# Self-gravitating fundamental strings and black holes

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## Abstract

The configuration of typical highly excited ($M \gg M_s \sim (\alpha')^{-1/2}$) string states is considered as the string coupling $g$ is adiabatically increased. The size distribution of very massive single string states is studied and the mass shift, due to long-range gravitational, dilatonic and axionic attraction, is estimated. By combining the two effects, in any number of spatial dimensions $d$, the most probable size of a string state becomes of order $\ell_s = \sqrt{2\,\alpha'}$ when $g^2 M/M_s \sim 1$. Depending on the dimension $d$, the transition between a random-walk-size string state (for low $g$) and a compact ($\sim \ell_s$) string state (when $g^2 M/M_s \sim 1$) can be very gradual ($d = 3$), fast but continuous ($d = 4$), or discontinuous ($d \geq 5$). Those compact string states look like nuggets of an ultradense state of string matter, with energy density $\rho \sim g^{-2} M_s^{d+1}$. Our results extend and clarify previous work by Susskind, and by Horowitz and Polchinski, on the correspondence between self-gravitating string states and black holes.

## I. INTRODUCTION

Almost exactly thirty years ago the study of the spectrum of string theory (known at the time as the dual resonance model) revealed [1] a huge degeneracy of states growing as an exponential of the mass. This led to the concept of a limiting (Hagedorn) temperature $T_{\rm Hag}$ in string theory. Only slightly more recently Bekenstein [2] proposed that the entropy of a black hole should be proportional to the area of its horizon in Planck units, and Hawking [3] fixed the constant of proportionality after discovering that black holes, after all, do emit thermal radiation at a temperature $T_{\rm Haw} \sim R_{\rm BH}^{-1}$.

When string and black hole entropies are compared one immediately notices a striking difference: string entropy[1] is proportional to the first power of mass in any number of spatial dimensions $d$, while black hole entropy is proportional to a $d$-dependent power of the mass, always larger than 1. In formulae:

$$S_s \sim \frac{\alpha' M}{\ell_s} \sim M/M_s \quad , \qquad S_{\rm BH} \sim \frac{\rm Area}{G_N} \sim \frac{R_{\rm BH}^{d-1}}{G_N} \sim \frac{(g^2\, M/M_s)^{\frac{d-1}{d-2}}}{g^2} \,, \tag{1.1}$$

where, as usual, $\alpha'$ is the inverse of the classical string tension, $\ell_s \sim \sqrt{\alpha' \hbar}$ is the quantum length associated with it[2], $M_s \sim \sqrt{\hbar/\alpha'}$ is the corresponding string mass scale, $R_{\rm BH}$ is the Schwarzschild radius associated with $M$:

$$R_{\rm BH} \sim (G_N\, M)^{1/(d-2)} \,, \tag{1.2}$$

and we have used that, at least at sufficiently small coupling, the Newton constant and $\alpha'$ are related via the string coupling by $G_N \sim g^2 (\alpha')^{(d-1)/2}$ (more geometrically, $\ell_P^{d-1} \sim g^2 \ell_s^{d-1}$).

Given their different mass dependence, it is obvious that, for a given set of the fundamental constants $G_N, \alpha', g^2$, $S_s > S_{\rm BH}$ at sufficiently small $M$, while the opposite is true at sufficiently large $M$. Obviously, there has to be a critical value of $M$, $M_c$, at which $S_s = S_{\rm BH}$. This observation led Bowick et al. [4] to conjecture that large black holes end up their Hawking-evaporation process when $M = M_c$, and then transform into a higher-entropy string state without ever reaching the singular zero-mass limit. This reasoning is confirmed [5] by the observation that, in string theory, the fundamental string length $\ell_s$ should set a minimal value for the Schwarzschild radius of any black hole (and thus a maximal value for its Hawking temperature). It was also noticed [4], [6], [7] that, precisely at $M = M_c$, $R_{\rm BH} = \ell_s$ and the Hawking temperature equals the Hagedorn temperature of string theory. For any $d$, the value of $M_c$ is given by:

$$M_c \sim M_s g^{-2} \,. \tag{1.3}$$

[1]The self-interaction of a string lifts the huge degeneracy of free string states. One then defines the entropy of a narrow band of string states, defined with some energy resolution $M_s \lesssim \Delta\, E \ll M$, as the logarithm of the number of states within the band $\Delta\, E$.

[2]Below, we shall use the precise definition $\ell_s \equiv \sqrt{2\alpha' \hbar}$, but, in this section, we neglect factors of order unity.

Susskind and collaborators [6], [8] went a step further and proposed that the spectrum of black holes and the spectrum of single string states be "identical", in the sense that there be a one to one correspondence between (uncharged) fundamental string states and (uncharged) black hole states. Such a "correspondence principle" has been generalized by Horowitz and Polchinski [9] to a wide range of charged black hole states (in any dimension). Instead of keeping fixed the fundamental constants and letting $M$ evolve by evaporation, as considered above, one can (equivalently) describe the physics of this conjectured correspondence by following a narrow band of states, on both sides of and through, the string $\rightleftharpoons$ black hole transition, by keeping fixed the entropy[3] $S = S_s = S_{\rm BH}$, while adiabatically[4] varying the string coupling $g$, i.e. the ratio between $\ell_P$ and $\ell_s$. The correspondence principle then means that if one increases $g$ each (quantum) string state should turn into a (quantum) black hole state at sufficiently strong coupling, while, conversely, if $g$ is decreased, each black hole state should "decollapse" and transform into a string state at sufficiently weak coupling. For all the reasons mentioned above, it is very natural to expect that, when starting from a black hole state, the critical value of $g$ at which a black hole should turn into a string is given, in clear relation to (1.3), by

$$g_c^2 \, M \sim M_s \, , \tag{1.4}$$

and is related to the common value of string and black-hole entropy via

$$g_c^2 \sim \frac{1}{S_{\rm BH}} = \frac{1}{S_s} \, . \tag{1.5}$$

Note that $g_c^2 \ll 1$ for the very massive states ($M \gg M_s$) that we consider. This justifies our use of the perturbative relation between $G_N$ and $\alpha'$.

In the case of extremal BPS, and nearly extremal, black holes the conjectured correspondence was dramatically confirmed through the work of Strominger and Vafa [10] and others [11] leading to a statistical mechanics interpretation of black-hole entropy in terms of the number of microscopic states sharing the same macroscopic quantum numbers. However, little is known about whether and how the correspondence works for non-extremal, non BPS black holes, such as the simplest Schwarzschild black hole[5]. By contrast to BPS states whose mass is protected by supersymmetry, we shall consider here the effect of varying $g$ on the mass and size of non-BPS string states.

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[3]One uses here the fact that, during an adiabatic variation of $g$, the entropy of the black hole $S_{\rm BH} \sim$ (Area)$/G_N \sim R_{\rm BH}^{d-1}/G_N$ stays constant. This result (known to hold in the Einstein conformal frame) applies also in string units because $S_{\rm BH}$ is dimensionless.

[4]The variation of $g$ can be seen, depending on one's taste, either as a real, adiabatic change of $g$ due to a varying dilaton background, or as a mathematical way of following energy states.

[5]For simplicity, we shall consider in this work only Schwarzschild black holes, in any number $d \equiv D - 1$ of non-compact spatial dimensions.

Although it is remarkable that black-hole and string entropy coincide when $R_{\rm BH} = \ell_s$, this is still not quite sufficient to claim that, when starting from a string state, a string becomes a black hole at $g = g_c$. In fact, the process in which one starts from a string state in flat space and increases $g$ poses a serious puzzle [6]. Indeed, the radius of a typical excited string state of mass $M$ is generally thought of being of order

$$R_s^{\rm rw} \sim \ell_s (M/M_s)^{1/2} \, , \tag{1.6}$$

as if a highly excited string state were a random walk made of $M/M_s = \alpha' M/\ell_s$ segments of length $\ell_s$ [12]. [The number of steps in this random walk is, as is natural, the string entropy (1.1).] The "random walk" radius (1.6) is much larger than the Schwarzschild radius for all couplings $g \leq g_c$, or, equivalently, the ratio of self-gravitational binding energy to mass (in $d$ spatial dimensions)

$$\frac{G_N \, M}{(R_s^{\rm rw})^{d-2}} \sim \left( \frac{R_{\rm BH}(M)}{R_s^{\rm rw}} \right)^{d-2} \sim g^2 \left( \frac{M}{M_s} \right)^{\frac{4-d}{2}} \tag{1.7}$$

remains much smaller than one (when $d > 2$, to which we restrict ourselves) up to the transition point. In view of (1.7) it does not seem natural to expect that a string state will "collapse" to a black hole when $g$ reaches the value (1.4). One would expect a string state of mass $M$ to turn into a black hole only when its typical size is of order of $R_{\rm BH}(M)$ (which is of order $\ell_s$ at the expected transition point (1.4)). According to Eq. (1.7), this seems to happen for a value of $g$ much larger than $g_c$.

Horowitz and Polchinski [13] have addressed this puzzle by means of a "thermal scalar" formalism [14]. Their results suggest a resolution of the puzzle when $d = 3$ (four-dimensional spacetime), but lead to a rather complicated behaviour when $d \geq 4$. More specifically, they consider the effective field theory of a complex scalar field $\chi$ in $d$ (spacetime) dimensions (with period $\beta$ in Euclidean time $\tau$), with mass squared $m^2(\beta) = (4\pi^2 \, \alpha'^2)^{-1} \, [G_{\tau\tau} \, \beta^2 - \beta_{\rm H}^2]$, where $G_{\mu\nu}$ is the string metric and $\beta_{\rm H}^{-1}$ the Hagedorn temperature. They took into account the effect of gravitational (and dilatonic) self-interactions in a mean field approximation. This leads to an approximate Hartree-like equation for $\chi(\boldsymbol{x})$, which admits a stable bound state, in some range $g_0 < g < g_c$, when $d = 3$. They interpret the size of the bound state "wave function" $\chi(\boldsymbol{x})$ as the "size of the string", and find that (in $d = 3$) this size is of order

$$\ell_\chi \sim \frac{1}{g^2 \, M} \, . \tag{1.8}$$

They describe their result by saying that "the string contracts from its initial (large) size", when $g \sim g_0 \sim (M/M_s)^{-3/4}$, down to the string scale when $g \sim g_c \sim (M/M_s)^{-1/2}$. This interpretation of the length scale $\ell_\chi$, characterizing the thermal scalar bound state, as "the size of the string" is unclear to us, because of the formal nature of the auxiliary field $\chi$ which has no direct physical meaning in Minkowski spacetime. Moreover, the analysis of Ref. [13] in higher dimensions is somewhat inconclusive, and suggests that a phenomenon of hysteresis takes place (when $d \geq 5$): the critical value of $g$ corresponding to the string $\rightleftharpoons$ black hole transition would be $g_0 \sim (M/M_s)^{(d-6)/4} > g_c$ for the direct process (string $\rightarrow$black hole), and $g_c$ for the reverse one. Finally, they suggest that, in the reverse process, a black hole becomes an excited string in an *atypical* state.

The aim of the present work is to clarify the string $\rightleftharpoons$ black hole transition by a direct study, in real spacetime, of the size and mass of a *typical* excited string, within the microcanonical ensemble of *self-gravitating* strings. Our results lead to a rather simple picture of the transition, in any dimension. We find no hysteresis phenomenon in higher dimensions. The critical value for the transition is (1.4), or (1.5) in terms of the entropy $S$, for both directions of the string $\rightleftharpoons$ black hole transition. In three spatial dimensions, we find that the size (computed in real spacetime) of a *typical self-gravitating* string is given by the random walk value (1.6) when $g^2 \leq g_0^2$, with $g_0^2 \sim (M/M_s)^{-3/2} \sim S^{-3/2}$, and by

$$R_{\rm typ} \sim \frac{1}{g^2\, M}\,, \tag{1.9}$$

when $g_0^2 \leq g^2 \leq g_c^2$. Note that $R_{\rm typ}$ smoothly interpolates between $R_s^{\rm rw}$ and $\ell_s$. This result confirms the picture proposed by Ref. [13] when $d = 3$, but with the bonus that Eq. (1.9) (which agrees with Eq. (1.8)) refers to a radius which is estimated directly in physical space (see below), and which is the size of a typical member of the microcanonical ensemble of self-gravitating strings. In all higher dimensions[6], we find that the size of a typical self-gravitating string remains fixed at the random walk value (1.6) when $g \leq g_c$. However, when $g$ gets close to a value of order $g_c$, the ensemble of self-gravitating strings becomes (smoothly in $d = 4$, but suddenly in $d \geq 5$) dominated by very compact strings of size $\sim \ell_s$ (which are then expected to collapse with a slight further increase of $g$ because the dominant size is only slightly larger than the Schwarzschild radius at $g_c$).

