_L &= \frac{1}{\sqrt{2T_2U_2}}(m_1 + m_2\overline{U} + n_1\overline{T} + n_2\overline{TU}) \\ p_R &= \frac{1}{\sqrt{2T_2U_2}}(m_1 + m_2\overline{U} + n_1 T + n_2 T\overline{U}) \ . \end{aligned} \tag{2.6}$$

These states are 1/4 BPS if either the left- or the right-moving component is a ground state [26,20,27], ie., $N_L = h_L^{(K3)} = 0$ or $N_R = h_R^{(K3)} = 0$, where $N_{L,R}$ denotes the oscillator number and $h_L^{(K3)} + h_R^{(K3)}$ the mass of excitations on the $K3$. That is, (suppressing any vacuum energy shifts) the level matching condition for BPS states reads

$$|p_L|^2 - |p_R|^2 \ \equiv \ m_1 n_2 - n_1 m_2 \ = \ \begin{cases} N_R + h_R^{(K3)} & : \ \ 1/4 \text{ BPS} \\ -N_L - h_L^{(K3)} & : \ \ 1/4 \text{ BPS} \\ 0 & : \ \ 1/2 \text{ BPS} \end{cases} \ , \tag{2.7}$$

which exhibits the dyonic nature of the 1/4 BPS states. Clearly, the 1/2 BPS states correspond to both left and right moving ground states, and the level matching condition is identically satisfied for the momenta (2.5) with $k_i = 0$.

3. Topological indices and $N=4$ supersymmetry

One–loop amplitudes that are sensitive only to BPS states must certainly very special. Indeed they must be proportional to certain "helicity traces" [2–6] (or generalizations thereof), in which the long multiplets cancel out. A canonical example for such a trace is given by [7]:

$$\langle \lambda^{2n} \rangle \ = \ \Big(\frac{\partial}{\partial v_L} + \frac{\partial}{\partial \overline{v}_R}\Big)^{2n} \underset{\substack{\text{all}\\ \text{sectors}}}{\mathrm{Str}} \Big[q^{L_0} \overline{q}^{\overline{L}_0} e^{v_L J_L^{(\mathrm{st})}} e^{\overline{v}_R J_R^{(\mathrm{st})}} \Big] \Big|_{v_L = \overline{v}_R = 0} \quad . \tag{3.1}$$

Here, $\lambda \equiv J_L^{(\mathrm{st})} + J_R^{(\mathrm{st})}$ denotes the helicity operator, where in each left and right moving sector $J^{(\mathrm{st})} = \frac{1}{2\pi i} \oint \widetilde{\psi^\mu} \psi^\mu$ is the zero mode of the fermion number current ("(st)" denotes the light-cone space-time part of the theory).

In order to recognize the saturation or vanishing of such traces more easily, it is convenient to map them to the RR sector of the theory, in which this becomes a simple question of saturation of fermionic zero modes. This map [28,29] is universal for a given number of left- and right-moving supersymmetries and otherwise does not depend on the background. For simplicity, we will write the relevant identity down only for the left-moving variables, understanding that an analogous identity holds independently also in the right-moving sector:

$$\begin{aligned} &\underset{\substack{\text{all}\\ \text{sectors}}}{\mathrm{Str}} \Big[q^{L_0} e^{v^{(\mathrm{st})} J^{(\mathrm{st})} + v^{(T2)} J^{(T2)} + v^{(K3)} J^{(K3)}} \Big] \\ &\quad = \ \underset{\mathrm{R}}{\mathrm{Tr}} \Big[(-1)^{J^{(\mathrm{st})} + J^{(T2)} + J^{(K3)}} q^{L_0} e^{\widehat{v}^{(\mathrm{st})} J^{(\mathrm{st})} + \widehat{v}^{(T2)} J^{(T2)} + \widehat{v}^{(K3)} J^{(K3)}} \Big] , \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} \widehat{v}^{(\mathrm{st})} &= \tfrac{1}{2} v^{(\mathrm{st})} + \tfrac{1}{2} v^{(T2)} + \tfrac{1}{2}\sqrt{2} v^{(K3)} \\ \widehat{v}^{(T2)} &= \tfrac{1}{2} v^{(\mathrm{st})} + \tfrac{1}{2} v^{(T2)} - \tfrac{1}{2}\sqrt{2} v^{(K3)} \\ \widehat{v}^{(K3)} &= \tfrac{1}{2}\sqrt{2} v^{(\mathrm{st})} - \tfrac{1}{2}\sqrt{2} v^{(T2)} \ . \end{aligned} \tag{3.3}$$

Above, $J^{(T2)}$ denotes the fermion number current in the T^2 sector and $J^{(K3)}$ the zero mode of the $U(1) \subset SU(2)$ current of the $N=4$ world-sheet superconformal algebra that is intrinsic to a sigma-model on $K3$. From this it is easy to see that in the type IIA compactification on $K3 \times T^2$, we need at least two current insertions in each of the left and right moving sectors in order to get a non-vanishing result.

There are however traces that are more general than the left-right symmetric helicity trace in (3.1), and involve arbitrarily high powers of a current insertion in one of the left- or right-moving sectors, for example:

$$
\begin{aligned}
B_{2K+4} \;\equiv\; & \big\langle\, (J_L^{(\mathrm{st})})^{2+2K} (J_R^{(\mathrm{st})})^2 \,\big\rangle \;=\; \Big(\frac{\partial}{\partial v_L}\Big)^{2+2K} \Big(\frac{\partial}{\partial \overline{v}_R}\Big)^2 \times \\
& \mathop{\mathrm{Tr}}_{\mathrm{RR}}\Big[(-1)^{\sum (J_L^i+J_R^i)} q^{L_0}\overline{q}^{\overline{L}_0} e^{\frac12 v_L (J_L^{(\mathrm{st})}+J_L^{(T2)}+\sqrt{2}J_L^{(K3)})} e^{\frac12 \overline{v}_R (J_R^{(\mathrm{st})}+J_R^{(T2)}+\sqrt{2}J_R^{(K3)})}\Big]\Big|_{v_L=\overline{v}_R=0} \\
=\;& |\eta(q)|^{-12} Z_{2,2}(T,U,q,\overline{q}) \Big(\frac{\partial}{\partial v_L}\Big)^{2+2K} \Big(\frac{\partial}{\partial \overline{v}_R}\Big)^2 \theta_1(\tfrac12 v_L)^2 \overline{\theta}_1(\tfrac12 \overline{v}_R)^2 \times \\
& \mathop{\mathrm{Tr}}_{\mathrm{RR}}\Big[(-1)^{J_L^{(K3)}+J_R^{(K3)}} q^{L_0}\overline{q}^{\overline{L}_0} e^{\frac12\sqrt{2}(v_L J_L^{(K3)}+\overline{v}_R J_R^{(K3)})}\Big]\Big|_{v_L=\overline{v}_R=0} \\
=\;& 2\eta(q)^{-6} Z_{2,2}(T,U,q,\overline{q}) \Big(\frac{\partial}{\partial v_L}\Big)^{2+2K} \mathcal{E}_{(\mathrm{st}\times T^2)}(\tfrac12 v_L,q)\,\mathcal{E}_{(K3)}(\tfrac12 v_L,q)\Big|_{v_L=0} \quad .
\end{aligned}
\tag{3.4}
$$

Here,

$$
Z_{2,2}(T,U,q,\overline{q}) \;=\; \sum_{p_L,p_R} q^{\frac12 |p_L|^2} \overline{q}^{\frac12 |p_R|^2} \tag{3.5}
$$

is the partition function of windings and momenta on the two-torus, and

$$
\begin{aligned}
\mathcal{E}_{(\mathrm{st}\times T^2)}(v,q) \;&=\; \Big(\frac{i\theta_1(v,q)}{\eta^3(q)}\Big)^2 \;\;=:\; \sum_{n\geq 0,\ell\in\mathbb{Z}} d(4n-\ell^2)\, q^n e^{\ell v} \\
\mathcal{E}_{(K3)}(v,q) \;&= 8\sum_{i=2}^{4} \Big(\frac{\theta_i(v,q)}{\theta_i(0,q)}\Big)^2 \;\;=:\; \sum_{n\geq 0,\ell\in\mathbb{Z}} e(4n-\ell^2)\, q^n e^{\ell v}
\end{aligned}
\tag{3.6}
$$

are the elliptic genera [30] of the space-time sector times T^2 and of the $K3$ surface, respectively. Like all elliptic genera, these are weak Jacobi modular forms, namely of weight -2 and index 1, and of weight 0 and index 1, respectively.

Even more general traces can be obtained by inserting the individual fermion number currents $J_L^{(\mathrm{st})}$, $J_L^{(T2)}$ and $J_L^{(K3)}$ independently. The generalized Riemanm identity (3.2) will then in general produce some product of the individual elliptic genera $\mathcal{E}_{(\mathrm{st})}(\widehat{v}^{(\mathrm{st})},q)$, $\mathcal{E}_{(T^2)}(\widehat{v}^{(T^2)},q)$ and $\mathcal{E}_{(K3)}(\widehat{v}^{(K3)},q)$. As we will see, for the amplitudes that we will consider, only one of those factors will be realized.

Elliptic genera depend holomorphically on q, which reflects that all non-zero modes in the right-moving sector cancel out due to world-sheet supersymmetry. Via the identity (3.2) (which is due to space-time supersymmetry), this is simultaneously a reflection of the fact that the long multiplets cancel out in the trace, independent of any deformations in the $K3$ moduli. While in fact the total number of BPS states may jump when varying the moduli, the weighted helicity sums count effectively net numbers of BPS states and so remain invariant. It is this index-like, topological nature of BPS saturated amplitudes what makes them special and their modular integrals exactly computable [28,2–7].

More specifically, for $K = 0$ only the $\chi(K3) = 24$ left and right moving ground states can contribute in the $K3$ sector, so that

$$\begin{aligned} B_4 \;&=\; \big\langle\, (J_L^{(\mathrm{st})})^2 (J_R^{(\mathrm{st})})^2 \,\big\rangle \;=\; 4\; Z_{2,2}(T,U,q,\overline{q}) \;\; \mathop{\mathrm{Tr}}_{K3}\Big[(-1)^{J_L^{(K3)}+J_R^{(K3)}}\, q^{L_0}\overline{q}^{\overline{L}_0}\Big] \\ &=\; 96\; Z_{2,2}(T,U,q,\overline{q}) \end{aligned} \tag{3.7}$$

gets only contributions from 1/2 BPS states and this is what underlies the R^2 coupling (1.1) [14,15].

