CERN-TH/99-183

HIGHER-SPIN CURRENT MULTIPLETS IN OPERATOR-PRODUCT EXPANSIONS

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Abstract

Various formulas for currents with arbitrary spin are worked out in general space-time dimension, in the free field limit and, at the bare level, in presence of interactions. As the n-dimensional generalization of the (conformal) vector field, the $(n/2-1)$-form is used. The two-point functions and the higher-spin central charges are evaluated at one loop. As an application, the higher-spin hierarchies generated by the stress-tensor operator-product expansion are computed in supersymmetric theories. The results exhibit an interesting universality.

CERN-TH/99-183
June, 1999

1 Introduction

This paper is devoted to the computation of some formulas for higher-spin tensor currents, and for their their two-point functions and supersymmetric hierarchies, which are generated by the operator-product expansion of the stress tensor [1, 2]. The spin and the space-time dimension are arbitrary, althoughy we occasionall focus on four dimensions for concreteness.

Higher-spin tensor currents [3] are unique in the free-field limit if they satisfy the conservation condition and are completely traceless [4]. In particular, the two-point functions define "higher-spin central charges" and, via a combination of supersymmetry and orthogonalization, the "higher-spin hierarchies" [4, 5, 6].

One purpose is to work out the general current multiplets defined by the stress-tensor OPE for supersymmetric theories [5, 6]. The result is that the higher-spin current multiplet is *universal*, the lowest-spin current being

$$\frac{8s(s-1)}{(s+1)(s+2)}\mathcal{J}_s^V+\frac{2s}{s+1}\mathcal{J}_s^F+\mathcal{J}_s^S,$$

no matter if the theory is N=4, N=2 or N=1 supersymmetric. Here $\mathcal{J}_s^{V,F,S}$ are spin-s currents of vector, fermion and scalar fields, in the normalization fixed in ref. [4].

While the highest-spin current of each multiplet is certainly universal ($8\mathcal{J}_s^V-2\mathcal{J}_s^F+\mathcal{J}_s^S$), because it has the same structure as the stress tensor, the universal character of the lowest-spin current is less obvious, but still an intrinsic property of supersymmetry.

This observation raises the interesting question about the preservation of such universality, in a form to be uncovered, when the interaction is turned on. We recall that the Ferrara–Gatto–Grillo–Nachtmann theorem [7, 8] is a sort of universality theorem for the spectrum of the anomalous dimensions of the higher-spin currents [5, 6]. For a brief review of these issues, see also [9].

The currents of this paper, worked out explicitly in the free-field limit, are easily extended to interacting theories, at the classical level and, which is the same, at the bare quantum level. This extension is done by replacing ordinary derivatives by covariant derivatives and extracting all traces out again. The resulting currents are non-conserved, both at the classical and quantum levels. In families of conformal field theories the violation of conservation is due to the anomalous dimension.

At the formal level various formulas for vector and fermion fields can be obtained from those of the scalar field by the simple replacements $s\rightarrow s-1$, $n\rightarrow n+2$ and $s\rightarrow s-2$, $n\rightarrow n+4$, respectively, where s is the spin and n is the space-time dimension. This reduces the computational effort by a factor 3 and justifies working in general dimension.

As the n-dimensional generalization of the (conformal) vector field, we take the $(n/2-1)$-form, whose field strength admits a decomposition into self-dual ans anti-self-dual component, useful to write down conformal currents of arbitrary spin.

The formulas elaborated here, which complete certain calculations of refs. [4, 5, 6], might have some relevance in domains different from the ones they are meant for, in particular in the theory of higher-spin fields [10], which however demands to work on Anti-de Sitter space.

In separate sections we treat: the scalar, fermion and vector fields, in sections 2, 3 and 5, respectively, where the higher-spin currents and their two-point functions are evaluated; the N=4 and N=2 hierarchies in sections 4, 6 and 7, where the mentioned universality is discussed and it is shown that it makes the N=1 hierarchy straightforward; useful formulas for projectors and conformal invariants (sect. 8). In section 9 we compute the contribution of the $(n/2-1)$-form to the first central charge c in the trace anomaly and in section 10 we collect the complete forms of the higher-spin currents. The general properties of the $(n/2-1)$-form in n dimensions are discussed in the appendix.

2 Scalar field

We write the relevant terms of the scalar current of even spin s as

$$\mathcal{J}^{(s)} = \sum_{k=0}^{s} a_k\ \partial^{(k)}\varphi\ \partial^{(s-k)}\varphi + b_k\ \delta_{..}\ \partial^{(k)}_\alpha\varphi\ \partial^{(s-k)}_\alpha\varphi + \text{“}\delta\delta\text{-terms”} + \cdots$$

with $a_k = a_{s-k}$, $b_k = b_{s-k}$, $a_0 = \frac{1}{2}$ and $b_0 = 0$. We omit the uncontracted indices (or replace them by dots). The symbol $\partial^{(k)}$ denotes a string of k derivatives; $\partial^{(k)}_\alpha$ is a string of k derivatives, one of which has index α. Complete symmetrization in the uncontracted indices is understood. In order to study conservation and tracelessness it is sufficient to make one and two indices explicit, respectively.

With one index explicit we have

$$\mathcal{J}^{(s)}_\mu = \frac{2}{s}\sum_{k=0}^{s}\Big\{ka_k\ \partial^{(k)}_\mu\varphi\ \partial^{(s-k)}\varphi + b_k\ \delta_{\mu\cdot}\ \partial^{(k)}_\alpha\varphi\ \partial^{(s-k)}_\alpha\varphi + (k-1)b_k\ \delta_{..}\ \partial^{(k)}_{\alpha\mu}\varphi\ \partial^{(s-k)}_\alpha\varphi + \cdots\Big\}$$

and with two indices explicit we have

$$\mathcal{J}^{(s)}_{\mu\nu} = \frac{2}{s(s-1)}\sum_{k=0}^{s}\Big\{ka_k\left[(k-1)\ \partial^{(k)}_{\mu\nu}\varphi\ \partial^{(s-k)}\varphi + (s-k)\ \partial^{(k)}_\mu\varphi\ \partial^{(s-k)}_\nu\varphi\right]$$
$$+b_k\left[\delta_{\mu\nu}\ \partial^{(k)}_\alpha\varphi\ \partial^{(s-k)}_\alpha\varphi + 2(k-1)b_k\ \delta_{\mu\cdot}\ \partial^{(k)}_{\alpha\nu}\varphi\ \partial^{(s-k)}_\alpha\varphi + 2(k-1)b_k\ \delta_{\nu\cdot}\ \partial^{(k)}_{\alpha\mu}\varphi\ \partial^{(s-k)}_\alpha\varphi\right]\Big\} + \cdots,$$

Here the dots stand for terms containing at least one Kronecker tensor with indices different from μ and ν.