Our results confirm and clarify the main idea of a correspondence between string states and black hole states [6], [8], [9], [13], and suggest that the transition between these states is rather smooth, with no apparent hysteresis, and with continuity in entropy, mass and typical size. It is, however, beyond the technical grasp of our analysis to compute any precise number at the transition (such as the famous factor 1/4 in the Bekenstein-Hawking entropy formula).

## II. SIZE DISTRIBUTION OF FREE STRING STATES

The aim of this section is to estimate the distribution function in size of the ensemble of free string states of mass $M$, i.e. to count how many massive string states have a given size $R$. This estimate will be done while neglecting the gravitational self-interaction. The effect of the latter will be taken into account in a later section.

Let us first estimate the distribution in size by a rough, heuristic argument based on the random walk model [12] of a generic excited string state. In string units ($\ell_s \sim M_s^{-1} \sim 1$), the geometrical shape (in $d$-dimensional space) of a generic massive string state can be roughly identified with a random walk of $M$ steps of unit length. We can constrain this random walk to stay of size $\lesssim R$ by considering a diffusion process, starting from a point source at the origin, in presence of an *absorbing* sphere $S_R$ of radius $R$, centered on the origin. In the

[6] With the proviso that the consistency of our analysis is open to doubt when $d \geq 8$.

continuous approximation, the kernel $K_t(\boldsymbol{x},\mathbf{0})$ giving the conditional probability density of ending, at time $t$, at position $\boldsymbol{x}$, after having started (at time 0) at the origin, without having ever gone farther from the origin than the distance $R$, satisfies: (i) the diffusion equation $\partial_t\, K_t = \Delta\, K_t$, (ii) the initial condition $K_0(\boldsymbol{x},\mathbf{0}) = \delta(\boldsymbol{x})$, and (iii) the "absorbing" boundary condition $K_t = 0$ on the sphere $S_R$. The kernel $K_t$ can be decomposed in eigenmodes,

$$K_t(\boldsymbol{x},\mathbf{0}) = \sum_n \psi_n(\boldsymbol{x})\, \psi_n(\mathbf{0})\, e^{-E_n\, t}\,, \tag{2.1}$$

where $\psi_n(\boldsymbol{x})$ is a normalized (real) $L^2$ basis ($\int d^d\, x\, \psi_n(\boldsymbol{x})\, \psi_m(\boldsymbol{x}) = \delta_{nm}$; $\sum\limits_n \psi_n(\boldsymbol{x})\, \psi_n(\boldsymbol{y}) = \delta(\boldsymbol{x}-\boldsymbol{y})$) satisfying

$$\Delta\, \psi_n = -E_n\, \psi\,, \tag{2.2}$$

in the interior, and vanishing on $S_R$. The total conditional probability of having stayed within $S_R$ after the time $t$ is the integral $\int_{B_R} d^d\, x\, K_t(\boldsymbol{x},\mathbf{0})$ within the ball $B_R : |\boldsymbol{x}| \leq R$. For large values of $t$, $K_t$ is dominated by the lowest eigenvalue $E_0$, and the conditional probability goes like $c_0\, e^{-E_0\, t}$, where $c_0$ is a numerical constant of order unity. The eigenvalue problem (2.2) is easily solved (in any dimension $d$), and the $s$-wave ground state can be expressed in terms of a Bessel function: $\psi_0(r) = N\, J_\nu(k_0\, r)/(k_0\, r)^\nu$ with $\nu = (d-2)/2$. Here, $k_0 = \sqrt{E_0}$ is given by the first zero $j_\nu$ of $J_\nu(z) : k_0 = j_\nu/R$. The important information for us is that the ground state energy $E_0$ scales with $R$ like $E_0 = c_1/R^2$, where $c_1 = \mathcal{O}(1)$ is a numerical constant. This scaling is evident for the Dirichlet problem(2.2), whatever be the shape of the boundary. Finally, remembering that the number of time steps is given by the mass, $t = M$, we expect the looked for conditional probability, i.e. the *fraction* of all string states at mass level $M$ which are of size $\lesssim R$, to be asymptotically of order

$$f(R) \sim e^{-c_1\, M/R^2}\,. \tag{2.3}$$

This estimate is expected to be valid when $M/R^2 \gg 1$, i.e. for string states which are much smaller (in size) than a typical random walk $R^2_{\rm rw} \sim M$ (but still larger than the string length, $R \gtrsim 1$). In the opposite limit, $R^2 \gg M$, the kernel $K_t(\boldsymbol{x},\mathbf{0})$ can be approximated by the free-space value $K_t^{(0)}(\boldsymbol{x},\mathbf{0}) = (4\,\pi\, t)^{-d/2}\, \exp(-\boldsymbol{x}^2/(4t))$, with $t = M$, so that the fraction of string states of size $\gtrsim R \gg R_{\rm rw}$ will be of order $\sim e^{-c_2\, R^2/M}$, with $c_2 = \mathcal{O}(1)$.

As the result (2.3) will be central to the considerations of this paper, we shall now go beyond the previous heuristic, random walk argument and derive the fraction of small string states by a direct counting of quantum string states. For simplicity, we shall deal with open bosonic strings ($\ell_s \equiv \sqrt{2\,\alpha'}$, $0 \leq \sigma \leq \pi$)

$$X^\mu(\tau,\sigma) = X^\mu_{\rm cm}(\tau,\sigma) + \widetilde{X}^\mu(\tau,\sigma)\,, \tag{2.4}$$

$$X^\mu_{\rm cm}(\tau,\sigma) = x^\mu + 2\,\alpha'\, p^\mu\, \tau\,, \tag{2.5}$$

$$\widetilde{X}^\mu(\tau,\sigma) = i\,\ell_s \sum_{n\neq 0} \frac{\alpha_n^\mu}{n}\, e^{-in\tau}\, \cos n\,\sigma\,. \tag{2.6}$$

Here, we have explicitly separated the center of mass motion $X^\mu_{\rm cm}$ (with $[x^\mu, p^\nu] = i\,\eta^{\mu\nu}$) from the oscillatory one $\widetilde{X}^\mu$ ($[\alpha^\mu_m, \alpha^\nu_n] = m\,\delta^0_{m+n}\,\eta^{\mu\nu}$). The free spectrum is given by $\alpha'\,M^2 = N-1$ where ($\alpha\cdot\beta \equiv \eta_{\mu\nu}\,\alpha^\mu\,\beta^\nu \equiv -\alpha^0\,\beta^0 + \alpha^i\,\beta^i$)

$$N = \sum_{n=1}^{\infty} \alpha_{-n}\cdot\alpha_n = \sum_{n=1}^{\infty} n\,N_n\,. \tag{2.7}$$

Here $N_n \equiv a^\dagger_n\cdot a_n$ is the occupation number of the $n^{\rm th}$ oscillator ($\alpha^\mu_n = \sqrt{n}\;a^\mu_n$,$[a^\mu_n, a^{\nu\dagger}_m] = \eta^{\mu\nu}\,\delta_{nm}$, with $n,m$ positive).

The decomposition (2.4)–(2.6) holds in any conformal gauge ($(\partial_\tau X^\mu \pm \partial_\sigma X^\mu)^2 = 0$). One can further specify the choice of worldsheet coordinates by imposing

$$n_\mu\,X^\mu(\tau,\sigma) = 2\alpha'(n_\mu\,p^\mu)\,\tau\,, \tag{2.8}$$

where $n^\mu$ is an arbitrary timelike or null vector ($n\cdot n \leq 0$) [15]. Eq. (2.8) means that the $n$-projected oscillators $n_\mu\,\alpha^\mu_m$ are set equal to zero. The usual "light-cone" gauge is obtained by choosing a fixed, null vector $n_\mu$. The light-cone gauge introduces a preferred ("longitudinal") direction in space, which is an inconvenience for defining the (rms) size of a massive string state. As we shall be interested in quasi-classical, very massive string states ($N \gg 1$) it should be possible to work in the "center of mass" gauge, where the vector $n^\mu$ used in Eq. (2.8) to define the $\tau$-slices of the world-sheet is taken to be the total momentum $p^\mu$ of the string. This gauge is the most intrinsic way to describe a string in the classical limit. Using this intrinsic gauge, one can covariantly[7] define the proper rms size of a massive string state as

$$R^2 \equiv \frac{1}{d}\,\langle(\widetilde{X}^\mu_\perp(\tau,\sigma))^2\rangle_{\sigma,\tau}\,, \tag{2.9}$$

where $\widetilde{X}^\mu_\perp \equiv \widetilde{X}^\mu - p^\mu(p\cdot\widetilde{X})/(p\cdot p)$ denotes the projection of $\widetilde{X}^\mu \equiv X^\mu - X^\mu_{\rm cm}(\tau)$ orthogonally to $p^\mu$, and where the angular brackets denote the (simple) average with respect to $\sigma$ and $\tau$. The factor $1/d$ in Eq. (2.9) is introduced to simplify later formulas. So defined $R$ is the rms value of the *projected size* of the string along an arbitrary, but fixed spatial direction. [We shall find that this projected size is always larger than $\sqrt{3\,\alpha'/2}$; i.e. string states cannot be "squeezed", along any axis, more than this.]

In the center of mass gauge, $p_\mu\,\widetilde{X}^\mu$ vanishes by definition, and Eq. (2.9) yields simply

$$R^2 = \frac{1}{d}\,\ell_s^2\,\mathcal{R}\,, \tag{2.10}$$

with

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[7] In an arbitrary conformal gauge, the definition (2.9) is gauge-dependent (in spite of the use of the orthogonal projection) because both the definition of $X^\mu_{\rm cm}(\tau)$, and that of the $(\sigma,\tau)$-averaging depend on the choice of world-sheet gauge. Even if we were using the (more intrinsic but more complicated) average with weight $\sqrt{-\det\,\gamma_{ab}}\,d\sigma\,d\tau = (\partial_\sigma\,X^\mu)^2\,d\sigma\,d\tau$, the dependence upon $X^\mu_{\rm cm}(\tau)$ would remain.

$$\mathcal{R} \equiv \sum_{n=1}^{\infty} \frac{\alpha_{-n} \cdot \alpha_n + \alpha_n \cdot \alpha_{-n}}{2 \, n^2} \, . \tag{2.11}$$

The squared-size operator $\mathcal{R}$, Eq. (2.11), contains the logarithmically infinite contribution $\sum 1/(2n)$. Without arguing with the suggestion that this contribution may have a physical meaning (see, e.g., [16]), we note here that this contribution is state-independent. We are interested in this work in the relative sizes of various highly excited, quasi-classical states. A concept which should reduce to the well-defined, finite rms size of a classical Nambu string in the classical limit. We shall therefore discard this state-independent contribution, i.e. work with the normal-ordered operator

$$: \mathcal{R} : = \sum_{n=1}^{\infty} \frac{a_n^\dagger \cdot a_n}{n} = \sum_{n=1}^{\infty} \frac{N_n}{n} \, . \tag{2.12}$$

We shall assume that we can work both in the center-of-mass (worldsheet) gauge ($p_\mu \, \alpha_m^\mu \to 0$) and in the center-of-mass (Lorentz) frame ($(p^\mu) = (M, \mathbf{0})$). This means that the scalar product in the level occupation number $N_n$ runs over the $d$ spatial dimensions: $N_n = a_n^\dagger \cdot a_n = \sum_{i=1}^{d} (a_n^i)^\dagger \, a_n^i$. The "wrong sign" time oscillators $\alpha_n^0$ are set equal to zero. The Virasoro constraints then imply, besides the mass formula $\alpha' \, M^2 = N - 1$, the usual sequence of constraints on physical states, $L_n \, |\phi\rangle = 0$, with $L_n = \frac{1}{2} \sum_m \sum_{i=1}^{d} \alpha_{n-m}^i \, \alpha_m^i$. These constraints mean that the $d$ oscillators $\alpha_n^i$ at level $n$ are not physically independent.