On the other hand, for $K > 0$ in (3.4) the states that contribute to the trace consist of right-moving ground states and arbitrary left-moving states – from what we said in the previous section, this precisely characterizes the 1/4 BPS states. For example, for $K = 1$ one has the following six-th order trace:†:

$$\begin{aligned} B_6 \;&=\; \big\langle\, (J_L^{(\mathrm{st})})^4 (J_R^{(\mathrm{st})})^2 \big\rangle \\ &= 12\, Z_{2,2}(T,U,q,\overline{q}) \;\; \mathop{\mathrm{Tr}}_{K3}\Big[(-1)^{J_L^{(K3)}+J_R^{(K3)}}\, (J_L^{(K3)})^2 q^{L_0}\overline{q}^{\overline{L}_0}\Big] \\ &= 192\, E_2(q)\, Z_{2,2}(T,U,q,\overline{q}) \ . \end{aligned} \tag{3.8}$$

The issue is now to identify physical amplitudes that contain these building blocks.

4. 1/2 and 1/4 BPS saturated amplitudes

As a warm–up, we study quartic interactions of the T and U moduli at one–loop order in type IIA compactified on $K3 \times T^2$:

$$\langle V_{\phi_1}(k_1) V_{\phi_2}(k_2) V_{\phi_3}(k_3) V_{\phi_4}(k_4)\rangle \ , \qquad \phi_i = T, U, \overline{T}, \overline{U} \ . \tag{4.1}$$

† Which has previously been calculated in the $\mathbf{Z}_2$ orbifold limit of $K3$ [15,7].

We want to extract from (4.1) the kinematical factor $(k_1k_3)(k_2k_4)$ (and permutations thereof), which corresponds to an one–loop corrections $\mathcal{A}$ to the term $(\partial_\mu\phi_1\partial^\mu\phi_3)(\partial_\nu\phi_2\partial^\nu\phi_4)$ in the effective action. These amplitudes receive non–vanishing contributions only from the $NSNS$–sector and not from the RNS, NSR and RR–sectors. We thus consider insertions of the moduli vertex operators:

$$\begin{aligned}V_T^{(0,0)}(k) &= \frac{2}{T-\overline{T}} : [\overline{\partial Z} + i(k\widetilde{\psi})\overline{\widetilde{\Psi}}(\overline{z})]\ [\partial Z + i(k\psi)\Psi(z)]\ e^{ik_\mu X^\mu(z,\overline{z})} : ,\\ V_U^{(0,0)}(k) &= \frac{-2}{U-\overline{U}} : [\overline{\partial} Z + i(k\widetilde{\psi})\widetilde{\Psi}(\overline{z})]\ [\partial Z + i(k\psi)\Psi(z)]\ e^{ik_\mu X^\mu(z,\overline{z})} : ,\end{aligned} \tag{4.2}$$

in the zero ghost picture with $\overline{Z} = \sqrt{T_2/2U_2}(X^4 + UX^5)$, $Z = \sqrt{T_2/2U_2}(X^4 + \overline{U}X^5)$, $\overline{\Psi} = \sqrt{T_2/2U_2}(\psi^4 + U\psi^5)$, $\Psi = \sqrt{T_2/2U_2}(\psi^4 + \overline{U}\psi^5)$. In this normalization we have: $\langle\Psi(z_1)\overline{\Psi}(z_2)\rangle_{\alpha,even} = -\frac{\theta_\alpha(z_{12},\tau)\theta_1'(0,\tau)}{\theta_\alpha(0,\tau)\theta_1(z_{12},\tau)}$, $\langle\Psi(z_1)\Psi(z_2)\rangle_{\alpha,even} = 0$, and for the kinematics we consider, the only non–vanishing fermionic contractions are those that lead to the helicity trace B_4 in (3.7). However, from the bosonic contractions$^\diamond$:

$$\begin{aligned}\langle\overline{\partial}Z(\overline{z}_1,z_1)\ \overline{\partial}Z(\overline{z}_2,z_2)\rangle &= p_R^2\\ \langle\overline{\partial}Z(\overline{z}_1,z_1)\ \partial Z(\overline{z}_2,z_2)\rangle &= p_Rp_L\\ \langle\overline{\partial}Z(\overline{z}_1,z_1)\ \partial\overline{Z}(\overline{z}_2,z_2)\rangle &= p_R\overline{p}_L\\ \langle\overline{\partial}Z(\overline{z}_1,z_1)\ \overline{\partial Z}(\overline{z}_2,z_2)\rangle &= |p_R|^2 - \frac{1}{2\pi\tau_2} + \frac{1}{2\pi^2}\partial^2_{\overline{z}_{21}}G_B\end{aligned} \tag{4.3}$$

($G_B \equiv -\ln|\chi|^2$), we have additional Narain momentum insertions, whose contributions are crucial for modular invariance. The resulting modular integrals‡ can be evaluated by using extensively the results of [32]. A typical example for a non–vanishing amplitude is:

$$\begin{aligned}U_2^4\ \mathcal{A}_{(\partial_\mu U\partial^\mu\overline{U})(\partial_\nu U\partial^\nu\overline{U})} &= T_2^2U_2^2\ \mathcal{A}_{(\partial_\mu T\partial^\mu\overline{T})(\partial_\nu U\partial^\nu\overline{U})}\\ &= \int\frac{d^2\tau}{\tau_2}\sum_{(p_L,p_R)}\tau_2^2\Big(|p_R|^2 - \frac{1}{2\pi\tau_2}\Big)\Big(|p_L|^2 - \frac{1}{2\pi\tau_2}\Big)q^{\frac12|p_L|^2}\overline{q}^{\frac12|p_R|^2}\\ &= -\frac{1}{4\pi^2}\,\mathcal{F}_1^{(T^2)}(T,U) + \frac{1}{4\pi^2}\,[1+\gamma_E - \ln(4\pi)]\ .\end{aligned} \tag{4.4}$$

$\diamond$ The third correlator may also contain a δ–function [31]. However, it would lead to manifestly non–covariant amplitudes.

$\ddagger$ Some of these integrals have to be regularized with an IR–regulator. This is described in the appendix of [32] and results in an extra constant contribution.

More generally, we find that all the non-vanishing amplitudes are proportional to $\mathcal{F}_1^{(T^2)}(T,U)$ (cf., (1.1)), which of course reflects that the helicity trace B_4 is sensitive only to 1/2 BPS states.

Now let us turn to more interesting scalar field interactions at the sixth derivative level. More specifically, we consider the following type of amplitudes:

$$\langle V_{\phi_1}(k_1)V_{\phi_2}(k_2)V_{\phi_3}(k_3)V_{\phi_4}(k_4)V_{\phi_5}(k_5)V_{\phi_6}(k_6)\rangle \ , \qquad \phi_i = T, U, \overline{T}, \overline{U} \tag{4.5}$$

and focus on the kinematics for which each modulus ϕ_i contributes one momentum k_i. This momentum structure can arise in three different ways: (i) four–fermionic contractions on both sides, giving rise to the helicity trace B_4, (ii) eight– and four–fermionic contractions on the right- and left-moving sides, respectively, or (iii) four– and eight–fermionic contractions on the right-moving and left-moving sides, respectively.

The main technical issue is the evaluation of the modular integrals, which is not entirely trivial and will be outlined in Appendix A. Performing these integrals, it turns out that there are two types of non–vanishing results, one type displaying again only 1/2 BPS states, the other however being sensitive to 1/4 BPS states. As an example for the first type, consider

$$\begin{aligned} U_2^6 \mathcal{A}^{(i)}_{(\partial_\mu U \partial^\mu \overline{U})(\partial_\nu U \partial^\nu \overline{U})(\partial_\rho \phi \partial^\rho \overline{\phi})} &= -\int \frac{d^2\tau}{\tau_2} \sum_{(p_L,p_R)} \tau_2^4 \Big(|p_L|^4 - \frac{2}{\pi\tau_2}|p_L|^2 + \frac{1}{2\pi^2\tau_2^2} \Big) \\ &\quad \times \Big(|p_R|^4 - \frac{2}{\pi\tau_2}|p_R|^2 + \frac{1}{2\pi^2\tau_2^2} \Big) \, q^{\frac{1}{2}|p_L|^2} \overline{q}^{\frac{1}{2}|p_R|^2} \\ &= \frac{1}{4\pi^4} \mathcal{F}_1^{(T^2)}(T,U) - \frac{1}{8\pi^4} \left[3 + 2\gamma_E - 2\ln(4\pi) \right] , \end{aligned} \tag{4.6}$$

where ϕ can be any of $\{T,U\}$. Since it involves the eight fermion contraction of type (i), which gives rise to B_4, this amplitude is obviously sensitive only to 1/2 BPS states. However, it can happen even for amplitudes with twelve-fermion contractions of types (ii) and (iii) that only 1/2 BPS states contribute. An example is given by the following two contributions to the same amplitude[♮]:

$$\begin{aligned}
\widetilde{\mathcal{A}}^{(ii)}_{(\partial_\mu U\partial^\mu\overline{U})(\partial_\nu U\partial^\nu\overline{U})(\partial_\rho\phi\partial^\rho\overline{\phi})} &= \int\frac{d^2\tau}{\tau_2}\sum_{(p_L,p_R)}\tau_2^4\Big(|p_R|^2-\frac{1}{2\pi\tau_2}\Big)\\
&\qquad\times\Big(|p_L|^4-\frac{2}{\pi\tau_2}|p_L|^2+\frac{1}{2\pi^2\tau_2^2}\Big)\widehat{\overline{E}}_2\ q^{\frac{1}{2}|p_L|^2}\overline{q}^{\frac{1}{2}|p_R|^2}\\
&= -\frac{3}{4\pi^4}\,\mathcal{F}_1^{(T^2)}(T,U)+\frac{3}{8\pi^4}\,[3+2\gamma_E-2\ln(4\pi)]\ ,\\
\widetilde{\mathcal{A}}^{(iii)}_{(\partial_\mu U\partial^\mu\overline{U})(\partial_\nu U\partial^\nu\overline{U})(\partial_\rho\phi\partial^\rho\overline{\phi})} &= \int\frac{d^2\tau}{\tau_2}\sum_{(p_L,p_R)}\tau_2^4\Big(|p_R|^4-\frac{2}{\pi\tau_2}|p_R|^2+\frac{1}{2\pi^2\tau_2^2}\Big)\\
&\qquad\times\Big(|p_L|^2-\frac{1}{2\pi\tau_2}\Big)\widehat{E}_2\ q^{\frac{1}{2}|p_L|^2}\overline{q}^{\frac{1}{2}|p_R|^2}\\
&= -\frac{3}{4\pi^4}\,\mathcal{F}_1^{(T^2)}(T,U)+\frac{3}{8\pi^4}\,[3+2\gamma_E-2\ln(4\pi)]\ .
\end{aligned}\tag{4.7}$$

Even though the (suitable regularized) second derivative of the elliptic genus appears in the form of $\widehat{E}_2\equiv E_2-\frac{3}{\pi\tau_2}$, the integral involving E_2 vanishes[*] in a non-trivial manner, so that basically only the non–harmonic part of $\widehat{E}_2$ contributes – this means that again only 1/2 BPS and no 1/4 BPS states contribute.

