WAVES ON NONCOMMUTATIVE SPACETIME AND GAMMA-RAY BURSTS

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Abstract Quantum group Fourier transform methods are applied to the study of processes on noncommutative Minkowski spacetime $[x^i, t] = \imath \lambda x^i$. A natural wave equation is derived and the associated phenomena of *in vacuo* dispersion are discussed. Assuming the deformation scale λ is of the order of the Planck length one finds that the dispersion effects are large enough to be tested in experimental investigations of astrophysical phenomena such as gamma-ray bursts. We also outline a new approach to the construction of field theories on the noncommutative spacetime, with the noncommutativity equivalent under Fourier transform to non-Abelianness of the 'addition law' for momentum in Feynman diagrams. We argue that CPT violation effects of the type testable using the sensitive neutral-kaon system are to be expected in such a theory.

1 Introduction

Quantum groups and their associated noncommutative geometry have been proposed as a candidate for the generalisation of geometry needed for Planck scale physics in [1, 2]. Using such methods there were provided models exhibiting the unification of quantum and gravity-like effects into a single system with a flat space quantum limit when a parameter $G \to 0$ and a classical but curved space limit when $\hbar \to 0$. Radically new phenomena at the Planck scale were also proposed, notably an extension of wave-particle duality between position and momentum via Fourier theory to a novel duality between quantum observables and states in which their roles

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could be interchanged. For the specific models in [1, 2] this observable-state duality was implemented through quantum group duality with the dual system of the same form but with different values of the parameters (i.e. a form of T-duality). In addition, in a different approach, it was proposed in [3] that noncommutative geometry in the form of q-deformation could provide an effective way to model Planck-scale quantum corrections to spacetime geometry. It was argued that field theory on such spacetimes would be more regular, with UV divergences appearing as poles at $q = 1$, while symmetries would be preserved as quantum group symmetries. More recently in Refs. [4, 5] an analysis of some candidate quantum-gravity phenomena was used to suggest that an effective large-distance description of some aspects of quantum gravity might be based on quantum symmetries and noncommutative geometry, while it was argued that at the Planck scale even more novel structures might be required. In particular, it was observed that the "classical-apparatus limit", which is fully consistent [6, 7] with ordinary Quantum Mechanics, is not accessible [8] in theories with gravitation, and this was used to suggest [5] that a fully developed quantum gravity should be based on a mechanics departing from ordinary quantum mechanics in such a way as to accommodate a new concept of apparatus and an accordingly modified relation between the apparatus and the system under observation.

In the present article we report progress toward the use of some of these ideas in a testable and workable approach to particle physics on noncommutative spacetime.

Since Planck-scale energies are so very far from present-day experiments we will be mostly attempting to model quantum gravity effects at distances much larger than the Planck length, postponing to future work the investigation of whether quantum-group ideas might prove useful for a description of physics at even shorter distances (perhaps all the way down to the Planck-length). Specifically, we are interested in noncommutative Minkowski space as a basis for an effective description of phenomena associated to a nontrivial "foamy" quantum gravity vacuum of the type considered by Hawking, Wheeler and others. When probed very softly such a space would appear as an ordinary Minkowski space, but probes of sufficiently high energy would be affected by the properties of the quantum-gravity foam and we attempt to model (at least some aspects of) the corresponding dynamics using a noncommutative Minkowski spacetime. In the present work we do not discuss the generalization necessary for a description of how the quantum-gravity foam affects spaces which are curved (non-Minkowski) at the classical level. Even for spaces which are Minkowski at the classical level a full quantum gravity of course would predict phenomena which could not be simply encoded in noncommutativity of Minkowski space and actually would not be exclusively associated to its foamy vacuum structure, but it is plausible that the most significant implications of quantum gravity for the low-energy (large-distance) physics of Minkowski spaces would be associated to some aspects of the Hawking-Wheeler foam.

The noncommutative Minkowski spacetime we consider here is the algebra

$$[x^i, t] = \imath \lambda \, x^i, \quad [x^i, x^j] = 0 \qquad (1)$$

where $i, j = 1, 2, 3$ and λ is a free length scale, which (as justified by the above physical motivation for our studies) we shall often implicitly assume to be closely related to the Planck length $L_p \sim 10^{-35} m$. We work in units such that the speed-of-light constant is $c = 1$. The algebra (1) can be interpreted as a version of Minkowski spacetime with noncommutative coordinates, see notably [9]. Such algebras in 3 dimensions can be found in [2] while a q-deformation version of them in 1+1 dimensions was further studied from a noncommutative spacetime point of view in [3]. They provide a compelling candidate for the type of spacetime in which we are physically interested because they have a natural interpretation in momentum space[10] and because they fit well the intuition emerging from certain heuristic analyses of the structure of the quantum gravity vacuum [4, 8]. They are also part of a 2-parameter family of algebras proposed for Planck-scale physics in [1].

Among the already-studied implications of adopting (1) is that the appropriate notion of Poincaré invariance under which it is covariant has to be modified and becomes in fact a quantum group[9] (using the notation P for momentum space)

$$U(so_{1,3}) \blacktriangleright\!\!\blacktriangleleft \mathbb{C}(P) \qquad (2)$$

of the bicrossproduct type introduced in [2] in the 3-dimensional Euclidean case. The paper [9] also showed that (2) was (nontrivially) isomorphic to the so-called κ-deformed Poincaré quantum group [11], which had been earlier introduced from another point of view. Ref. [9] identified (1) as the spacetime on which κ-Poincaré acts covariantly. The introduction of a noncommutative-geometric point of view in which the κ-Poincaré indeed acts covariantly on a suitable κ-Minkowski spacetime (1) was the main result in [9]. The paper also solved the problem of finding the coordinate algebra dual to the κ-Poincaré algebra and allowed it to be identified it with an otherwise unconnected proposal in [12].

Preliminary, but to some extent heuristic, analyses of the physical implications of the deformed or κ-Poincaré proposal have led to interesting hypotheses, most notably the possibility of modified dispersion relations [13, 14]. Since recent progress in the phenomenology of gamma-ray bursts [15] and other astrophysical phenomena [16, 17] renders experimentally accessible [18] such modified dispersion relations, there is strong motivation in verifying whether the analyses reported in [13, 14] can be seen as part of a wider systematic analysis of particle-physics phenomena in the noncommutative Minkowski spacetime (1). This is one of the primary objectives of the present paper. We shall rely on a different approach suggested by [19] that makes use more directly of the structure[9] of (1) itself as a quantum group in its own right. Its coproduct

structure here is

$$\Delta t = t \otimes 1 + 1 \otimes t, \quad \Delta x^i = x^i \otimes 1 + 1 \otimes x^i \tag{3}$$

