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CLIC Note 404
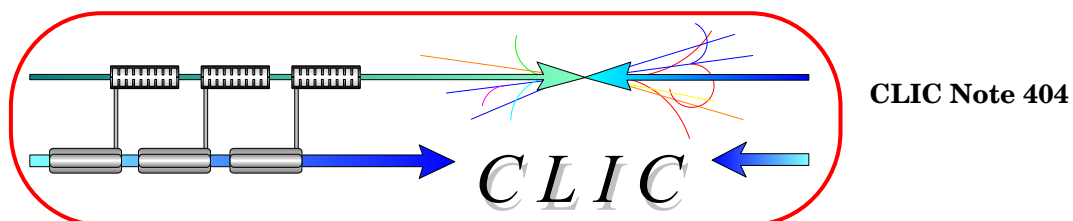
Calculation of the Energy Loss of a Single Bunch
in a Traveling Wave Structure

W. Wuensch

Abstract

The energy lost by a single bunch in traversing a travelling wave structure is derived in this note. The relationship between standing wave and traveling wave loss factors and the energy loss as function of group velocity will emerge from the derivation.

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Calculation of the Energy Loss of a Single Bunch in a Traveling Wave Structure

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The energy lost by a single bunch in traversing a travelling wave structure is derived in this note. The relationship between standing wave and traveling wave loss factors and the energy loss as function of group velocity will emerge from the derivation.

Although a traveling wave structure is being considered, the derivation will be made by considering the excitation of the resonances of a traveling wave structure with short circuited input and output couplers. Since no fields arrive at output short circuit before the bunch has already left the structure and the fields travel away from the input short circuit, the energy loss of the beam cannot be affected by the presence of the short circuits - thus an analysis using resonances as eigenfunctions is equivalent to one using traveling waves as eigenfunctions. The motivation for considering a short circuited structure is that the loss factor is defined for resonances [1]. The loss factor is used rigorously as it is defined. The energy lost by the beam is determined by calculating the energy deposited in the resonances.

In order to simplify the calculation, a structure with a constant v_g across its entire passband is considered. Only a small error is incurred by ignoring dispersion at the extremities of the passband since most of the beam/structure interaction occurs near the synchronous crossing (the synchronous crossing is the intersection of the speed of light line with the dispersion curve). The justification for the constant v_g structure simplification will be clarified within the derivation. Phase advance and frequency are related through the relation,

$$v_g = \frac{\partial \omega}{\partial \beta} \qquad\qquad 1$$

For a period (cell) length l in the traveling wave structure,

$$\varphi = \beta l \qquad\qquad 2$$

Consequently,

$$v_g = 2\pi l \frac{\partial f}{\partial \varphi}$$

$$\partial f = \frac{v_g}{2\pi l} \partial \varphi \qquad\qquad 3$$

$$\partial \varphi = \frac{2\pi l}{v_g} \partial f$$

The Brillouin diagram of a constant v_g structure based on these relations is shown in figure 1. The Brillouin curve intersects the speed of light line at the synchronous frequency f_s and the synchronous phase advance φ_s. The bandwidth of the structure is,

$$f_{span} = \frac{v_g}{2l} \qquad\qquad 4$$

This is the inverse of twice the transit time of a cell length.