Summarizing, the first kind of sextic couplings has exactly the same 1/2 BPS structure as the quartic couplings discussed above.[•]

On the other hand, in the following examples, where only one type of contraction contributes (either of type *(ii)* or type *(iii)*), we see an interesting new structure

---

♮ The normalization of the vertex operators (4.2) is absorbed into $\widetilde{\mathcal{A}}$.

∗ Up to a term $\frac{T_2}{4\pi^3}c(0)$, which is absorbed in $\ln(\eta(T))$.

• It is known [33] that some of the six-derivative couplings are related to the four-derivative ones by field redefinitions. The similarity of the results in (4.7) and (4.4) may be partly related to that. Moreover, their kinematical structure coincides with some of the six-derivative couplings that arise from expansions of Born-Infeld actions [34], and so may be also reproduced from simple D-brane interactions.

emerging. More specifically, we find*:

$$
\begin{aligned}
&\widetilde{\mathcal{A}}^{(ii)}_{(\partial_\mu T\partial^\mu U)(\partial_\nu T\partial^\nu U)(\partial_\rho\phi\partial^\rho\overline{\phi})} \\
&\qquad = \int \frac{d^2\tau}{\tau_2} \sum_{(p_L,p_R)} \tau_2^4 \Big(|p_R|^2 - \frac{1}{2\pi\tau_2}\Big) p_L^4 \ \widehat{\overline{E}}_2 \ q^{\frac{1}{2}|p_L|^2} \overline{q}^{\frac{1}{2}|p_R|^2} \\
&\qquad = -\frac{4}{\pi^2} T_2^2 U_2^2 \ \Big(\partial_U + \frac{2}{U-\overline{U}}\Big)\Big(\partial_T + \frac{2}{T-\overline{T}}\Big) \mathcal{G}_1(T,U) \\
&\widetilde{\mathcal{A}}^{(ii)}_{(\partial_\mu T\partial^\mu U)(\partial_\nu T\partial^\nu U)(\partial_\rho T\partial^\rho\overline{U})} \\
&\qquad = \int \frac{d^2\tau}{\tau_2} \sum_{(p_L,p_R)} \tau_2^4 \overline{p}_R^2 p_L^4 \ \widehat{\overline{E}}_2 \ q^{\frac{1}{2}|p_L|^2} \overline{q}^{\frac{1}{2}|p_R|^2} \\
&\qquad = \frac{4}{\pi^2} T_2^3 U_2 \ \Big(\partial_T + \frac{4}{T-\overline{T}}\Big)\Big(\partial_T + \frac{2}{T-\overline{T}}\Big) \mathcal{G}_1(T,U) \ ,
\end{aligned}
\tag{4.8}
$$

and:

$$
\begin{aligned}
&\widetilde{\mathcal{A}}^{(iii)}_{(\partial_\mu T\partial^\mu \overline{U})(\partial_\nu T\partial^\nu \overline{U})(\partial_\rho\phi\partial^\rho\overline{\phi})} \\
&\qquad = \int \frac{d^2\tau}{\tau_2} \sum_{(p_L,p_R)} \tau_2^4 \Big(|p_L|^2 - \frac{1}{2\pi\tau_2}\Big) \overline{p}_R^4 \ \widehat{E}_2 \ q^{\frac{1}{2}|p_L|^2} \overline{q}^{\frac{1}{2}|p_R|^2} \\
&\qquad = -\frac{4}{\pi^2} T_2^2 U_2^2 \ \Big(\partial_{\overline{U}} - \frac{2}{U-\overline{U}}\Big)\Big(\partial_T + \frac{2}{T-\overline{T}}\Big) \mathcal{G}_2(T,\overline{U}) \\
&\widetilde{\mathcal{A}}^{(iii)}_{(\partial_\mu T\partial^\mu \overline{U})(\partial_\nu T\partial^\nu \overline{U})(\partial_\rho T\partial^\rho U)} \\
&\qquad = \int \frac{d^2\tau}{\tau_2} \sum_{(p_L,p_R)} \tau_2^4 p_L^2 \overline{p}_R^4 \ \widehat{E}_2 \ q^{\frac{1}{2}|p_L|^2} \overline{q}^{\frac{1}{2}|p_R|^2} \\
&\qquad = \frac{4}{\pi^2} T_2^3 U_2 \ \Big(\partial_T + \frac{4}{T-\overline{T}}\Big)\Big(\partial_T + \frac{2}{T-\overline{T}}\Big) \mathcal{G}_2(T,\overline{U}) \ .
\end{aligned}
\tag{4.9}
$$

Furthermore$^\nabla$

$$
\begin{aligned}
\widetilde{\mathcal{A}}^{(iii)}_{(\partial_\mu T\partial^\mu \overline{T})(\partial_\nu T\partial^\nu \overline{U})(\partial_\rho\phi\partial^\rho\overline{\phi})} &= \int \frac{d^2\tau}{\tau_2} \sum_{(p_L,p_R)} \tau_2^4 \Big(|p_L|^2 - \frac{1}{2\pi\tau_2}\Big)\Big(|p_R|^2 - \frac{3}{2\pi\tau_2}\Big) \\
&\qquad \times \overline{p}_R^2 \ \widehat{E}_2 \ q^{\frac{1}{2}|p_L|^2} \overline{q}^{\frac{1}{2}|p_R|^2} \\
&= -\frac{1}{\pi^2} T_2 U_2 \ \mathcal{G}_2(T,\overline{U}) \ .
\end{aligned}
\tag{4.10}
$$

$\star$ However: $\widetilde{\mathcal{A}}^{(i)}_{(\partial_\mu T\partial^\mu U)(\partial_\nu T\partial^\nu U)(\partial_\rho\phi\partial^\rho\overline{\phi})} = 0 = \widetilde{\mathcal{A}}^{(i)}_{(\partial_\mu T\partial^\mu U)(\partial_\nu T\partial^\nu U)(\partial_\rho T\partial^\rho\overline{U})}$.

∇ But, $\widetilde{\mathcal{A}}^{(i)}_{(\partial_\mu T\partial^\mu \overline{T})(\partial_\nu T\partial^\nu \overline{U})(\partial_\rho\phi\partial^\rho\overline{\phi})} = 0$.

One can easily check that there is no tree-level contribution to this kind of BPS saturated amplitudes, so they are exact up to one loop order (possible higher loop and non-perturbative corrections are suppressed anyway in the limit that we consider, $S \to i\infty$).

Importantly, what characterizes these amplitudes is a prepotential:

$$
\begin{aligned}
\mathcal{G}_1(T,U) \;=\;& \frac{\zeta(-1)}{2} c(0) + \frac{i}{4\pi}\frac{c(0)}{U-\overline{U}} + \frac{i}{4\pi}\frac{1}{T-\overline{T}} + \frac{3}{2\pi^2}\frac{1}{(T-\overline{T})(U-\overline{U})} \\
& - \frac{3}{\pi^2(T-\overline{T})}\,\partial_U \sum_{l>0} \mathcal{L}i_1\Big(e^{2\pi i l U}\Big) - \frac{3}{\pi^2(U-\overline{U})}\,\partial_T \sum_{k>0} \mathcal{L}i_1\Big(e^{2\pi i k T}\Big) \\
& + \sum_{(k,l)>0} c(kl)\; \mathcal{L}i_{-1}\Big(e^{2\pi i(kT+lU)}\Big)\;,
\end{aligned}
\tag{4.11}
$$

and similarly for $\mathcal{G}_2(T,\overline{U})$ in the chamber $T_2 > U_2$. Here $\zeta(-1) = -\frac{1}{12}$, and the polylogarithms are defined by $\mathcal{L}i_a(z) = \sum_{p>0} z^p p^{-a}$ for $a>0$ and $\mathcal{L}i_a(z) = (z\partial_z)^{|a|}\frac{1}{1-z}$ for $a<0$; in particular, $\mathcal{L}i_1(e^z) = -\ln(1-e^z)$, $\mathcal{L}i_0(e^z) = \frac{e^z}{1-e^z}$ and $\mathcal{L}i_{-1}(e^z) = \frac{e^z}{(1-e^z)^2}$. Moreover the sum runs over the positive roots $k>0,\ l\in\mathbb{Z}\ \ \wedge\ \ k=0,\ l>0$, and the coefficients are defined by:

$$
\sum_n c(n)q^n \;:=\; E_2(q) \;\equiv\; 1 - 24q - 72q^2 + \dots\ . \tag{4.12}
$$

This must be the derivative of some combination of the elliptic genera in (3.6), but because of the uniqueness of the quasi-modular form of weight two, it is unclear at this point of exactly which elliptic genus:† $E_2 = \frac{1}{4}{\partial_v}^2\mathcal{E}_{(K3)}(v/2,q)|_{v=0} = \frac{1}{2}{\partial_v}^4\mathcal{E}_{(\mathrm{st}\times T^2)}(v/2,q)|_{v=0} = \frac{1}{96}{\partial_v}^4\mathcal{E}_{(\mathrm{st}\times T^2)}(v/2,q)\times\ \mathcal{E}_{(K3)}(v/2,q)|_{v=0}$. While the distinction is not important here, it will be more relevant later on when we will discuss the generalization to $(2K+4)$-point amplitudes.

Note that since $c(-1)=0$, there is no singularity in the T,U moduli space and this reflects the impossibility of states becoming massless. Note also that $\mathcal{G}_1(T,U)$ has weight 2 under T–and U–duality, respectively, and indeed we find that (4.11) and its

† It was the equality of these expressions that has been misleading us to some conclusions that we have presented in a previous version of this paper.

holomorphic/anti-holomorphic cousin can be rewritten in terms of a simple product involving (regularized) Eisenstein functions:

$$\begin{aligned}\mathcal{G}_1(T,U) &= -\frac{1}{24}\widehat{E}_2(T)\widehat{E}_2(U) \\ \mathcal{G}_2(T,\overline{U}) &= -\frac{1}{24}\widehat{E}_2(T)\widehat{E}_2(\overline{U}) \ .\end{aligned} \tag{4.13}$$

These intriguing identities exhibit a factorization that is not manifest in (4.11). We can furthermore obtain both of these prepotentials (and their complex conjugates) by taking mixed derivatives of the following function:

$$\mathcal{H}(T,U) \ = \ -6\,\ln(T_2|\eta(T)|^4)\cdot\ln(U_2|\eta(U)|^4)\ , \tag{4.14}$$

which in this sense appears to be a more fundamental function[◇] for the six-point amplitudes we consider here. It is the analog of the 1/2 BPS free energy $\mathcal{F}_1^{(T^2)}(T,U)$ in (1.1), the difference being that the $\ln\eta$'s are multiplied rather than added; by adjusting possible integration constants we see that $\mathcal{H}(T,U)$ is essentially the square of $\mathcal{F}_1^{(T^2)}(T,U)$.

