Backreaction in Cosmological Models

Christian Sicka[1], *Thomas Buchert*[2,1] *and Martin Kerscher*[1]

[1] *Ludwig–Maximilians–Universität, Theresienstraße 37, D–80333 München, Germany*
[2] *Theory Division, CERN, CH–1211 Genève 23, Switzerland*
emails: sicka,buchert,kerscher@theorie.physik.uni-muenchen.de

1.1 The backreaction

The basic equations for the description of structure formation in cosmology are Einstein's laws for gravitationally interacting systems or, with restrictions, Newton's law of gravity. Taking the Universe to be filled with a self–gravitating pressure–less fluid ('dust'), one has to deal with a nonlinear system of differential equations. To simplify this system of differential equations one commonly assumes special symmetries. Then, for each set of symmetries, we obtain a class of solutions.

For simplicity and for some philosophical reasons, most cosmological models studied today are those based on the assumption of homogeneity and isotropy. Observationally one can find evidence that supports these assumptions on very large scales, the strongest being the almost isotropy of the Cosmic Microwave Background radiation after assigning the whole dipole to our proper motion relative to this background. However, on small and on intermediate scales up to several hundreds of Mpcs, there are strong deviations from homogeneity and isotropy [5]. Here the problem arises how to relate the observations with the homogeneous and isotropic models. The usual proposal for solving this problem is to assume that Friedmann–Lemaître models describe the mean observables. Such mean values may be identified with spatial averages. For Newtonian fluid dynamics the averaging procedure has been discussed in detail in [2]. The key difference between the averaged models and the standard model can be related to the fact that the spatial average of a tensor field $\mathcal{A}$ over a domain comoving with the fluid (a "Lagrangian domain") does not commute with the time evolution of that tensor field:

$$d_t\langle\mathcal{A}\rangle_{\mathcal{D}} - \langle d_t\mathcal{A}\rangle_{\mathcal{D}} = \langle\mathcal{A}\theta\rangle_{\mathcal{D}} - \langle\theta\rangle_{\mathcal{D}}\langle\mathcal{A}\rangle_{\mathcal{D}} \; . \tag{1}$$

$\langle\mathcal{A}\rangle_{\mathcal{D}}$ denotes the Euclidean spatial average of $\mathcal{A}$ over a domain $\mathcal{D}$, and θ the local expansion rate.

By averaging Raychaudhuri's equation for a dust matter model and using the commutation rule (1) one obtains a differential equation for the averaged expansion rate. This equation can also be written in a form similar to the standard Friedmann equation, but now with the domain dependent scale factor $a_{\mathcal{D}} = V_{\mathcal{D}}^{1/3}$; $V_{\mathcal{D}} = |\mathcal{D}|$, featuring an additional 'backreaction' term $\mathcal{Q}$:

$$3\frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} + 4\pi G\langle\varrho\rangle_{\mathcal{D}} - \Lambda = \mathcal{Q} \quad \text{with} \quad \mathcal{Q} := \frac{2}{3}(\langle\theta^2\rangle_{\mathcal{D}} - \langle\theta\rangle_{\mathcal{D}}^2) + 2\langle\omega^2 - \sigma^2\rangle_{\mathcal{D}} \; . \tag{2}$$

σ denotes the rate of shear and ω the rate of rotation of an infinitesimal fluid element.

As soon as inhomogeneities are present in the domain, the rate of shear, the rate of rotation and the expansion–rate are nonzero. Hence, $\mathcal{Q} \neq 0$ and the domain–dependent scale factor $a_{\mathcal{D}}$ will behave in a different way compared with the scale factor of a homogeneous–isotropic Friedmann cosmology.

1.2 Spatially compact cosmologies without boundary

The backreaction term can also be written in the following form after the application of Gauß' theorem:

$$\mathcal{Q} = \frac{1}{V_{\mathcal{D}}} \int_{\partial\mathcal{D}} (\mathbf{u}(\nabla \cdot \mathbf{u}) - (\mathbf{u} \cdot \nabla)\mathbf{u}) \cdot \mathrm{d}\mathbf{S} - \frac{2}{3} \left(\frac{1}{V_{\mathcal{D}}} \int_{\partial\mathcal{D}} \mathbf{u} \cdot \mathrm{d}\mathbf{S} \right)^2 \ , \tag{3}$$

with the surface $\partial\mathcal{D}$ bounding the domain $\mathcal{D}$ and the surface-element $\mathrm{d}\mathbf{S}$; $\mathbf{u}$ denotes the peculiar velocity field.

If we choose the compact domain $\mathcal{D}$ to be the whole universe, then backreaction vanishes for all models with closed 3–spaces (compact without boundary). For example all toroidal models with local inhomogeneities can be treated globally like Friedmann models. The homogeneous density in the Friedmann models is then simply the density averaged over the whole space. From the same argument we conclude that no backreaction is present in N–body simulations on the periodicity scale. However, this can be accomplished only within the framework of Newtonian theory. In General Relativity the situation is more involved due to the presence of an averaged contribution from the Ricci curvature, and the line of arguments above is not conclusive in that case [3].