A constant v_g structure N cells long will have N discrete resonances that are evenly spaced over the passband. The spacing in phase is,
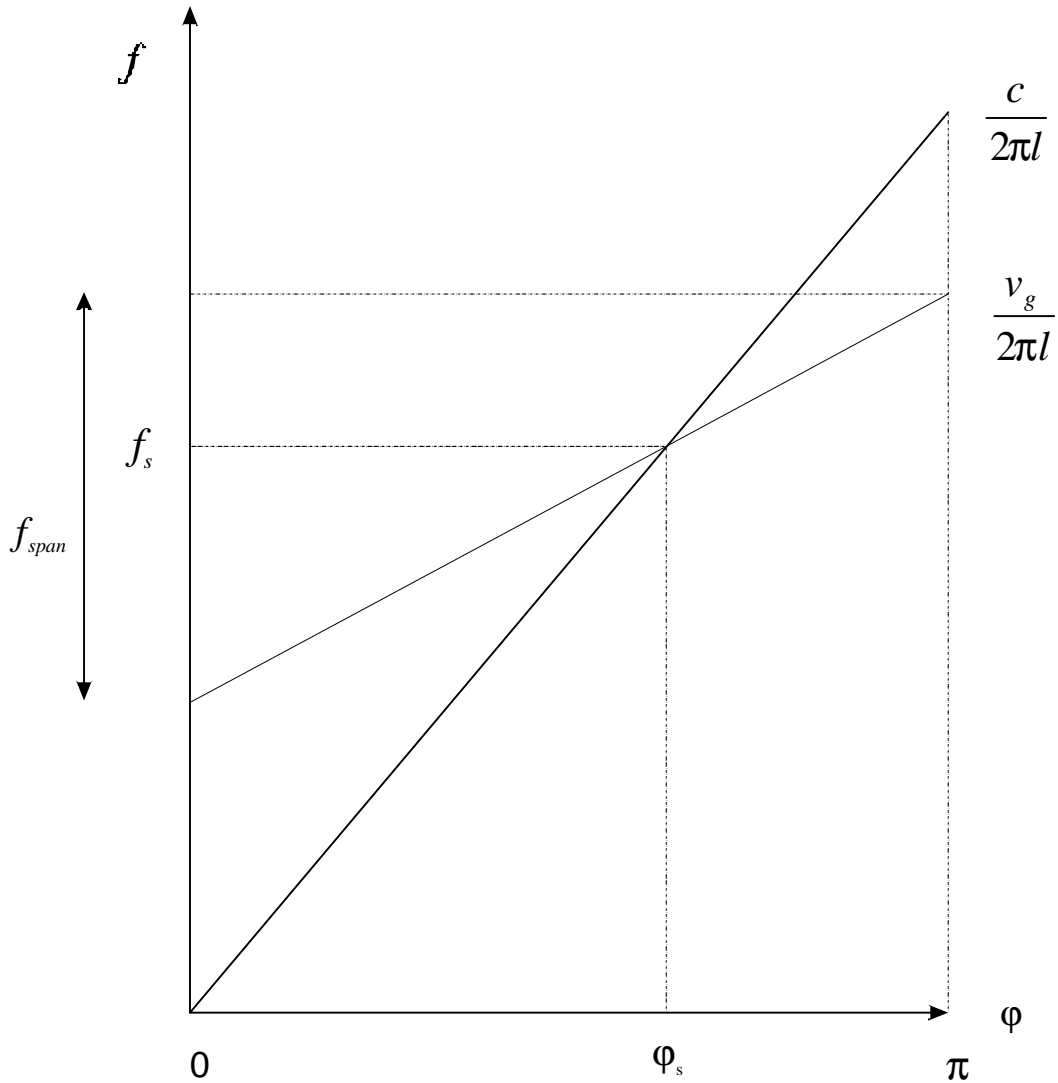
Figure 1: Brillouin diagram of a constant v_g structure.

$$\varphi_{spacing} = \frac{\pi}{N} \qquad\qquad 5$$

In frequency,

$$f_{spacing} = \frac{1}{N}\frac{v_g}{2l} \qquad\qquad 6$$

The structure is assumed to be tuned and short-circuited in such a way that a resonance, number n_s and referred to as the synchronous resonance, falls on the synchronous crossing. The Brillouin diagram in the area around the synchronous resonance is shown in figure 2.