The problem we would like to solve is to count the number of physical states, in the Fock space of the center-of-mass oscillators $\alpha_n^i$, having some fixed values of $N$ and $\mathcal{R}$ (we henceforth work only with the normal-ordered operator (2.12) without adorning it with the : : notation). The Virasoro constraints make this problem technically quite difficult. However, we know from the exact counting of physical states (without size restriction) in the light-cone gauge that the essential physical effect of the Virasoro constraints is simply to reduce the number of independent oscillators at any level $n$ from $d = D - 1$ (in the center-of-mass gauge) to $d - 1 = D - 2$. If we (formally) consider $d$ as a large parameter[8], this change in the number of effective free oscillators should have only a small fractional effect on any other coarse-grained, counting problem. We shall assume that this is the case, and solve the much simpler counting problem where the $d$ oscillators $\alpha_n^i$ are considered as independent[9]. To solve this problem we pass from a microcanonical problem (fixed values

[8]It would be interesting to see if one can technically implement a large $d$ approach to our counting problem.

[9]We tried to work in the light-cone gauge, with $d-1$ independent oscillators. However, the necessary inclusion of the longitudinal term $M^{-2}(p \cdot \widetilde{X})^2$in (2.9), which is quadratic in the longitudinal oscillators $\alpha_n^- = (p^+)^{-1} \, L_n^{\rm transverse}$, leads to a complicated, interacting theory of the $d-1$ transverse oscillators.

of $N$ and $\mathcal{R}$) to a grand canonical one (fixed values of some thermodynamical conjugates of $N$ and $\mathcal{R}$). Let us introduce the formal "partition function"

$$Z_d(\beta, \gamma) \equiv \sum_{\{N_n^i\}} \exp(-\beta \, N \, [N_n^i] - \gamma \, \mathcal{R} \, [N_n^i]) \, , \tag{2.13}$$

where the sum runs over all sequences (labelled by $n \geq 1$ and $i = 1, \ldots, d$) of independent occupation numbers $N_n^i = (a_n^i)^\dagger \, a_n^i = 0, 1, 2, \ldots$, and where $N \, [N_n^i]$ and $\mathcal{R} \, [N_n^i]$ are defined by Eqs. (2.7), and (2.12), with $N_n \equiv \sum_{i=1}^{d} N_n^i$. Note that (2.13) is not the usual thermodynamical partition function, and that $\beta$ is not the usual inverse temperature. Indeed, $\beta$ is a formal conjugate to $N \simeq \alpha' \, M^2$ and not to the energy $M$. In particular, because the degeneracy grows exponentially with $M$ (and not $M^2$) its Laplace transform (2.13) is defined for arbitrary values of $\beta$. We associate with the definition (2.13) that of a formal grand canonical ensemble of configurations, with the probability

$$p \, [\{N_n^i\}] = Z_d^{-1} \, (\beta, \gamma) \, \exp(-\beta \, N \, [N_n^i] - \gamma \, \mathcal{R} \, [N_n^i]) \tag{2.14}$$

of realization of the particular sequence $N_n^i$ of occupation numbers. The mean values of $N \, [N_n^i]$ and $\mathcal{R} \, [N_n^i]$ in this ensemble are

$$\overline{N} = -\frac{\partial \, \psi_d \, (\beta, \gamma)}{\partial \, \beta} \; , \; \overline{\mathcal{R}} = -\frac{\partial \, \psi_d \, (\beta, \gamma)}{\partial \, \gamma} \, , \tag{2.15}$$

where we denote

$$\psi_d \, (\beta, \gamma) \equiv \ln \, Z_d \, (\beta, \gamma) \, . \tag{2.16}$$

The second derivatives of the thermodynamical potential $\psi_d \, (\beta, \gamma)$ give the fluctuations of $N$ and $\mathcal{R}$ in this grand canonical ensemble:

$$\overline{(\Delta N)^2} = \frac{\partial^2 \, \psi_d \, (\beta, \gamma)}{\partial \, \beta^2} \; , \; \overline{(\Delta \mathcal{R})^2} = \frac{\partial^2 \, \psi_d \, (\beta, \gamma)}{\partial \, \gamma^2} \, . \tag{2.17}$$

Let us define as usual the entropy $S(\beta, \gamma)$ as the logarithm of the number of string configurations having values of $N$and $\mathcal{R}$ equal to $\overline{N}$ and $\overline{\mathcal{R}}$, Eqs. (2.15), within the precision of the rms fluctuations (2.17) [17]. This definition means that, in the saddle-point approximation, $Z_d \, (\beta, \gamma) \simeq \exp \, [S - \beta \, \overline{N} - \gamma \, \overline{\mathcal{R}}]$, i.e.

$$\psi_d \, (\beta, \gamma) \simeq S(\beta, \gamma) - \beta \, \overline{N} - \gamma \, \overline{\mathcal{R}} \, , \tag{2.18}$$

or

$$S \simeq \psi_d \, (\beta, \gamma) - \beta \, \frac{\partial \, \psi_d \, (\beta, \gamma)}{\partial \, \beta} - \gamma \, \frac{\partial \, \psi_d \, (\beta, \gamma)}{\partial \, \gamma} \, . \tag{2.19}$$

In other words, the entropy $S(\overline{N}, \overline{\mathcal{R}})$ is the Legendre transform of $\psi_d \, (\beta, \gamma)$.

Because of the (assumed) independence of the $d$ oscillators in (2.13), one has

$$Z_d\,(\beta,\gamma) = \prod_{n=1}^{\infty}\,[1 - e^{-(\beta n + \gamma/n)}]^{-d}\,, \tag{2.20}$$

i.e.

$$\psi_d\,(\beta,\gamma) = d\,\psi_1\,(\beta,\gamma)\,, \tag{2.21}$$

with

$$\psi_1\,(\beta,\gamma) = -\sum_{n=1}^{\infty}\,\ln\left[1 - \exp\left(-\beta n - \frac{\gamma}{n}\right)\right]\,. \tag{2.22}$$

We shall check *a posteriori* that we are interested in values of $\beta$ and $\gamma$ such that

$$\beta \ll \sqrt{\beta\,\gamma} \ll 1\,. \tag{2.23}$$

For such values, one can approximate the discrete sum (2.22) by a continuous integral over $x = \beta\,n$. This yields

$$\psi_1\,(\beta,\gamma) = \frac{I(\delta)}{\beta}\ \text{, where } \delta \equiv \sqrt{\beta\,\gamma}\,, \tag{2.24}$$

and

$$I(\delta) \equiv -\int_0^{\infty} dx\,\ln\,[1 - e^{-(x\,+\,\delta^2/x)}]\,. \tag{2.25}$$

As $\delta = \sqrt{\beta\,\gamma} \ll 1$, we can try to expand $I(\delta)$ in powers of $\delta$: $I(\delta) = I(0) + \delta\,I'(0) + o(\delta)$. [Though the integral (2.25) is expressed in terms of $\delta^2$, its formal expansion in powers of $\delta^2$ leads to divergent integrals.] The zeroth-order term is $I(0) = -\int_0^{\infty} dx\,\ln\,(1 - e^{-x}) = \pi^2/6$, while

$$I'(0) = \lim_{\delta\to 0}\left[-2\int_0^{\infty}\frac{du}{u}\;\frac{\delta}{e^{\delta(u+1/u)} - 1}\right] = -2\int_0^{\infty}\frac{du}{u^2+1} = -\pi\,.$$

Hence, using (2.21),

$$\psi_d\,(\beta,\gamma) = \frac{1}{\beta}\left[C - D\,\sqrt{\beta\,\gamma} + o\left(\sqrt{\beta\,\gamma}\right)\right] \tag{2.26}$$

with

$$C = \frac{\pi^2}{6}\,d\ ,\ \ D = \pi\,d\,. \tag{2.27}$$

[The notation $D$ in (2.27) should not be confused with the space-time dimension $d+1$.] The thermodynamic potential (2.26) corresponds to the mean values

$$\overline{N} \simeq \frac{C - \frac{1}{2}\,D\,\sqrt{\beta\,\gamma}}{\beta^2}\ ,\ \ \overline{\mathcal{R}} \simeq \frac{D}{2\,\delta}\,, \tag{2.28}$$

and to the entropy

$$S \simeq \frac{2\,C - D\,\delta}{\beta} \simeq 2\,\sqrt{C\,\overline{N}}\left[1 - \frac{D^2}{8\,C\,\overline{\mathcal{R}}}\right] , \tag{2.29}$$

i.e.

$$S \simeq 2\,\pi\,\sqrt{\frac{d}{6}\,\overline{N}}\left[1 - \frac{3d}{4}\,\frac{1}{\overline{\mathcal{R}}}\right] \simeq 2\,\pi\left(\frac{d\,\alpha'}{6}\right)^{\frac{1}{2}} M\left[1 - \frac{3}{4}\,\frac{\ell_s^2}{R^2}\right] . \tag{2.30}$$

The lowest-order term $2\,\pi\,\sqrt{d\,\overline{N}/6}$ is the usual (Hardy-Ramanujan) result for $d$ independent oscillators, without size restriction. The factor in bracket, $1-(3/4)(\ell_s^2/R^2)$, with $\ell_s^2 = 2\,\alpha'$, gives the fractional reduction in the entropy brought by imposing the size constraint $R^2 \simeq d^{-1}\,\ell_s^2\,\overline{\mathcal{R}}$. Under the conditions (2.23) the fluctuations (2.17) are fractionally small. More precisely, Eqs. (2.17) yield

$$\frac{\overline{(\Delta N)^2}}{\overline{N}^2} \sim \beta \sim \frac{M_s}{M} , \qquad \frac{\overline{(\Delta\,\mathcal{R})^2}}{\overline{\mathcal{R}}^2} \sim \frac{\beta}{\delta} \sim \frac{(R^2/\ell_s^2)}{(M/M_s)} . \tag{2.31}$$

As said above, though we worked under the (physically incorrect) assumption of $d$ independent oscillators at each level $n$, we expect the result (2.29) to be correct when $d \gg 1$. [We recall that the exact result for $S$ in absence of size restriction is $2\,\pi\,\sqrt{(d-1)\,\overline{N}/6}$.] Note the rough physical meanings of the auxiliary quantities $\beta$, $\gamma$ and $\delta$: $\beta \sim (\overline{N})^{-1/2} \sim (M/M_s)^{-1}$, $\delta \sim \overline{\mathcal{R}}^{-1} \sim \ell_s^2/R^2$, $\gamma \sim (\overline{N})^{1/2}/\overline{\mathcal{R}}^2 \sim M\,\ell_s^5/R^4$.

Summarizing, the main result of the present section is that the number(“degeneracy”) of free string states of mass $M$ and size $R$ (within the narrow bands defined by the fluctuations (2.17)) is of the form

$$\mathcal{D}\,(M,R) \sim \exp\left[c\,(R)\,a_0\,M\right] , \tag{2.32}$$

where $a_0 = 2\,\pi\,((d-1)\,\alpha'/6)^{1/2}$ and

$$c\,(R) = \left(1 - \frac{c_1}{R^2}\right)\left(1 - c_2\,\frac{R^2}{M^2}\right) , \tag{2.33}$$

with the coefficients $c_1$ and $c_2$ being of order unity in string units. [We have added, for completeness, in Eq. (2.32) the factor $1 - c_2\,R^2/M^2$ which operates when one considers very “large” string states, $R^2 \gg R_{\rm rw}^2$ (as discussed below Eq. (2.3)).] The coefficient $c\,(R)$ gives the fractional reduction in entropy brought by imposing a size constraint. Note that (as expected) this reduction is minimized when $c_1\,R^{-2} \sim c_2\,R^2/M^2$, i.e. for $R \sim R_{\rm rw} \sim \ell_s\,\sqrt{M/M_s}$. [The absolute reduction in degeneracy is only a factor $\mathcal{O}(1)$ when $R \sim R_{\rm rw}$.] Note also that $c\,(R) \to 0$ both when $R \sim \ell_s$ and when $R \sim \ell_s(M/M_s)$. [The latter value corresponding to the vicinity of the leading Regge trajectory $J \sim \alpha'\,M^2$.]