Tracelessness up to “δ-terms” relates a_k ad b_k:

$$b_k = -\frac{k(s-k)}{2s+n-4}a_k \tag{1}$$

where n is the space-time dimension. In deriving (1) it should be remembered that a_k and b_k are symmetric under $k \to s-k$. Finally, conservation determines the coefficients a_k recursively, by the condition

$$ka_k + (s-k+1)a_{k-1} + 2b_k + 2b_{k-1} = 0,$$

which gives, using (1),

$$a_k = -\frac{(s-k+1)(2s-2k+n-2)}{k(2k+n-4)} a_{k-1}, \tag{2}$$

solved by

$$a_k = \frac{(-1)^k}{2} \frac{\binom{s}{k}\binom{s+n-4}{k+\frac{n}{2}-2}}{\binom{s+n-4}{\frac{n}{2}-2}}. \tag{3}$$

We write our formulas in a convenient way for n even, although the results are valid for n odd also. In four dimensions we get, in particular,

$$a_k = \frac{(-1)^k}{2} \binom{s}{k}^2,$$

which can be checked in the case of the stress tensor and the spin-4 scalar current worked out in ref. [4].

Summarizing, we have

$$\mathcal{J}^{(s)} = \frac{1}{2\binom{s+n-4}{\frac{n}{2}-2}} \sum_{k=0}^{s} (-1)^k \binom{s}{k} \binom{s+n-4}{k+\frac{n}{2}-2} \left[\partial^{(k)}\varphi \, \partial^{(s-k)}\varphi - \frac{k(s-k)}{n+2s-4} \, \delta_{..} \, \partial_\alpha^{(k)}\varphi \, \partial_\alpha^{(s-k)}\varphi \right] \tag{4}$$

plus terms containing at least two Kronecker tensors.

Now we compute the two-point functions. The form of the correlator is unique, expressible in terms of suitable projectors or conformal tensors (see section 8 and [4]). The space-time structure, a sum of various terms constructed with x_μ and $\delta_{\mu\nu}$, contains a term $x_1 \cdots x_{2s}/|x|^{4s+2n-4}$ with no Kronecker tensor. Unicity assures that it is sufficient to compute the coefficient of such a term in the two-point functions, which reduces the effort considerably.

The scalar propagator reads

$$\langle \varphi(x) \; \varphi(0) \rangle = \frac{\Gamma\left(\frac{n}{2}\right)}{2(n-2)\pi^{\frac{n}{2}}} \frac{1}{|x|^{n-2}}.$$

We have

$$\langle \partial^p\varphi \, \partial^{s-p}\varphi(x) \quad \partial^q\varphi \, \partial^{s-q}\varphi(0) \rangle = \frac{x_1 \cdots x_{2s}}{|x|^{4s+2n-4}} \frac{2^{2s-4}(2s)! \left[\left(\frac{n}{2}-2\right)! \right]^2}{\pi^n} \times \left\{ \frac{\binom{p+q+\frac{n}{2}-2}{\frac{n}{2}-2}\binom{2s-p-q+\frac{n}{2}-2}{\frac{n}{2}-2}}{\binom{2s}{p+q}} + \frac{\binom{s-p+q+\frac{n}{2}-2}{\frac{n}{2}-2}\binom{s+p-q+\frac{n}{2}-2}{\frac{n}{2}-2}}{\binom{2s}{s-p+q}} \right\}$$

plus δ-terms. A factor $(-1)^2$ is omitted, since s is even for the moment. Therefore the two-point function is

$$\langle \mathcal{J}^{(s)}(x) \; \mathcal{J}^{(s)}(0) \rangle = \frac{x_1 \cdots x_{2s}}{|x|^{4s+2n-4}} \frac{1}{\pi^n} 2^{2s-5}(2s)! \left[\left(\frac{n}{2}-2\right)! \right]^2 \; F[s,n],$$

where $F[s,n]$ is a double sum of an expression in binomial coefficients. Precisely,

$$F[s,n] = \sum_{p,q=0}^{s} \frac{(-1)^{p+q}\binom{s}{p}\binom{s}{q}}{\binom{2s}{p+q}\binom{s+n-4}{\frac{n}{2}-2}^2} \binom{s+n-4}{p+\frac{n}{2}-2} \binom{s+n-4}{q+\frac{n}{2}-2} \binom{p+q+\frac{n}{2}-2}{\frac{n}{2}-2} \binom{2s-p-q+\frac{n}{2}-2}{\frac{n}{2}-2}. \tag{5}$$

The result is

$$F[s,n] = \frac{(2s+n-4)!\ s!}{(2s)!(s+n-4)!}, \tag{6}$$

which we have checked with a computer up to s and n equal to 50. The calculation can be split in two steps. First one considers the sum

$$\sum_{q=0}^{s} (-1)^q \frac{\binom{s}{q}\binom{s+n-4}{q+\frac{n}{2}-2}}{\binom{2s}{p+q}} \binom{p+q+\frac{n}{2}-2}{\frac{n}{2}-2} \binom{2s-p-q+\frac{n}{2}-2}{\frac{n}{2}-2} = (-1)^p \frac{s!(s+n-4)!}{(2s)!\left[(\frac{n}{2}-2)!\right]^2}.$$

This is the most difficult sum that we meet here. Observe that under the replacement $p \to s-p$ the sum picks up a factor $(-1)^s$. We have used Mathematica to compute this sum symbolically for $p=0$ (and $p=s$). The symbolic calculation, equivalent to the use of the available talbes, is however more difficult for the other values of p. Once more, we have checked the result numerically for other values up to $(s,n)=(50,50)$ and all values of $p \leq s$ in this range. We do not know how to rigorously show that the simple p-dependence of this sum is the claimed one. This would suffice to complete the proof. All our general results for the two-point functions rely on the sum above, while the other sums that we meet can be treated with the known tables or with a symbolic calculation. A cross-check of (6) will come from supersymmetry.

