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The Forward–Backward Asymmetry in NNLO QCD *

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THE FORWARD–BACKWARD ASYMMETRY IN NNLO QCD

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Abstract

We have recently calculated the second-order QCD corrections to the forward–backward asymmetry in e^+e^- annihilation. Here we recall the results and compare them to others in the literature.

Experimental measurements of the forward–backward and left–right forward–backward asymmetries in e^+e^- annihilation to fermions provide some of the best determinations of the weak mixing angle $\sin^2\theta_{eff}$ [1]. In particular, the forward–backward asymmetry of b quarks is measured with a precision of about 2%, allowing an extraction of $\sin^2\theta_{eff}$ with almost per mille accuracy. However, since we are dealing with quarks in the final state, we must ensure that QCD corrections, both perturbative and non-perturbative, are understood to at least the same precision. For the perturbative corrections, this requires working to at least next-to-next-to-leading order (NNLO).

To date there have been two $\mathcal{O}(\alpha_S^2)$ calculations, both in the massless approximation and using a slightly different definition of the asymmetry than the experimental measurements, which use the thrust axis rather than the quark direction. The classic calculation of Altarelli and Lampe [2] determined the $\mathcal{O}(\alpha_S^2)$ coefficient numerically and found it to be small. This result has been the basis of all the experimental analyses since. However, the recent analytical calculation by Ravindran and van Neerven [3] obtained a coefficient about four times bigger. This discrepancy is comparable to the size of the experimental errors and needs to be resolved before the final electroweak fits to the LEP1 data can be made. The $\mathcal{O}(\alpha_S^2)$-calculation using the experimentally-used thrust axis definition, would also be highly desirable.

We have recently performed a numerical calculation of the $\mathcal{O}(\alpha_S^2)$ corrections to the forward–backward asymmetry [4]. Anticipating the result, given below, we can say that to the precision required by experiment we confirm the result of Ravindran and van Neerven and therefore rule out the result of Altarelli and Lampe. However, we do have a theoretically-important difference compared to Ravindran and van Neerven, in that we find that the forward–backward asymmetry contains terms enhanced by logarithms of the quark mass. Even though these terms are numerically tiny for realistic quark masses, as a point of principle it means that the forward–backward asymmetry of massless quarks is not perturbatively calculable and non-perturbative fragmentation functions have to be introduced.

We also calculated for the first time the corrections using the thrust axis definition rather than the quark direction. These lie approximately midway between the results of Refs. [2] and [3] for the quark axis definition.

We here only briefly sketch the method and give the final result, and refer the reader to Ref. [4] for more details.

The simplest definition of the b-quark[1] forward–backward asymmetry A_{FB} is

$$A_{FB} = \frac{N_F - N_B}{N_F + N_B} \quad , \tag{1}$$

where N_F and N_B are the number of b quarks observed in the forward and backward hemispheres, respectively.

The axis that identifies the forward direction can be defined in a variety of ways. In this paper we explicitly consider two different definitions: the b-quark direction, and the thrust axis direction, which we denote by A^b_{FB} and A^T_{FB} respectively.

According to the definition in Eq. (1), A_{FB} can be expressed in an equivalent way in terms of the cross section

$$\frac{\mathrm{d}\sigma(e^+e^- \to b + X)}{\mathrm{d}x \; \mathrm{d}\!\cos\theta} \tag{2}$$

for inclusive b-quark production, where x is the fraction of the electron energy carried by the b quark and θ is the angle between the electron momentum and the direction defining the forward hemisphere (both energies and angles are defined in the centre-of-mass frame).

Starting from the distribution in Eq. (2), we can introduce the forward and backward cross sections σ_F and σ_B:

$$\sigma_F \equiv \int_0^1 \mathrm{d}\!\cos\theta \int_0^1 \mathrm{d}x \, \frac{\mathrm{d}\sigma}{\mathrm{d}x \; \mathrm{d}\!\cos\theta} \; , \qquad \sigma_B \equiv \int_{-1}^0 \mathrm{d}\!\cos\theta \int_0^1 \mathrm{d}x \, \frac{\mathrm{d}\sigma}{\mathrm{d}x \; \mathrm{d}\!\cos\theta} \; , \tag{3}$$

and the symmetric and antisymmetric cross sections σ_S and σ_A:

$$\sigma_S = \sigma_F + \sigma_B \; , \qquad \sigma_A = \sigma_F - \sigma_B \; . \tag{4}$$