which expresses the addition law on (1). As noted already in [9], this addition law is valid but is not itself covariant under the deformed Poincaré algebra, hence this algebra necessarily takes a back seat in the new approach. We deal separately with translation and (classical) Lorentz covariance. However, this new approach based directly on the intrinsic structure of (1) allows us to make substantial progress toward a formalism in which all computations are not significantly more complicated than in a corresponding ordinary theory in a commutative spacetime. Important tools are provided by the availability of a 4-dimensional translation-invariant differential calculus[20] (which is not possible in the κ-Poincaré covariant setting) and by the quantum group Fourier transform which was worked out for our particular algebra in [10]. The latter is defined by the additive quantum group structure and allows one to rewrite structures living on noncommutative spacetime as structures living on a commutative (but nonabelian) "energy-momentum" space. Since our emphasis is on the structure of the space-time (1) we find it convenient to write formulas in terms of the length scale λ rather than the dimensionful parameter κ of the κ-Poincaré approach. Because of the transparent relation $\lambda = \hbar\kappa^{-1}$ we do not expect our choice of conventions to create any confusion; however, for good measure, we shall occasionally refer back to the "κ" notation and emphasize that some of the structures we consider are frequently denominated in the literature as κ-Minkowski spacetime and κ-Poincaré group.

In the next Section we provide the basic elements of the mathematics used for our proposal: we discuss the analogues of functions and a differential calculus on the deformed Minkowski spacetime (1) and then use the above-mentioned Fourier transform to introduce the deformed momentum space and the wave equation for deformed Minkowski spacetime. In Section 3 we discuss the physical interpretation of some of the new structures present in deformed Minkowski spacetime. In Section 4 we provide a possible scenario in which (1) arises as the quantum system associated to spacetime itself as the phase space of some 'pregeometry'. In Section 5, also using the Fourier theory, we sketch out a proposal for a new approach to the construction of field theories in noncommutative spacetimes. In Section 6 we elaborate on the phenomenology associated to *in vacuo* dispersion and CPT violation. Finally in Section 7 we summarize our results and set up an agenda for future work.

2 Functions, Differential Calculus, Momentum space and Wave Equation

A key ingredient of our proposal (and a general feature of the particular quantum groups in [1]) is that all functions in x^i, t can be treated as if classical under a normal ordering prescription. Thus, we consider general functions on the deformed Minkowski spacetime (1) as elements of the algebra of the form $:\psi(\vec{x}, t):$ where ψ is a usual function in 4 variables and where by convention the t generator is taken to the right.

The translation coproduct (3) implies a natural 4-dimensional translation-invariant calculi of differentials[20] spanned by $\mathrm{d}x^i, \mathrm{d}t$. In noncommutative geometry the differential calculi are not usually unique but in the 1+1 dimensional case of (1) there are in fact two possibilities discussed in [20]; we chose one of these to extend to our 4-dimensional case, namely with the relations

$$[x^i, \mathrm{d}x^j] = 0, \quad [t, \mathrm{d}x^i] = 0, \quad [x^\mu, \mathrm{d}t] = \imath\lambda\mathrm{d}x^\mu. \tag{4}$$

The corresponding partial derivatives are defined by

$$\mathrm{d}\psi = (\partial_\mu\psi)\mathrm{d}x^\mu \tag{5}$$

and take the form

$$\partial_i : \psi(\vec{x}, t) :=: \frac{\partial}{\partial x^i}\psi(\vec{x}, t) :, \quad \partial_0 : \psi(\vec{x}, t) :=: (\imath\lambda)^{-1}\left(\psi(\vec{x}, t) - \psi(\vec{x}, t - \imath\lambda)\right) : \tag{6}$$

This means that the associated noncommutative differential geometry of our deformed Minkowski space behaves in practice like the usual differentials in position and like a lattice in the time direction. On the other hand, t is an operator and there is no fixed lattice in this noncommutative geometry. Concerning the nature of the time-direction lattice it would be tempting here to redefine, say, $i\lambda = \mu$ so that ∂_0 appears like a usual lattice derivative. However, if μ is real then (1) tell us that for hermitian x, t would have to be antihermitian. So in any conventional ideas of measurement it would have imaginary eigenvalues. One would then be displacing imaginary time values by real μ. Since we prefer to envisage real eigenvalues for t we are forced to take λ real. In fact the "$\imath$" here is not so alarming. On analytic functions we obtain as $\lambda \to 0$ the usual differential just as well as for a real displacement, so this is an equally valid deformation even if a little unfamiliar. In fact its meaning is that this ∂_0 *is a lattice differential in Euclidean space* and just appears as above after Wick rotation. We recall that frequently in theoretical physics certain constructions look more natural in Euclidean space and are only viewed in Minkowski space after Wick rotation. This would appear to be such a situation.

Next we consider integration in our non-commutative spacetime. A natural translation-invariant choice is [3]

$$\int : \psi := \int \mathrm{d}^3\vec{x} \; \mathrm{d}t \; \psi(\vec{x}, t) \tag{7}$$

in terms of usual integration of the underlying function. It is such that the integral of a partial derivative of a suitably decaying function ψ vanishes.

We now consider the momentum space dual under non-Abelian Fourier transform to the Minkowski spacetime (1). Note first of all that Fourier theory is usually considered for Abelian groups but the nonAbelian case can be handled just as well using modern (quantum group) methods. Thus, if P is some nonAbelian matrix group then its algebra of coordinate functions $\mathbb{C}(P)$ can be regarded as a (commutative) quantum group or Hopf algebra. Its dual (cocommutative) Hopf algebra is the enveloping algebra $U(\mathfrak{p})$ where $\mathfrak{p}$ is the Lie algebra of P and Fourier transform provides maps $\mathbb{C}(P) \to U(\mathfrak{p})$ and vice-versa. This is a completely canonical construction[21], but it does oblige us to regard the enveloping algebra $U(\mathfrak{p})$ as the 'coordinates' of some noncommutative space if we want to think of Fourier theory as mapping functions on one space to 'functions' on some dual space (this is why usual Fourier theory is restricted to Abelian groups so that the dual is a usual and not noncommutative space).

Our Minkowski spacetime (1), (3) is such an enveloping algebra and is therefore connected by nonAbelian Fourier theory precisely to functions on a classical but nonAbelian momentum group, namely the group P of matrices of the form

$$\begin{pmatrix} e^{\lambda\omega} & k_1 & k_2 & k_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{8}$$

One may then compute the canonical Fourier theory using the integral (7) and the canonical element for the duality pairing[21]. It comes out as cf.[10]

$$T : \mathrm{Mink}_\lambda \to \mathbb{C}(P), \quad T(: \psi :)(\vec{k}, \omega) = \int \mathrm{d}^3\vec{x} \; \mathrm{d}t \; e^{i\vec{k}\cdot\vec{x}} e^{i\omega t} \psi(e^{\lambda\omega}\vec{x}, t) \tag{9}$$

where the integral is over usual commuting functions. The canonical property of Fourier theory comes out as

$$T(\partial_i\psi) = -T(\psi)i k_i e^{-\lambda\omega}, \quad T(\partial_0\psi) = -T(\psi) \, i \, \frac{1 - e^{-\lambda\omega}}{\lambda}. \tag{10}$$

We can also work with the generator of ∂_0 as

$$\partial_t : \psi :=: \frac{\partial}{\partial t}\psi :, \quad T(\partial_t\psi) = -T(\psi)i\omega. \tag{11}$$

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These formulae emerge naturally from noncommutative geometry and of course become usual Fourier theory when $\lambda \to 0$. Here

$$(\vec{k}, \omega)(\vec{k}', \omega') = (\vec{k} + e^{\lambda \omega} \vec{k}', \omega + \omega'), \quad (\vec{k}, \omega)^{-1} = (-\vec{k} e^{-\lambda \omega}, -\omega) \tag{12}$$

are the group law and inversion in the nonAbelian momentum group.