The sum can be written more simply as

$$\sum_{q=0}^{s} (-1)^q \frac{(p+q+k)!(2s-p-q+k)!}{q!(s-q)!(q+k)!(s-q+k)!} = (-1)^p,$$

where $k = n/2 - 2$.

This result reduces the computation of $F[s,n]$ to a sum of the form

$$\sum_{k=0}^{s} \binom{s}{k} \binom{s+n-4}{k+\frac{n}{2}-2} = \frac{(2s+n-4)!}{\left[(s+\frac{n}{2}-2)!\right]^2}. \tag{7}$$

Summarizing,

$$\langle \mathcal{J}^{(s)}(x)\ \mathcal{J}^{(s)}(0)\rangle = \frac{x_1 \cdots x_{2s}}{|x|^{4s+2n-4}} \frac{2^{2s-5}(2s+n-4)!\ s!\left[(\frac{n}{2}-2)!\right]^2}{\pi^n (s+n-4)!} + \delta\text{-terms}. \tag{8}$$

Changing the normalization appropriately, we have checked the agreement with the results of [4] for $s = 0, 2, 4$ in $n = 4$. The above formula, derived for even spin, extends to odd spin also. This will be shown later on, using supersymmetry.

The two normalizations, of this paper and of ref. [4], are related as follows. Here we have, from formula (4):

$$\mathcal{J}^{(s)} = \frac{1}{2} \frac{(2s+n-4)!\ (\frac{n}{2}-2)!}{(s+n-4)!\ (s+\frac{n}{2}-2)!} \varphi \partial^s \varphi + \text{total derivatives},$$

having used (7). In ref. [4] we had

$$\mathcal{J}^{(s)}_{[4]} = \varphi \overleftrightarrow{\partial}^s \varphi + \text{total derivatives} = 2^s \varphi \partial^s \varphi + \text{total derivatives},$$

and therefore

$$\mathcal{J}^{(s)} = \frac{1}{2^{s+1}} \frac{(2s+n-4)!\,(\frac{n}{2}-2)!}{(s+n-4)!\,(s+\frac{n}{2}-2)!} \mathcal{J}^{(s)}_{[4]},$$

so that

$$\langle \mathcal{J}^{(s)}_{[4]}(x)\ \mathcal{J}^{(s)}_{[4]}(0)\rangle = \frac{x_1 \cdots x_{2s}}{|x|^{4s+2n-4}} \frac{2^{4s-3} s!(s+n-4)!\left[(s+\frac{n}{2}-2)!\right]^2}{\pi^n (2s+n-4)!} \tag{9}$$

plus δ-terms.

3 Fermion

We now repeat the construction for the free fermion. Here s can be either even or odd. The current can be written as

$$\mathcal{J}^{(s)} = \sum_{k=0}^{s-1} a_k\ \partial^{(k)}\bar{\psi}\ \gamma\ \partial^{(s-k-1)}\psi + b_k\ \delta_{\cdot\cdot}\ \partial^{(k)}_{\alpha}\bar{\psi}\ \gamma\ \partial^{(s-k-1)}_{\alpha}\psi + \text{“}\delta\delta\text{-terms”} + \cdots$$

with $a_k = (-1)^{s+1} a_{s-k-1}$, $b_k = (-1)^{s+1} b_{s-k-1}$, $a_0 = 1$ and $b_0 = 0$. With one index explicit we have

$$\begin{aligned}\mathcal{J}^{(s)}_{\mu} = \frac{1}{s}\sum_{k=0}^{s-1} \Big\{ a_k\ &\Big[k\partial^{(k)}_{\mu}\bar{\psi}\ \gamma\ \partial^{(s-k-1)}\psi + \partial^{(k)}\bar{\psi}\ \gamma_{\mu}\ \partial^{(s-k-1)}\psi + (s-k-1)\partial^{(k)}\bar{\psi}\ \gamma\ \partial^{(s-k-1)}_{\mu}\psi\Big] \\ +b_k\ &\Big[2\delta_{\mu\cdot}\ \partial^{(k)}_{\alpha}\bar{\psi}\ \gamma\ \partial^{(s-k-1)}_{\alpha}\psi + (k-1)\delta_{\cdot\cdot}\ \partial^{(k)}_{\alpha\mu}\bar{\psi}\ \gamma\ \partial^{(s-k-1)}_{\alpha}\psi \\ &+ \delta_{\cdot\cdot}\ \partial^{(k)}_{\alpha}\bar{\psi}\ \gamma_{\mu}\ \partial^{(s-k-1)}_{\alpha}\psi + (s-k-1)\delta_{\cdot\cdot}\ \partial^{(k)}_{\alpha}\bar{\psi}\ \gamma\ \partial^{(s-k-2)}_{\alpha\mu}\psi\Big]\Big\} + \cdots\end{aligned}$$

We do not write the expression with two indices explicit, since by now this should be straightforward.

Tracelessness implies

$$b_k = -\ \frac{k(s-k-1)}{(2s+n-4)} a_k$$

and conservation gives

$$a_k\ k + a_{k-1}(s-k) + 2b_k + 2b_{k-1}\ = 0.$$

We have the recursion relation

$$a_k = -\frac{(s-k)(2s-2k+n-2)}{k(2k+n-2)} a_{k-1},$$

which is the same as (2) with $s \to s-1$ and $n \to n+2$, solved by

$$a_k = (-1)^k \frac{\binom{s-1}{k}\binom{s+n-3}{k+\frac{n}{2}-1}}{\binom{s+n-3}{\frac{n}{2}-1}}.$$

In particular, we have in four dimensions

$$a_k = \frac{(-1)^k}{s+1} \binom{s-1}{k} \binom{s+1}{k+1}.$$

Summarizing, the fermion current reads

$$\mathcal{J}^{(s)} = \sum_{k=0}^{s-1} \frac{(-1)^k \binom{s-1}{k} \binom{s+n-3}{k+\frac{n}{2}-1}}{\binom{s+n-3}{\frac{n}{2}-1}} \left[\partial^{(k)} \bar{\psi} \; \gamma \; \partial^{(s-k-1)} \psi - \frac{k(s-k-1)}{2s+n-4} \; \delta_{..} \; \partial_\alpha^{(k)} \bar{\psi} \; \gamma \; \partial_\alpha^{(s-k-1)} \psi \right] \tag{10}$$

plus terms with at least two Kronecker tensors. This expression matches the complete formulas worked out in [4] up to spin 5.