We can then write the forward–backward asymmetry as

$$A_{FB} = \frac{\sigma_A}{\sigma_S} \quad . \tag{5}$$

In order to calculate this ratio perturbatively, we first separate the contributions to the cross sections into three classes: flavour non-singlet (NS), flavour singlet (S), and interference (or triangle) (Tr) (see Ref. [4]) for their precise definition). We thus write the cross sections as

$$\sigma_S = \sigma_{S,NS} + \sigma^{(2)}_{S,S} + \sigma^{(2)}_{S,Tr} + \mathcal{O}(\alpha_{\mathrm{S}}^3) \; , \tag{6}$$

$$\sigma_A = \sigma_{A,NS} + \sigma^{(2)}_{A,Tr} + \mathcal{O}(\alpha_{\mathrm{S}}^3) \; . \tag{7}$$

[1]Throughout this paper we explicitly consider the case of the b-quark. The results for the charm quark can be simply obtained by properly replacing the mass and couplings of the massive quark.

In this notation, up to $\mathcal{O}(\alpha_S)$ there are only non-singlet contributions. Thus, $\sigma^{(2)}_{S,S}, \sigma^{(2)}_{S,Tr}$ and $\sigma^{(2)}_{A,Tr}$ are proportional to α_S^2. There are no singlet contributions to the antisymmetric cross section σ_A.

The forward–backward asymmetry is decomposed in a similar way. Expanding the ratio σ_A/σ_S up to $\mathcal{O}(\alpha_S^2)$, we write

$$A^{(2)}_{FB} = A^{(2)}_{FB,NS} + \frac{\sigma^{(0)}_A}{\sigma^{(0)}_S}\left(\frac{\sigma^{(2)}_{A,Tr}}{\sigma^{(0)}_A} - \frac{\sigma^{(2)}_{S,Tr}}{\sigma^{(0)}_S} - \frac{\sigma^{(2)}_{S,S}}{\sigma^{(0)}_S}\right) , \tag{8}$$

where $A^{(2)}_{FB,NS}$ denotes the non-singlet component:

$$A^{(2)}_{FB,NS} = \frac{\sigma_{A,NS}}{\sigma_{S,NS}} . \tag{9}$$

The triangle contributions give non-universal (i.e. non-factorizable) corrections to both the symmetric and antisymmetric cross sections. They are calculated in Ref. [2] for the b-quark axis definition and found to be very small. To our knowledge their contribution to the thrust axis definition has never been calculated, but we expect it to be similarly small. We therefore neglected it, i.e. $\sigma^{(2)}_{S,Tr}$ and $\sigma^{(2)}_{A,Tr}$ in Eq. (8), from our calculation.

The singlet contribution to the symmetric cross section, σ_S, is logarithmically enhanced in the small-mass limit and proportional to $\alpha_S^2 \ln^3 Q^2/m_b^2$. An approximate expression for it, denoted by $F^{\rm Branco}$, was used in Ref. [2]. It is calculated exactly to $\mathcal{O}(\alpha_S^2)$ in Refs. [5, 6], and the leading and next-to-leading logarithms are summed to all orders in α_S in Ref. [6].

In some sense the singlet component is a 'background' to the forward–backward asymmetry measurement and, in fact, in the experimental analyses (see e.g. Ref. [7]) it is statistically subtracted using Monte Carlo event generators. We therefore neglected it, i.e. $\sigma^{(2)}_{S,S}$ in Eq. (8), from our calculation.

Before describing the calculation of $A^{(2)}_{FB,NS}$, we take a slight diversion to discuss the contribution to it from four-b final states. Let us first point out a basic fact. The four-b process contributes to both the b-quark cross sections σ_S and σ_A and the e^+e^- total cross section. However, they appear with different multiplicity factors in the two cases. In the case of the e^+e^- total cross section the multiplicity factor is simply equal to unity. In the contribution to the *inclusive* b-quark cross sections σ_S and σ_A, these terms count twice since there are two b quarks in the final state. This observation is important in understanding the results for the non-singlet component of the symmetric cross section σ_S discussed shortly.