5. Partition functions

The holomorphic prepotential (4.11) is one of the main results of this paper. Its appearance hints at the existence of a yet unknown superspace formulation of the theory, in which it might figure as an effective lagrangian. It resembles the "Borcherds" type prepotentials that arise in other contexts [2,35,36,5], where non-negative polylogarithms appear instead. However, that difference is not important and simply to be attributed to the mass dimension of the couplings.

The structurally more profound feature is that $\mathcal{G}_1(T,U)$ intrinsically mixes the Kähler and complex structure sectors, and this is specifically tied to the 1/4 BPS

◇ Switching on all the moduli should promote $\ln(U_2|\eta(U)|^4)$ to the logarithm of some $SO(22,6,\mathbb{Z})$ modular form.

states. Indeed, when restricted to the subset of 1/2 BPS states (which corresponds to the terms with $kl=0$), the sum in (4.11) nicely separates into decoupled pieces:

$$\begin{aligned}\mathcal{G}_1(T,U) \quad &\xrightarrow[(kl)=0]{1/2BPS} \quad \sum_{k=1}^{\infty}\frac{e^{2\pi ikT}}{(1-e^{2\pi ikT})^2}+\sum_{l=1}^{\infty}\frac{e^{2\pi ilU}}{(1-e^{2\pi ilU})^2}+\dots\\ &= \ -\frac{1}{2\pi i}\big[\partial_T\ln\eta(T)+\partial_U\ln\eta(U)\big]+\dots\\ &= \ -\frac{1}{4\pi i}\big(\partial_T+\partial_U\big)\,\mathcal{F}_1^{(T^2)}(T,U)+\dots\ ,\end{aligned} \tag{5.1}$$

which in turn can be written manifestly in terms of the 1/2 BPS spectrum using [37,1]:

$$\mathcal{F}_1^{(T^2)}(T,U) \ = \ -\tfrac{1}{12}\sum_{m_1n_2-n_1m_2=0}\ln|p_R|^2=-\tfrac{1}{12}\sum_{m_1n_2-n_1m_2=0}\ln|p_L|^2\ . \tag{5.2}$$

We thus explicitly see that the mixing terms in $\mathcal{G}_1(T,U)$ (or $\mathcal{G}_2(T,\overline{U})$) correspond to the 1/4 BPS states and originate from the presence of $E_2(\overline{q})$ (or $E_2(q)$) in the integrand. Its effect is to shift the 1/2 BPS level matching condition, $|p_L|^2=|p_R|^2$, to the 1/4 BPS condition: $|p_L|^2=|p_R|^2+kl$ (or $|p_R|^2=|p_L|^2+kl$).

Using the product representation (4.13) and the well-known sum formulas of the Eisenstein series, we can represent the prepotentials in a form that generalizes the 1/2 BPS sum (5.2):

$$\begin{aligned}-24\,\mathcal{G}_1(T,U)\Big|_{\substack{holom.\\ piece}} \ &= \ E_2(T)E_2(U) \ = \sum_{\substack{(N_1,N_2)\neq(0,0)\\(M_1,M_2)\neq(0,0)}}\frac{1}{(N_2+N_1T)^2(M_2+M_1U)^2}\\ &\qquad\qquad\qquad\quad = \frac{1}{2T_2U_2}\sum_{m_1n_2-n_1m_2=0}\frac{\gamma(m_i,n_i)}{\overline{p}_L^2}\ ,\\ -24\,\mathcal{G}_2(T,\overline{U})\Big|_{\substack{holom.\\ piece}} \ &= \ E_2(T)E_2(\overline{U}) \ = \frac{1}{2T_2U_2}\sum_{m_1n_2-n_1m_2=0}\frac{\gamma(m_i,n_i)}{p_R^2}\ ,\end{aligned} \tag{5.3}$$

with $m_1=N_2M_2, m_2=N_2M_1, n_1=N_1M_2, n_2=N_1M_1$. Since there are in general many different $\{M_i,N_i\}$ that contribute to a given set $\{m_i,n_i\}$, the coefficients $\gamma(m_i,n_i)$ are in general larger than one, and this must be so since otherwise the sums would be counting (just like (5.2)) exactly the 1/2 BPS states.

Note that while $\mathcal{F}_1^{(T^2)}(T,U)$ has been written in (5.2) as a sum over 1/2 BPS states circulating in loops, it has also an interpretation in terms of world-sheet instantons

[8]; this is exhibited by the instanton expansion in the first line of eq. (5.1). Such a view-point is indeed more natural in the path integral formulation, where $\mathcal{F}_1^{(T^2)}(T,U)$ is seen as counting holomorphic maps from a toroidal world-sheet into the target space T^2.

The additional mixing terms proportional to $e^{2\pi ikT}e^{2\pi ilU}$, which are due to the 1/4 BPS states, must have an analogous instantonic interpretation, however involving holomorphic (and anti-holomorphic) maps that couple together both Kähler and complex structure sectors. Such configurations can presumably be obtained via T-duality from string networks [38,39], in which the 1/4 BPS states have a simple geometric representation.†

6. Generalization to an infinite sequence of prepotentials

We will now discuss the generalization of the results of section 4 to $(2K{+}4)$–point amplitudes with $K+1$ pairs of T and U moduli, besides one pair of moduli, $(\phi,\overline{\phi})$ with $\phi=\{T,U\}$. Performing four-fermion contractions in the left-moving sector and $4K{+}4$ fermion contractions in the right-moving sector, integrating over the location of the vertex operators and subsequently applying the Riemann identity (see Appendix B for some of the details), we eventually find:

$$\begin{aligned}\widetilde{\mathcal{A}}^{[2K+4]}_{(\partial_\nu T\partial^\nu U)^{K+1}(\partial_\rho\phi\partial^\rho\overline{\phi})} \;=\; &\frac{1}{4}\int\frac{d^2\tau}{\tau_2}\sum_{(p_L,p_R)}\Big(|p_R|^2-\frac{1}{2\pi\tau_2}\Big)(\tau_2 p_L)^{2K+2}q^{\frac12|p_L|^2}\overline{q}^{\frac12|p_R|^2}\\ &\times\Big(\frac{\partial}{\partial\overline{v}}\Big)^{2K+2}\Big[e^{-\frac{\overline{v}^2}{4\pi\tau_2}}\mathcal{E}_{(\mathrm{st}\times T^2)}(\overline{v},\overline{q})\Big]\Big|_{\overline{v}=0}\ .\end{aligned} \tag{6.1}$$

The most harmonic part then evaluates to:

$$-\frac{1}{\pi^2}(-2T_2U_2)^{K+1}\Big(\partial_T+\frac{2K}{T-\overline{T}}\Big)\Big(\partial_U+\frac{2K}{U-\overline{U}}\Big)\mathcal{G}_1^{[2K+4]}(T,U)\ , \tag{6.2}$$

† By correctly identifying the variables, we can map the mass formula for (wrapped) triple string junctions to the mass formula (2.6), ie., $M^2_{BPS}=\sum_{i=1}^3 T_{p_i,q_i}=\max\{|Z^+|^2,|Z^-|^2\}$. Here, p_i,q_i are the charges of the i-th link of the junction, T_{p_i,q_i} the corresponding tension and $Z^\pm$ the central charges of (2.4). Our results thus amount to counting such string junctions.

with prepotentials $\mathcal{G}_1^{[2K+4]}(T,U)$ that form an infinity sequence given by:

$$\mathcal{G}_1^{[2K+4]}(T,U)\Big|_{\substack{harm.\\ piece}} = \frac{1}{2}\zeta(1-2K)\, c^{[2K+4]}(0) + \sum_{(k,l)>0} c^{[2K+4]}(kl)\ \mathcal{L}i_{1-2K}\Big(e^{2\pi i(kT+lU)}\Big). \tag{6.3}$$

The counting functions for these are simply:

$$\sum_n c^{[2K+4]}(n) q^n \ = \ \Big(\frac{\partial}{\partial v}\Big)^{2K+2} \mathcal{E}_{(\mathrm{st}\times T^2)}(v,q)\Big|_{v=0} \ = \ \Big(\frac{\partial}{\partial v}\Big)^{2K+2} \Big(\frac{i\theta_1(v,q)}{\eta^3(q)}\Big)^2\Big|_{v=0}, \tag{6.4}$$

In fact, we can concisely assemble all the prepotentials into a single generating function:

$$\begin{aligned} \widehat{\mathcal{G}}_1(T,U,V) \ &= \ \sum_{K=0}^{\infty} \frac{1}{(2K)!} V^{2K} \mathcal{G}_1^{[2K+4]}(T,U) \\ &= \sum_{(k,l,m)>0} d(4kl-m^2)\ \mathcal{L}i_3\Big(e^{2\pi i(kT+lU)} e^{mV}\Big)\ , \end{aligned} \tag{6.5}$$

where $d(4n-m^2)$ are the expansion coefficients (3.6) of $\mathcal{E}_{(\mathrm{st})\times T^2}(v,q)$.

Note that what appears here is the elliptic genus of the space-time sector times T^2, and not the elliptic genus of $K3$ as one might have expected.$^\diamond$ This is indeed a bit surprising, since the Riemann identities typically mix *all* the internal and space-time sectors (cf., (3.3)).

However, our amplitudes do not (as usual) amount to fermion number current insertions, but to more complicated fermionic contractions, and the results of Appendix B show that, in the net result, the elliptic genus of $K3$ happens to cancel out. Heuristically one may say that this is because our correlators probe only the T^2 sector of the theory because they involve only T and U.

7. Non-perturbative results for the N=4 heterotic string

So far we have been dealing with perturbative quantities in the type IIA string on $K3\times T^2$. The interesting issue now is to map these via duality to the heterotic

$\diamond$ and as we had mis-stated in a previous version of this paper. It does however appear in subsequent work [40] which deals with graviphoton amplitudes.

string on T^6, by identifying [16,17]:

$$
\begin{aligned}
T &= S_{\rm het} \equiv \frac{\theta}{2\pi} + \frac{4\pi i}{{g_{\rm het}}^2} \equiv a + ie^{-\Phi} \\
S &= T_{\rm het} \equiv B^{\rm het}_{45} + i\sqrt{|G^{\rm het}|} \\
U &= U_{\rm het} \equiv (G^{\rm het}_{45} + i\sqrt{|G^{\rm het}|})/G^{\rm het}_{44} \ ,
\end{aligned}
\tag{7.1}
$$

where $T_{\rm het}, U_{\rm het}$ correspond to the two-torus in $T^6 = T^4 \times T^2$.

The perturbative T-dependence that we have been computing before will thus give non-perturbative information about the heterotic string. Remember that we have been suppressing non-perturbative corrections in the type IIA string by going to weak coupling, by sending $S = T_{\rm het} \to i\infty$. This corresponds to the decompactification limit of the heterotic two-torus.