We now proceed with the computation of the two-point function of the spin-s current. The basic ingredient is

$$\begin{aligned} &\langle \partial^{(p)} \bar{\psi} \; \gamma \; \partial^{(s-p-1)} \psi \; (x) \quad \partial^{(q)} \bar{\psi} \; \gamma \; \partial^{(s-q-1)} \psi \; (0) \rangle \\ &= \frac{x_1 \cdots x_{2s}}{|x|^{4s+2n-4}} \frac{(-1)^{s-1} 2^{2s+\frac{n}{2}-3}}{\pi^n} \left(s-p+q+\frac{n}{2}-2 \right)! \left(s+p-q+\frac{n}{2}-2 \right)!, \end{aligned}$$

the fermion propagator being

$$\langle \psi(x) \, \bar{\psi}(0) \rangle = \frac{\Gamma\left(\frac{n}{2}\right)}{2\, \pi^{\frac{n}{2}}} \frac{\not{x}}{|x|^n}.$$

The two-point function is equal to

$$\frac{x_1 \cdots x_{2s}}{|x|^{4s+2n-4}} \frac{1}{\pi^n} 2^{2s+\frac{n}{2}-3} \left[\left(\frac{n}{2} - 1 \right)! \right]^2 (2s-2)! \; F[s-1, n+2].$$

The same sum as in the scalar case appears, and we conclude

$$\langle \mathcal{J}^{(s)}(x) \; \mathcal{J}^{(s)}(0) \rangle = \frac{x_1 \cdots x_{2s}}{|x|^{4s+2n-4}} \frac{2^{2s+\frac{n}{2}-3} (2s+n-4)! \; (s-1)! \left[\left(\frac{n}{2}-1 \right)! \right]^2}{\pi^n (s+n-3)!} \tag{11}$$

plus δ-terms. It can be checked that the sign is in agreement with reflection positivity and for this purpose it is worth to observe that the odd-spin currents are "purely imaginary" in our notation. The above formula agrees with the results of [4] up to spin 5 included.

Let us relate the two normalizations. Here we have, from (10) and (7):

$$\mathcal{J}^{(s)} = \frac{(2s+n-4)! \left(\frac{n}{2}-1 \right)!}{(s+n-3)! \left(s+\frac{n}{2}-2 \right)!} \bar{\psi} \gamma \partial^{s-1} \psi + \text{total derivatives},$$

while in ref. [4] we had

$$\mathcal{J}_{[4]}^{(s)} = \bar{\psi} \gamma \overleftrightarrow{\partial}^{s-1} \psi + \text{total derivatives} = 2^{s-1} \bar{\psi} \gamma \partial^{s-1} \psi + \text{total derivatives},$$

and therefore

$$\langle \mathcal{J}_{[4]}^{(s)}(x) \; \mathcal{J}_{[4]}^{(s)}(0) \rangle = \frac{x_1 \cdots x_{2s}}{|x|^{4s+2n-4}} \frac{2^{4s+\frac{n}{2}-5} (s+n-3)! (s-1)! \left[\left(s+\frac{n}{2}-2 \right)! \right]^2}{\pi^n (2s+n-4)!}. \tag{12}$$

4 The simplest hierarchy: hypermultiplet

The currents that we use are in the normalization of [6, 4], namely

$$\mathcal{J}_s^F = \bar{\psi}\gamma \overleftrightarrow{\partial}^{s-1}\psi + \text{impr.}, \qquad \mathcal{A}_s^F = \bar{\psi}\gamma_5\gamma \overleftrightarrow{\partial}^{s-1}\psi + \text{impr.},$$
$$\mathcal{J}_s^S = 2\bar{A}_i \overleftrightarrow{\partial}^{s} A_i + \text{impr.},$$

where $i = 1, 2$ and "impr." stands for the improvement terms.

The sypersymmetric transformation relates currents of the same multiplet according to the formulas

$$\mathcal{J}_{2s}^S \rightarrow -4\ \mathcal{A}_{2s+1}^F,$$
$$\mathcal{J}_{2s}^F \rightarrow -2\ \mathcal{A}_{2s+1}^F, \qquad \mathcal{A}_{2s-1}^F \rightarrow -2\ \mathcal{J}_{2s}^F +\ \mathcal{J}_{2s}^S$$

The general current multiplet of the hierarchy is made of the three currents $\Omega_s = (\Lambda_s, \mathcal{A}_{s+1}^F, \Xi_{s+2} = -2\mathcal{J}_{s+2}^F + \mathcal{J}_{s+2}^S)$, where

$$\Lambda_s = -\frac{s+1}{4(2s+1)}\left(\frac{2s}{s+1}\mathcal{J}_s^F + \mathcal{J}_s^S\right) \tag{13}$$

(s even), fixed by imposing orthogonality between the currents with the same spin, Λ_s and Ξ_s. The results of [6] are correctly reproduced.