After summing and squaring the Feynman diagrams for four-b production, we obtain two types of contribution: *i*) those that are identical to the $b\bar{b}q\bar{q}$ final state but with the other quark q replaced by an untriggered-on b quark, and *ii*) those that are genuine interference terms arising from the fact that the two antiquarks are indistinguishable, called the E-term in Ref. [8]. The squared diagrams of type *i*) are lumped together with the corresponding terms from $b\bar{b}q\bar{q}$ in the singlet ($\sigma^{(2)}_{S,S}$ in Eq. (6)), non-singlet ($\sigma_{S,NS}$ and $\sigma_{A,NS}$ in Eqs. (6) and (7)) or triangle ($\sigma^{(2)}_{S,Tr}$ and $\sigma^{(2)}_{A,Tr}$ in Eqs. (6) and (7)) contributions. The squared diagrams of type *ii*), which give a universal (i.e. factorizable) correction to both the antisymmetric and symmetric cross sections, can be considered part of the non-singlet contributions.

It is not entirely clear how four-quark final states are actually treated in the different experimental analyses, i.e. the extent to which they are genuinely measuring the inclusive cross sections. Often some vague statement like "a four-b final state is more likely to be tagged than a two-b one, but less than twice as likely" is made. To know what to calculate one must understand the corrections that are applied for this difference in tagging efficiency, which are not usually explicitly stated in the papers. In the absence of a unique experimental procedure and of a definitive statement from the experiments on what they are measuring, we make this ambiguity explicit by multiplying the E-term by an arbitrary weight factor W_E. An inclusive definition would correspond to $W_E = 2$ (each b quark contributing once), while an exclusive definition (the cross section for events containing at least one b quark) would correspond to $W_E = 1$. Since the forward–backward asymmetry is defined to be the asymmetry of a differential cross section, it is clear that we must use the *same* cross section definition in the numerator and denominator, i.e. that W_E must be the same in the symmetric and antisymmetric cross sections.

Having defined the weight factor W_E for the E-term, we can define the following symmetric and antisymmetric cross sections

$$\sigma_{S,NS}(W_E) = \sigma_{S,NS}(W_E = 0) + W_E\, \sigma_S^{(0)} \int E_S \ , \qquad (10)$$

$$\sigma_{A,NS}(W_E) = \sigma_{A,NS}(W_E = 0) + W_E\, \sigma_A^{(0)} \int E_A \ , \qquad (11)$$

where $\int E_S$ and $\int E_A$ denote the integral of the symmetric and antisymmetric E-term, respectively. We recall that the 'truly' inclusive cross sections in Eq. (4) correspond to the definition with $W_E = 2$, i.e. $\sigma_{S,NS} = \sigma_{S,NS}(W_E = 2)$ and $\sigma_{A,NS} = \sigma_{A,NS}(W_E = 2)$.

The $\mathcal{O}(\alpha_S^2)$-calculation of the cross sections in Eqs. (10, 11) and of the corresponding forward–backward asymmetry in the case of a finite b-quark mass is extremely complicated, and we are not able to perform it. It is thus convenient to separate the calculation into a piece that is finite (although still cumbersome) in the massless limit and a simpler piece that is not. Then, the finite piece can be more easily computed in the massless approximation, while the simpler non-finite piece can be computed in the massive theory.

It is possible to show [4] that the inclusive definition, with $W_E = 2$, results in an antisymmetric cross section σ_A (or, analogously, $\sigma_{A,NS}$) that is finite in the massless limit, at least at $\mathcal{O}(\alpha_S^2)$. However, in the same limit, the inclusive symmetric cross section is divergent at $\mathcal{O}(\alpha_S^2)$, even if we only consider its non-singlet component. The corrections to (the non-singlet component of) the forward–backward asymmetry itself must therefore also be divergent in the massless limit.

This final statement remains true for *any* value of $W_E > 0$. For example, with $W_E = 1$, the non-singlet part of the symmetric cross section is finite, but the antisymmetric cross section contains logarithmically-enhanced terms.