The phase slip per cell, $\Delta\varphi$, between the speed of light line and the resonance n_s+1 is found by first finding A,
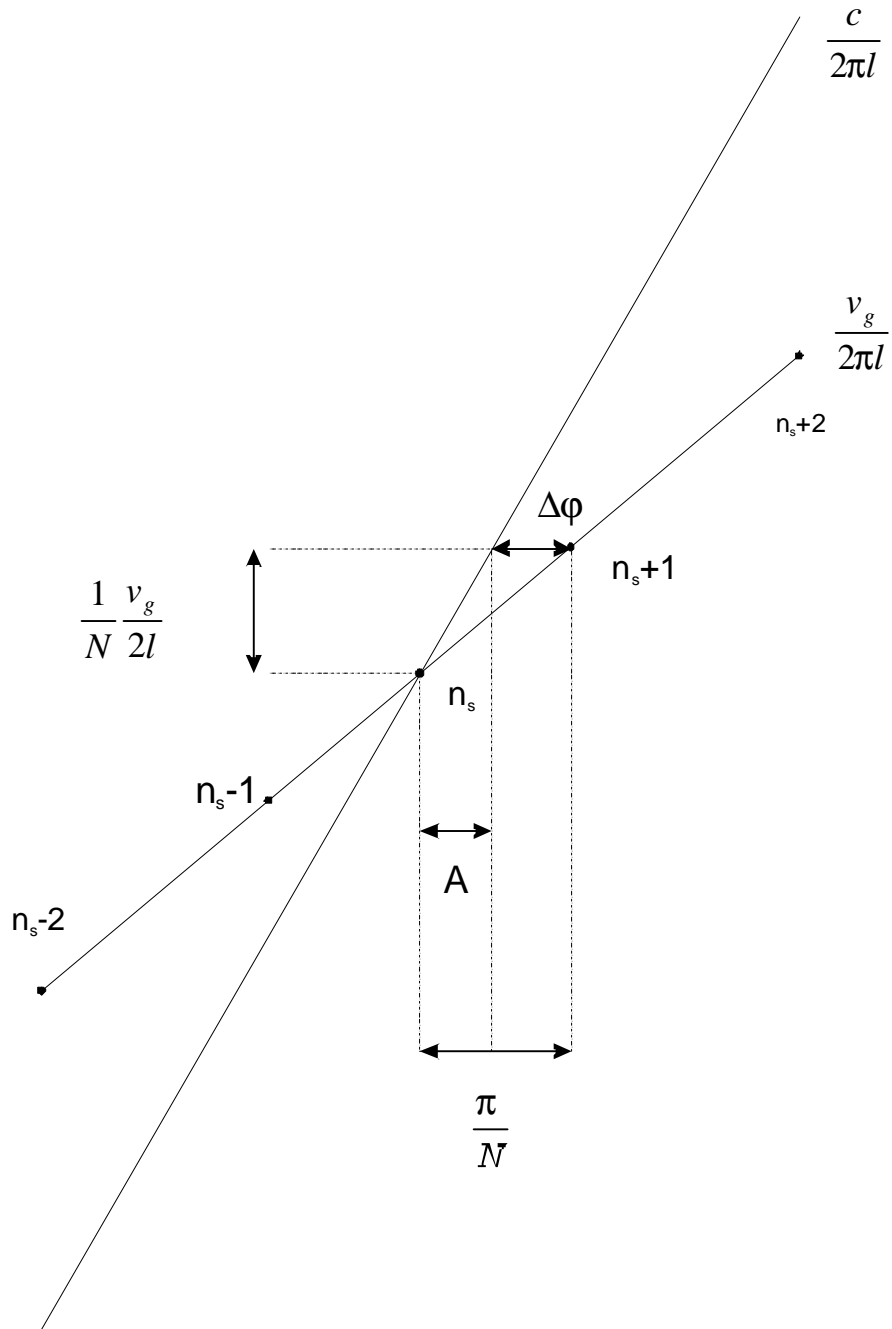
Figure 2: Zoom in on the Brillouin diagram near the synchronous crossing

$$A = \frac{2\pi l}{c}\left(\frac{1}{N}\frac{v_g}{2l}\right)$$

$$= \frac{\pi}{N}\frac{v_g}{c}$$

7

Then,

3

$$\Delta\varphi = \frac{\pi}{N} - A$$

$$= \frac{\pi}{N}\left(1 - \frac{v_g}{c}\right) \qquad\qquad 8$$

The phase slip per cell is a linear function of resonance number *n* and is given by,

$$\Delta\varphi_n = \frac{\pi}{N}\left(1 - \frac{v_g}{c}\right)(n - n_s) \qquad\qquad 9$$

The total phase slip as a function of resonance number, over the length of the N cell structure is,

$$\Delta\varphi_{total_n} = \pi\left(1 - \frac{v_g}{c}\right)(n - n_s) \qquad\qquad 10$$

The excitation of each resonance is a function of the phase slip and a derivation is given in appendix A. The derivation assumes that the resonances differ only in phase advance and the fields within the cells are always the same - an assumption already implicitly made by ignoring dispersion and assuming a constant v_g structure. Again, since the beam/structure interaction is sharply peaked near the synchronous crossing, this introduces a negligible error.

Recalling A.3, the normalized excitation spectrum,

$$E_n = \frac{e^{i\pi\left(1 - \frac{v_g}{c}\right)(n - n_s)} - 1}{i\pi\left(1 - \frac{v_g}{c}\right)(n - n_s)} \qquad\qquad 11$$

The energy in each resonance is,

$$U_n = E_n E_n^*$$

$$= \frac{\left(e^{i\pi\left(1 - \frac{v_g}{c}\right)(n - n_s)} - 1\right)\left(e^{-i\pi\left(1 - \frac{v_g}{c}\right)(n - n_s)} - 1\right)}{\left(\pi\left(1 - \frac{v_g}{c}\right)(n - n_s)\right)^2} \qquad\qquad 12$$

$$= \frac{2\left(1 - \cos\left(\pi\left(1 - \frac{v_g}{c}\right)(n - n_s)\right)\right)}{\left(\pi\left(1 - \frac{v_g}{c}\right)(n - n_s)\right)^2}$$