5 Vector field and its generalization to arbitrary dimension

The vector field A_μ is not conformal in dimension different from 4. As the generalization of the four-dimensional vector field to arbitrary (even) dimension, it is better to take the $(n/2-1)$-form $A_{\mu_1\ldots\mu_{l-1}}$, where $s = 2l$. This theory is indeed conformal. The $l-1$ indices are completely antisymmetrized and this "vector" has the usual gauge-invariance. The field strength reads

$$F_{\mu_1\ldots\mu_l} = \partial_{\mu_1}A_{\mu_2\ldots\mu_l} - \partial_{\mu_2}A_{\mu_1\mu_3\ldots\mu_l} - \partial_{\mu_3}A_{\mu_2\mu_1\mu_4\ldots\mu_l} + \cdots$$

and the lagrangian is

$$\mathcal{L} = \frac{1}{n}F^2_{\mu_1\ldots\mu_l} \equiv \frac{1}{n}F^2.$$

The stress tensor,

$$T_{\mu\nu} = F_{\mu\{\alpha\}}F_{\nu\{\alpha\}} - \frac{1}{n}\delta_{\mu\nu}F^2,$$

is traceless, as claimed. Here $\{\alpha\}$ denotes collectively a string of $l-1$ indices and repeated $\{\alpha\}$'s denote contracted indices. The contraction is done with the identity in the space of tensors with $(l-1)$-antisymmetric indices (see the Appendix).

We can now proceed to construct the higher-spin currents. As we explain in the Appendix, the four-dimensional formalism extends straightforwardly to arbitrary dimension multiple of 4, but also l=odd does not present problems if complex fields are considered. We therefore assume that l=even and the results will extend immediately to l=odd. In particular, the stress tensor can be written as

$$T_{\mu\nu} = 2F^+_{\mu\{\alpha\}}F^-_{\nu\{\alpha\}},$$

without care about the positions of the indices μ and ν. The same can be said of the currents with arbitrary spin. Conservation and tracelessness are evident.

We write the relevant terms of the current of spin s as

$$\mathcal{J}^{(s)} = \sum_{k=0}^{s-2} a_k\ \partial^{(k)}F^+_{\{\alpha\}}\ \partial^{(s-k-2)}F^-_{\{\alpha\}} + b_k\ \delta_{..}\ \partial^{(k)}_\beta F^+_{\{\alpha\}}\ \partial^{(s-k-2)}_\beta F^-_{\{\alpha\}} + \text{“}\delta\delta\text{-terms”} + \cdots,$$

with $a_k = (-1)^s a_{s-k}$, $b_k = (-1)^s b_{s-k}$, $a_0 = 1$ and $b_0 = 0$. $F^\pm_{\{\alpha\}}$ are the self-dual and anti-self-dual components of the field strength, where one index is not written explicitly (for example, $F^+_{\{\alpha\}}F^-_{\{\alpha\}}$ stands for $F^+_{\{\alpha\}\mu}F^-_{\{\alpha\}\nu}$ making the two indices, μ and ν, explicit). Here $\{\alpha\}$ denotes a string of (contracted) $l-1$ indices, as before.

With one index explicit we have

$$\begin{aligned}\mathcal{J}^{(s)}_\mu = \frac{1}{s}\sum_{k=0}^{s-2}\Big\{ & a_k\Big[k\partial^{(k)}_\mu F^+_{\{\alpha\}}\ \partial^{(s-k-2)}F^-_{\{\alpha\}} \\ & +(s-k-2)\partial^{(k)}F^+_{\{\alpha\}}\ \partial^{(s-k-2)}_\mu F^-_{\{\alpha\}} + 2\partial^{(k)}F^+_{\{\alpha\}\mu}\ \partial^{(s-k-2)}F^-_{\{\alpha\}}\Big] \\ & +b_k\ \Big[2\delta_{\mu\cdot}\ \partial^{(k)}_\beta F^+_{\{\alpha\}}\ \partial^{(s-k-2)}_\beta F^-_{\{\alpha\}} + 2\delta_{..}\ \partial^{(k)}_\beta F^+_{\{\alpha\}\mu}\ \partial^{(s-k-2)}_\beta F^-_{\{\alpha\}} \\ & +(k-1)\delta_{..}\ \partial^{(k)}_{\beta\mu}F^+_{\{\alpha\}}\ \partial^{(s-k-2)}_\beta F^-_{\{\alpha\}} + \ (s-k-3)\delta_{..}\ \partial^{(k)}_\beta F^+_{\{\alpha\}}\ \partial^{(s-k-2)}_{\beta\mu}F^-_{\{\alpha\}}\Big]\Big\}.\end{aligned}$$

One obtains the same relations as in the scalar case, with $s \to s-2$ and $n \to n+4$. Therefore

$$a_k = (-1)^k \frac{\binom{s-2}{k}\binom{s+n-2}{k+\frac{n}{2}}}{\binom{s+n-2}{\frac{n}{2}}}, \qquad b_k = -\frac{k(s-k-2)}{2s+n-4}a_k. \tag{14}$$

In four dimensions, in particular,

$$a_k = \frac{2(-1)^k}{(s+1)(s+2)}\binom{s-2}{k}\binom{s+2}{k+2},$$

which can be checked in the cases worked out explicitly in ref. [4].

Summarizing, we have

$$\mathcal{J}^{(s)} = \sum_{k=0}^{s-2}\frac{(-1)^k\binom{s-2}{k}\binom{s+n-2}{k+\frac{n}{2}}}{\binom{s+n-2}{\frac{n}{2}}}\left[\partial^{(k)}F^+_{\{\alpha\}}\ \partial^{(s-k-2)}F^-_{\{\alpha\}} - \frac{k(s-k-2)}{2s+n-4}\ \delta_{..}\ \partial^{(k)}_\beta F^+_{\{\alpha\}}\ \partial^{(s-k-2)}_\beta F^-_{\{\alpha\}}\right]$$

plus terms containing at least two Kronecker tensors.

For the calculation of the two-point functions we need the result

$$\begin{aligned}&\langle\partial^{(p)}F^+_{\{\alpha\}}\partial^{(s-p-2)}F^-_{\{\alpha\}}(x)\ \partial^{(q)}F^+_{\{\beta\}}\partial^{(s-q-2)}F^-_{\{\beta\}}\rangle = \\ &\qquad \frac{x_1\cdots x_{2s}}{|x|^{4s+2n-4}}\frac{(-1)^s 2^{2s-7}n^2(n-2)!(2s-4)!}{\pi^n}\frac{\binom{s-p+q+\frac{n}{2}-2}{\frac{n}{2}}\binom{s+p-q+\frac{n}{2}-2}{\frac{n}{2}}}{\binom{2s-4}{s-p+q-2}},\end{aligned}$$

up to δ-terms as usual. The derivation of this formula is lengthy, but does not present particular difficulties. See the appendix for the propagator and other details.