The divergences in the non-singlet components correspond to logarithmically-enhanced terms $\alpha_S^2 \ln Q^2/m_b^2$ coming from the E-term in the triple-collinear limit, i.e. when three fermions of the four-quark final state become simultaneously parallel. The integral of the symmetric E-term is calculated numerically in Ref. [4] and, neglecting corrections of $\mathcal{O}(m_b/Q)$, the final result is

$$\int E_S = C_F \left(C_F - \frac{C_A}{2} \right) \left(\frac{\alpha_S}{2\pi} \right)^2 \left[2 \left(\frac{13}{4} - \frac{\pi^2}{2} + 2\zeta_3 \right) \ln \frac{Q^2}{m_b^2} - 8.1790 \pm 0.0013 \right] , \tag{12}$$

where the analytic coefficient in front of $\ln Q^2/m_b^2$ is proportional to the integral of the non-singlet Altarelli–Parisi probability $P_{q\bar{q}}^{NS}(z, \alpha_S)$ (see, for instance, Ref. [9]):

$$\int_0^1 \mathrm{d}z \, P_{q\bar{q}}^{NS}(z, \alpha_S) = \left(\frac{\alpha_S}{2\pi} \right)^2 C_F \left(C_F - \frac{1}{2} C_A \right) \left(\frac{13}{4} - \frac{\pi^2}{2} + 2\zeta_3 \right) , \tag{13}$$

and the constant term is the result of our numerical calculation.

Having pointed out that the symmetric E-term is divergent in the massless limit, it is very simple to show how the divergence appears in the inclusive symmetric cross section. According to the definition of the non-singlet component of σ_S, the virtual diagrams that contribute to $\sigma_{S,NS}$ are exactly those that contribute to the e^+e^- total cross section. As for the real diagrams, they only differ by the contributions coming from the E-term. In the total cross section, the E-term enters with a multiplicity factor $W_E = 1$, and its divergence is cancelled by that of the virtual diagrams. In the inclusive b-quark cross section, the multiplicity factor of the E-term is $W_E = 2$ and, thus, the cancellation of the divergence with the virtual terms is spoiled.

This argument also allows us to directly compute the $\mathcal{O}(\alpha_S^2)$-correction to Eq. (10). Exploiting the fact that the massless QCD correction to $\sigma_{S,NS}(W_E = 1)$ is equal to the correction $R_{e^+e^-}$ to the total cross section, we write

$$\sigma_{S,NS}(W_E) = \sigma_S^{(0)} \left[R_{e^+e^-} + (W_E - 1) \int E_S + \mathcal{O}(\alpha_S^3) \right] \ . \tag{14}$$

Then, we obtain an explicit expression for $\sigma_{S,NS}(W_E)$ by simply introducing in Eq. (14) our result in Eq. (12) for $\int E_S$ and the well-known result [10] for $R_{e^+e^-}$. In particular, for the inclusive symmetric cross section we obtain

$$\sigma_{S,NS} = \sigma_{S,NS}(W_E = 2) = \sigma_S^{(0)} \left[R_{e^+e^-} + \int E_S + \mathcal{O}(\alpha_S^3) \right] \ . \tag{15}$$

Since both $\sigma_{A,NS}(W_E = 2)$ and $\sigma_{S,NS}(W_E = 1)$ are finite when $m_b \to 0$, we can use the dependence on W_E to construct an unphysical observable that is finite in the massless limit:

$$A_{FB}^{(2);\text{finite}} \equiv \frac{\sigma_{A,NS}(W_E = 2)}{\sigma_{S,NS}(W_E = 1)} \ . \tag{16}$$

This observable is the ratio of the antisymmetric part of the *inclusive* cross section and the symmetric part of the *exclusive* cross section. Thus $A_{FB}^{(2);\text{finite}}$ is unphysical in the sense that it is not the forward–backward asymmetry of a single differential cross section. Nonetheless the definition in Eq. (16) helps us to perform a massless calculation. The physical result for $W_E = 2$ is then given by

$$A_{FB,NS}^{(2)} = A_{FB}^{(2);\text{finite}} - A_{FB}^{(0)} \int E_S \ , \tag{17}$$

where $\int E_S$ is the integral of the symmetric E-term, given in Eq. (12).