## III. MASS SHIFT OF STRING STATES DUE TO SELF-GRAVITY

In this section we shall estimate the mass shift of string states (of mass $M$ and size $R$) due to the exchange of the various long-range fields which are universally coupled to the string: graviton, dilaton and axion. As we are interested in very massive string states, $M \gg M_s$, in extended configurations, $R \gg \ell_s$, we expect that massless exchange dominates the (state-dependent contribution to the) mass shift.

The evaluation, in string theory, of (one loop) mass shifts for massive states is technically quite involved, and can only be tackled for the states which are near the leading Regge trajectory [18]. [Indeed, the vertex operators creating these states are the only ones to admit a manageable explicit oscillator representation.] As we consider states which are very far from the leading Regge trajectory, there is no hope of computing exactly (at one loop) their mass shifts. We shall resort to a semi-classical approximation, which seems appropriate because we consider highly excited configurations. As a starting point to derive the mass-shift in this semi-classical approximation we shall use the classical results of Ref. [19] which derived the effective action of fundamental strings. The one-loop exchange of $g_{\mu\nu}$, $\varphi$ and $B_{\mu\nu}$ leads to the effective action

$$I^{\rm eff} = I_0 + I_1 \, , \tag{3.1}$$

where $I_0$ is the free (Nambu) string action ($d^2\,\sigma_1 \equiv d\sigma_1\,d\tau_1$, $\gamma_1 = -\det\,\gamma_{ab}\,(X^\mu(\sigma_1,\tau_1))$)

$$I_0 = -T \int d^2\,\sigma_1\,\sqrt{\gamma_1}\, , \tag{3.2}$$

and $I_1$ the effect of the one-loop interaction ($X_1^\mu \equiv X^\mu(\sigma_1,\tau_1), \ldots$)

$$I_1 = 2\pi \int\!\!\int d^2\sigma_1\,d^2\sigma_2\,G_F(X_1 - X_2)\,\sqrt{\gamma_1}\,\sqrt{\gamma_2}\;C_{\rm tot}(X_1,X_2)\, , \tag{3.3}$$

where $G_F$ is Feynman's scalar propagator ($\Box\,G_F(x) = -\delta^D(x)$), and $C_{\rm tot}(X_1,X_2) = J_\varphi(X_1)\cdot J_\varphi(X_2) + J_g(X_1)\cdot J_g(X_2) + J_B(X_1)\cdot J_B(X_2)$ comes from the couplings of $\varphi$, $g_{\mu\nu}$ and $B_{\mu\nu}$ to their corresponding world-sheet sources (indices suppressed; spin-structure hidden in the dot product). The exchange term $C_{\rm tot}$ takes, in null (conformal) coordinates $\sigma^\pm = \tau \pm \sigma$, the simple left-right factorized form [19]

$$\sqrt{\gamma_1}\,\sqrt{\gamma_2}\;C_{\rm tot}(X_1,X_2) = 32\,G_N\,T^2\,(\partial_+ X_1^\mu\;\partial_+ X_{2\mu})(\partial_- X_1^\nu\;\partial_- X_{2\nu})\, . \tag{3.4}$$

Here, $T = (2\,\pi\,\alpha')^{-1}$ is the string tension, $G_N$ is Newton's constant[10] and $\partial_\pm = \partial/\partial\sigma^\pm = \frac{1}{2}\,(\partial_\tau \pm \partial_\sigma)$. Let us define $P_\pm^\mu = P_\pm^\mu(\sigma^\pm)$ by ($\ell_s = \sqrt{2\,\alpha'}$ as above)

$$2\,\partial_\pm\,X^\mu = \ell_s\,P_\pm^\mu\, , \tag{3.5}$$

so that, for an open (bosonic) string (with $\alpha_0^\mu \equiv \ell_s\,p^\mu$),

$$P_\pm^\mu = \sum_{-\infty}^{+\infty}\;\alpha_n^\mu\,e^{-in\sigma^\pm}\, . \tag{3.6}$$

[10]Normalized, in any dimension, by writing the Einstein action as $(16\,\pi\,G_N)^{-1}\int d^D\,x\,\sqrt{g}\,R(g)$.

Using the definition (3.5) and inserting the Fourier decomposition of $G_F$ yields

$$I_1 = \frac{4\,G_N}{\pi} \int \frac{d^D\,k}{(2\pi)^D}\,\frac{1}{k^2 - i\,\varepsilon} \int\int d^2\sigma_1\,d^2\sigma_2\,(P_+(X_1)\cdot P_+(X_2))(P_-(X_1)\cdot P_-(X_2)) \\ \times\, e^{ik\cdot(X_1 - X_2)}\,, \tag{3.7}$$

where one recognizes the insertion of two gravitational vertex operators $V^{\mu\nu}(k;X) = P_+^\mu(X)\,P_-^\nu(X)\,e^{ik\cdot X}$ at two different locations on the worldsheet, and with two opposite momenta for the exchanged graviton[11]. [Note that the exchanged graviton is off-shell.] It is convenient to use the Virasoro constraints $(P_\pm^\mu(X))^2 = 0$ to replace in Eq. (3.6) $P_\pm(X_1)\cdot P_\pm(X_2) = -\frac{1}{2}\,(\Delta\,P_\pm^\mu)^2$where $\Delta\,P_\pm^\mu \equiv P_\pm^\mu(X_1) - P_\pm^\mu(X_2)$. It is important to note that the zero mode contribution $\alpha_0^\mu$ drops out of $\Delta\,P_\pm^\mu$ (i.e. $\Delta\,P_\pm^\mu = \Delta\,\widetilde{P}_\pm^\mu$ is purely oscillatory).

Writing that the correction $I_1$ to the effective action $I^{\rm eff}$ (which gives the vacuum persistence amplitude; see, e.g., Eq. (7) of [19]) must correspond to a phase shift $-\int \delta E\,dt = -\int \delta M\,dX_{\rm cm}^0$, in the center-of-mass frame of the string, yields (with the normalization (2.5)) the link $I_1 = -\ell_s^2 \int d\tau\,M\,\delta M = -\frac{1}{2}\,\ell_s^2 \int d\tau\,\delta M^2$. Let us also define $\Delta\,X^\mu \equiv X_1^\mu - X_2^\mu$ and decompose it in its zero-mode part $\Delta\,X_{\rm cm}^\mu = \ell_s^2\,p^\mu(\tau_1 - \tau_2)$ and its oscillatory part $\Delta\,\widetilde{X}^\mu = \widetilde{X}_1^\mu - \widetilde{X}_2^\mu$. Finally, the mass-shift can be read from

$$\int d\tau\,\delta M^2 = -\frac{2\,G_N}{\pi\,\ell_s^2} \int \frac{d^D\,k}{(2\pi)^D}\,\frac{1}{k^2 - i\varepsilon} \int\int d^2\sigma_1\,d^2\sigma_2\,e^{i\,\ell_s^2\,k\cdot p\,(\tau_1-\tau_2)}\,W(k,1,2)\,, \tag{3.8}$$

where (1 and 2 being short-hands for $(\tau_1,\sigma_1)$ and $(\tau_2,\sigma_2)$, respectively)

$$W(k,1,2) = (\Delta\,P_+^\mu(1,2))^2\,(\Delta\,P_-^\nu(1,2))^2\,e^{ik\cdot\Delta\widetilde{X}(1,2)}\,. \tag{3.9}$$

Interpreted at the quantum level, the classical result (3.8) gives (modulo some ordering problems, which are, however, fractionally negligible when considering very massive states) the mass-shift $\delta\,M_N^2$ of the string state $|N\rangle$ when replacing $W(k,1,2)$, on the right-hand side of Eq. (3.8), by the quantum average $\langle N|\,W(k,1,2)\,|N\rangle$. Here, we shall mainly be interested in the real part of $\delta\,M^2$, obtained by replacing $(k^2 - i\varepsilon)^{-1}$ by the principal part of $(k^2)^{-1}$ (denoted simply $1/k^2$), i.e. the Feynman Green's function $G_F(x)$ by the half-retarded-half-advanced one $G_{\rm sym}(x)$. [The imaginary part of $\delta\,M^2$ gives the decay rate, i.e. the total rate of emission of massless quanta.] As $L_0 - 1$ is the "Hamiltonian" that governs the $\tau$-evolution of an open string, the vanishing of $(L_0 - 1)\,|N\rangle$ for any physical state ensures that $\langle N|\,W(k,1,2)\,|N\rangle$ is $\tau$-translation invariant, i.e. that it depends only on the difference $\tau_{12} \equiv \tau_1 - \tau_2$, and not on the average $\overline{\tau} \equiv \frac{1}{2}\,(\tau_1 + \tau_2)$. This means that the double worldsheet integration $d^2\sigma_1\,d^2\sigma_2 = d\tau_1\,d\sigma_1\,d\tau_2\,d\sigma_2 = d\overline{\tau}\,d\tau_{12}\,d\sigma_1\,d\sigma_2$ on the right-hand side of Eq. (3.8) contains a formally infinite infra-red "volume" factor $\int d\,\overline{\tau}$ which precisely cancels the integral $\int d\,\tau$ on the left-hand side to leave a finite answer for $\delta\,M^2$.

It is also important to note the good ultraviolet behaviour of Eq. (3.8). The ultraviolet limit $k \to \infty$ corresponds to the coincidence limit $(\tau_2,\sigma_2) \to (\tau_1,\sigma_1)$ on the world-sheet.

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[11]For simplicity, we call "graviton" the exchanged particle, which is a superposition of the graviton, the dilaton and the axion.

Let us define $u \equiv \sigma_1^+ - \sigma_2^+$, $v \equiv \sigma_1^- - \sigma_2^-$ and consider the coincidence limit $u \to 0$, $v \to 0$. In this limit the vertex insertion factors $(\Delta P_+)^2 (\Delta P_-)^2$ tend to zero like $u^2 v^2$, while the Green's function blows up like $[(\Delta X)^2]^{-(D-2)/2} \propto (uv)^{-(D-2)/2}$. The resulting integral, $\int du\, dv\, (uv)^{-(D-6)/2}$, has its first ultraviolet pole when the space-time dimension $D = d + 1 = 8$. This means probably that in dimensions $D \geq 8$ the exchange of massive modes (of closed strings) becomes important. Our discussion, which is limited to considering only the exchange of massless modes, is probably justified only when $D < 8$.

Following the (approximate) approach of Section 2 we shall estimate the average mass shift $\delta M^2(R)$ for string states of size $R$ by using the grand canonical ensemble with density matrix

$$\rho \equiv (Z_d(\beta,\gamma))^{-1} \exp(-\beta N - \gamma :\mathcal{R}:) \,, \tag{3.10}$$

where the operators $N$ and $:\mathcal{R}:$, defined by Eqs. (2.7) and (2.12), belong to the Fock space built upon $d$ sequences of string oscillators $\alpha_n^i$ (formal "center-of-mass" oscillators). For any quantity $Q$ (built from string oscillators) we denote the grand canonical average as $\langle Q \rangle_{\beta,\gamma} \equiv \mathrm{tr}\,(Q\,\rho)$. Using the $\tau$-shift invariance mentioned above, Eq. (3.8) yields

$$\delta M^2_{\beta,\gamma} = -\frac{2\,G_N}{\pi\,\ell_s^2} \int \frac{d^d k\; d\omega}{(2\pi)^{d+1}\,[\boldsymbol{k}^2 - \omega^2 - i\varepsilon]} \int d\tau_{12}\, d\sigma_1\, d\sigma_2\, e^{-i\,\ell_s^2 M \omega\,\tau_{12}} \langle W(k,1,2)\rangle_{\beta,\gamma} \,, \tag{3.11}$$

where we have separated $k^\mu$ in its center-of-mass components $k^0 = \omega$, $k^i = \boldsymbol{k}$, and where $\tau_{12} \equiv \tau_1 - \tau_2$ as above.