The result involves a binomial sum similar to the one of the fermion and scalar currents. Precisely,

$$\langle \mathcal{J}^{(s)}(x)\,\mathcal{J}^{(s)}(0)\rangle = \frac{x_1\cdots x_{2s}}{|x|^{4s+2n-4}}\frac{2^{2s-7}n^2(n-2)!(2s-4)!}{\pi^n}F[s-2,n+4] = \frac{x_1\cdots x_{2s}}{|x|^{4s+2n-4}}\frac{2^{2s-7}n^2(n-2)!}{\pi^n}\frac{(2s+n-4)!(s-2)!}{(s+n-2)!}. \tag{15}$$

We now check the values of [4]. We can write

$$\mathcal{J}^{(s)} = \frac{(\frac{n}{2})!(2s+n-4)!}{(s+n-2)!\,(s+\frac{n}{2}-2)!}F^+_{\{\alpha\}}\partial^{s-2}F^-_{\{\alpha\}} + \text{total derivs.} = -\frac{2^{2-s}\,(\frac{n}{2})!(2s+n-4)!}{(s+n-2)!\,(s+\frac{n}{2}-2)!}\mathcal{J}^{(s)}_{[4]},$$

where we recall that $\mathcal{J}^{(s)}_{[4]} = -F^+_{\{\alpha\}}\overleftrightarrow{\partial}^{s-2}F^-_{\{\alpha\}}$ plus improvement terms. The two-point function in the notation of [4] reads

$$\langle \mathcal{J}^{(s)}_{[4]}(x)\,\mathcal{J}^{(s)}_{[4]}(0)\rangle = \frac{x_1\cdots x_{2s}}{|x|^{4s+4}}\frac{2^{4s-9}(n-2)!}{\pi^n\,\left[(\frac{n}{2}-1)!\right]^2}\frac{(s-2)!(s+n-2)!\left[(s+\frac{n}{2}-2)!\right]^2}{(2s+n-4)!}. \tag{16}$$

6 The N=4 hierarchy

Now we construct the complete N=4 hierarchy, according to the results of [5]. We normalize the currents as in [4, 5]:

$$\begin{aligned} \mathcal{J}^V_s &= -F^+_\alpha\overleftrightarrow{\partial}^{s-2}F^-_\alpha + \text{impr.}, & \mathcal{J}^F_s &= \frac{1}{2}\bar{\lambda}_i\gamma\overleftrightarrow{\partial}^{s-1}\lambda_i + \text{impr.}, \\ \mathcal{A}^V_s &= -F^+_\alpha\overleftrightarrow{\partial}^{s-2}F^-_\alpha + \text{impr.}, & \mathcal{A}^F_s &= \frac{1}{2}\bar{\lambda}_i\gamma_5\gamma\overleftrightarrow{\partial}^{s-1}\lambda_i + \text{impr.}, \\ \mathcal{J}^S_s &= A_{ij}\overleftrightarrow{\partial}^{s}A_{ij} + B_{ij}\overleftrightarrow{\partial}^{s}B_{ij} + \text{impr.}, \end{aligned} \tag{17}$$

where $i = 1, \ldots 4$, $\mathcal{J}$ and $\mathcal{A}$ are even-spin and odd-spin currents, respectively. The supersymmetry operation is

$$\begin{aligned} \mathcal{A}^V_{2s-1} &\rightarrow -4\ \mathcal{J}^V_{2s} + \frac{1}{4}\ \mathcal{J}^F_{2s}, & \mathcal{A}^F_{2s-1} &\rightarrow -16\ \mathcal{J}^V_{2s} - 2\ \mathcal{J}^F_{2s} + 2\ \mathcal{J}^S_{2s}, \\ \mathcal{J}^V_{2s} &\rightarrow -4\ \mathcal{A}^V_{2s+1} + \frac{1}{4}\ \mathcal{A}^F_{2s+1}, & \mathcal{J}^F_{2s} &\rightarrow -16\ \mathcal{A}^V_{2s+1} - 2\ \mathcal{A}^F_{2s+1}, \\ \mathcal{J}^S_{2s} &\rightarrow -6\ \mathcal{A}^F_{2s+1}. \end{aligned}$$

The general structure of the multiplet is

$$\begin{aligned} \Sigma_s &= a_s\ \mathcal{J}^V_s + b_s\ \mathcal{J}^F_s + c_s\ \mathcal{J}^S_s \\ \Lambda_{s+1} &= -4(a_s + 4b_s)\ \mathcal{A}^V_{s+1} + \frac{1}{4}(a_s - 8b_s - 24c_s)\ \mathcal{A}^F_{s+1} \end{aligned}$$

$$
\begin{aligned}
\Xi_{s+2} &= 12(a_s + 8b_s + 8c_s)\ \mathcal{J}^V_{s+2} - \frac{3}{2}(a_s - 8c_s)\ \mathcal{J}^F_{s+2} + \frac{1}{2}(a_s - 8b_s - 24c_s)\ \mathcal{J}^S_{s+2} \\
\Omega_{s+3} &= 3(a_s + 16b_s + 24c_s)\left(-8\ \mathcal{A}^V_{s+3} + \mathcal{A}^F_{s+3}\right) \\
\Upsilon_{s+4} &= 6(a_s + 16b_s + 24c_s)\left(8\ \mathcal{J}^V_{s+4} - 2\ \mathcal{J}^F_{s+4} + \mathcal{J}^S_{s+4}\right),
\end{aligned}
$$

where s is any even integer number. In addition, the stress tensor is a current multiplet in itself and is in practice Υ_2. Note that the capital Greek letters are not used here in the same way as in ref. [5].