The natural plane waves associated to a point $(\vec{k}, \omega)$ in momentum space are provided by the inverse Fourier transform of left-translation invariant delta-functions at $(\vec{k}, \omega)^{-1}$, which come out as

$$\psi_{\vec{k}, \omega} = e^{\imath \vec{k} \cdot \vec{x}} e^{\imath \omega t}, \tag{13}$$

i.e. a plane wave in our deformed Minkowski spacetime. These respect the group law on momentum space in the sense

$$\psi_{(\vec{k}, \omega)(\vec{k}', \omega')} = \psi_{\vec{k}, \omega} \psi_{\vec{k}', \omega'} \tag{14}$$

so that, in particular, the wave in the reverse direction in momentum space is

$$\psi_{(\vec{k}, \omega)^{-1}} = e^{-\imath \vec{k} e^{-\lambda \omega} \cdot \vec{x}} e^{-\imath \omega t} = e^{-\imath \omega t} e^{-\imath \vec{k} \cdot \vec{x}}, \tag{15}$$

i.e., another plane wave in our deformed Minkowski space time (note, however, the order of the generators.)

We are now ready to obtain the appropriate dispersion relations for such waves. By definition these are constraints in momentum space P which should be Lorentz invariant. Because our momentum space is a nonAbelian group, not the usual $\mathbb{R}^{1,3}$, the appropriate action of the Lorentz algebra is not the usual one. Rather, there is a particular action of the Lorentz algebra on the momentum group P which is used in the semidirect product algebra (2) of the deformed Poincaré quantum group in [9]. We clearly should use this action. It is [9]

$$M_i = -\epsilon_{imn} k_m \frac{\partial}{\partial k_n}, \quad N_i = k_i \frac{\partial}{\partial \omega} - \left(\frac{\lambda}{2} \vec{k}^2 + \frac{(1 - e^{2\lambda \omega})}{2\lambda}\right) \frac{\partial}{\partial k_i} + \lambda k_i k_j \frac{\partial}{\partial k_j} \tag{16}$$

for the action of the standard rotation and boost generators. These are the vector fields on P corresponding to the action on generators given in [9].

From (16) one finds the appropriate constraint which has the right limit and which is both Lorentz invariant and invariant under group inversion (12) to be

$$\lambda^{-2} \left(e^{\lambda \omega} + e^{-\lambda \omega} - 2\right) - \vec{k}^2 e^{-\lambda \omega} = m^2. \tag{17}$$

The operator corresponding under Fourier theory (10) to the left hand side in momentum space is $-\Box$, where

$$\Box : \psi :=: -\lambda^{-2} \left(\psi(\vec{x}, t + \imath \lambda) + \psi(\vec{x}, t - \imath \lambda) - 2\psi(\vec{x}, t)\right) - \sum \partial_i^2 \psi(\vec{x}, t + \imath \lambda) : \tag{18}$$

i.e.,

$$\Box = (\partial_0^2 - \sum \partial_i^2)L$$

where $L : \psi(\vec{x},t) :=: \psi(\vec{x}, t + \imath\lambda) :$ is the shift operator and ∂_0 is the derivative in (6). It is easy to see that the plane waves (13) are eigenfunctions with eigenvalue given by the left hand side of (17). Also, from the bicrossproduct construction of the deformed Poincaré algebra (2) in [9] it is known that a Lorentz-invariant expression in momentum space necessarily corresponds to a Casimir from the deformed Poincaré point of view.

It is of course important to be able to construct wave packets from our plane-wave solutions. To construct a wave packet we should average over waves with some density function a, for example a might be a Gaussian centred at the origin and then translated to be centred at some average spatial momentum $\vec{k}_0$ (and trivial in the energy direction). In more conceptual terms the wave-packet is the inverse Fourier transform of the translated a. In addition, the composite waves are constrained to obey the dispersion relation. The noncommutative analogue is therefore

$$\psi_{a,\vec{k}_0}(\vec{x},t) = \int \mathrm{d}^3\vec{k} \ e^{\lambda\omega} a((\vec{k}_0, \omega_0)(\vec{k}, \omega)^{-1})\psi_{(\vec{k},\omega)^{-1}} \tag{19}$$

where ω is a function of $\vec{k}$ according to the dispersion relation (17). Similarly for ω_0. The choice of a a left-invariant delta function recovers a pure on shell plane wave. With care one may also change the variable $(\vec{k}, \omega)^{-1}$ of integration to $(\vec{k}, \omega)$.

Let us note that while this is the natural definition from the mathematical point of view, the physical applications to which we put our wave-packet might dictate other choices based on the same pattern. Thus, in the above we have used the left-translation invariant integral $\int \mathrm{d}\omega \mathrm{d}^3\vec{k} e^{\lambda\omega}$ required by the quantum-group Fourier theory in the present conventions. It is also possible that one might prefer to build a wave-packet using an integration invariant under the deformed action of Lorentz transformations. This would be with integration measure

$$\int \mathrm{d}\omega \ \mathrm{d}^3\vec{k} \ e^{-3\lambda\omega} \tag{20}$$

so that $\int M_i(f) = 0, \quad \int N_i(f) = 0$ (where M_i, N_i are the rotation and boost vector fields) if f is sufficiently rapidly decaying at infinity. In other conventions or some other applications one might also need right-invariant integration measure

$$\int \mathrm{d}\omega \ \mathrm{d}^3\vec{k} \tag{21}$$

so that $\int f_{(\vec{k}_0, \omega_0)} = \int f$, where $f_{(\vec{k}_0, \omega_0)}(\vec{k}, \omega) = f((\vec{k}, \omega)(\vec{k}_0, \omega_0))$. These choices will be discussed further in Section 5.