We shall estimate the grand canonical average $\langle W \rangle_{\beta,\gamma}$ in a semi-classical approximation in which we neglect some of the contributions linked to the ordering of the operator $W$, but take into account the quantum nature of the d ensity matrix $\rho$, Eq. (3.10). The discreteness of the Fock states built from the$(a_n^i)^\dagger$, and the Planckian nature of $\rho$ will play a crucial role in the calculation below. [By contrast, a purely classical calculation would be awkward and ill-defined because of the problem of defining a measure on classical string configurations, and because of the Rayleigh-Jeans ultraviolet catastrophe.] To compute $\langle W \rangle_{\beta,\gamma}$ it is convenient to define it as a double contraction of the coefficient of $\zeta^1_{\mu_1}\ \zeta^2_{\mu_2}\ \zeta^3_{\mu_3}\ \zeta^4_{\mu_4}$ in the exponentiated version of $W$:

$$W_\zeta \equiv : \exp\left[\zeta^1_{\mu_1}\,\Delta P_+^{\mu_1} + \zeta^2_{\mu_2}\,\Delta P_+^{\mu_2} + \zeta^3_{\mu_3}\,\Delta P_-^{\mu_3} + \zeta^4_{\mu_4}\,\Delta P_-^{\mu_4} + ik\cdot\Delta\widetilde{X}\right] : \tag{3.12}$$

We shall define our ordering of $W$ by working with the normal ordered exponentiated operator (3.12) (and picking the term linear in$\zeta^1_{\mu_1}\ \zeta^2_{\mu_2}\ \zeta^3_{\mu_3}\ \zeta^4_{\mu_4}$). The average $\langle W_\zeta \rangle_\beta = \mathrm{tr}\,(W_\zeta\,\rho)$ (where, to ease the notation, we drop the extra label $\gamma$) can be computed by a generalization of Bloch's theorem. Namely, if $A$ denotes any operator which is linear in the oscillators $\alpha_n^i$, we have the results

$$\langle e^A \rangle_\beta = \exp\left[\frac{1}{2}\,\langle A^2 \rangle_\beta\right] \ ; \ \langle : e^A : \rangle_\beta = \exp\left[\frac{1}{2}\,\langle : A^2 : \rangle_\beta\right] \,, \tag{3.13}$$

as well as their corollaries

$$\langle e^A \rangle_0 = \exp\left[\frac{1}{2}\,\langle A^2 \rangle_0\right] \ ; \ e^A = : e^A : \, \exp\left[\frac{1}{2}\,\langle A^2 \rangle_0\right] \,, \tag{3.14}$$

where $\langle W\rangle_0$ denotes the vacuum average (obtained in the zero temperature limit $\beta^{-1} \to 0$). The simplest way to prove these results is to use coherent-state methods [20] (see also,Ref. [15] and the Appendix 7.A of Ref. [21]). For instance, to prove the second equation (3.13) it is sufficient to consider a single oscillator and to check that (denoting $q = e^{-\epsilon}$, with $\epsilon = \beta\, n + \gamma/n$ (label $n$ suppressed), so that $Z = (1-q)^{-1}$, and $|b) \equiv \exp{(b a^\dagger)}\, |0\rangle$)

$$\begin{aligned}\langle e^{c_1 a^\dagger}\, e^{c_2 a}\rangle_\beta &= Z^{-1}\,\mathrm{tr}\,(e^{c_1 a^\dagger}\, e^{c_2 a}\, q^{a^\dagger a}) = Z^{-1} \int \frac{d^2\, b}{\pi}\, e^{-b^* b} (b|\, e^{c_1 a^\dagger}\, e^{c_2 a}\, q^{a^\dagger a}\, |b) \\ &= (1-q) \int \frac{d^2\, b}{\pi}\, e^{-b^* b} (b|\, e^{c_1 b^*}\, e^{c_2 a}\, |qb) \\ &= (1-q) \int \frac{d^2\, b}{\pi}\, e^{-(1-q) b^* b}\, e^{c_1 b^* + c_2 q b} = e^{c_1 c_2 q/(1-q)}\,,\end{aligned} \tag{3.15}$$

and to recognize that $q/(1-q) = [e^\epsilon - 1]^{-1}$ is the Planckian mean occupation number $\langle a^\dagger\, a\rangle_\beta$.

If we apply the second Eq. (3.13) to an expression of the type$W_\zeta \;=: \exp\left(\sum_{i=1}^{4} \zeta_i\, A_i + B\right)$ :, one gets a Wick-type expansion for the coefficient (say $W_{1234}$) of $\zeta_1\, \zeta_2\, \zeta_3\, \zeta_4$:

$$\begin{aligned}W_{1234} = e^{\frac{1}{2}[BB]}\, (&[A_1 A_2]\, [A_3 A_4] + 2 \text{ terms} + [A_1 B]\, [A_2 B]\, [A_3 A_4] + 5 \text{ terms} \\ &+ [A_1 B]\, [A_2 B]\, [A_3 B]\, [A_4 B])\,,\end{aligned} \tag{3.16}$$

where $[AB]$ denotes the "thermal" contraction $[AB] \equiv \langle :AB: \rangle_\beta$.

The looked-for grand canonical average of $W(k,1,2)$, Eq. (3.9), is given by replacing $B = ik\cdot\Delta\, \widetilde{X}$ and $A_1 = A_2 = \Delta\, P_+^\mu$, $A_3 = A_4 = \Delta\, P_-^\nu$ in Eq. (3.16). This leads to

$$\langle W\rangle_{\beta,\gamma} = e^{-\frac{1}{2}\langle :(k\cdot\Delta \widetilde{X})^2: \rangle_{\beta,\gamma}} \left\{ \langle :(\Delta\, P_+)^2: \rangle_{\beta,\gamma}\, \langle :(\Delta\, P_-)^2: \rangle_{\beta,\gamma} + \ldots \right\}\,, \tag{3.17}$$

where the ellipsis stand for other contractions (which will be seen below to be subleading).

The calculation of the various contractions $[AB] \equiv \langle :AB: \rangle_\beta$ in Eqs. (3.16), (3.17) is easily performed by using the basic contractions among the oscillators $a_n$,$a_m^\dagger$ $(n, m > 0)$ (which are easily derived from the definition (3.10) of the density matrix)

$$\langle :a_n^i\, (a_m^j)^\dagger: \rangle_{\beta,\gamma} = \langle :(a_m^j)^\dagger\, a_n^i: \rangle_{\beta,\gamma} = \frac{\delta^{ij}\, \delta_{nm}}{e^{\epsilon_n} - 1}\,, \tag{3.18}$$

where $\epsilon_n = \beta\, n + \gamma/n$. The other contractions $[aa]$ and $[a^\dagger a^\dagger]$ vanish. In terms of the $\alpha$-oscillators, the basic contraction reads $[\alpha_n^i\, \alpha_m^j] = \delta^{ij}\, \delta_{n+m}^0\, |n|/(\exp(\epsilon_{|n|}) - 1)$, where now $n$ and $m$ can be negative(but not zero). Using these basic contractions, and the oscillator expansion (2.6) of $\widetilde{X}^\mu$ (and noting that, in the center-of-mass frame only the spatial components of $\widetilde{X}^\mu$ survive) one gets

$$\langle :(\boldsymbol{k}\cdot\Delta\, \widetilde{\boldsymbol{X}})^2: \rangle_{\beta,\gamma} = 2\, \boldsymbol{k}^2\, \ell_s^2 \sum_{n=1}^{\infty} \frac{x_n(1,2)}{n(e^{\epsilon_n} - 1)}\,, \tag{3.19}$$

with

$$x_n\,(1,2) = \cos^2\, n\,\sigma_1 + \cos^2\, n\,\sigma_2 - 2\,\cos\, n\,\sigma_1\,\cos\, n\,\sigma_2\,\cos\, n\,\tau_{12}\,. \tag{3.20}$$

Similarly, the oscillator expansion (3.6) yields

$$\langle : \Delta\, P_+)^2 : \rangle_{\beta,\gamma} = 4\,d\,\sum_{n=1}^{\infty} \frac{n\,p_n^+(1,2)}{e^{\epsilon_n} - 1}\,, \tag{3.21}$$

with

$$p_n^+(1,2) = 1 - \cos\, n(\sigma_1^+ - \sigma_2^+) = 1 - \cos\, n(\tau_{12} + \sigma_1 - \sigma_2)\,. \tag{3.22}$$

The result for $(\Delta\, P_-)^2$ is obtained by changing $\sigma^+ \to \sigma^-$ in Eq. (3.22) (i.e. $\sigma_1 - \sigma_2 \to -\sigma_1 + \sigma_2$).

We can estimate the values of the right-hand sides of Eqs. (3.19) and (3.21) by using the following "statistical" approximation. In the parameter range discussed in Section 2 the basic sums $\sum n^{\pm 1}\,(e^{\epsilon_n} - 1)^{-1}$, appearing in (3.19), (3.21), see their values dominated by a large interval, $\Delta\, n \gg 1$, around some $n_0 \gg 1$, of values of $n$, so that one can, with a good approximation, replace the discrete sum over $n$ by a formal continuous integral over a real parameter. In such a continuous approximation one can integrate by parts to show that any "oscillatory" integral of the type $\int dn\, f(n)\,\cos\, n\,\sigma \;=\; [n^{-1}\, f(n)\,\sin\, n\,\sigma] - \int dn\, n^{-1}\, f'(n)\,\sin\, n\,\sigma$ is, because of the factors $n^{-1}$, numerically much smaller than the non-oscillatory one $\int dn\, f(n)$. [Here $\sigma$ denotes some combination of $\sigma_1$ and $\sigma_2$, like $2\sigma_1$, $2\sigma_2$, $\sigma_1 \pm \sigma_2$.] Alternatively, we can say that, for generic values of $\sigma_1$ and $\sigma_2$, one can treat in Eqs. (3.20) or (3.22) $\cos\, n\,\sigma_1$ and $\cos\, n\,\sigma_2$ as statistically independent random variables with zero average. Within such an approximation one can estimate (3.19) by replacing $x_n(1,2)$ by 1 (because $\cos^2\, n\,\sigma_1 + \cos^2\, n\,\sigma_2 = 1 + \frac{1}{2}\,(\cos\, 2n\,\sigma_1 + \cos\, 2n\,\sigma_2)$). Similarly, one can estimate (3.21) by replacing $p_n^{\pm} \to 1$. The resulting estimates of (3.19) and (3.21) introduce exactly the grand canonical averages of the quantities $:\mathcal{R}:$ and $N$:

$$\frac{1}{2}\,\langle : (\boldsymbol{k}\cdot\Delta\,\widetilde{\boldsymbol{X}})^2 : \rangle_{\beta,\gamma} \simeq \boldsymbol{k}^2\,\langle R^2\rangle_{\beta,\gamma}\,, \tag{3.23}$$

$$\langle : (\Delta\, P_+)^2 : \rangle_{\beta,\gamma} \simeq \langle : (\Delta\, P_-)^2 : \rangle_{\beta,\gamma} \simeq 4\,\langle N\rangle_{\beta,\gamma} \simeq 4\,\alpha'\, M^2\,. \tag{3.24}$$

Furthermore, one can check that the other contractions (like $[\Delta\, P_+\,\Delta\, P_-]$ or $[\Delta\, P_+\; k\cdot\Delta\,\widetilde{X}]$) entering Eq. (3.17) are all of the "oscillatory" type which is expected to give subleading contributions.