Three features of equation 12 are particularly important. The first is that the energy in the synchronous resonance is equal to one: $U_{ns}=1$. The second is that the function is sharply peaked around the synchronous resonance, as shown in figure 3. The third is that the energy spectrum, expressed in resonance number, is *independent of N* and consequently of the length of the structure! Expressed in

frequency the energy spectrum narrows for increasing N, but since the resonance spectrum is denser by the same factor the energy remains constant in the nth resonance away from synchronism.
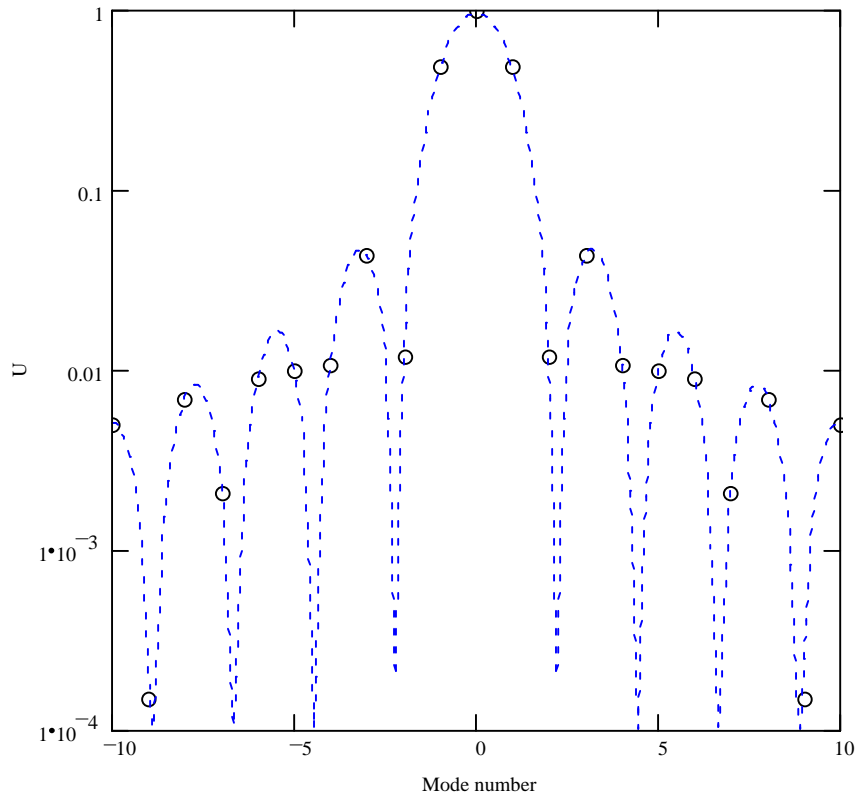

Figure 3: Plot of U_n around the synchronous resonance for $v_g/c=0.1$. For visual clarity the energy spectrum with n expressed as a continuous variable is superimposed on the resonances - BUT - caveat emptor, the continuous spectrum has no physical significance in this derivation!

The actual - as opposed to the normalized - energy deposited in the synchronous resonance is given by the classical standing wave loss factor. For simplicity the normalized energy is used throughout this derivation.

The total energy lost by a single bunch compared to the energy lost to the synchronous resonance is given by,

$$\frac{U_{total}}{U_{n_s}} = \frac{1}{U_{n_s}} \sum_{n=1}^{N} U_n = \sum_{n=1}^{N} U_n \qquad 13$$

In order to simplify the analysis, the synchronous resonance can be removed from the summation and the rest of the summation can be split into two summations that are made on either side of the synchronous resonance. This implies the change of variable $m=n-n_s$. Since U_n decreases quickly away from the synchronous resonance, the summations can be taken to infinity so that they can be expressed analytically,

$$\frac{U_{total}}{U_{n_s}} = 1 + 2\left(\frac{2}{\pi^2\left(1-\dfrac{v_g}{c}\right)^2}\sum_{m=1}^{\infty}\frac{1-\cos\left(\pi\left(1-\dfrac{v_g}{c}\right)m\right)}{m^2}\right)$$