3   Physical Interpretation

It is of course necessary to discuss the relation between the algebra of the deformed Minkowski spacetime (1) and the physically measured time and position of events. It is tempting to associate to our formal normal ordering prescription an operative prescription in which the coordinates can be treated conventionally provided one always measures the time coordinates first. This is in fact what one would expect based on an analogy with similar normal-ordering prescriptions in ordinary quantum-mechanics frameworks. While in the following we do assume that there exists some form of measurement procedure in an unknown theory of quantum gravity allowing us to treat our coordinates conventionally, we want to emphasize that the nature of the observables associated to our operators must be somewhat different from the observables of ordinary quantum mechanics. We expect such differences especially because we have a time operator, while ordinary quantum mechanics only involves a time parameter[3]. The observables of ordinary quantum mechanics are measured in correspondence with a value of the time parameter, and at least the observable associated to our time operator does not appear suited for this type of operative definition. Thus, one may attempt to treat the system with operators x_i, t quantum mechanically (we give an example in the next section) but the time variable for that quantum mechanics would have to be different from the operator t. The two times would at some point need to be reconciled within a more complete and unknown theory of quantum gravity. In fact the problem we are facing here is nothing else than another version of the "problem of time" encountered in one form or another in any approach to quantum gravity, although not always immediately evident within some of the more abstract formalisms.

There is probably no reason to be surprised of these difficulties of ordinary quantum mechanics. In fact, the conceptual analysis of measurements procedures [8, 24, 25] for candidate quantum-gravity observables has been used [5, 26] to argue that the mechanics on which quantum gravity is based should not be exactly the one of ordinary quantum mechanics. The new mechanics should accommodate a somewhat different relationship between "system" and "measuring apparatus", and should take into account the fact that the limit in which the apparatus behaves classically is not accessible[4] once gravitation is turned on. The issue of separation between 'observer' and 'observed', which is likely to play a central role in the new mechanics, has already been explored to some extent from the point of view of the necessary formalisms in Refs. [1] and in more recent works such as [27]. In general measurement is seen as an interaction

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[3]The time appearing in the evolution equations of ordinary quantum mechanics is indeed only a parameter. One can attempt to construct in some way a "time of arrival operator" (see, *e.g.*, Ref. [22]) but in general there is no self-adjoint operator canonically conjugate to the total energy, if the energy spectrum is bounded from below [23]

[4]In ordinary (non-gravitational) quantum mechanics the limiting procedure allowing to consider classical apparatus requires an infinite-mass limit [6, 7], which turns out to be inconsistent with the structure of gravitational measurements [5, 8, 24, 25].

between aspects of the system labelled by macroscopic or classical 'handles' and the microscopic quantum system. To formulate this properly one first needs to have a way to 'identify' the macroscopic, typically geometric, aspects (such as the separation between two devices) within the overall quantum system, which is precisely the task of noncommutative geometry.

For our present purposes, in trying to envisage a type of setup that would allow to treat our coordinates conventionally, let us consider the measurement of the speed of a particle travelling along a straight-line trajectory. Assuming the space-time points were identified in our laboratory by a grid of clock-detector pairs and that one was able to set up the emission of the particle from position P_0 at time T_0, the speed could be measured in two ways: by measuring the clock time needed by the particle to reach a given detector in the grid or by measuring the position (detector triggering) of the particle at a given time of the (sychronized) clocks of our grid. While any definite statement must await the development of a consistent measurement theory for quantum gravity (and in particular for the class of models we are considering), we expect that within our proposal these two ways of measuring the speed would be significantly different and it appears plausible that our normal ordering prescription would correspond to the second method, the one in which a chosen clock readout triggers the detectors to determine the position of the particle at that time.

4   An example of pregeometry quantum system

While for the effective theory viewpoint that we advocate the details of the underlying physics are not directly relevant, it might nevertheless be useful to have at least an intuitive picture of the fact that our noncommutative Minkowski spacetime should emerge from quantum gravity. We assume that only certain macroscopic modes of the unknown quantum gravity theory survive at the level of our effective description and, for the sake of discussion, that these form an effective quantum mechanical system underlying the $[x^i, t]$ noncommutativity relations. We call this the *pregeometry quantum system*. It should be considered as still an effective description of some unknown quantum gravity theory but one which is slightly deeper than the operative prescription for handling x^i, t in terms of classical functions in the preceding sections.

We should stress that our operative prescriptions for handling x, t as well as for scattering in terms of classical momentum and energy $\vec{k}, \omega$ do not require us necessarily to develop this extra layer of 'pregeometry' for our model. Moreover, the best description of the effective 'pregeometry' may not be a quantum one at all. Nevertheless, the conventional way of thinking about noncommutative algebras is in terms of quantum mechanics and hence it is natural to provide, for completeness, at least a sketch of one example of a suitable quantum system that could serve as a link between our operative description and the unknown quantum gravity theory.

To approach this question, not knowing a complete quantum gravity theory, we can nev-

ertheless explore some mathematical possibilities. This is akin to using classical topology to distinguish different *a priori* possible classical solutions of a complex system, but in our case in an algebraic or quantum mechanical setting. Thus we would like to ask about different possibilities to extend the algebra to a quantum system with additional p_i generators and suitable commutation relations between position and momenta subject to some *a priori* assumptions.

This question was explored and answered in one spatial dimension in [1]. Thus, if we are given a variable x which we deem to be position and a variable π which we deem *a priori* to be some kind of 'momentum' variable and ask for *all possible* commutation relations such that the addition law in phase space $\mathbb{R}^2$ extends to the quantum system as a quantum group A extending x, π in the sense

$$\mathbb{C}[\pi] \to A \to \mathbb{C}[x]$$

as a Hopf algebra extension (here $\mathbb{C}[x]$ denotes functions in one variable x, etc.) then one finds (coming out of the analysis) a two-parameter family of possibilities[1] for A, namely

$$[x, \pi] = \imath \hbar_0 (1 - e^{-\frac{x}{\rho}}) \tag{22}$$

where $\hbar_0, \rho$ are the two parameters, with the coproduct

$$\Delta x = x \otimes 1 + 1 \otimes x, \quad \pi = \pi \otimes 1 + e^{-\frac{x}{\rho}} \otimes \pi. \tag{23}$$

This is the 2-parameter 'Planck-scale Hopf algebra' $\mathbb{C}[\pi] \blacktriangleright\!\!\triangleleft_{\hbar_0,\rho} \mathbb{C}[x]$ introduced in [1] in this way. Of course, that A should *a priori* be a Hopf algebra extension is a conceptual assumption which may well not be true. I.e. we are not absolutely forced to take this form of A, it is merely a mathematically natural class of possibilities. Moreover, whereas in [1] the two parameters were intepreted as the physical $\hbar$ and the gravitational length scale of the background geometry, in our case they are the parameters of the pregeometry system with an unknown relationship to the actual physical parameters of the unknown quantum gravity theory. Likewise, we do not suppose that π is exactly the physical momentum of the theory. Rather, we are merely using the mathematical formalism of quantum mechanics to build a deeper model behind the commutation relations (1).