Inserting the results (3.23), (3.24) into Eqs. (3.17) and (3.11) leads to a trivial integral over $\tau_{12}$ ($\int d\,\tau_{12}\,\exp(-i\,\ell_s^2\, M\,\omega\,\tau_{12}) = 2\pi\,\delta(\omega)/(\ell_s^2\, M)$) and, hence, to the following result for $\delta\, M = \delta\, M^2/(2M)$

$$\delta\, M_{\beta,\gamma} \simeq -4\,\pi\, G_N\, M^2 \int \frac{d^d\, k}{(2\pi)^d}\,\frac{e^{-\boldsymbol{k}^2 R^2}}{\boldsymbol{k}^2 - i\,\varepsilon}\,. \tag{3.25}$$

The imaginary part of $\delta\, M$ is easily seen to vanish in the present approximation. Finally,

$$\delta M_{\beta,\gamma} \simeq -c_d\, G_N\, \frac{M^2}{R^{d-2}}\,, \tag{3.26}$$

with the (positive[12]) numerical constant

$$c_d = \left[\frac{d-2}{2}\,(4\pi)^{\frac{d-2}{2}}\right]^{-1}\,, \tag{3.27}$$

equal to $1/\sqrt{\pi}$ in $d = 3$.

The result (3.26) was expected in order of magnitude, but we found useful to show how it approximately comes out of a detailed calculation of the mass shift which incorporates both relativistic and quantum effects. It shows clearly that perturbation theory breaks down, even at arbitrarily small coupling, for sufficiently heavy and compact strings. Let us also point out that one can give a simple statistical interpretation of the calculation (3.16) of the normal-ordered vertex operator $W(k,1,2)$, with the basic contractions (3.18). The result of the calculation would have been the same if we had simply assumed that the oscillators $a_n^i$ were classical, complex random variables with a Gaussian probability distribution $\propto \exp\left[-\frac{1}{2}\,(e^{\epsilon_n}-1)\,|a_n^i|^2\right]$. This equivalence underlies the success of the classical random walk model of a generic excited string state. The fact that the random walk must be made of $M/M_s$ independent steps is linked to the fact that the Planckian distribution of mean occupation numbers, $\overline{N}_n = (\exp(\beta\, n + \gamma/n) - 1)^{-1}$ is sharply cut off when $n \gtrsim \beta^{-1}$, i.e., from Eq. (2.28), when $n \gtrsim M/M_s$. More precisely, using the same "statistical" approximation as above, one finds that the slope correlator $\langle : \partial_\sigma\, \widetilde{X}^i(\tau,\sigma_1)\, \partial_\sigma\, \widetilde{X}^j(\tau,\sigma_2) : \rangle_{\beta,\gamma}$ decays quite fast when $|\sigma_2 - \sigma_1| \gtrsim M_s/M$.

Finally, let us mention that, by using the same tools as above, one can compute the imaginary part of the mass shift $\delta\, M = \delta\, M_{\rm real} - i\,\Gamma/2$, i.e. the total decay rate $\Gamma$ in massless quanta. The quantity $\Gamma$ is, in fact, easier to define rigorously in string theory because, using $(k^2 - i\,\varepsilon)^{-1} = PP(k^2)^{-1} + i\pi\,\delta(k^2)$ in (3.8), it is given by an integral where the massless quanta are all on-shell. When $\gamma = 0$ (a consistent approximation for a result dominated by $n \sim \beta^{-1}$) one can use the covariant formalism with $D = d+1$ oscillators to find,after replacing a discrete sum over $n$ by an integral over $\omega$,

$$\Gamma = c'_d\, \frac{G_N}{M\,\ell_s^2} \int d\omega\, \omega^{d-2} \left(\frac{n}{e^{\beta n}-1}\right)^2\,, \tag{3.28}$$

where $c'_d$ is a numerical constant, and where $n = M\,\ell_s^2\,\omega/2$. The spectral decomposition of the total power radiated by the $\beta$-ensemble of strings is then simply deduced from (3.28) by adding a factor $\hbar\,\omega$ in the integrand:

$$P = c'_d\, \frac{G_N}{M\,\ell_s^2} \int d\omega\, \omega^{d-1} \left(\frac{n}{e^{\beta n}-1}\right)^2\,. \tag{3.29}$$

[12]The sign $\delta\, M < 0$ was classically clear (even when taking into account relativistic effects), say in $d = 3$, from the starting formulas (3.3), (3.4) where $G_{\rm sym}(x) = (4\pi)^{-1}\,\delta(x^2) > 0$ and $4(\partial_+\, X_1 \cdot \partial_+\, X_2)\,(\partial_-\, X_1 \cdot \partial_-\, X_2) = (\partial_+\,\Delta\, X)^2\,(\partial_-\,\Delta\, X)^2 > 0$ because $\partial_\pm \Delta\, X^\mu$ is purely spacelike in the center-of-mass frame. The same conclusion would hold in the light-cone gauge.

The results (3.28), (3.29) agree with corresponding results (for closed strings) in the second reference [8] and in [22] (note, however, that the factor $M^2$ in the equation (3.2) of [22] should be $M$ and that the constant contains $G_N$ and powers of $\ell_s$). The integrals (3.28), (3.29) are dominated by $n \sim \beta^{-1}$, i.e. $\omega \sim M_s$. This gives for the integrated quantities:

$$\Gamma \sim g^2 M \ , \ P \sim g^2 M M_s \, . \tag{3.30}$$

The second equation (3.30) means that the mass of a highly excited string decays exponentially, with half-evaporation time

$$\tau_{\rm evap}^{\rm string} \equiv M/P \sim \frac{\ell_s}{g^2} \, . \tag{3.31}$$

Let us anticipate on the next section and note that, at the transition $\lambda \equiv g^2 M/M_s \sim 1$ between string states and black hole states, not only the mass and the entropy are (in order of magnitude at least) continuous, but also the various radiative quantities: total luminosity $P$, half-evaporation time $\tau_{\rm evap}$, and peak of emission spectrum. Indeed, for a black hole decaying under Hawking radiation the temperature is $T_{\rm BH} \sim R_{\rm BH}^{-1}$ and

$$P_{\rm BH} \sim R_{\rm BH}^{-2} \sim \ell_s^{-2} \, \lambda^{-2/(d-2)} \ , \ \tau_{\rm evap}^{\rm BH} \sim R_{\rm BH} \, S_{\rm BH} \sim \ell_s \, g^{-2} \, \lambda^{d/(d-2)} \, . \tag{3.32}$$

## IV. ENTROPY OF SELF-GRAVITATING STRINGS

In the present section we shall combine the main results of the previous sections, Eqs. (2.32) and (3.26), and heuristically extend them at the limit of their domain of validity. We consider a narrow band of string states that we follow when increasing adiabatically the string coupling $g$, starting from $g = 0$[13]. Let $M_0$, $R_0$ denote the "bare" values (i.e. for $g \to 0$) of the mass and size of this band of states. Under the adiabatic variation of $g$, the mass and size, $M$, $R$, of this band of states will vary. However, the entropy$S(M,R)$ remains constant under this adiabatic process: $S(M,R) = S(M_0,R_0)$. We assume, as usual, that the variation of $g$ is sufficiently slow to be reversible, but sufficiently fast to be able to neglect the decay of the states. We consider states with sizes $\ell_s \ll R_0 \ll M_0$ for which the correction factor,

$$c\,(R_0) \simeq (1 - c_1 \, R_0^{-2}) \, (1 - c_2 \, R_0^2/M_0^2) \, , \tag{4.1}$$

in the entropy

$$S(M_0,R_0) = c\,(R_0) \, a_0 \, M_0 \, , \tag{4.2}$$

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[13] Alternatively, we can consider the coupling as an adjustable parameter (it is so in perturbation theory) and just follow how different physical quantities change as $g$ is varied, whithout pretending that the change takes place in physical time.

is near unity. [We use Eq. (2.32) in the limit $g \to 0$, for which it was derived.] Because of this reduced sensitivity of $c\,(R_0)$ on a possible direct effect of $g$ on $R$ (i.e. $R(g) = R_0 + \delta_g\, R$), the main effect of self-gravity on the entropy (considered as a function of the actual values $M$, $R$ when $g \neq 0$) will come from replacing $M_0$ as a function of $M$ and $R$. The mass-shift result (3.26) gives $\delta\, M = M - M_0$ to first order in $g^2$. To the same accuracy[14], (3.26) gives $M_0$ as a function of $M$ and $R$:

$$M_0 \simeq M + c_3\, g^2\, \frac{M^2}{R^{d-2}} = M \left(1 + c_3\, \frac{g^2\, M}{R^{d-2}}\right) , \tag{4.3}$$

where $c_3$ is a positive numerical constant.

Finally, combining Eqs. (4.1)–(4.3) (and neglecting, as just said, a small effect linked to $\delta_g\, R \neq 0$) leads to the following relation between the entropy, the mass and the size (all considered for self-gravitating states, with $g \neq 0$)

$$S(M,R) \simeq a_0\, M \left(1 - \frac{1}{R^2}\right) \left(1 - \frac{R^2}{M^2}\right) \left(1 + \frac{g^2\, M}{R^{d-2}}\right) . \tag{4.4}$$

For notational simplicity, we henceforth set to unity the coefficients $c_1$, $c_2$ and $c_3$. There is no loss of generality in doing so because we can redefine $\ell_s$, $R$ and $g$ to that effect, and use the corresponding (new) string units. The main point of the present paper is to emphasize that, for a given value of the total energy $M$ (and for some fixed value of $g$), the entropy $S(M,R)$ has a non trivial dependence on the radius $R$ of the considered string state. Eq. (4.3) exhibits two effects varying in opposite directions: (i) self-gravity favors small values of $R$ (because they correspond to larger values of $M_0$, i.e. of the "bare" entropy), and (ii) the constraint of being of some fixed size $R$ disfavors both small ($R \ll \sqrt{M}$) and large ($R \gg \sqrt{M}$) values of $R$. For given values of $M$ and $g$, the most numerous (and therefore most probable) string states will have a size$R_*(M;g)$ which maximizes the entropy $S(M,R)$. Said differently, the total degeneracy of the complete ensemble of self-gravitating string states with total energy $M$ (and no *a priori* size restriction) will be given by an integral (where $\Delta\, R$ is the rms fluctuation of $R$ given by Eq. (2.17))

$$\mathcal{D}(M) \sim \int \frac{dR}{\Delta\, R}\, e^{S(M,R)} \sim e^{S(M,R_*)} \tag{4.5}$$

which will be dominated by the saddle point $R_*$ which maximizes the exponent.

The value of the most probable size $R_*$ is a function of $M$, $g$ and the space dimension $d$. To better see the dependence on $d$, let us first consider the case (which we generically assume) where the correction factors in Eq. (4.4) (parentheses on the right-hand-side) are very close to unity so that

$$S(M,R) \simeq a_0\, M\, (1 - V(R))\, , \tag{4.6}$$

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[14] Actually, Eq. (4.3) is probably a more accurate version of the mass-shift formula because it exhibits the real mass $M$ (rather than the bare mass $M_0$) as the source of self-gravity.

where

$$V(R) = \frac{1}{R^2} + \frac{R^2}{M^2} - \frac{g^2\, M}{R^{d-2}}\,. \tag{4.7}$$

One can think of $V(R)$ as an effective potential for $R$. The most probable size $R_*$ must minimize $V(R)$. This effective potential can be thought of as the superposition of: (i) a centrifugal barrier near $R = 0$ (coming from the result (2.30)), (ii) an harmonic potential (forbidding the large values of $R$), and (iii) an attractive (gravitational) potential. When $g^2$ is small the minimum of $V(R)$ will come from the competition between the centrifugal barrier and the harmonic potential and will be located around the value $R_*^{-2} \simeq R_*^2/M^2$, i.e. $R_* \simeq \sqrt{M} = R_{\rm rw}$. This random walk value will remain (modulo small corrections) a local minimum of $V(R)$ (i.e. a local maximum of $S(M,R)$) as long as $g^2\, M/R_*^{d-2} \ll R_*^{-2}$, i.e. for $g^2 \ll g_0^2$ with

$$g_0^2 \equiv M^{\frac{d-6}{2}}\,. \tag{4.8}$$

More precisely, working perturbatively in $g^2$, the minimization of $V(R)$ yields

$$R_* \simeq \sqrt{M} \left(1 - \frac{d-2}{8}\,\frac{g^2}{g_0^2}\right)\,. \tag{4.9}$$

Note that, when $g^2 \ll g_0^2$, the value of $V(R)$ at this local minimum is of order $V_{\rm min} \simeq +\,2R_*^{-2} \simeq +\,2M^{-1}$, i.e. that it corresponds to a saddle-point entropy $S(M,R_*) \simeq a_0\, M(1 - V_{\rm min}) \simeq a_0\, M - \mathcal{O}(1)$ which differs essentially negligibly from the "bare" entropy $a_0\, M$ ($\gg 1$). To study what happens when $g^2$ further increases let us consider separately the various dimensions $d \geq 3$. We shall see that the special value $g_0^2$, Eq. (4.8), is significant (as marking a pre-transition, before the transition to the black hole state) only for $d = 3$. For $d \geq 4$, the only special value of $g^2$ is the critical value

$$g_c^2 \sim M^{-1}\,, \tag{4.10}$$

around which takes place a transition toward a state more compact than the usual random walk one.