$$= 1 + \frac{4}{\pi^2\left(1-\dfrac{v_g}{c}\right)^2}\left(\sum_{m=1}^{\infty}\frac{1}{m^2} - \sum_{m=1}^{\infty}\frac{\cos\left(\pi\left(1-\dfrac{v_g}{c}\right)m\right)}{m^2}\right)$$

$$= 1 + \frac{4}{\pi^2\left(1-\dfrac{v_g}{c}\right)^2}\left(\zeta(2) - \left(\frac{\pi^2}{6} - \frac{\pi}{2}\left(\pi\left(1-\dfrac{v_g}{c}\right)\right) + \frac{\pi^2\left(1-\dfrac{v_g}{c}\right)^2}{4}\right)\right)$$

$$= \frac{2}{1-\dfrac{v_g}{c}} \qquad\qquad 14$$

It is important to note that the summations were solved by extending the sums to infinity rather than by converting the sums to integrals. This is a direct consequence of the independence on N of the excitation spectrum expressed in resonance number - the excitation spectrum does not go to a continuum as N increases to infinity. The spectrum narrows in frequency just as quickly as the mode spectrum becomes denser.

The physical meanings of the numerator and denominator in equation 14 are now considered. The factor two in the numerator is precisely the ratio between the 'traveling wave R'/Q' and the 'standing wave R'/Q'. This is perhaps most easily explained by considering the conundrum,

"If a standing wave is the sum of a forward and a backward wave, what is a forward wave the sum of?"

It is in fact the sum of a number of standing wave resonances, the energy spectrum of which is given in equation 12. The resonant mode explanation of why the 'traveling wave R'/Q' is a factor two higher than the 'standing wave R'/Q' is that energy is deposited in more than one resonance. The superposition of the resonances result in the field pattern that we identify as a forward wave. The synthesis of a traveling wave from a sum of standing waves is already evident in a two cell structure. Such a structure has two modes, a 0 mode (symmetric) and a π (anti-symmetric) mode. If these modes are excited with equal amplitudes, the fields cancel in alternating cells with a period equal to the frequency split. This means the energy flows back and forth, from one cell to the other - creating a travelling wave.

The criterion for using the 'traveling wave R'/Q' to calculate energy loss is that a structure has a large number of modes. In this case the approximation of extending the summations to infinity remains valid. Resonances must be considered individually using the standing wave loss factor in structures with few cells.

The denominator is the much discussed 'group velocity factor', and shows that the single bunch energy loss is a factor,

$$\frac{1}{1-\dfrac{v_g}{c}} \qquad\qquad 15$$

higher than that expected using simply the synchronous traveling mode loss factor.

The results produced by computer programs can be clearly identified with terms found in this derivation. A frequency domain calculation will give the standing wave loss factor of the synchronous resonance. A time domain calculation made using a single exciting bunch will give the result expressed in equation 14.

REFERENCES

[1] P. Wilson, "Introduction to Wakefields and Wake Potentials", SLAC-pub 4547.

ACKNOWLEDGEMENTS

APPENDIX A, THE EXCITATION SPECTRUM CALCULATED FROM PHASE SLIP

Since acceleration and deceleration are the same process, the excitation amplitude spectrum as a function of phase slip is derived by considering acceleration. The phase slip per unit distance is given by,

$$\Delta\varphi'_n = \frac{\Delta\varphi_n}{l}$$

A.1

The total voltage gained across a section is given by,

$$
\begin{aligned}
Vacc_n &= \int_0^{Nl} e^{iz\Delta\varphi'_n} dz \\
&= \frac{e^{i\Delta\varphi'_n} - 1}{i\Delta\varphi'} \\
&= Nl\left(\frac{e^{i\pi\left(1-\frac{v_g}{c}\right)(n-n_s)} - 1}{i\pi\left(1-\frac{v_g}{c}\right)(n-n_s)} \right)
\end{aligned}
$$

A.2

This result is often referred to as the correction for the time of flight.
Dividing by the structure length Nl,

$$E_n = \frac{e^{i\pi\left(1-\frac{v_g}{c}\right)(n-n_s)} - 1}{i\pi\left(1-\frac{v_g}{c}\right)(n-n_s)}$$

A.3

Rewriting,

$$E_n = \frac{\sin\left(\pi\left(1-\frac{v_g}{c}\right)(n-n_s)\right)}{\pi\left(1-\frac{v_g}{c}\right)(n-n_s)} - i\frac{1-\cos\left(\pi\left(1-\frac{v_g}{c}\right)(n-n_s)\right)}{\pi\left(1-\frac{v_g}{c}\right)(n-n_s)}$$

A.4