We conventionally fix the normalization so that the scalar component of the highest-spin current has coefficient 1:

$$
6(a_s + 16b_s + 24c_s) = 1.
$$

Secondly, we impose that multiplets with different spins be orthogonal. The conditions $\langle \Upsilon_s \, \Sigma_s \rangle = 0$ and $\langle \Sigma_{s+2} \, \Xi_{s+2} \rangle = 0$ give, respectively,

$$
\begin{aligned}
0 &= \frac{(s+1)(s+2)}{4s(s-1)} a_s - 4\frac{s+1}{s} b_s + 6c_s, \\
0 &= \frac{(s+3)(s+4)}{8(s+1)(s+2)}(a_s + 8b_s + 8c_s)a_{s+2} - \frac{s+3}{s+2}(a_s - 8c_s)b_{s+2} + (a_s - 8b_s - 24c_s)c_{s+2}.
\end{aligned}
$$

The three conditions are solved uniquely by a_s, b_s, c_s such that

$$
\Sigma_s = \frac{(s+1)(s+2)}{96(2s+1)(2s+3)}\left[\frac{8s(s-1)}{(s+1)(s+2)}\mathcal{J}^V_s + \frac{2s}{s+1}\mathcal{J}^F_s + \mathcal{J}^S_s\right], \tag{18}
$$

in agreement with the results of [5]. The remaining orthogonality relationships are automatically satisfied, which is also a cross-check of the formula for $F[s,n]$.

The orthogonalized two-point functions define the so-called "higher-spin central charges", in particular

$$
\langle \Sigma_s(x)\, \Sigma_s(0) \rangle = \frac{2^{4s-10}(s!)^2(s+1)!(s+2)!}{9(2s+3)!}\left(\frac{x_1 \cdots x_{2s}}{|x|^{4s+4}} + \delta\text{-terms}\right). \tag{19}
$$

Note that the curious $s \to \infty$ limit

$$
\Sigma_\infty = \frac{1}{384}\left(8\mathcal{J}^V_s + 2\mathcal{J}^F_s + \mathcal{J}^S_s\right).
$$

7 The N=2 vector multiplet

We conclude the applications with the hierarchy of the N=2 vector multiplet [6]. The fermion and vector currents are the same as in (17) (with $i = 1, 2$ now) and the scalar currents are

$$
\mathcal{J}^S_s = M \overleftrightarrow{\partial}^s M + N \overleftrightarrow{\partial}^s N + \text{impr.}, \qquad \mathcal{A}^S_s = -2iM \overleftrightarrow{\partial}^s N + \text{impr.}. \tag{20}
$$

We also have odd-spin currents for scalar fields. We determine their two-point functions using supersymmetry, as in [6].

The supersymmetry operation is

$$\begin{aligned}
\mathcal{J}^S_{2s} &\to -2\ \mathcal{A}^F_{2s+1} + 2\ \mathcal{A}^S_{2s+1}, & \mathcal{A}^S_{2s-1} &\to -2\ \mathcal{J}^F_{2s} + 2\ \mathcal{J}^S_{2s},\\
\mathcal{J}^F_{2s} &\to -8\ \mathcal{A}^V_{2s+1} + \ \mathcal{A}^S_{2s+1}, & \mathcal{A}^F_{2s-1} &\to -8\ \mathcal{J}^V_{2s} + \ \mathcal{J}^S_{2s},\\
\mathcal{J}^V_{2s} &\to -2\ \mathcal{A}^V_{2s+1} + \frac{1}{4}\ \mathcal{A}^F_{2s+1}, & \mathcal{A}^V_{2s-1} &\to -2\ \mathcal{J}^V_{2s} + \frac{1}{4}\ \mathcal{J}^F_{2s}.
\end{aligned}$$

Writing

$$\begin{aligned}
\Sigma_s &= \ a_s\ \mathcal{J}^V_s + b_s\ \mathcal{J}^F_s + c_s\ \mathcal{J}^S_s,\\
\Lambda_{s+1} &= -2(a_s+4b_s)\ \mathcal{A}^V_{s+1} + \frac{1}{4}(a_s-8c_s)\ \mathcal{A}^F_{s+1} + (b_s+2c_s)\ \mathcal{A}^S_{s+1},\\
\Xi_{s+2} &= \frac{a_s+8b_s+8c_s}{4}\left(8\mathcal{J}^V_{s+2} - 2\ \mathcal{J}^F_{s+2} + \ \mathcal{J}^S_{s+2}\right),
\end{aligned}$$

($\mathcal{J} \leftrightarrow \mathcal{A}$ when s is odd) we impose the normalization $a_s + 8b_s + 8c_s = 4$ and orthogonality ($\langle\Sigma_{s+1}\ \Lambda_{s+1}\rangle = \langle\Sigma_s\ \Xi_s\rangle = 0$). The result is

$$\Sigma_s = \frac{(s+2)}{4(2s+1)}\left[\frac{8s(s-1)}{(s+1)(s+2)}\mathcal{J}^V_s + \frac{2s}{s+1}\mathcal{J}^F_s + \mathcal{J}^S_s\right]. \qquad (21)$$

Comparing (13), (18) and (21) we observe an unexpected universality of the higher-spin hierarchy. The lowest spin is always

$$\frac{8s(s-1)}{(s+1)(s+2)}\mathcal{J}^V_s + \frac{2s}{s+1}\mathcal{J}^F_s + \mathcal{J}^S_s, \qquad (22)$$

and theories with different supersymmetric contents just differ in restrictions such as the parity of s, the presence or absence of the vector currents $\mathcal{J}^V_s$, the overall normalization, etc.. For example, s takes any integer value in the N=2 vector multiplet, but only even values in the N=4 multiplet and in the hypermultiplet.

Observe that the highest-spin component of the current multiplets $(8\mathcal{J}^V_s - 2\mathcal{J}^F_s + \mathcal{J}^S_s)$ is also universal and independent of the spin. This follows from the universality of the stress tensor. The highest-spin currents are annihilated, by definition, by the supersymmetry transformation and the unique structure with this property is that of the stress tensor.

In conclusion, the infinite OPE algebra of (supersymmetric) conformal field theory in higher dimensions is uniquely described by the simple, universal, structure (22). It is plausible that the universality that we have uncovered is much deeper. It should be an intrinsic property of supersymmetry, since the highest-spin/lowest-spin two-point function ($\langle\Upsilon_s\,\Sigma_s\rangle$ for N=4 and $\langle\Xi_s\,\Sigma_s\rangle$ for N=2) is proportional to the difference $2N_V - 4N_F + N_S$ between the number of bosonic and fermionic degrees of freedom ($N_{V,F,S}$ being the number of vectors, Dirac fermions and real scalars, respectively).