### A. $d = 3$

Let us first consider the (physical) case $d = 3$, for which $g_0^2 \sim M^{-3/2} \ll g_c^2 \sim M^{-1}$. In that case, when $g^2$ becomes larger than $g_0^2$, the (unique) local minimum of $V(R)$ slowly shifts towards values of $R$ lower than $R_{\rm rw}$ and determined by the competition between the centrifugal barrier $1/R^2$ and the gravitational potential$-g^2\, M/R$.

In the approximation where we use the linearized form (4.6), (4.7), and where (for $g^2 \gg g_0^2$) we neglect the term $R^2/M^2$, the most probable size $R_*$ is

$$R_*^{(d=3)} \simeq \frac{2}{g^2\, M}\ , \ \text{when } M^{-3/2} \ll g^2 \ll M^{-1}\,. \tag{4.11}$$

Note that as $g^2$ increases between $M^{-3/2}$ and $M^{-1}$, the most probable size $R_*^{(d=3)}$ smoothly interpolates between $R_{\rm rw}$ and a value of order unity, i.e. of order the string length. Note also that $V_{\rm min} \simeq -g^2\, M/(2R_*) \simeq -g^4\, M^2/4$ remains smaller than one when $g^2 \lesssim M^{-1}$ so that the saddle-point entropy $S(M, R_*) \simeq a_0\, M(1 - V_{\rm min})$ never differs much from the "bare" value $a_0\, M$.

When $g^2$, in its increase, becomes comparable to $M^{-1}$, the radius becomes of order one and it is important to take into account the (supposedly) more exact expression (4.4) (in which the factor $(1 - R^{-2})$ plays the crucial role of cutting off any size $R \leq 1$). If we neglect, as above, the term $R^2/M^2$ (which is indeed even more negligible in the region $R \sim 1$) but maximize the factored expression (4.4), we find that the most probable size $R_*$ reads

$$R_*^{(d=3)} \simeq \frac{1 + \sqrt{1 + 3\,\lambda^2}}{\lambda} \ , \ \text{when } g^2 \gg M^{-3/2} \, , \tag{4.12}$$

where we recall the definition

$$\lambda \equiv g^2\, M \, . \tag{4.13}$$

When $\lambda \ll 1$, the result (4.12) reproduces the simple linearized estimate (4.11). When $\lambda \gtrsim 1$, Eq. (4.12) says that the most probable size, when $g^2$ increases, tends to a limiting size ($R_\infty = \sqrt{3}$) slightly larger than the minimal one ($R_{\rm min} = 1$) corresponding to zero entropy. [Note that even for the formal asymptotic value $R_\infty = \sqrt{3}$, the reduction in entropy due to the factor $1 - R^{-2}$ is only 2/3.] On the other hand, the fractional self-gravity $G_N\, M/R_*$ (which measures the gravitational deformation away from flat space), or the corresponding term in Eq. (4.4), continues to increase with $g^2$ as

$$\frac{\lambda}{R_*} = \frac{\lambda^2}{1 + \sqrt{1 + 3\,\lambda^2}} \, . \tag{4.14}$$

The right-hand side of Eq. (4.14) becomes unity for $\lambda = \sqrt{5} = 2.236$. The picture suggested by these results is that the string smoothly contracts, as $g$ increases, from its initial random walk size down to a limiting compact state of size slightly larger than $\ell_s$. For some value of $\lambda$ of order unity (may be between 1 and 2; indeed, even for $\lambda = 1$ the size $R_* = 2$ and the self-gravity $\lambda/R_* = 0.5$ suggest one may still trust a compact string description) the self-gravity of this compact string state will become so strong that one expects it to collapse to a black hole state. We recall that, as emphasized in Refs. [4], [6], [7], [8], [9], [13], when $\lambda \sim 1$, the mass of the string state matches (in order of magnitude) that of a (Schwarzschild) black hole with Bekenstein-Hawking entropy equal to the string entropy $S$.

### B. $d = 4$

When $d = 4$, the argument above Eq. (4.9) suggests that the random-walk size remains the most probable size up to $g^2 \lesssim g_0^2 \sim M^{-1}$, i.e. up to $\lambda \lesssim 1$. A more accurate approximation to the most probable size $R_*$, when $\lambda < 1$, is obtained by minimizing exactly $V(R)$, Eq. (4.7). This yields

$$R_*^{(d=4)} \simeq M^{1/2}\,(1-\lambda)^{1/4}\ , \ \text{when } \lambda < 1\,. \tag{4.15}$$

This shows that the size will decrease, but one cannot trust this estimate when $\lambda \to 1^-$. To study more precisely what happens when $\lambda \sim 1$ we must take into account the more exact factorized form (4.4). Let us now neglect the $R^2/M^2$ term and consider the approximation

$$S^{(d=4)}(M,R) \simeq a_0\, M \left(1-\frac{1}{R^2}\right)\left(1+\frac{\lambda}{R^2}\right)\,. \tag{4.16}$$

The right-hand side of Eq. (4.15) has a maximum only for $\lambda > 1$, in which case

$$R_*^{(d=4)} \simeq \left(\frac{2\lambda}{\lambda-1}\right)^{1/2} \quad \text{when } \lambda > 1\,. \tag{4.17}$$

If we had taken into account the full expression (4.4) the two results (4.15), (4.17), valid on each side of $\lambda = 1$, would have blended in a result showing that around[15] $\lambda = 1$ the most probable size *continuously* interpolates between $R_{\rm rw}$ and a size of order $\ell_s$. Note that, according to Eq. (4.17), as $\lambda$ becomes $\gg 1$, $R_*^{(d=4)}$ tends to a limiting size ($R_\infty = \sqrt{2}$) slightly larger than $R_{\rm min} = 1$ (corresponding to zero entropy). When $\lambda > 1$ the fractional self-gravity of the compact string states reads

$$\frac{\lambda}{R_*^2} = \frac{\lambda-1}{2}\,. \tag{4.18}$$

As in the case $d = 3$, one expects that for some value of $\lambda$ strictly larger than 1, the self-gravity of the compact string state will become so strong that it will collapse to a black hole state. Again the mass, size and entropy match (in order of magnitude) those of a black hole when $\lambda \sim 1$. The only difference between $d = 4$ and $d = 3$ is that the transition to the compact state, though still continuous, is sharply concentrated around $\lambda = 1$ instead of taking place over the extended range $M^{-1/2} \lesssim \lambda \lesssim 1$.

### C. $d \geq 5$

When $d \geq 5$, the argument around Eq. (4.8) shows that the random walk size $R_{\rm rw} \simeq \sqrt{M}$ is a consistent local maximum of the entropy in the whole domain $g^2 \ll g_0^2$, i.e. for $\lambda \equiv g^2\, M \ll M^{\frac{d-4}{2}}$, which allows values $\lambda \gg 1$. However, a second, disconnected maximum of the entropy, as function of$R$, could exist. To investigate this we consider again (4.4), when neglecting the $R^2/M^2$ term (because we are interested in other possible solutions with small sizes):

$$\frac{S(M,R)}{a_0\, M} \simeq (1-x)\,(1+\lambda\, x^\nu) \equiv s(x)\,, \tag{4.19}$$

[15] The transition takes place in the range $|\lambda - 1| \sim M^{-2/3}$ corresponding to $R_* \sim M^{1/3}$.

where we have defined $x \equiv R^{-2}$ and $\nu \equiv (d-2)/2$. By studying analytically the maxima and inflection points of $s(x)$, one finds that, in the present case where $\nu = (d-2)/2 > 1$, there are two critical values $\lambda_1 < \lambda_2$ of the parameter $\lambda \equiv g^2 M$. The first one,

$$\lambda_1 = \left(\frac{\nu+1}{\nu-1}\right)^{\nu-1} \;,\; R_1 = x_1^{-1/2} = \left(\frac{\nu+1}{\nu-1}\right)^{1/2} > 1 \;,\; s_1 = 1 - \left(\frac{\nu-1}{\nu+1}\right)^2 \;, \tag{4.20}$$

corresponds to the birth (through an inflection point) of a maximum and a minimum of the function $s(R)$. Because $s_1 < 1$ is strictly lower than the usual random walk maximum with $s(R_{\rm rw}) \simeq 1 - 2/M \simeq 1$, the local maximum near $R \sim 1$ of the entropy, which starts to exist when $\lambda > \lambda_1$, is, at first, only metastable with respect to $R_{\rm rw}$. However, there is a second critical value of $\lambda$, $\lambda_2 > \lambda_1$, defined by

$$\lambda_2 = \nu \left(\frac{\nu}{\nu-1}\right)^{\nu-1} \;,\; R_2 = x_2^{-1/2} = \left(\frac{\nu}{\nu-1}\right)^{\frac{1}{2}} > 1 \;,\; s_2 = 1\,. \tag{4.21}$$

When $\lambda > \lambda_2$ the local maximum near $R \sim 1$ of the entropy has $s(R) > 1$, i.e. it has become the global maximum of the entropy, making the usual random walk local maximum only metastable. Therefore,when $\lambda > \lambda_2$ the most probable string state is a very compact state of size comparable to $\ell_s$. Formally, this new global maximum exists for any $\lambda \gtrsim 1$ and tends, when $\lambda \to \infty$, toward the limiting location $R_\infty = ((\nu+1)/\nu)^{1/2} > 1$, i.e. slightly (but finitely) above the minimum size $R = 1$. However, as in the cases $d \leq 4$, the self-gravity of the stable compact string state will become strong when $\lambda \gtrsim 1$, so that it is expected to collapse (for some $\lambda_c > \lambda_2$) to a black hole state. As in the cases $d \leq 4$, the mass, size and entropy of this compact string state match those of a black hole. The big difference with the cases $d \leq 4$ is that the transition between the (stable) random walk typical configuration and the (stable) compact one is discontinuous. Our present model suggests that (when $\nu \equiv (d-2)/2 > 1$) a highly excited single string system can exist, when $\lambda > \lambda_2$, in two different stable typical states: (i) a dilute state of typical size $R_{\rm rw} \simeq \sqrt{M}$ and typical mean density $\rho \sim M/R_{\rm rw}^d \sim M^{-\nu} \ll 1$, and (ii) a condensed state of typical size $R \sim 1$ and typical mean density (using $\lambda \sim 1$): $\rho \sim M \sim g^{-2} \gg 1$. We shall comment further below on the value $\rho \sim g^{-2}$ of the dense state of string matter.

## V. DISCUSSION

Technically, the main new result of the present work is the (dimension independent) estimate[16] $c(R) = 1 - c_1/R^2$, with $c_1 \simeq (3/4)\,\ell_s^2 = 3\alpha'/2$, of the factor giving the decrease in the entropy $(2\pi((d-1)\,\alpha'/6)^{1/2}\,M)$ of a narrow band of very massive (open[17]) string states

[16] In spite of our efforts in Section II, this result remains non rigorous and open to $\mathcal{O}(1/d)$ fractional corrections because of the difficulty to define a good quantum operator representing the mean radius of a string state.