More specifically, while the Kaluza-Klein modes (labelled by m_i in (2.6)) remain KK modes in the heterotic string, the type IIA windings around 1-cycles of T^2 (labelled by n_i) turn into magnetically charged wrapping modes of the heterotic fivebrane‡ around 5-cycles in T^6. Alternatively, one may imagine wrapping the fivebrane first around the sub-torus T^4, to yield a string in six dimensions that is dual to the heterotic string [16,44,45,17,26]. The type IIA windings n_i are then the same as the windings of this dual string around 1-cycles of the remaining T^2 on the heterotic side. In total we thus have dyonic bound states of wrapped fivebranes of charge m_i with KK modes of momentum n_j, which are 1/4 BPS if $m_1 n_2 - m_2 n_1 \neq 0$ and 1/2 BPS if the DZW product vanishes. The windings and momenta are exchanged by S-duality, which is a non-perturbative symmetry from the heterotic string point of view, but a perturbative one from either the type IIA string or from the heterotic fivebrane point of view [21,46].

However, in analogy to the type IIA side, it is more natural to interpret the prepotentials in terms of instanton series. Quite generally, world-sheet instantons are mapped under the duality to space-time instantons, and indeed contributions of the form $e^{2\pi i k S_{\rm het}}$ correspond [14] to gravitational fivebrane instantons that arise from winding the heterotic fivebrane around the whole of T^6.

‡ Since we are at a generic point in the Narain moduli space, where there are no non-abelian gauge symmetries, this is the neutral heterotic fivebrane [41–43] with zero size, or a "small instanton" [26].

As far as the $U_{\rm het}$ dependence is concerned (which simply describes KK excitations), it is actually more interesting to convert $U_{\rm het} \to T_{\rm het}$, by making use of the $T_{\rm het} - U_{\rm het}$ exchange symmetry of the heterotic string. The purely $T_{\rm het}$ dependent terms then describe heterotic world-sheet instanton contributions, and the mixed terms in the prepotentials

$$\begin{aligned} \mathcal{G}_1^{[2K+4]}(S_{\rm het},T_{\rm het})\Big|_{\substack{harm.\\ piece}} &\sim \left(\frac{\partial}{\partial V}\right)^{2K+2} \widehat{\mathcal{G}}_1(S_{\rm het},T_{\rm het},V)\Big|_{V\to 0} \\ &\sim \sum_{(k,l)>0} c^{[2K+4]}(kl)\,\mathcal{L}i_{1-2K}\Big(e^{2\pi i k S_{\rm het}}e^{2\pi i l T_{\rm het}}\Big) \end{aligned} \tag{7.2}$$

must therefore be due to bound states or superpositions of fivebrane instantons with world-sheet instantons. In particular, the remarkable factorization property (4.13) of the prepotential for the six-point couplings,

$$\mathcal{G}_1^{[6]}(S_{\rm het},T_{\rm het})\Big|_{\substack{holom.\\ piece}} = -\frac{1}{24}\left(1-24\sum_{k>0}k\frac{e^{2\pi i k S_{\rm het}}}{1-e^{2\pi i k S_{\rm het}}}\right)\left(1-24\sum_{l>0}l\frac{e^{2\pi i l T_{\rm het}}}{1-e^{2\pi i l T_{\rm het}}}\right), \tag{7.3}$$

tells us that the fivebrane and world-sheet instanton sectors that contribute to these couplings must be essentially independent.

In fact, by investigating the dependence on the coupling constants we find that the prepotentials $\mathcal{G}_1^{[2K+4]}$ correspond to non-perturbative corrections to $(2K+4)$-point amplitudes at K-loop order in the heterotic string, so that the world-sheet instantons are of genus $g \le K$.[♮]

Note that world-sheet instantons on top of a fivebrane can also be viewed as gauge instantons in the world-volume theory of the fivebranes [47]. More specifically, it is known that a stack of $Q_5 = k$ heterotic fivebranes has $Sp(2k)$ gauge symmetry [26,48]. Accordingly it has among other terms:

$$\int d^6x\,\left(B \wedge \mathop{\rm tr}_{Sp(2k)} F\wedge F + \frac{1}{{g_{5br}}^2}\mathop{\rm tr}_{Sp(2k)} F^2\right)$$

on its world-volume, the first term being necessary for anomaly cancellation [49]. It is known [16,17] that the "space-time" coupling of the T^4-wrapped fivebrane (or

♮ The prepotentials $\mathcal{G}_1^{[2K+4]}$ are thus analogs of the well-known prepotentials F_g that arise (in $N=2$ supersymmetric theories) at one-loop order in the heterotic string but at g loops on the type II side [8,9,10,12].

dual string, that is) is equal to the world-sheet coupling of the fundamental string, which means: $\frac{1}{{g_{5br}}^2} = \sqrt{|G^{\rm het}|}$. Comparing to (7.1), we thus see that a charge $Q_1 = l$ instanton on top of a charge $Q_5 = k$ fivebrane will give an additional factor of $e^{2\pi i l T_{\rm het}}$ besides $e^{2\pi i k S_{\rm het}}$, and this is what finally gives a particularly interesting physical interpretation of the $S_{\rm het} - T_{\rm het}$ mixing terms in (7.2). Moreover the instantons must break one-half of the supersymmetries on the fivebrane, so that the total configuration has only 1/4 unbroken supersymmetries.

Something non-trivial may then be learned for these gauge theories from the numerical values of the coefficients of the mixing terms in (7.2). In analogous situations such coefficients count either isolated instantons, or Euler numbers of the moduli spaces if the instantons are not isolated. Most likely the coefficients mean something similar here too, and in particular $c^{[2K+4]}(kl)$ should carry information about the cohomology of moduli spaces of charge l, genus K instantons in the $Sp(2k)$ gauge theories on T^6 (the exchange symmetry in k and l would relate this to charge k instantons in $Sp(2l)$ gauge theories). We hope to present a more complete discussion elsewhere.

Acknowledgements

We would like to thank I. Antoniadis, A. Brandhuber, B. de Wit, K. Förger, Y. Oz, S. Rey, A. Sen, H. Verlinde, N.P. Warner and in particular J. Harvey, E. Kiritsis and P. Mayr for discussions. We also would like to thank C. Vafa for valuable comments about the manuscript. Moreover we thank the NATO Advanced Study Institute at Cargèse for kind invitation and providing a pleasant atmosphere.

Appendix A. Generalized world–sheet torus integrals

In this section we outline how to evaluate world–sheet torus integrals of the following form:

$$\begin{aligned} &\frac{\partial^{q_1+q_2+q_3+q_4}}{\partial\Lambda_1^{q_1}\partial\Lambda_2^{q_2}\partial\Lambda_3^{q_3}\partial\Lambda_4^{q_4}} \int \frac{d^2\tau}{\tau_2}\, \tau_2^r \frac{1}{\tau_2^s} \sum_{(p_L,p_R)} e^{\pi i \tau |p_L|^2} e^{-\pi i \overline{\tau} |p_R|^2} \\ &\times e^{\Lambda_1 \overline{p}_R + \Lambda_2 p_R + \Lambda_3 \overline{p}_L + \Lambda_4 p_L}\; e^{-\frac{1}{2\pi\tau_2}(\Lambda_1\Lambda_2+\Lambda_2\Lambda_3+\Lambda_3\Lambda_4+\Lambda_4\Lambda_1)}\; f_k(\overline{q}) g_l(q)\Big|_{\Lambda_i=0}\,, \end{aligned} \tag{A.1}$$

where $f_k(\overline{q})=\sum\limits_n c(n)\overline{q}^n$ and $g_l(q)=\sum\limits_m d(m)q^m$ are modular functions of weights k,l, respectively, and the integers obey $r,s\geq 0$, $q_1,q_2,q_3,q_4\geq 0$. Modular invariance of the integrand requires:

$$\begin{aligned}q_1+q_2-r+s+k&=0\\ q_3+q_4-r+s+l&=0\ .\end{aligned}\tag{A.2}$$

The integral (A.1) can be performed with the orbit decomposition method of [50]. In the following we will discuss only the non–degenerate orbit I_1, as the degenerate orbit has already been evaluated in ref. [32].

In the same reference, the non–degenerate orbit I_1 with $g_l(q)=1, f_k(\overline{q})\neq 1$ has been worked out as well. A general feature of I_1 is that the T and U moduli always appear in pairs in the poly-logarithms that are either completely holomorphic or anti-holomorphic, i.e., either (T,U) and/or $(\overline{T},\overline{U})$ appear.

Before we go to the general case, let us discuss the example $f_k(\overline{q})=1, g_l(q)\neq 1$, which is in fact what we need in section 4:

$$\begin{aligned}&\frac{\partial^{q_1+q_2+q_3+q_4}}{\partial\Lambda_1^{q_1}\partial\Lambda_2^{q_2}\partial\Lambda_3^{q_3}\partial\Lambda_4^{q_4}}\int\frac{d^2\tau}{\tau_2}\ \tau_2^r\frac{1}{\tau_2^s}\sum_{(p_L,p_R)}\ e^{\pi i\tau|p_L|^2}e^{-\pi i\overline{\tau}|p_R|^2}\\ &\times e^{\Lambda_1\overline{p}_R+\Lambda_2p_R+\Lambda_3\overline{p}_L+\Lambda_4p_L}\ e^{-\frac{1}{2\pi\tau_2}(\Lambda_1\Lambda_2+\Lambda_2\Lambda_3+\Lambda_3\Lambda_4+\Lambda_4\Lambda_1)}\ g_l(q)\Big|_{\Lambda_i=0}\ .\end{aligned}\tag{A.3}$$

The presence of a *holomorphic* function $g_l(q)$ (which is in contrast to the usually considered situation) has as consequence that now mixed holomorphic/anti-holomorphic pairs of moduli appear in the arguments of the poly-logarithms, i.e., $(T,\overline{U})$ and/or $(\overline{T},U)$. After introducing

$$\begin{aligned}b&=p^2-\frac{i(\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4)p}{\pi\sqrt{2T_2U_2}}-\frac{1}{8}\frac{(\Lambda_1-\Lambda_2+\Lambda_3-\Lambda_4)^2}{\pi^2T_2U_2}\\ \varphi&=p(kT_1+lU_1)\\ &\quad+\frac{1}{2\pi\sqrt{2T_2U_2}}\Big[(kT_2-lU_2)\Lambda_1+(-kT_2+lU_2)\Lambda_2+(-kT_2-lU_2)\Lambda_3+(kT_2+lU_2)\Lambda_4\Big]\end{aligned}\tag{A.4}$$

plus the function

$$\widetilde{I}_1(\alpha,\beta)=\frac{2}{\sqrt{\beta b}}e^{-2\pi(kT_2-lU_2)\sqrt{\alpha\beta b}}\ e^{-2\pi i\varphi}\ ,\tag{A.5}$$

we obtain the following closed expression for I_1 in the chamber $T_2 > U_2$:

$$I_1 = \frac{\partial^{q_1+q_2+q_3+q_4}}{\partial\Lambda_1^{q_1}\partial\Lambda_2^{q_2}\partial\Lambda_3^{q_3}\partial\Lambda_4^{q_4}} \times \\ \times \sum_{\substack{k>0 \\ l\in\mathbb{Z}}} \sum_{p\neq 0} d(-kl) \Big[\frac{T_2U_2}{\pi(kT_2-lU_2)^2}\Big]^r \frac{1}{(\pi T_2U_2b)^s} \frac{(-1)^{r+s}\partial^{r+s}}{\partial\alpha^r\partial\beta^s} \widetilde{I}_1(\alpha,\beta)\Big|_{\substack{\alpha=1 \\ \beta=1}, \Lambda_i=0} . \tag{A.6}$$