1.3 Effects of backreaction on intermediate scales

On intermediate scales the Newtonian 'dust' approximation is well–established in models of structure formation. With the inhomogeneous Friedmann equation (2) we have a tool to quantify the time evolution of the scale factor of spatial domains in the Universe and to relate them to the global background expansion. Unfortunately, the dynamical evolution of the backreaction term is not known. Therefore, we use the Eulerian linear approximation and the 'Zel'dovich approximation' to estimate the effect of the backreaction term. Such a calculation is only well–defined, after all what has been said, if we assume that the Universe can be approximated by spatially flat space sections and that the inhomogeneities are subjected to periodic boundary conditions on some large scale. In this way the calculation of the backreaction effect on scales below the periodicity scale can be investigated. We want to emphasize that no conclusion about the global value of backreaction is possible, because it vanishes by assumption. This, of course, restricts the generality and shows the need to go to a general relativistic investigation. Russ et al. [6] have attempted this in a recent work. However, their assumptions to start with are too restrictive, so that a vanishing backreaction already follows from their basic equations (see [3] [4]).

1.3.1 Backreaction in Eulerian linear theory

We use the linear approximation theory to calculate the time evolution of the velocity field. For an Einstein–de–Sitter background and in the initial stages of structure formation the backreaction term $\mathcal{Q}$ decreases [4]:

$$\mathcal{Q}^{\ell} = a^{-1}\mathcal{Q}_0 \ \ \text{with} \ \ a = \left(\frac{t}{t_0}\right)^{2/3} \ . \tag{4}$$

The dimensionless contribution to the expansion may be estimated by $\mathcal{Q}/(4\pi G\varrho_H)$, quantifying the impact of the backreaction $\mathcal{Q}$ in comparison to the background density ϱ_H. This dimensionless backreaction grows proportionally to the variance of the density contrast field $\delta^2_{\mathrm{r.m.s.}} \propto a^2$. For this type of linearization we have to assume that the deformation of the comoving volume is negligible. We can relax this assumption if we work in the Lagrangian picture. That is why the 'Zel'dovich approximation' is a better tool in this context.

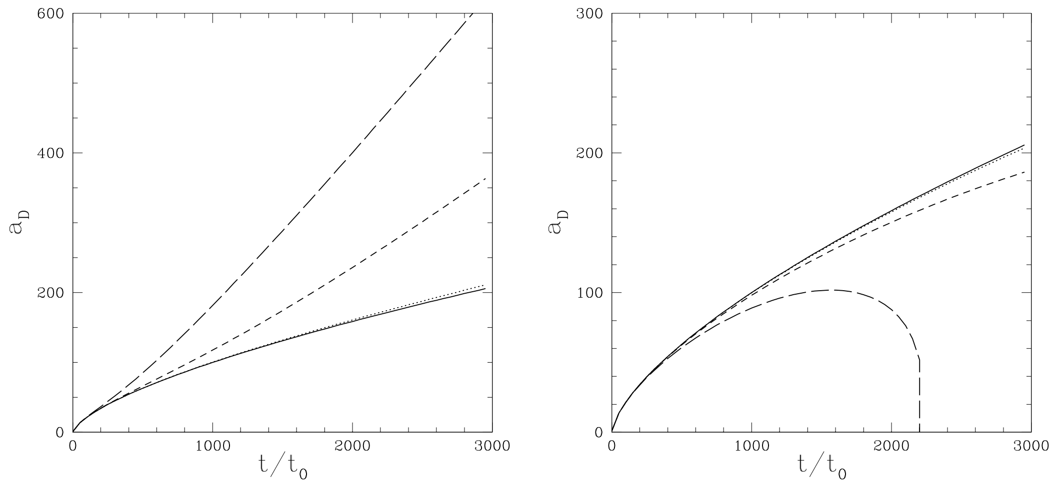

Figure 1: In the left panel the time evolution of the scale factor $a_{\mathcal{D}}$ is shown for $\langle I_0\rangle_{\mathcal{D}_0} = 0$, $\langle II_0\rangle_{\mathcal{D}_0} = 10^{-6}$, and $\langle III_0\rangle_{\mathcal{D}_0} = 10^{-8}$ (dotted), $\langle III_0\rangle_{\mathcal{D}_0} = 10^{-6}$ (short dashed), $\langle III_0\rangle_{\mathcal{D}_0} = 10^{-5}$ (long dashed). In the right panel we used $\langle I_0\rangle_{\mathcal{D}_0} = 0$, $\langle II_0\rangle_{\mathcal{D}_0} = -10^{-6}$, and $\langle III_0\rangle_{\mathcal{D}_0} = -10^{-8}$ (dotted), $\langle III_0\rangle_{\mathcal{D}_0} = -10^{-7}$ (short dashed), $\langle III_0\rangle_{\mathcal{D}_0} = -10^{-6}$ (long dashed). The solid line corresponds to an Einstein–de–Sitter universe, i.e. $\mathcal{Q} = 0$.