In any event, motivated by this one-dimensional analysis, as an example of a pregeometry quantum system for our 3+1-dimensional Minkowski space we take three independent copies of (22), i.e. we add generators π_i, say, where $i = 1, 2, 3$, with the relations and coproduct

$$[x^i, \pi_j] = \delta^i{}_j \imath \hbar_0 (1 - e^{-\frac{x^i}{\rho}}), \quad [\pi_i, \pi_j] = 0,$$

$$\Delta \vec{x} = \vec{x} \otimes 1 + 1 \otimes \vec{x}, \quad \Delta \pi_i = \pi_i \otimes 1 + e^{-\frac{x^i}{\rho}} \otimes \pi_i$$

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Within this larger algebra we identify our noncommutative Minkowski space as generated by the x^i and

$$t = \sum_i \pi_i$$

in the limit

$$\rho, \hbar_0 \to \infty, \quad \frac{\hbar_0}{\rho} = \lambda.$$

Note that the role of the π_i here is as 'auxiliary time variables' with their sum giving the time of the Minkowski theory. One in fact expects something unusual like this when one considers the asymmetric (and to date still problematic) treatment of time in canonical quantum gravity; there one considers the spatial fields and their conjugates on each time-slice and tries to reconstruct the spacetime time afterwards. In addition, the commuting π_i corresponds to the absence of spatial curvature in the noncommutative Minkowski-space. One can certainly envisage more complex models where an additional parameter enters into nontrivial commutation relations between the π_i as well.

Although this is just one example of a pregeometry quantum system, it shows how the Minkowski space algebra (1) might arise as the limiting case of commutation relations which have a more familiar 'quantum mechanical' form. (And if one wants to render the commutation relations in an even more canonical form one need only change to $\tilde{\pi}_i = (1 - e^{-\frac{x^i}{\rho}})^{-1} \pi_i$ at the expense of a more unnatural coproduct in terms of $x_i, \tilde{\pi}_i$.)

Given such a picture, one can now explore, at least tentatively, certain issues. First of all, as a genuine quantum system in [1] one has natural hermiticity properties

$$x^{i*} = x^i, \quad \pi_i^* = \pi_i$$

giving a Hopf *-algebra. We see that if we want to have a quantum-mechanical interpretation of the noncommutativity of our Minkowski-space then we should take λ real when t is hermitian. This forces us to the imaginary finite differences in ∂_0 in Section 2. Alternatively we could replace $\imath\lambda$ by μ here and in Section 2 for a more conventional 'lattice differential' in ∂_0 but would then have to take t antihermitian for a quantum mechanical picture. This situation is not unlike quantum field theory where for a proper mathematical foundation it is best to Wick rotate to imaginary time. One might therefore expect that this should be an effective remnant of the problem of Wick rotation in the unknown quantum gravity theory.

One also has a natural 'Schroedinger type' representation on wavefunctions $\phi(\vec{x})$ with x^i acting by multiplication and $\pi_i = -\imath\hbar_0(1 - e^{-\frac{x^i}{\rho}})\frac{\partial}{\partial x^i}$, etc. The implied representation of our Minkowski space is

$$x^i \cdot \phi = x^i \phi, \quad t \cdot \phi = -\imath\lambda \sum_i x_i \frac{\partial}{\partial x^i} \phi \tag{24}$$

i.e., t acts as an infinitesimal scale transformation. The x^i is hermitian and t indeed hermitian with respect to a certain inner product.

One has uncertainties in the simultaneous measurement of x^i, π_i and other familiar quantum effects. Of course it implies the obvious uncertainty due to (1) but potentially further uncertainties as well, depending on the ultimate physical interpretation of the individual π_i. Similarly, if one takes as in [1] the Hamiltonian $\vec{\pi}^2/2m$, one has dynamics on the pregeometry quantum system consisting of a particle moving more and more slowly as it approaches the origin (in a manner not unlike the approach to a black hole event horizon[1]). Of course, the formal time variable pertaining to this discussion of the pregeometry quantum system should not be confused with the operator t defined in (24) from the pregeometry momentum operators π_i.

We emphasize again that we are providing these comments solely for illustrative purposes. Of course, experiments suitable for exploring the nature of such a pregeometry system are well beyond our reach. In principle one would first devise experiments to confirm (or falsify) the models at the "geometry level" and only once this level was well established one could hope to devise even more refined experiments to test models of the "pregeometry level". Since technology only very recently [18, 26, 28, 29] became advanced enough for a few very preliminary experimental investigations of the "geometry level", all considerations concerning the "pregeometry level" must indeed be considered as purely illustrative.

Finally, as well as the example discussed above based on 'extension theory' there are other more naive approaches to the pregeometry quantum system one could also consider. For example, for any Hopf algebra H there is a canonical semidirect product $H{\bowtie}H^*$, the Weyl algebra or so-called Heisenberg double, see [21]. It is easy enough to compute in our case as generated by x^μ, p_μ with the commutation relations given in [9] as the action of the p_μ on the x^ν as part of the action there of the deformed Poincaré quantum group. While probably playing some role, we do not take it as the pregeometry quantum system itself because as a 'quantisation' it treats time on the same footing as the space (which is not really appropriate even when quantising a single relativistic particle). The Weyl algebra also does not have a coproduct or other interesting mathematical properties to characterise it in place of that. We defer the discussion of this to further work in which, particularly, the relationship between any pregeometry quantum system and quantum mechanics *on* the noncommutative Minkowski space (which are different questions) should be explored.

5 Quantum field theory on noncommutative spacetime via non-Abelian energy-momentum space

In this section we point out the possibility of a new approach to the construction of field theories on a noncommutative space-time. Previous attempts at a satisfactory definition of field

theory in a non-commutative space-time have had only limited success. At a rather formal level some progress has been made, but eventually one was confronted with the difficulties involved in generalizing to a noncommutative space-time some of the operators and other tools required for a field theory.[5] Here we observe that these difficulties could be evaded by exploiting the fact that quantum group Fourier transform allows us, as we have already seen, to rewrite structures living on noncommutative spacetime as structures living on a classical (but nonAbelian) "energy-momentum" space. If one is content to evaluate everything in energy-momentum space, this observation gives the opportunity to by-pass all problems directly associated with the non-commutativity of space-time. We are confident that eventually a compelling space-time formulation of field theories on noncommutative geometries will emerge, but in the meantime we restrict ourselves to energy-momentum space where the underlying noncommutativity manifests itself only through the "curvature" (nonabelianness of the group) of the space. Note that this approach does not work for *any* noncommutative spacetime but for all those where the spacetime coordinate algebra is the enveloping algebra of a Lie algebra, with the Lie algebra generators regarded 'up side down' as noncommuting coordinates[19].

Because of the viewpoint we are advocating, within our approach field theories are not naturally described in terms of a Lagrangian. We resort directly to a Feynman-diagramatic formulation.[6] In principle, according to our proposal a given ordinary field theory can be "deformed" into a counterpart living in a suitable noncommutative spacetime not by fancy quantum group methods but simply by the appropriate modification of the momentum-space Feynman rules to those appropriate for a nonAbelian group. The quantum group concepts are, however, required in order to do this in a manner consistent with the (noncommutative) geometry of space-time, for example to consistently obtain predictions for cross sections from the amplitudes evaluated using the nonAbelian Feynman rules.