Finally, for the general case where $f_k, g_l \neq 1$, the expressions (A.5) and (A.6) are modified to

$$I_1(\alpha,\beta) = \frac{2}{\sqrt{\beta b}} e^{-2\pi\sqrt{(kT_2+lU_2)^2+4T_2U_2(n-kl)}\sqrt{\alpha\beta b}}\ e^{-2\pi i\varphi} \tag{A.7}$$

and

$$I_1 = \frac{\partial^{q_1+q_2+q_3+q_4}}{\partial\Lambda_1^{q_1}\partial\Lambda_2^{q_2}\partial\Lambda_3^{q_3}\partial\Lambda_4^{q_4}} \sum_n \sum_{\substack{k>0 \\ l\in\mathbb{Z}}} \sum_{p\neq 0} c(n)d(n-kl) \times \\ \times \Big[\frac{T_2U_2}{\pi[(kT_2+lU_2)^2+4T_2U_2(n-kl)]}\Big]^r \frac{1}{(\pi T_2U_2b)^s} \frac{(-1)^{r+s}\partial^{r+s}}{\partial\alpha^r\partial\beta^s} I_1(\alpha,\beta)\Big|_{\substack{\alpha=1 \\ \beta=1}, \Lambda_i=0} , \tag{A.8}$$

The last equation represents the result for the non–degenerate orbit of (A.1). We see that the amount of holomorphic/anti–holomorphic mixing is determined by $m = n-kl$ and is absent for $m = 0$. This reflects in our context that 1/4 BPS states can mix holomorphic and anti-holomorphic sectors, in contrast to the 1/2 BPS states.

Appendix B. Fermionic contractions and bipartite Graphs

In this section we want to calculate the following correlator of $2N$ real fermions (where $N \equiv (2K+2)$)

$$\sum_{\substack{\text{spin} \\ \text{structures } \alpha}} \int dz_1 \ldots \int dz_N\ \langle \psi^{i_1}(z_1)\psi^{j_1}(z_1)\ldots\psi^{i_N}(z_N)\psi^{j_N}(z_N)\rangle_\alpha\ , \tag{B.1}$$

which appears in the amplitudes $\widetilde{\mathcal{A}}^{[2K+4]}_{(\partial_\nu T\partial^\nu U)^{K+1}(\partial_\rho\phi\partial^\rho\overline{\phi})}$ in (6.1). For these amplitudes, where the left–moving sector is saturated with four fermionic insertions, the computations reduce to the right–moving part only; this is in line with the considerations about elliptic genera of section 3, and of course a reflection of the fact that only 1/4 BPS states contribute.

We therefore expect (B.1) to be given by a $(2K+2)$–fold derivative of an elliptic genus, and by considering modular invariance we see that this genus should be a Jacobi form of weight -2. There are only two natural candidates for it, namely either the total elliptic genus

$$\begin{aligned}\mathcal{E}(v_1, v_2, q) &= \mathcal{E}_{(\mathrm{st}\times T^2)}[\tfrac{1}{2}(v_1+v_2), q]\ \mathcal{E}_{(K3)}[\tfrac{1}{2}(v_1-v_2), q] \\ &= \frac{i\theta_1[\frac{1}{2}(v_1+v_2), q]}{\eta^3(q)} \frac{i\theta_1[\frac{1}{2}(v_1+v_2), q]}{\eta^3(q)} \mathcal{E}_{(K3)}\left[\tfrac{1}{2}(v_1-v_2), q\right] ,\end{aligned} \tag{B.2}$$

or just only the weight -2 factor of it, which is $\mathcal{E}_{(\mathrm{st}\times T^2)}$.

To fix this ambiguity, our strategy will be to do the fermion contractions in (B.1) for a few values of K, integrate these and sum over the spin structures, and then compare the results with the two candidate genera.

To deal with the fermionic contractions, we first decompose the correlator (B.1) into a product of N two–point functions:

$$\langle \psi^i(z_1)\psi^j(z_2)\rangle_\alpha = \delta^{ij}\ \frac{\theta_\alpha(z_{12},\tau)\theta_1'(0,\tau)}{\theta_\alpha(0,\tau)\theta_1(z_{12},\tau)} . \tag{B.3}$$

There are in general many of such partitions, and denoting contractions $\langle\psi^i(z_i)\psi^j(z_j)\rangle$ by (ij), the pattern looks like:

$$\begin{aligned} [2,2,2\ldots,2,2] &\sim (12)^2(34)^2\ldots(N-1,N)^2 , \\ [2,2\ldots,2,4] &\sim (12)(23)(34)(41)\times(56)^2\ldots(N-1,N)^2 , \\ \vdots \qquad & \qquad \vdots \quad \vdots \quad \vdots \\ [N] &\sim (12)(23)(34)\ldots(N1) . \end{aligned} \tag{B.4}$$

Note that these contractions form cycles, and the idea is to perform the integrations and spin structure sums just for one (canonically ordered) representative of each cycle class, P (above, the cycle class is indicated on the left). Indeed, as the ordering of the positions z_i influences the z_i–integrations in (B.1), each cycle class in (B.4) will in general lead, after integration, to a different modular function $g^P_\alpha(\overline{q})$. Subsequently will then need to multiply the result for each cycle class with the appropriate combinatorial factor.

Moreover, we need to sum over the spin structures α, which amounts to folding the $g_\alpha^P(\overline{q})$ into the right–moving part of the partition function, which then yields new functions $G^P(\overline{q})$. We find it easiest to write this map in terms of a $\mathbb{Z}_N$ orbifold limit of the $K3$:

$$G^P(\overline{q}) = \sum_{(h,g)\neq(0,0)} \frac{\eta(\overline{q})^{-6}}{\Theta\left[{1+h \atop 1+g}\right]\Theta\left[{1-h \atop 1-g}\right]} \sum_{(\alpha,\beta)} g_\alpha^P(\overline{q})\ \Theta^2\left[{\alpha \atop \beta}\right](\overline{q})\Theta\left[{\alpha-h \atop \beta-g}\right](\overline{q})\Theta\left[{\alpha+h \atop \beta+g}\right](\overline{q})\ , \tag{B.5}$$

Taking all this together, we see that (B.1) may be written in the following way:†

$$\begin{aligned}\sum_\alpha \int dz_1 \ldots \int dz_N\ \langle \psi^{i_1}(z_1)\psi^{j_1}(z_1)\ldots\psi^{i_N}(z_N)\psi^{j_N}(z_N)\rangle_\alpha \\ = (2\tau_2)^{N-1}\pi^N \sum_{\substack{\text{cycle}\\ \text{classes } P}} n_P\ G^P(\overline{q})\ ,\end{aligned} \tag{B.6}$$

where n_P denote the combinatorial factors that count the number of permutations in (B.4).

Let us now compute the (quasi)-modular forms G^P for some low values of $N \equiv 2K+2$:

4 fermion contractions ($N=2$): There is only the possibility $(12)^2$, which yields:

$$g_\alpha^{[2]}(\tau) = \frac{1}{3}(E_2 + e_\alpha)\ , \tag{B.7}$$

where $e_\alpha(q) \ = -24q\frac{d}{dq}\ln\theta_\alpha(q) - E_2(q)$. Inserted in (B.5), this gives◇:

$$G^{[2]} = -12\ . \tag{B.8}$$

8 fermion contractions ($N=4$): There are two possibilities, namely, (i) $(12)^2(34)^2$ and (ii) $(12)(23)(34)(41)$ which integrate to

$$\begin{aligned} g_\alpha^{[2,2]}(\tau) &= \frac{1}{3^2}(E_2+e_\alpha)^2\ ,\\ g_\alpha^{[4]}(\tau) &= \frac{1}{9}(E_4 - e_\alpha^2)\ ,\end{aligned} \tag{B.9}$$

† Correlators similar to (B.1) appear in N gauge boson amplitudes with fermionized currents. In that case the combinations (B.4) correspond to the different invariants $(\mathrm{Tr}F^{2n})^m\ , 2nm = N$, which arise in the decomposition of the N gauge boson amplitude. For $N=4$ this has been studied before in the literature [51].

◇ We perform the calculation in an $\mathbb{Z}_2$ orbifold limit of $K3$. However the calculations can be easily generalized to $\mathbb{Z}_N$ orbifold limits along [52].

respectively. Inserting into (B.5) leads to:[‡]

$$
\begin{aligned}
G^{[2,2]} &= -8E_2 \ , \\
G^{[4]} &= 0 \ .
\end{aligned} \tag{B.10}
$$

In the following we will explicitly display the spin-structure dependent correlator only for the last combination in (B.4), namely $g_\alpha^{[N]}(q)$, as the others are combinations of correlators with less fermions.

12 fermion contractions ($N=6$): The chain (12)(23)(34)(45)(56)(61) gives after the z_i–integration

$$
g_\alpha^{[6]}(\tau) \ = \ \frac{2}{45}(E_6 - E_4 e_\alpha) \ . \tag{B.11}
$$

Altogether, after the orbifold sum we obtain:

$$
\begin{aligned}
G^{[2,2,2]} &= -4(E_2^2 + E_4) \ , \\
G^{[2,2,4]} &= -\frac{8}{3}E_4 \ , \\
G^{[6]} &= -\frac{8}{5}E_4 \ .
\end{aligned} \tag{B.12}
$$

16 fermion contractions ($N=8$): Similarly as before:

$$
g_\alpha^{[8]}(\tau) \ = \ \frac{2^8}{5040}(-\frac{1}{16}E_4^2 - \frac{5}{48}e_\alpha^4 + \frac{1}{6}E_4 e_\alpha^2) \ , \tag{B.13}
$$

and

$$
\begin{aligned}
(12)^2(34)^2(56)^2(78)^2 &\qquad G^{[2,2,2,2]} = -\tfrac{8}{9}(2E_2^3 + 6E_2E_4 + E_6) \\
(12)(23)(34)(41)\times(56)(67)(78)(85) &\qquad G^{[4,4]} = -\tfrac{8}{9}E_6 \\
(12)(23)(34)(45)(56)(61)\times(78)^2 &\qquad G^{[2,6]} = -\tfrac{8}{15}(E_6 + E_2E_4) \\
(12)(23)(34)(41)\times(56)^2(78)^2 &\qquad G^{[2,2,4]} = -\tfrac{8}{9}(E_6 + 2E_2E_4) \\
(12)(23)(34)(45)(56)(67)(78)(81) &\qquad G^{[8]} = -\tfrac{8}{21}E_6 \ .
\end{aligned} \tag{B.14}
$$