1.3.2 Backreaction in the 'Zel'dovich approximation'

Zel'dovich's approximation is a subcase of solutions to a Lagrangian first–order perturbation approach [7][1]; for the (dimensionless) trajectories of the fluid elements we have:

$$f^Z(\mathbf{X}, t) = a(t)(\mathbf{X} + \xi(t)\nabla_0\psi(\mathbf{X})) \ . \tag{5}$$

Here, $\nabla_0\psi(\mathbf{X})$ denotes the initial displacement field, $\xi(t)$ is a universal function of time and $\mathbf{X}$ is the initial position of the fluid particles. The 'Zel'dovich approximation' is known to give a good picture for structure formation also in the nonlinear regime until shell–crossing takes place.

It is useful to calculate the scaled backreaction term $a_{\mathcal{D}}^6\mathcal{Q}$ in the 'Zel'dovich approximation', which can be written in a compact form [4]:

$$\begin{aligned} a_{\mathcal{D}}^6\mathcal{Q} = a^6\dot{\xi}^2 \quad & \Big(2\langle II_0\rangle_{\mathcal{D}_0} - \tfrac{2}{3}\langle I_0\rangle_{\mathcal{D}_0}^2 + \\ & + \xi(6\langle III_0\rangle_{\mathcal{D}_0} - \tfrac{2}{3}\langle I_0\rangle_{\mathcal{D}_0}\langle II_0\rangle_{\mathcal{D}_0}) + \xi^2(2\langle I_0\rangle_{\mathcal{D}_0}\langle III_0\rangle_{\mathcal{D}_0} - \tfrac{2}{3}\langle II_0\rangle_{\mathcal{D}_0}^2)\Big) \ , \end{aligned} \tag{6}$$

with $\langle I_0\rangle_{\mathcal{D}_0}, \langle II_0\rangle_{\mathcal{D}_0}, \langle III_0\rangle_{\mathcal{D}_0}$ being the first, the second and the third scalar invariants of the tensor field $\psi_{|ij}$ averaged over the initial domain $\mathcal{D}(t_0)$; '$_|$' denotes the derivation with respect to Lagrangian coordinates.

Using the approximation (6) for the backreaction $\mathcal{Q}$ we solve the differential equation (2) for $a_{\mathcal{D}}$ numerically. The results for various initial displacement fields are shown in Fig. 1 for an Einstein–de–Sitter background model with $a = (t/t_0)^{2/3}$. We have chosen $\langle I_0\rangle_{\mathcal{D}_0} = 0$ in both plots, corresponding to a mean over–density of zero in the domain $\mathcal{D}_0$. If no backreaction is present, such domains should follow the expansion of the background model. However, with $\langle II_0\rangle_{\mathcal{D}_0} \neq 0$ and $\langle III_0\rangle_{\mathcal{D}_0} \neq 0$ we have $\mathcal{Q} \neq 0$ in general. The accelerated expansion visible in the left plot and the accelerated collapse in the right plot is triggered only by the small deviations in the initial displacement field as described by $\langle II_0\rangle_{\mathcal{D}_0}$ and $\langle III_0\rangle_{\mathcal{D}_0}$.

Summarizing, we have shown that the backreaction significantly influences the dynamics of Lagrangian domains in the 'Zel'dovich approximation'. Backreaction can invoke a dynamics that ressembles cosmologies with a cosmological constant. Indeed every homogeneous and isotropic Λ–model can be approximated with a backreaction model for suitably chosen initial conditions. Considering Gaussian random fields we will estimate the effect for generic initial displacement fields [4]. Backreaction may, on the other hand, provide us with a new class of collapse models that outperform the standard 'top–hat' model (which excludes backreaction by its restriction to spherical symmetry). Such models may deepen our understanding of the dynamics on cluster scales. Finally, backreaction may substantiate the problem of missing mass: for domains dominated by shear fluctuations, backreaction can act as a "dynamical dark matter" component.

Acknowledgements

This work is supported by the 'Sonderforschungsbereich für Astroteilchenphysik SFB 375 der DFG'. We thank Claus Beisbart for valuable comments and discussions.

References

[1] T. Buchert, Lagrangian theory of gravitational instability of Friedmann–Lemaître cosmologies and the "Zel'dovich approximation" MNRAS **254**, 729, 1992.

[2] T. Buchert and J. Ehlers, Averaging inhomogeneous Newtonian cosmologies. A&A **320**, 1, 1997.

[3] T. Buchert, On average properties of inhomogeneous fluids in general relativity I: dust cosmologies. G.R.G. in press (gr-qc/9906015), 1999.

[4] T. Buchert, M. Kerscher and C. Sicka, Backreaction of inhomogeneities on the expansion: a dynamical approach to cosmic variance. in preparation, 1999.

[5] M. Kerscher, J. Schmalzing, T. Buchert and H. Wagner, Fluctuations in the IRAS 1.2 Jy Catalogue. A&A **333**, 1, 1998.

[6] H. Russ, M.H. Soffel, M. Kasai and G. Börner, Age of the Universe: influence of the inhomogeneities on the global expansion-factor Phys. Rev. D **56**, 2044, 1997.

[7] Ya.B. Zel'dovich , Fragmentation of a homogeneous medium under the action of gravitation. Astrophysics **6**, 164, 1970.