While we postpone to future work (also because some of the relevant mathematics is only at an early stage of development) the detailed discussion of examples of such field-theoretical models, in the rest of this section we give some general guidelines to be followed in constructing the type of field theories we are proposing.

Let us start with the Feynman rules. As mentioned the guiding principle of our proposal for the construction of deformed field theories is the replacement with quantum-group counterparts of those group-theoretic elements which characterize the structures relevant for ordinary field theories. Accordingly, the propagator $D(\vec{k}, \omega)$ of a scalar particle will be essentially given by the

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[5]In particular, interesting studies of field theories in certain other noncommutative geometries were reported in the two preprints [30] and [31] which appeared while we were completing the writeup of the present article. Our own approach is completely different from these works (in fact they are based on different methods which would not seem to apply to our particular spacetime (1) at all.)

[6]This also implies that the description of certain non-perturbative effects (the ones not obtainable as infinite sums of Feynman diagrams) might not be possible within our energy-momentum space formulation.

inverse of the operator in the dispersion relation Eq. (17), i.e. in place of $D = (\omega^2 - \vec{k}^2 - m^2)^{-1}$ we take

$$D_\lambda = \left(\lambda^{-2}(e^{\lambda\omega} + e^{-\lambda\omega} - 2) - e^{-\lambda\omega}\vec{k}^2 - m^2 \right)^{-1} . \tag{25}$$

The Feynman rules for vertices that do not involve the momenta of incoming/outgoing particles remain unchanged. For example, in "Φ^4" theory the 4-point vertex is still simply given by the coupling constant

$$\Gamma = g \to \Gamma_\lambda = g . \tag{26}$$

Vertices which involve the momenta of incoming/outgoing particles and in particular those that require to sum the momenta of pairs of particles must be rewritten also taking into account the rule (12) for combining momenta in our deformed Minkowski spacetime. We postpone a full discussion until we will be ready to discuss more complex field theories. It is clear, however, that when our momenta are nonAbelian there will be a fundamental difference between scattering particle 1 with particle 2 and scattering particle 2 with particle 1; even for trivial scattering the total momentum of particle 1 plus particle 2 or particle 2 plus particle 1 (where addition is replaced by our nonAbelian group law) will be different in the two cases. This is a new physical effect which we are predicting, which is therefore difficult to lay down the rules for in advance. In the first instance one should simply do scattering computations according to all distinct order-of-addition rules, to see which fit best with a given set of actual scattering experiments. One could also express ignorance of the new effect by averaging over the different orderings of the momenta. Such an averaging procedure might even be the correct choice at least in cases involving indistinguishable particles.

The Feynman diagrams involving integration over loop momenta will also reflect the underlying non-commutativity of spacetime and nonAbelian nature of energy-momentum space, through the measure of integration on the latter. As mentioned in our discussion of wave packets in Section 2, there are at least three candidates for the measure of integration in energy-momentum space (*i.e.* for loop integration); the left-invariant, right-invariant and Lorentz invariant measures. All three coincide classically but in our noncommutative theory we have to choose. Fortunately, all the measures have a similar form

$$\int \mathrm{d}\omega \ \mathrm{d}^3\vec{k} \ e^{\alpha\lambda\omega} \tag{27}$$

for suitable $\alpha = 1, 0, -3$. One can therefore proceed, for example, with α regarded as a parameter to be fitted by comparison with experminent.

Since Feynman rules come in fact from an analysis of the scattering of wave-packets, the obvious choice suggested by Section 2 is the left-invariant one $\alpha = 1$. However, we would prefer

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to leave the choice open at the present stage. Future work might show that only some (perhaps only one) of these candidates leads to renormalizable theories. Actually, it is plausible that some of these measures might lead to finite theories, since the exponential suppression of high-energy modes might be sufficient to eliminate all ultraviolet problems. We postpone investigation of these issues to future publications, but let us emphasize here that these issues that confront us because we have lost the equivalence between left-invariant, right-invariant and Lorentz invariant measures are more complex examples of the type of issues that one encounters, *e.g.* when allowing P-parity violation in particle physics (which actually turns out to be the scenario preferred by *Nature*). The loss of P-parity introduces the arbitrariness between "V-A" and "V+A" behaviour which can only be settled by experiments. In our case besides experiments also the requirement of mathematical consistency might be useful in identifying the correct measure. Future more in-depth investigations of this approach might uncover additional requirements to be satisfied by the integration measure, thereby reducing the number of choices available.

Having sketched out our approach to deformed Feynman rules let us close this Section with some comments on obtaining cross sections from the amplitudes calculated using the nonAbelian Feynman rules. This is of course a necessary step since experimental data are compared to cross sections. The usual formulas cannot be naively applied in our case since the derivation of cross sections from amplitudes must now be done consistently with the measurement of solid angles etc in the noncommutative spacetime. This could be the most delicate part of our approach because it is the part where we cannot fully confine the analysis within energy-momentum space. We indicate here only a general strategy that could be adopted. First of all, using our principle of normal ordering we consider normal ordered spacetime expressions as identified with their classical counterparts for the purposes of specifying solid angles, etc. Using this identification one is left with the task of obtaining a consistent deformation of the standard cross-section formulas. Let us discuss the elements of novelty required by our framework within the specific example of a scattering process with two particles in the initial state and two particles in the final state. The relevant standard cross section formula in the ordinary commutative Minkowski spacetime is

$$
d\sigma \;=\; \frac{d^3\vec{k}_{f,1}}{16\pi^3\omega_{f,1}}\frac{d^3\vec{k}_{f,2}}{16\pi^3\omega_{f,2}}\frac{|\mathcal{M}(1_i+2_i\to 1_f+2_f)|^2}{|v_{i,1}-v_{i,2}|}\int \frac{d^3\vec{q}_1}{16\pi^3\omega_{i,1}}\frac{d^3\vec{q}_2}{16\pi^3\omega_{i,2}} \tag{28}
$$
$$
|\Phi_{\vec{k}_{i,1}}(q_1)|^2|\Phi_{\vec{k}_{i,2}}(q_2)|^2\,16\pi^4\,\delta^{(4)}\Big((\vec{q}_1,\omega_{i,1})+(\vec{q}_2,\omega_{i,2})-(\vec{k}_{f,1},\omega_{f,1})-(\vec{k}_{f,2},\omega_{f,2})\Big)\;,
$$

where $(\vec{k}_{i,1},\omega_{i,1})$ and $(\vec{k}_{i,2},\omega_{i,2})$ (respectively $v_{i,1}$ and $v_{i,2}$) denote energy-momentum (respectively velocity) of the particles in the initial state, $d^3k_{f,1}$ and $d^3k_{f,2}$ are infinitesimal volume elements in the space of momenta of the particles in the final state, $\Phi_{\vec{k}_{i,1}}(\vec{q})$ and $\Phi_{\vec{k}_{i,2}}(\vec{q})$ are the momentum-space wave functions of the particles in the initial states, which are assumed to be sharply peaked around $\vec{q}\sim\vec{k}_{i,1}$ and $\vec{q}\sim\vec{k}_{i,2}$ respectively.