20 fermion contractions ($N=10$):

$$
g_\alpha^{[10]}(\tau) = \frac{-2^{10}}{362880}(\frac{11}{12}E_4E_6 + \frac{5}{6}E_6 e_\alpha^2 + \frac{7}{4}E_4^2 e_\alpha) \ . \tag{B.15}
$$

‡ We remark that while the last type of contractions leads to a vanishing result for $K3 \times T^2$ vacua, it gives a non-zero contribution in the corresponding computation in the heterotic string and so leads to additional kinematics.

and

$$
\begin{array}{rl}
(12)^2(34)^2(56)^2(78)^2(9,10)^2 & G^{[2,2,2,2,2]}=-\frac{4}{27}(5E_2^4+30E_2^2E_4+10E_2E_6+9E_4^2) \\
(12)(23)(34)(41)\times(56)(67)(78)(89)(9,10)(10,5) & G^{[4,6]}=-\frac{16}{45}E_4^2 \\
(12)(23)(34)(41)\times(56)(67)(78)(85)\times(9,10)^2 & G^{[2,4,4]}=-\frac{8}{27}(2E_4^2+E_2E_6) \\
(12)(23)(34)(45)(56)(67)(78)(81)\times(9,10)^2 & G^{[2,8]}=-\frac{16}{945}(18E_4^2+\frac{15}{2}E_2E_6) \\
(12)(23)(34)(45)(56)(61)\times(78)^2(9,10)^2 & G^{[2,2,6]}=-\frac{8}{45}(3E_4^2+2E_2E_6+E_2^2E_4) \\
(12)(23)(34)(41)\times(56)^2(78)^2(9,10)^2 & G^{[2,2,2,4]}=-\frac{8}{9}(E_4^2+E_2E_6+E_2^2E_4) \\
(12)(23)(34)(45)(56)(67)(78)(89)(9,10)(10,1) & G^{[2,2,2,2,2]}=-\frac{8}{45}E_4^2\ .
\end{array}
\tag{B.16}
$$

24 fermion contractions ($N=12$):

$$
g_\alpha^{[12]}(\tau)=\frac{2}{4455}(-\frac{34}{63}E_6^2-\frac{1}{5}E_4^3+\frac{92}{63}E_6e_\alpha^3+\frac{11}{5}E_4^2e_\alpha^2)\ , \tag{B.17}
$$

and

$$
\begin{array}{rl}
(12)^2(34)^2(56)^2(78)^2(9,10)^2(11,12)^2 & G^{[2,2,2,2,2,2]}=-\frac{1}{27}(8E_2^5+80E_2^3E_4+40E_2^2E_6 \\
 & \qquad +72E_2E_4^2+16E_4E_6) \\
(12)(23)(34)(41)\times(56)(67)(78)(89)(9,10)(10,11)(11,12)(12,1) & G^{[4,8]}=-\frac{136}{945}E_4E_6 \\
(12)(23)(34)(45)(56)(61)\times(78)(89)(9,10)(10,11)(11,12)(12,1) & G^{[6,6]}=-\frac{32}{225}E_4E_6 \\
(12)(23)(34)(41)\times(56)(67)(78)(81)\times(9,10)(10,11)(11,12)(12,1) & G^{[4,4,4]}=-\frac{8}{27}E_4E_6 \\
(12)(23)(34)(41)\times(56)(67)(78)(85)\times(9,10)^2(11,12)^2 & G^{[2,2,4,4]}=-\frac{8}{81}(E_2^2E_6+4E_2E_4^2+4E_4E_6) \\
(12)(23)(34)(41)\times(56)(67)(78)(89)(9,10)(10,1)\times(11,12)^2 & G^{[2,4,6]}=-\frac{16}{135}(E_2E_4^2+2E_4E_6) \\
(12)(23)(34)(45)(56)(67)(78)(89)(9,10)(10,1)\times(11,12)^2 & G^{[2,10]}=-\frac{8}{135}(E_2E_4^2+\frac{41}{21}E_4E_6) \\
(12)(23)(34)(45)(56)(67)(78)(81)\times(9,10)^2(11,12)^2 & G^{[2,2,8]}=-\frac{8}{189}(E_2^2E_6+\frac{24}{5}E_2E_4^2+\frac{22}{5}E_4E_6) \\
(12)(23)(34)(45)(56)(61)\times(78)^2(9,10)^2(11,12)^2 & G^{[2,2,2,6]}=-\frac{8}{15}(\frac{1}{9}E_2^3E_4+\frac{1}{3}E_2^2E_6+E_2E_4^2+\frac{5}{9}E_4E_6) \\
(12)(23)(34)(41)\times(56)^2(78)^2(9,10)^2(11,12)^2 & G^{[2,2,2,2,4]}=-\frac{8}{27}(\frac{4}{3}E_2^3E_4+2E_2^2E_6+4E_2E_4^2+\frac{5}{3}E_4E_6) \\
(12)(23)(34)(45)(56)(67)(78)(89)(9,10)(10,11)(11,12)(12,1) & G^{[12]}=-\frac{736}{10395}E_4E_6\ .
\end{array}
\tag{B.18}
$$

In order to assemble these modular functions into (B.6), we still need to find the combinatorial factors n_P. For this we employ a graphical method, somewhat similar in spirit to that what was used in the second reference of [51]. Indeed all the contractions can be represented by graphs, which can be labelled by their cycle structure.

More precisely, we need to consider graphs with two kinds of vertices, one kind referring to the moduli T and the other to the moduli U. That is, the first kind of vertices correspond to operators $(k \cdot \psi)\Psi$, while the second kind corresponds to $(k \cdot \psi)\overline{\Psi}$. Charge conservation for $\Psi, \overline{\Psi}$ and the kinematical structure of the form $(\partial_\mu T \partial^\mu U)^{K+1}$ then implies that only contractions between the two sets of vertices are allowed; furthermore the contractions must form loops made from alternating sequences of $\psi\overline{\psi}$ and $\Psi\overline{\Psi}$ propagators – see the figure.

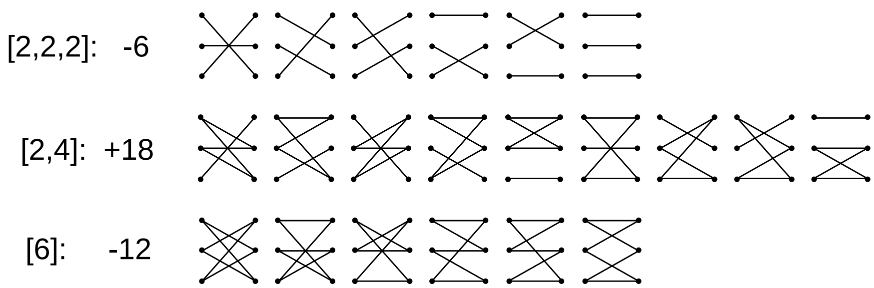

Fig.1: *Bipartite graphs relevant for $K = 2$. Each point on the left of a diagram corresponds to an operator $(k \cdot \psi)\Psi$, while on the right it correspond to $(k \cdot \psi)\overline{\Psi}$. Each loop has to be counted twice, reflecting the two ways to assign to it an alternating sequence of $\psi\overline{\psi}$ and $\Psi\overline{\Psi}$ propagators. The cycle structure also determines the signs.*

Such "bipartite" graphs are characterized by incidence matrices of the block from

$$I = \begin{pmatrix} 0 & R \\ R^t & 0 \end{pmatrix}$$

where $R = \sum_{i,j}(r_i + r_j)$ and where $\{r_i\}$ are all the permutations of the columns of the $N/2 = K + 1$ dimensional identity matrix. Clearly there are in total $((K + 1)!)^2$ such graphs. Each of such graphs needs now to be classified with respect to its cycle structure, which also determines its sign (given by the signature of the permutation).

Being pragmatic, we generated these graphs up to $N = 12$ with Mathematica (which gives a formidable number), and decomposed them in terms of their cycle structure. In this way, we obtained the following list of combinatorial coefficients:

$$
\begin{aligned}
N=4: &\quad n_{[2,2]}=2, n_{[4]}=-2 \\
N=6: &\quad n_{[2,2,2]}=-6,\ n_{[2,4]}=18,\ n_{[6]}=-12 \\
N=8: &\quad n_{[2,2,2,2]}=24, n_{[2,2,4]}=-144, \\
&\quad n_{[4,4]}=72, n_{[2,6]}=192, n_{[8]}=-144 \\
N=10: &\quad n_{[2,2,2,2,2]}=-120, n_{[2,2,2,4]}=1200, n_{[2,4,4]}=-1800, \\
&\quad n_{[2,2,6]}=-2400, n_{[4,6]}=2400, n_{[2,8]}=3600, \\
&\quad n_{[10]}=-2880 \\
N=12: &\quad n_{[2,2,2,2,2,2]}=720, n_{[2,2,2,2,4]}=-10800, n_{[2,2,4,4]}=32400, \\
&\quad n_{[2,2,2,6]}=28800, n_{[4,4,4]}=-10800, n_{[2,4,6]}=-86400, \\
&\quad n_{[2,2,8]}=-64800, n_{[6,6]}=28800, n_{[4,8]}=64800, \\
&\quad n_{[2,10]}=103680, n_{[12]}=-86400.
\end{aligned}
$$

Inserting these together with our expressions for the $G^P(q)$ into (B.6), then produces combinations of Eisenstein series that *exactly* match the following derivatives of $\mathcal{E}_{(\mathrm{st}\times T^2)}(v,q)$:

$$
\sum_{\substack{\text{cycle}\\ \text{classes } P}} n_{P(K)} G^P_{(K)}(\overline{q}) \;=\; \Big(\frac{\partial}{\partial v}\Big)^{2K+2}\Big(\frac{i\theta_1(\frac{1}{2}v,q)}{\eta^3(q)}\Big)^2\Big|_{v=0} \qquad ,
$$

for $K=0,\ldots,5$. By the uniqueness of the Jacobi forms this makes clear what the relevant elliptic genus is, and in particular that the elliptic genus of $K3$ cancels out.

References

[1] C. Vafa, *A Stringy test of the fate of the conifold,* Nucl. Phys. **B447** 252 (1995), `hep-th/9505023`.

[2] J.A. Harvey and G. Moore, *Algebras, BPS states, and strings,* Nucl. Phys. **B463** 315 (1996), `hep-th/9510182`.

[3] C. Bachas and E. Kiritsis, *F^4 terms in N=4 string vacua,* Nucl. Phys. Proc. Suppl. **55B** 194 (1997), `hep-th/9611205`.