Even in the massless limit numerical two-loop calculations are prohibitively difficult to set up. Fortunately there is a cancellation between the genuinely two-loop effects in the ratio on the right-hand-side of Eq. (16), which does allow its numerical evaluation. The total contribution can be written as

$$A_{FB}^{(2);\text{finite}} = \frac{\sigma_A^{(0)} + \sigma_A^{(1);\text{one-loop}} + \sigma_A^{(1);\text{tree}} + \sigma_A^{(2);\text{two-loop}} + \sigma_A^{(2);\text{one-loop}} + \sigma_A^{(2);\text{tree}}(W_E = 2)}{\sigma_S^{(0)} + \sigma_S^{(1);\text{one-loop}} + \sigma_S^{(1);\text{tree}} + \sigma_S^{(2);\text{two-loop}} + \sigma_S^{(2);\text{one-loop}} + \sigma_S^{(2);\text{tree}}(W_E = 1)} \ . \tag{18}$$

The $\mathcal{O}(\alpha_S)$-contributions come from the one-loop cross sections $\sigma^{(1);\text{one-loop}}$ for the two-parton process $e^+e^- \to b\bar{b}$ and the tree-level cross sections $\sigma^{(1);\text{tree}}$ for the three-parton process $e^+e^- \to b\bar{b}g$. Similarly the non-singlet $\mathcal{O}(\alpha_S^2)$-contributions from the two-parton, three-parton and four-parton final states are denoted by $\sigma^{(2);\text{two-loop}}$, $\sigma^{(2);\text{one-loop}}$ and $\sigma^{(2);\text{tree}}$ respectively. Of course, the dependence on W_E enters only through the four-parton terms $\sigma_A^{(2);\text{tree}}(W_E = 2)$ and $\sigma_S^{(2);\text{tree}}(W_E = 1)$.

Each of the cross sections is separately divergent, so they have to be regularized in some way before being combined together. In any regularization scheme that preserves the helicity conservation of massless QCD[2] (for example, dimensional regularization), we have the properties

$$\frac{\sigma_A^{(1);\text{one-loop}}}{\sigma_A^{(0)}} = \frac{\sigma_S^{(1);\text{one-loop}}}{\sigma_S^{(0)}}, \qquad \frac{\sigma_A^{(2);\text{two-loop}}}{\sigma_A^{(0)}} = \frac{\sigma_S^{(2);\text{two-loop}}}{\sigma_S^{(0)}}, \tag{19}$$

so that if we expand the ratio in Eq. (18) up to $\mathcal{O}(\alpha_S^2)$, the two-loop corrections cancel, and we obtain

$$\begin{aligned} A_{FB}^{(2);\text{finite}} &= \frac{\sigma_A^{(0)}}{\sigma_S^{(0)}}\Bigg[1 + \left(1 - \frac{\sigma_S^{(1)}}{\sigma_S^{(0)}}\right)\left(\frac{\sigma_A^{(1)}}{\sigma_A^{(0)}} - \frac{\sigma_S^{(1)}}{\sigma_S^{(0)}}\right) \\ &\quad + \frac{\sigma_A^{(2);\text{one-loop}}}{\sigma_A^{(0)}} - \frac{\sigma_S^{(2);\text{one-loop}}}{\sigma_S^{(0)}} + \frac{\sigma_A^{(2);\text{tree}}(W_E = 2)}{\sigma_A^{(0)}} - \frac{\sigma_S^{(2);\text{tree}}(W_E = 1)}{\sigma_S^{(0)}}\Bigg], \end{aligned} \tag{20}$$

where $\sigma_A^{(1)}$ and $\sigma_S^{(1)}$ are the complete contributions to the antisymmetric and symmetric cross sections at $\mathcal{O}(\alpha_S)$, $\sigma^{(1)} = \sigma^{(1);\text{one-loop}} + \sigma^{(1);\text{tree}}$. The first line can be calculated analytically, but the second line is too complicated to be able to, so must be done numerically. Since the two-loop terms have cancelled, this has the structure of a NLO three-jet calculation, as first noticed by Altarelli and Lampe [2]. Thus the calculation can be performed using known techniques (we use the dipole-formalism version of the subtraction method [11]).