The deformation of the formula (28) requires various elements of our formalism. Most notably, the energy-momentum conservation enforced by the δ function must be implemented consistently with the nonAbelianess of our energy-momentum space, and this brings in again some choices with respect to the ordering of the various momenta entering the sums. The usual problem of choosing the measure of integration is also present here, but one would expect this ambiguity to be settled by a requirement of consistency with the choice of measure adopted for loop-integrals in Feynman diagrams and in the definition of wave packets. In particular, at present it appears legitimate to proceed taking measures according to the left-invariance advocated in Section 2. Finally the wave functions $\Phi_{\vec{k}_{i,1}}, \Phi_{\vec{k}_{i,2}}$ appear in a very simple way in equation (28), but this is the result [32] of the simplicity of the procedure for the construction of two-particle wave packets in ordinary Minkowski spacetime. In our case we have already constructed 1-particle wave packets in Section 2, modulo some possible variations. The usual definition (as an approximation) for multiple-wave packets is as the tensor product of 1-particle wave packets, i.e. this in itself presents no problem in our formalism. E.g.

$$\psi_{a_1,\vec{k}_1,a_2,\vec{k}_2} = \psi_{a_1,\vec{k}_1} \otimes \psi_{a_2,\vec{k}_2}. \tag{29}$$

The ordering of the addition of momenta for in and out states in a scattering corresponds to the ordering of such tensor products. For identical particles one could again perform some form of symmetrization to express our ignorance of which particle should be on the left and which on the right factor, but in any case nontrivial implications for the cross-section formula are to be expected.

6 Phenomenology

6.1 Phenomenology of deformed dispersion relations

The deformed dispersion relation (17) can have important implications even though the deformation is only minute (it is proportional to λ, which we expect to be close to the Planck length). While we derived (17) for scalar particles, and a rigorous analysis of spin-1 particles must still await some developments on the mathematics side, it appears quite plausible that the same dispersion relation, which is primarily dictated by the deformed symmetries present in our approach, would also apply to photons. This would lead to an effect of energy dependence of the speed of photons which is large enough for observation in experiments involving the gamma rays we collect from astrophysical sources.

In clarifying the origin of this energy-dependence of the speed of massless particles let us start by observing that within the stated assumption of existence of a practical measurement procedure allowing to treat normal ordered expressions conventionally it is legitimate to describe the physical wave velocity of our noncommutative plane waves (13) according to the conventional

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formula

$$v_i = \frac{d\omega}{dk_i}.$$ (30)

One may analyse this in terms of a wave packet or, equivalently, by thinking about one wave (13) at a time. When traveling a distance $\vec{L}$ in time T the wave still completes $n = (\vec{k} \cdot \vec{L} + \omega T)/2\pi$ cycles as usual. Hence if we vary $\vec{k}$ with the same number of cycles, the arrival time varies by

$$\delta T = -\frac{(\vec{L} + T\vec{v})}{\omega} \cdot \delta\vec{k}$$

as usual, with v_i defined by (30). This is arranged so that

$$e^{\imath\vec{k}\cdot\vec{x}}e^{\imath\omega t}|_{\vec{L},T} = e^{\imath(\vec{k}+\delta\vec{k})\cdot\vec{x}}e^{\imath(\omega+\vec{v}\cdot\delta\vec{k})t}|_{\vec{L},T+\delta T}$$

when one replaces the noncommutative coordinates $\vec{x}, t$ by their measured values as shown. Note that one would obtain quite different answers due to the noncommutativity of the generators without the normal ordering assumption for the comparison with measured values. This provides at least some justification for (30) within the present framework; a fuller justification would presumably come out of a more detailed model of an actual measuring apparatus within a more complete theory.

With this justification, we may combine (13) and (17) to finds that the velocity of massless particles is given by

$$v_i = \frac{d\omega}{dk_i} = \frac{\lambda k_i}{\lambda^2 \vec{k}^2 + \frac{\lambda\omega}{|\lambda\omega|}\sqrt{\lambda^2\vec{k}^2}} .$$ (31)

Consequently, the speed of massless particles is given by

$$v = \frac{1}{1 + \frac{\lambda\omega}{|\lambda\omega|}\sqrt{\lambda^2\vec{k}^2}} = e^{-\lambda\omega} \simeq 1 - \lambda\omega ,$$ (32)

where on the right-hand side we expanded for small ω ($\omega \ll \lambda^{-1}$).

This velocity law for massless particles Eq. (32) was already considered in some studies [13, 14, 18] based on the κ-Poincaré symmetries and studies based on Liouville non-critical String Theory [18, 33]. The fact that we also encounter this velocity law is of course not surprising since (in the sense clarified in Section 1) our approach is consistent with a background κ-Poincaré symmetry. It is significant however that, thanks to the quantum group Fourier transform methods, we could for the first time discuss corresponding "plane waves" (13) and thereby justify Eq. (32) as fully deserving its physical interpretation as velocity law. Instead, in previous κ-Poincaré approaches this velocity law was only suggested at a rather heuristic level starting from the properties of a Casimir and using formal manipulations with generators p_μ which, although commuting among themselves, were viewed as part of a noncommutative

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deformed Poincaré algebra, and with formulae such as $v_i = dp_0/dp_i$ assumed formally. By replacing these p_μ by the underlying energy-momentum space with points $(\vec{k}, \omega)$ we are able to compute with the latter, which are numbers and not formal generators. And we are able to give at least some justification for (30) through the properties of the plane waves (13) now at our disposal. Also notice that our wave equation was not obtained simply using a Casimir, which would have not fixed it or its corresponding dispersion relation uniquely; we also demanded that the dispersion relation be invariant under group inversion in energy-momentum space.

The velocity law (32) is a significant prediction of our proposal since recent progress in the phenomenology of gamma-ray bursts [15] and other astrophysical phenomena [16, 17] renders experimentally accessible [18] the investigation of certain modified velocity laws, including the ones of type Eq. (32). As explained in Ref. [18], these experimental tests are actually rather simple. In fact, according to (32), two signals respectively of energy ω and $\omega + \delta\omega$ emitted simultaneously from the same astrophysical source in travelling a distance L acquire a "relative time delay" $|\delta t|$ given by

$$|\delta t| \sim \lambda \, \delta\omega \frac{L}{c} \ . \tag{33}$$

This time delay can be detected if $\delta\omega$ and L are large whilst the time scale over which the signal exhibits time structure is small. These conditions are in particular met by certain gamma-ray bursts. We recall that these bursts involve [34] typical photon energies in the range $0.1 - 100$ MeV and time structure down to the millisecond scale has been observed in the light curves. According to Eq. (33) a signal with millisecond time structure in a burst of photons with energies spread over a range of order 10 MeV coming from a distance of order 10^{10} light years[7] would be sensitive to λ of order $10^{-35}m \sim L_p$.