[4] C. Bachas, C. Fabre, E. Kiritsis, N.A. Obers and P. Vanhove, *Heterotic / type I duality and D-brane instantons,* Nucl. Phys. **B509** 33 (1998), `hep-th/9707126`; E. Kiritsis and N.A. Obers, *Heterotic type I duality in $d < 10$-dimensions, threshold corrections and D-instantons,* JHEP**10** 004 (1997), `hep-th/9709058`.

[5] W. Lerche and S. Stieberger, *Prepotential, mirror map and F theory on K3,* Adv. Theor. Math. Phys. **2** 1105 (1998), `hep-th/9804176`;
W. Lerche, S. Stieberger and N.P. Warner, *Quartic gauge couplings from K3 geometry,* `hep-th/9811228`; *Prepotentials from symmetric products,* `hep-th/9901162`.

[6] R. Gopakumar and C. Vafa, *M-theory and topological strings. 1,2* `hep-th/9809187`, `hep-th/9812127`.

[7] E. Kiritsis, *Introduction to nonperturbative string theory,* `hep-th/9708130`; *Duality and instantons in string theory,* `hep-th/9906018`.

[8] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Holomorphic anomalies in topological field theories,* Nucl. Phys. **B405** 279 (1993), `hep-th/9302103`; *Kodaira–Spencer theory of gravity and exact results for quantum string amplitudes,* Commun. Math. Phys. **165** 311 (1994), `hep-th/9309140`.

[9] D. Ghoshal and C. Vafa, *$C = 1$ string as the topological theory of the conifold,* Nucl. Phys. **B453** 121 (1995), `hep-th/9506122`.

[10] I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, *Topological amplitudes in string theory,* Nucl. Phys. **B413** 162 (1994), `hep-th/9307158`;
M. Marino and G. Moore, *Counting higher genus curves in a Calabi-Yau manifold,* Nucl.Phys. **B543** 592 (1999), `hep-th/9808131`.

[11] J.A. Harvey and G. Moore, *Exact gravitational threshold correction in the FHSV model,* Phys. Rev. **D57** 2329 (1998), `hep-th/9611176`.

[12] A. Klemm and E. Zaslow, *Local mirror symmetry at higher genus,* `hep-th/9906046`.

[13] See e.g., *Essays on mirror manifolds*, (S. Yau, ed.), International Press 1992; *Mirror symmetry II*, (B. Greene et al, eds.), International Press 1997.

[14] J.A. Harvey and G. Moore, *Five-brane instantons and R^2 couplings in N=4 string theory,* Phys. Rev. **D57** 2323 (1998), `hep-th/9610237`.

[15] A. Gregori, E. Kiritsis, C. Kounnas, N.A. Obers, P.M. Petropoulos and B. Pioline, *R^2 corrections and nonperturbative dualities of $N = 4$ string ground states,* Nucl. Phys. **B510** 423 (1998), `hep-th/9708062`.

[16] M.J. Duff and R.R. Khuri, *Four-dimensional string/string duality,* Nucl. Phys. **B411** 473 (1994), `hep-th/9305142`. M.J. Duff, *Strong/weak coupling duality from the dual string,* Nucl. Phys. **B442** 47 (1995), `hep-th/9501030`.

[17] M.J. Duff, J.T. Liu and J. Rahmfeld, *Four-dimensional string-string-string triality,* Nucl. Phys. **B459** 125 (1996), `hep-th/9508094`.

[18] N. Berkovits and C. Vafa, *N=4 topological strings,* Nucl. Phys. **B433** 123 (1995), `hep-th/9407190`.

[19] J. Maldacena, G. Moore and A. Strominger, *Counting BPS black holes in toroidal type-II string theory,* `hep-th/9903163`.

[20] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy,* Phys. Lett. **B379** 99 (1996), `hep-th/9601029`.

[21] J.H. Schwarz and A. Sen, *Duality symmetries of 4-D heterotic strings,* Phys. Lett. **B312** 105 (1993), `hep-th/9305185`.

[22] S. Ferrara, C. Kounnas and M. Porrati, *General Dimensional Reduction Of Ten-Dimensional Supergravity And Superstring,* Phys. Lett. **181B** 263 (1986).

[23] E. Kiritsis and C. Kounnas, *Perturbative and nonperturbative partial supersymmetry breaking: N=4 $\rightarrow$ N=2 $\rightarrow$ N=1,* Nucl. Phys. **B503** 117 (1997), `hep-th/9703059`.

[24] G. Lopes Cardoso, G. Curio, D. Lüst, T. Mohaupt and S. Rey, *BPS spectra and nonperturbative gravitational couplings in N=2, N=4 supersymmetric string theories,* Nucl. Phys. **B464** 18 (1996), `hep-th/9512129`.

[25] L. Girardello, M. Porrati and A. Zaffaroni, *Heterotic Type II string duality and the H monopole problem,* Int. J. Mod. Phys. **A11** 4255 (1996), `hep-th/9508056`.

[26] E. Witten, *Small instantons in string theory,* Nucl. Phys. **B460** 541 (1996), hep-th/9511030.

[27] R. Dijkgraaf, E. Verlinde and H. Verlinde, *Counting dyons in N=4 string theory,* Nucl. Phys. **B484** 543 (1997), hep-th/9607026;
R. Dijkgraaf, G. Moore, E. Verlinde and H. Verlinde, *Elliptic genera of symmetric products and second quantized strings,* Commun. Math. Phys. **185** 197 (1997), hep-th/9608096.

[28] W. Lerche, *Elliptic Index And Superstring Effective Actions,* Nucl. Phys. **B308** 102 (1988).

[29] W. Lerche, A.N. Schellekens and N.P. Warner, *Ghost Triality And Superstring Partition Functions,* Phys. Lett. **B214** 41 (1988); *Lattices And Strings,* Phys. Rept. **177** 1 (1989).

[30] A. Schellekens and N. Warner, *Anomalies, characters and strings,* Nucl. Phys. **B287** (1987) 317;
E. Witten, *Elliptic genera and quantum field theory,* Commun. Math. Phys. **109** (1987) 525;
W. Lerche, B.E.W. Nilsson, A.N. Schellekens and N.P. Warner, Nucl. Phys. **B299** (1988) 91.

[31] J.A. Minahan, *One Loop Amplitudes On Orbifolds And The Renormalization Of Coupling Constants,* Nucl. Phys. **B298** 36 (1988);
P. Mayr and S. Stieberger, *Dilaton, antisymmetric tensor and gauge fields in string effective theories at the one loop level,* Nucl. Phys. **B412** 502 (1994), hep-th/9304055;
K. Förger, B.A. Ovrut, S.J. Theisen and D. Waldram, *Higher derivative gravity in string theory,* Phys. Lett. **B388** 512 (1996), hep-th/9605145.

[32] K. Förger and S. Stieberger, *String amplitudes and N=2, $d = 4$ prepotential in heterotic $K3 \times T^2$ compactifications,* Nucl. Phys. **B514** 135 (1998), hep-th/9709004;
K. Foerger and S. Stieberger, *Higher derivative couplings and heterotic type I duality in eight dimensions,* Nucl. Phys. **B559** (1999) 277, hep-th/9901020.

[33] F. Gonzalez-Rey, B. Kulik, I.Y. Park and M. Rocek, *Selfdual effective action of N=4 superYang-Mills,* Nucl. Phys. **B544** 218 (1999), hep-th/9810152.

[34] R. de Mello Koch, A. Paulin-Campbell and J.P. Rodrigues, *Nonholomorphic corrections from three-branes in F theory,* hep-th/9903029.

[35] T. Kawai, *N=2 heterotic string threshold correction, K3 surface and generalized Kac-Moody superalgebra,* Phys. Lett. **B372** 59 (1996), `hep-th/9512046`; *String duality and enumeration of curves by Jacobi forms,* `hep-th/9804014`.

[36] G. Moore, *String duality, automorphic forms, and generalized Kac-Moody algebras,* Nucl. Phys. Proc. Suppl. **67** 56 (1998), `hep-th/9710198`.

[37] H. Ooguri and C. Vafa, *Geometry of N=2 strings,* Nucl. Phys. **B361** (1991) 469; S. Ferrara, C. Kounnas, D. Lüst and F. Zwirner, *Duality invariant partition functions and automorphic superpotentials for (2,2) string compactifications,* Nucl. Phys. **B365** (1991) 431.

[38] J.H. Schwarz, *Lectures on superstring and M theory dualities: Given at ICTP Spring School and at TASI Summer School,* Nucl. Phys. Proc. Suppl. **55B** 1 (1997), `hep-th/9607201`.

[39] A. Sen, *String network,* JHEP **03** 005 (1998), `hep-th/9711130`.

[40] A. Hammou and J. Morales, *Fivebrane instantons and higher derivative couplings in type I theory,* `hep-th/9910144`.

[41] S. Rey, *The confining phase of superstrings and axionic strings,* Phys. Rev. **D43** 526 (1991).

[42] M.J. Duff and J.X. Lu, *Elementary five-brane solutions of $D = 10$ supergravity,* Nucl. Phys. **B354** 141 (1991).

[43] C.G. Callan, J.A. Harvey and A. Strominger, *World sheet approach to heterotic instantons and solitons,* Nucl. Phys. **B359** 611 (1991); *Worldbrane actions for string solitons,* Nucl. Phys. **B367** 60 (1991).

[44] A. Sen, *String string duality conjecture in six-dimensions and charged solitonic strings,* Nucl. Phys. **B450** 103 (1995), `hep-th/9504027`.

[45] J.A. Harvey and A. Strominger, *The heterotic string is a soliton,* Nucl. Phys. **B449** 535 (1995), `hep-th/9504047`.

[46] A. Sen, *Strong-weak coupling duality in four-dimensional string theory,* Int. J. Mod. Phys. **A9** 3707 (1994), `hep-th/9402002`.

[47] M.R. Douglas, *Branes within branes,* `hep-th/9512077`.

[48] M.R. Douglas, *Gauge fields and D-branes,* J. Geom. Phys. **28** 255 (1998), `hep-th/9604198`.

[49] J. Mourad, *Anomalies of the SO(32) five-brane and their cancellation*, Nucl. Phys. **B512** 199 (1998), `hep-th/9709012`.

[50] L.J. Dixon, V. Kaplunovsky and J. Louis, *Moduli dependence of string loop corrections to gauge coupling constants,* Nucl. Phys. **B355** 649 (1991).

[51] D.J. Gross, J.A. Harvey, E. Martinec and R. Rohm, *Heterotic string theory (2). The interacting heterotic string*, Nucl. Phys. **B267** (1986) 75;
W. Lerche, B.E.W Nilsson, A.N. Schellekens and N.P. Warner, *Anomaly cancelling terms from the elliptic genus*, Nucl. Phys. **B299** (1988) 91;
J. Ellis, P. Jetzer and L. Mizrachi, *One–loop corrections to the effective field theory*, Nucl. Phys. **B303** (1988) 1.

[52] S. Stieberger, *(0,2) heterotic gauge couplings and their M-theory origin*, Nucl. Phys. **B541**, 109 (1999), `hep-th/9807124`.