We are finally ready to present our numerical results. We start with the unphysical, but finite, quantity defined in Eq. (16), and separate out the different colour factors, as in Refs. [2, 3]:

$$\begin{aligned} A_{FB}^{(2);\text{finite};b} &= A_{FB}^{(0)}\Bigg[1 - \frac{\alpha_S}{2\pi}\left(1 - \frac{\alpha_S}{2\pi}\frac{3}{2}C_F\right)\left(\frac{3}{2}C_F\right) \\ &\qquad + \left(\frac{\alpha_S}{2\pi}\right)^2 C_F\left(CC_F + NN_C + TT_RN_f\right)\Bigg], \end{aligned} \tag{21}$$

with $\alpha_S \equiv \alpha_S(Q^2)$. Our numerical results are shown in Table 1, in comparison with the previous calculations. It is clear that we disagree badly with the results of Altarelli and Lampe [2], but are in excellent agreement with Ravindran and van Neerven [3], who give the coefficients analytically. However, we should recall that $A_{FB}^{(2);\text{finite}}$, as given in Eq. (21), is not the forward–backward asymmetry of a definite cross section. The physical forward–backward asymmetry must have subtracted

[2]Note that the relations (19) are explicitly violated for massive quarks.

b-quark axis	C	N	T
AL [2]	4.4 ± 0.5	-10.3 ± 0.3	5.68 ± 0.04
RvN [3]	$\frac{3}{8} = 0.375$	$-\frac{123}{8} = -15.375$	$\frac{11}{2} = 5.5$
Our Calculation	0.3765 ± 0.0038	-15.3769 ± 0.0034	5.5002 ± 0.0008

Table 1: *Results for the coefficients of the $\mathcal{O}(\alpha_S^2)$ correction to the finite part of the forward–backward asymmetry with the b-quark axis definition, Eqs. (17, 21).*

from Eq. (21) the logarithmically-enhanced term of Eq. (17), which is not present in the result of Ref. [3]. Thus it seems that they have somehow computed the unphysical $A_{FB}^{(2);\mathrm{finite}}$ rather than the forward–backward asymmetry. In fact, their expression for the correction to the symmetric cross section ($f_T + f_L$ in their Eqs. (31) and (32)) is actually equal to our $\sigma_{S,NS}(W_E = 1)$ in Eq. 14. So, the fact that their result for $A_{FB}^{(2)}$ agrees with our $A_{FB}^{(2);\mathrm{finite}}$ means that we confirm their result [12, 3] for the inclusive antisymmetric cross section $\sigma_A^{(2)} = \sigma_A^{(2)}(W_E = 2)$ (f_A in Eq. (33) of Ref. [3]).

Using our numerical program it is straightforward to calculate the forward–backward asymmetry with any other axis definition (or cuts, for example on the value of the thrust). With the thrust axis definition, we obtain

$$A_{FB}^{(2);\mathrm{finite};T} = A_{FB}^{(0)}\left[1 - \frac{\alpha_S}{2\pi}\left(1 - \frac{\alpha_S}{2\pi}\frac{3}{2}C_F\right)(1.34 C_F) + \left(\frac{\alpha_S}{2\pi}\right)^2 C_F\left(C C_F + N N_C + T T_R N_f\right)\right], \quad (22)$$

with $\alpha_S \equiv \alpha_S(Q^2)$ and the coefficients given in Table 2. The logarithmically-enhanced piece that has to be added to this is identical to that in the b-quark axis definition, namely Eqs. (17, 12). It is worth noting that the difference between the two definitions is the same size and in the same direction as at $\mathcal{O}(\alpha_S)$, leading to an overall difference of 0.8% for $\alpha_S \sim 0.12$.

We finally recall that we include an arbitrary factor W_E in front of the

thrust axis	C	N	T
Our Calculation	-3.7212 ± 0.0065	-9.6011 ± 0.0049	4.4144 ± 0.0006

Table 2: *Results for the coefficients of the $\mathcal{O}(\alpha_S^2)$ correction to the finite part of the forward–backward asymmetry with the thrust axis definition, Eqs. (17, 22).*

four-b contribution to account for the way in which it is treated in the experimental analyses. For a fully inclusive definition, in which each b quark contributes once, W_E should be set equal to 2, while for an exclusive definition, W_E should be set equal to 1. Our final result for the non-singlet component of the forward–backward asymmetry, is then:

$$A^{(2)}_{FB,NS}(W_E) \equiv \frac{\sigma_{A,NS}(W_E)}{\sigma_{S,NS}(W_E)} = A^{(2);\mathrm{finite}}_{FB} - A^{(0)}_{FB}\left[(1-\tfrac{1}{2}W_E)\left(2\int E_A - \int E_S\right) + \tfrac{1}{2}W_E \int E_S\right] , \tag{23}$$