Already available data [16] rule out values of λ of order $10^{-33}m$ in Eq. (32), and planned experiments should achieve sensitivity to values of λ of order $\sim 10^{-35}m \sim L_p$ within a few years [17].

Let us also emphasize that in our proposal the "v" appearing in Eq. (32) is naturally interpreted as the expectation of a velocity operator in some underlying "prequantized pregeometry". In particular, this implies that a sample of massless particles of energy ω would have average speed given by the $v(\omega)$ of Eq. (32) but there would also be a certain spread $\sigma_v(\omega)$ in the speeds of individual particles within the sample. Since we do not have any very definite knowledge of the structure of the pregeometry quantum system we are unable to make definite predictions for $\sigma_v(\omega)$, but we hope experimentalists will find motivation in our analysis to search for this effect. Additional motivation for this particular type of experimental investigations comes from analogous effects encountered in other quantum gravity motivated studies [35].

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[7]The cosmological origin of at least some GRBs has been recently established [15].

6.2 CPT violation

Another class of recently proposed quantum gravity phenomena have to do with violations of CPT invariance. This is a rather general prediction of quantum gravity [36], since most approaches involve some elements of non-locality (so that one of the hypotheses of the "CPT theorem" does not hold) and/or decoherence. What is remarkable is that certain quantum gravity approaches predict violations of CPT invariance large enough to be detectable by exploiting the properties of the very delicate neutral-kaon system.

In this Subsection we observe that the proposal we are putting forward in the present Article hosts a mechanism of CPT violation. Although a detailed study of CPT violation within our approach will require the development of the mathematical tools mentioned in the preceding Section, we shall also provide here some evidence suggesting that this CPT violation might be tested using the neutral-kaon system.

The root of CPT violation within our approach resides in the discretization of time (in the sense clarified earlier). Actually, CPT invariance is not necessarily "lost": it can in fact be traded for a novel invariance, which we could see as a deformed CPT invariance. Our (quantum) deformation of Minkowski spacetime leads to deformation of P and T transformations. The situation of CPT transformations in our proposal is somewhat analogous to the deformations of Lorentz invariance considered in [13, 14, 18], whose experimental signature would be a violation of ordinary Lorentz invariance [18], but at the fundamental level can be described by replacing the Lorentz symmetries with a deformed version of Lorentz symmetries [13, 14].

In characterizing the deformed CPT invariance which is consistent with our approach it is important to notice that in our approach a particle with charges, say, α, β, γ and momentum $(\vec{k}, \omega)$ has as antiparticle a particle of charges $-\alpha, -\beta, -\gamma$ and momentum not $(-\vec{k}, -\omega)$ but

$$(\vec{k}, \omega)^{-1} = (-\vec{k}e^{\lambda\omega}, -\omega) . \tag{34}$$

from (12). Correspondingly in the loop integrals of our momentum-space field theory particles and antiparticles do not contribute in a totally symmetric way. This is also evident when comparing the positive values of energy and the negative values energy which are consistent with a given momentum $(\vec{k}, \omega)$ according to Eq. (17) and (25). That is, *if* one takes a usual splitting of momentum into spatial momentum $\vec{k}$ and energy ω and carries this over to the experimental interpretation one can expect to observe the modification in the group inversion as a breakdown of ordinary CPT invariance.

It may be that such a breaking of ordinary CPT invariance turns out to be consistent with quantum mechanics, *i.e.* the violations of CPT invariance may be described as terms in an (effective) Hamiltonian which governs otherwise ordinary evolution equations of quantum mechanics. While this seems rather probable a definite statement will have to wait more detailed analyses;

in fact, at present one cannot exclude that the novel elements of our approach (particularly the peculiar nature of time) could lead to evolution equations not exactly of the type expected within ordinary quantum mechanics. We have discussed this possibility already in Section 3. On the other hand, at present, we are setting up only a framework for an effective low-energy description of certain quantum gravity effects and the fact that the full quantum gravity might require departures from ordinary quantum mechanics does not necessarily imply that its effective low-energy descriptions should already incorporate this property. We emphasize this point because other approaches to quantum gravity lead to violations of CPT invariance which cannot be accommodated within the formalism of ordinary quantum mechanics [37, 38, 39].

The difference between breaking ordinary CPT invariance within quantum mechanics [39, 40] and outside quantum mechanics has been emphasized in work on the neutral-kaon system, and is accessible experimentally [29]. The type of breaking of ordinary CPT invariance which we expect to emerge in future developments of the approach here proposed would also be distinguishable from other proposals because of the fact that here CPT invariance is replaced by a "deformed CPT invariance" whose predictions could be tested experimentally.

7   Summary and outlook

The analysis here reported had two objectives which we can now restate more succinctly using the discussion that preceded. The first objective was the one of putting on firmer ground recent ideas on the possibility that the quantum-group formalism might allow a consistent formulation of theories with deformed dispersion relations of the type which can now be tested [18] using recent progress in the phenomenology of gamma-ray bursts [15] and other astrophysical phenomena [16, 17]. In Sections 2, 3 and 6 this more rigorous analysis was given together with the first elements of a possible measurement theory for noncommutative spacetimes. We hope that having established more firmly the possibility of a consistent formalism for the mentioned deformed dispersion relations we will provide additional motivation for experimentalists to look for this new effect.

Our second objective was to point out the possibility of a new approach to the construction of field theories on noncommutative spacetimes of the type here considered (those where a quantum-geometry transformation[19] to a classical but nonAbelian energy-momentum group is possible), and to discuss some of the issues arising. This was done in Section 5. While several mathematical and interpretational developments are still required for us to be able to use this approach for the construction of a meaningful model, we believe that the procedure here outlined can provide a useful starting point for future work in this direction. As mentioned in Section 5, it appears likely that some of these field theories would be well-behaved in the ultraviolet (they would not require regularization of ultraviolet divergences). Additional motivation for this research

programme should come from the fact that by constructing (if this indeed turns out to be possible) a consistent model of particle physics according to the guidelines described in Section 5 we might then have a formalism which allows direct/explicit calculation of the mentioned in-vacuo dispersion effects and CPT-violation effects. The magnitude of these effects could be related directly and calculably to λ, while in other quantum-gravity formalisms believed to host these effects the evaluation of the magnitude of the effects directly from the original theory turns out to be too difficult (but one is able to identify in the theory the structures required for the effects of interest and phenomenological models [33, 35, 39, 41] can then be made to parametrize the magnitude of the effects).

Of course it would also be interesting to investigate further the idea that our noncommutative Minkowski spacetime could be used to model some properties of the "foamy vacuum" of quantum gravity. A natural framework for such studies appears to be provided by Canonical/Loop quantum gravity [42].

Acknowledgements

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