[17] For technical simplicity, we have restricted our attention to open bosonic strings. We could have dealt with closed bosonic strings by doubling the oscillators, but the level matching condition,

$M \gg (\alpha')^{-1/2}$, when considering only string states of size $R$ (modulo some fractionally small grand-canonical-type fluctuations). We have also justified (by dealing explicitly with relativistic and quantum effects in a semi-classical approximation) and refined (by computing the numerical coefficient, Eq. (3.27)) the naive estimate, $\delta M = -c_d \, G_N \, M^2/R^{d-2}$, of the mass shift of a massive string state due to the exchange of long range fields (graviton, dilaton and axion). [The exchange of these fields is expected to be the most important one both because very excited string states tend to be large, and because the corresponding interactions are attractive and cumulative with the mass.]

Conceptually, the main new result of this paper concerns the most probable state of a very massive single[18] self-gravitating string. By combining our estimates of the entropy reduction due to the size constraint, and of the mass shift we come up with the expression (4.4) for the logarithm of the number of self-gravitating string states of size $R$. Our analysis of the function $S(M,R)$ clarifies the correspondence [6], [8], [9], [13] between string states and black holes. In particular, our results confirm many of the results of [13], but make them (in our opinion) physically clearer by dealing directly with the size distribution, in real space, of an ensemble of string states. When our results differ from those of [13], they do so in a way which simplifies the physical picture and make even more compelling the existence of a correspondence between strings and black holes. For instance, [13] suggested that in $d=5$ there was a phenomenon of hysteresis, with a critical value $g_0^2 \sim M^{-1/2}$ for the string $\rightarrow$ black hole transition, and a different critical value $g_c^2 \sim M^{-1} \ll g_0^2$ for the inverse transition: black hole $\rightarrow$ string. Also, [13] suggested that in $d>6$, most excited string states would never form black holes. The simple physical picture suggested[19] by our results is the following: In any dimension, if we start with a massive string state and increase the string coupling $g$, a *typical* string state will, eventually, become more compact and will end up, when $\lambda_c = g_c^2 \, M \sim 1$, in a "condensed state" of size $R \sim 1$, and mass density $\rho \sim g_c^{-2}$. Note that the basic reason why small strings, $R \sim 1$, dominate in any dimension the entropy when $\lambda \sim 1$ is that they descend from string states with bare mass $M_0 \simeq M(1+\lambda/R^{d-2}) \sim 2M$ which are exponentially more numerous than less condensed string states corresponding to smaller bare masses.

The nature of the transition between the initial "dilute" state and the final "condensed"

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$N_L = N_R$, would have complicated the definition of the grand canonical ensemble we used. We expect that our results (whichare semi-classical) apply (with some numerical changes) to open or closed superstrings.

[18]We consider states of a single string because, for large values of the mass, the single-string entropy approximates the total entropy up to subleading terms.

[19]Our conclusions are not rigourously established because they rely on assuming the validity of the result (4.4) beyond the domain ($R^{-2} \ll 1$, $g^2 \, M/R^{d-2} \ll 1$) where it was derived. However, we find heuristically convincing to believe in the presence of a reduction factor of the type $1 - R^{-2}$ down to sizes very near the string scale. Our heuristic dealing with self-gravity is less compelling because we do not have a clear signal of when strong gravitational field effects become essential.

one depends on the value of the space dimension $d$. [As explained below Eq. (4.4), we henceforth set to unity, by suitable redefinitions of $\ell_s$, $R$ and $g$, the coefficients $c_1$, $c_2$ and $c_3$.] In $d = 3$, the transition is gradual: when $\lambda < M^{-1/2}$ the size of a typical state is $R_*^{(d=3)} \simeq M^{1/2}(1 - M^{1/2}\,\lambda/8)$, when $\lambda > M^{1/2}$ the typical size is $R_*^{(d=3)} \simeq (1+(1+3\,\lambda^2)^{1/2})/\lambda$. In $d = 4$, the transition toward a condensed state is still continuous, but most of the size evolution takes place very near $\lambda = 1$: when $\lambda < 1$, $R_*^{(d=4)} \simeq M^{1/2}(1-\lambda)^{1/4}$, and when $\lambda > 1, R_*^{(d=4)} \simeq (2\lambda/(\lambda-1))^{1/2}$, with some smooth blending between the two evolutions around $|\lambda - 1| \sim M^{-2/3}$. In $d \geq 5$, the transition is discontinuous (like a first order phase transition between, say, gas and liquid states). Barring the consideration of metastable (supercooled) states, on expects that when $\lambda = \lambda_2 \simeq \nu^\nu/(\nu-1)^{\nu-1}$ (with $\nu = (d-2)/2$), the most probable size of a string state will jump from $R_{\rm rw}$ (when $\lambda < \lambda_2$) to a size of order unity (when $\lambda > \lambda_2$).

Let us, for definiteness, write down in more detail what happens in $d = 3$. After maximization over $R$, the entropy of a self-gravitating string is given, when $M^{-3/2} \ll g^2 \ll M^{-1}$, by

$$S(M) = S(M, R_*(M)) \simeq a_0\, M \left[1 + \frac{1}{4}\,(g^2\, M)^2\right] . \tag{5.1}$$

By differentiating $S$ with respect to $M$, one finds the temperature of the ensemble of highly excited single string states of mass $M$:

$$T \simeq T_{\rm Hag}\left(1 - \frac{3}{4}\,(g^2\, M)^2\right) , \tag{5.2}$$

with $T_{\rm Hag} \equiv a_0^{-1}$. Eq. (5.2) explicitly exhibits the modification of the Hagedorn temperature due to self-gravity (in agreement with results of [13] obtained by a completely different approach). Note that, both in Eqs. (5.1) and (5.2), the self-gravity modifications are fractionally of order unity at the transition $g^2\, M \sim 1$.

One can think of the "condensed" state of (single) string matter, reached (in any $d$) when $\lambda \sim 1$, as an analog of a neutron star with respect to an ordinary star (or a white dwarf). It is very compact (because of self gravity) but it is stable (in some range for $g$) under gravitational collapse. However, if one further increases $g$ or $M$ (in fact, $\lambda = g^2\, M$), the condensed string state is expected (when $\lambda$ reaches some $\lambda_3 > \lambda_2$, $\lambda_3 = \mathcal{O}(1)$) to collapse down to a black hole state (analogously to a neutron star collapsing to a black hole when its mass exceeds the Landau-Oppenheimer-Volkoff critical mass). Still in analogy with neutron stars, one notes that general relativistic strong gravitational field effects are crucial for determining the onset of gravitational collapse; indeed, under the "Newtonian" approximation (4.4), the condensed string state could continue to exist for arbitrary large values of $\lambda$.

It is interesting to note that the value of the mass density at the formation of the condensed string state is $\rho \sim g^{-2}$. This is reminiscent of the prediction by Atick and Witten [23] of a first-order phase transition of a self-gravitating thermal gas of strings, near the Hagedorn temperature[20], towards a dense state with energy density $\rho \sim g^{-2}$ (typical of a

[20]Note that, by definition, in our *single* string system, the formal temperature $T = (\partial S/\partial M)^{-1}$

genus-zero contribution to the free energy). Ref. [23] suggested that this transition is first-order because of the coupling to the dilaton. This suggestion agrees with our finding of a discontinuous transition to the single string condensed state in dimensions $\geq 5$ (Ref. [23] work in higher dimensions, $d = 25$ for the bosonic case). It would be interesting to deepen these links between self-gravitating single string states and multi-string states.

Assuming the existence (confirmed by the present work) of a dense state of self-gravitating string matter with energy density $\rho \sim g^{-2}$, it would be fascinating to be able to explore in detail (with appropriate, strong gravity tools) its gravitational dynamics, both in the present context of a single, isolated object ("collapse problem"), and in the cosmological context (problem of the origin of the expansion of the universe).

Let us come back to the consequences of the picture brought by the present work for the problem of the end point of the evaporation of a Schwarzschild black hole and the interpretation of black hole entropy. In that case one fixes the value of $g$ (assumed to be $\ll 1$) and considers a black hole which slowly looses its mass via Hawking radiation. When the mass gets as low as a value[21] $M \sim g^{-2}$, for which the radius of the black hole is of order one (in string units), one expects the black hole to transform (in all dimensions) into a typical string[22] state corresponding to $\lambda = g^2 M \sim 1$, which is a dense state (still of radius $R \sim 1$). This string state will further decay and loose mass, predominantly via the emission of massless quanta, with a quasi thermal spectrum with temperature $T \sim T_{\rm Hagedorn} = a_0^{-1}$ (see Eq. (3.29) and Refs. [8], [22]) which smoothly matches the previous black hole Hawking temperature. This mass loss will further decrease the product $\lambda = g^2 M$, and this decrease will, either gradually or suddenly, cause the initially compact string state to inflate to much larger sizes. For instance, if $d \geq 4$, the string state will quickly inflate to a size $R \sim \sqrt{M}$. Later, with continued mass loss, the string size will slowly shrink again toward $R \sim 1$ until a remaining string of mass $M \sim 1$ finally decays into stable massless quanta. In this picture, the black hole entropy acquires a somewhat clear statistical significance (as the degeneracy of a corresponding typical string state) *only* when $M$ and $g$ are related by $g^2 M \sim 1$. If we allow ourselves to vary (in a Gedanken experiment) the value of $g$ this gives a potential statistical significance to any black hole entropy value $S_{\rm BH}$ (by choosing $g^2 \sim S_{\rm BH}^{-1}$). We do not claim, however, to have a clear idea of the direct statistical meaning of $S_{\rm BH}$ when $g^2 S_{\rm BH} \gg 1$. Neither do we clearly understand the fate of the very large space (which could be excited in many ways) which resides inside very large classical black holes of radius $R_{\rm BH} \sim (g^2 S_{\rm BH})^{1/(d-1)} \gg 1$. The fact that the interior of a black hole of given mass could be

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is always near the Hagedorn temperature.

[21]Note that the mass at the black hole $\rightarrow$ string transition is larger than the Planck mass $M_P \sim (G_N)^{-1/2} \sim g^{-1}$ by a factor $g^{-1} \gg 1$.

[22]A check on the single-string dominance of the transition black hole $\rightarrow$ string is to note that the single string entropy $\sim M/M_s$ is much larger than the entropy of a ball of radiation $S_{\rm rad} \sim (RM)^{d/(d+1)}$ with size $R \sim R_{\rm BH} \sim \ell_s$ at the transition.

*arbitrarily* large[23], and therefore arbitrarily complex, suggests that black hole physics is not exhausted by the idea (confirmed in the present paper) of a reversible transition between string-length-size black holes and string states.

On the string side, we also do not clearly understand how one could follow in detail (in the present non BPS framework) the "transformation" of a strongly self-gravitating string state into a black hole state.

Finally, let us note that we expect that self-gravity will lift nearly completely the degeneracy of string states. [The degeneracy linked to the rotational symmetry, i.e. $2J+1$ in $d=3$, is probably the only one to remain, and it is negligible compared to the string entropy.] Therefore we expect that the separation $\delta E$ between subsequent (string and black hole) energy levels will be exponentially small: $\delta E \sim \Delta M \exp(-S(M))$, where $\Delta M$ is the canonical-ensemble fluctuation in $M$. Such a $\delta E$ is negligibly small compared to the radiative width $\Gamma \sim g^2 M$ of the levels. This seems to mean that the discreteness of the quantum levels of strongly self-gravitating strings and black holes is very much blurred, and difficult to see observationally.

## ACKNOWLEDGEMENTS

This work has been clarified by useful suggestions from M. Douglas, K. Gawedzki, M. Green, I. Kogan, G. Parisi, A. Polyakov, and (last but not least) M. Vergassola. We wish also to thank A. Buonanno for collaboration at an early stage and D. Gross, J. Polchinski, A. Schwarz, L. Susskind and A. Vilenkin for discussions. T.D. thanks the Theory Division of CERN, Gravity Probe B (Stanford University), and the Institute for Theoretical Physics (Santa Barbara) for hospitality. Partial support from NASA grant NAS8-39225 is acknowledged. G.V. thanks the IHES for hospitality during the early, crucial stages of this work.

[23]E.g., in the Oppenheimer-Snyder model, one can join an arbitrarily large closed Friedmann dust universe, with hyperspherical opening angle $0 \leq \chi_0 \leq \pi$ arbitrarily near $\pi$, onto an exterior Schwarzschild spacetime of given mass $M$.

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