where $A^{(2);\mathrm{finite}}_{FB}$ is given in Eqs. (21, 22) and Tables 1 and 2, $\int E_S$ is given in Eq. (12), and (see Appendix B of Ref. [4])

$$2\int E_A - \int E_S = \left(\frac{\alpha_S}{2\pi}\right)^2 C_F(C_F - \tfrac{1}{2}C_A)\big(0.3620 \pm 0.0007\big), \quad \text{quark axis}, \tag{24}$$

$$2\int E_A - \int E_S = \left(\frac{\alpha_S}{2\pi}\right)^2 C_F(C_F - \tfrac{1}{2}C_A)\big(0.1144 \pm 0.0009\big), \quad \text{thrust axis}. \tag{25}$$

Note that the combinations of E-term contributions in Eqs. (24) and (25) are finite in the massless limit (see the discussion in Appendix B of Ref. [4]).

Putting all these numbers together, and setting $N_f = 5$, we write the forward–backward asymmetry according to the two definitions as:

$$A^{(2);b}_{FB,NS}(W_E) = A^{(0)}_{FB}\left[1 - 0.318\alpha_S - 0.973\alpha_S^2 + W_E\alpha_S^2\left(0.00405\ln\frac{Q^2}{m_b^2} - 0.0240\right)\right], \tag{26}$$

$$A^{(2);T}_{FB,NS}(W_E) = A^{(0)}_{FB}\left[1 - 0.284\alpha_S - 0.676\alpha_S^2 + W_E\alpha_S^2\left(0.00405\ln\frac{Q^2}{m_b^2} - 0.0233\right)\right], \tag{27}$$

with $\alpha_S \equiv \alpha_S(Q^2)$. Note that the logarithmically-enhanced term, $\ln Q^2/m_b^2$, is present for any physical ($W_E > 0$) value of W_E.

Putting in an explicit value for α_S, we summarize the total QCD correction according to the various available calculations in Table 3. We continue to neglect all terms that vanish in the massless limit. Since in the existing experimental analyses (see for example Ref. [7]), the known $\mathcal{O}(\alpha_S)$ correction for the thrust axis definition was included, together with the Altarelli and Lampe quark axis value for the $\mathcal{O}(\alpha_S^2)$ corrections, we do the same in Table 3.

We find that the difference between the Ravindran and van Neerven calculation and ours is numerically irrelevant, being smaller than 10^{-4} for b quarks and $\sim 2.5\times10^{-4}$ for c quarks. Therefore at the numerical precision required by current or any foreseen experiments, we agree with their result – the difference is only one

	AL [2] quark axis	RvN [3] quark axis	Our Calculation quark axis	Our Calculation thrust axis
Correction, $A_{FB}^{(2)}/A_{FB}^{(0)}$	0.962	0.952	0.952	0.956

Table 3: *Total QCD correction to the forward–backward asymmetry in the small-mass limit, with* $\alpha_S = 0.12$*. In each case, the thrust axis definition is used for the* $\mathcal{O}(\alpha_S)$ *correction and the definition shown is used for the* $\mathcal{O}(\alpha_S^2)$ *correction, as discussed in the text.*

of principle. The difference between the Altarelli and Lampe calculation and ours for the quark axis definition is more significant though, at around 1%. However, the error in their calculation and the effect of using the thrust axis definition partially cancel, and the total difference is around 0.6%.

We should also mention the important fact, discussed in Ref. [7], that the experimental procedures introduce a bias towards more two-jet-like events. This actually decreases the size of the QCD corrections considerably, so our number should be considered as an upper bound on the final difference.

In Ref. [4] we try to estimate the remaining uncertainties in the forward–backward asymmetry, bearing in mind that while the 2% precision of current experiments is close to their final limit, a future linear collider could be capable of experimental errors on the left–right forward–backward asymmetry of order 0.1% [13]. We found several sources of uncertainty that all contribute at the few per mille level. While this is certainly sufficient for the current precision of the data, matching the precision of a future linear collider measurement could be extremely difficult. It is likely that this could only be done by making even more stringent two-jet cuts in order to work in a region in which the corrections and their uncertainties are smaller.

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