This observation extends immediately to N=1 hierarchies. Indeed, N=1 supersymmetric multiplets contain only two types of fields (vector, fermion for the vector multiplet and fermion, scalar for the scalar multiplet), and the universality of the highest-spin current, together with orthogonality, implies immediately the universality of the lowest-spin current. This fact is not obvious, instead, in extended supersymmetry. The current multiplets are

$$\begin{array}{c}\frac{8s(s-1)}{(s+1)(s+2)}\mathcal{J}^V_s+\frac{2s}{s+1}\mathcal{J}^F_s\\ 8\mathcal{J}^V_{s+1}-2\ \mathcal{J}^F_{s+1}\end{array},\qquad\qquad \begin{array}{c}\frac{2s}{s+1}\mathcal{J}^F_s+\mathcal{J}^S_s\\ -2\ \mathcal{J}^F_{s+1}+\ \mathcal{J}^S_{s+1}\end{array},$$

for the vector and scalar multiplets, respectively, up to normalization factors. Here s takes any integer value.

A byproduct of the calculations of this section is the extension of the scalar two-point functions, (8) and (9), which we have derived for even spin, to odd spin (and complex scalar fields), with the normalization specified in (20). This result is implied by orthogonality of the odd-spin components of the N=2 vector-field current multiplets.

8 Formulas for projectors and conformal tensors

In this section we work out formulas to express the two-point functions in the notation of [4] and write them down in complete form (including the δ-terms).

We start from the projectors $\prod^{(s)}_{\mu_1\cdots\mu_s,\nu_1\cdots\nu_s}$, which are defined as the unique $2s$ indiced polynomials of degree $2s$ in derivatives, completely symmetric, conserved and traceless in $\mu_1\cdots\mu_s$ and $\nu_1\cdots\nu_s$. We write them in compact notation as $\prod^{(s)}_{\{\mu\},\{\nu\}}$ and we expand them as

$$\prod\nolimits^{(s)}_{\{\mu\},\{\nu\}}=\sum_{k=0}^{\left[\frac{s}{2}\right]}a_k\pi^k_{\bar\mu\bar\mu}\pi^k_{\bar\nu\bar\nu}\pi^{s-2k}_{\bar\mu\bar\nu}. \tag{23}$$

Here $\pi^k_{\bar\mu\bar\mu}$ denotes a string of k projectors $\pi_{\alpha\beta}=\partial_\alpha\partial_\beta-\delta_{\alpha\beta}\Box$, with indices from the set $\{\mu\}$. The normalization is conventionally fixed so that $a_0=1$. Expression (23) is automatically conserved and we have to impose tracelessness. For this purpose, we can make two indices of the set $\{\mu\}$, say α and β, explicit:

$$\begin{aligned}\prod\nolimits^{(s)}_{\{\mu\},\{\nu\}}=\sum_{k=0}^{\left[\frac{s}{2}\right]}a_k\pi^k_{\bar\nu\bar\nu}\Big[&2k\ \pi_{\alpha\beta}\pi^{k-1}_{\bar\mu\bar\mu}\pi^{s-2k}_{\bar\mu\bar\nu}+2k(s-2k)(\pi_{\alpha\bar\mu}\pi_{\beta\bar\nu}+\pi_{\alpha\bar\nu}\pi_{\beta\bar\mu})\pi^{k-1}_{\bar\mu\bar\mu}\pi^{s-2k-1}_{\bar\mu\bar\nu}\\ &+4k(k-1)\pi_{\alpha\bar\mu}\pi_{\beta\bar\mu}\pi^{k-2}_{\bar\mu\bar\mu}\pi^{s-2k}_{\bar\mu\bar\nu}+(s-2k)(s-2k-1)\pi_{\alpha\bar\nu}\pi_{\beta\bar\nu}\pi^{k}_{\bar\mu\bar\mu}\pi^{s-2k-2}_{\bar\mu\bar\nu}\Big].\end{aligned}$$

Tracing in α and β we get the recursion relation

$$a_k=-a_{k-1}\frac{(s-2k+2)(s-2k+1)}{2k(2s-2k+n-3)} \tag{24}$$

in generic dimension n. The solution is

$$a_k=(-1)^k\frac{(2s-2k+n-4)!\ s!\ (s+\frac{n}{2}-2)!}{k!\ (s-2k)!\ (s-k+\frac{n}{2}-2)!\ (2s+n-4)!},$$

which correctly reproduces the values of [4] for $n=4$ and $s=0,\ldots,5$.

It can be shown, using the known tables, that

$$\sum_{k=0}^{[\frac{s}{2}]} a_k = \frac{2^s (s+n-4)!\,(s+\frac{n}{2}-2)!}{(\frac{n}{2}-2)!(2s+n-4)!}$$

and the general space-time structure of the two-point function $\langle \mathcal{J}^{(s)}_{\{\mu\}}(x)\,\mathcal{J}^{(s)}_{\{\nu\}}(0)\rangle$ of the spin-s currents is

$$\prod\nolimits^{(s)}_{\{\mu\},\{\nu\}}\left(\frac{1}{|x|^{2n-4}}\right) = \frac{x_1\cdots x_{2s}}{|x|^{4s+2n-4}}\frac{2^{3s}(2s+n-3)!(s+n-4)!\,(s+\frac{n}{2}-2)!}{(n-3)!\,(\frac{n}{2}-2)!(2s+n-4)!} + \delta\text{-terms}.$$

This allows us to convert the formulas of the previous sections, (8), (9), (11), (12), (15) and (16), in the more elegant notation of [4]. For example (19) can be re-expressed as

$$\langle \Sigma_s(x)\,\Sigma_s(0)\rangle = \frac{2^{s-10}(s+1)!(s+2)!}{9(2s+1)(2s+3)!}\prod\nolimits^{(s)}\left(\frac{1}{|x|^4}\right).$$

An equivalent way to express the two-point functions is by using the conformal tensor $\mathcal{I}^{(s)}_{\mu_1\cdots\mu_s,\nu_1\cdots\nu_s}(x)$, first introduced by Ferrara *et al.* in refs. [2], which we write in compact form as

$$\mathcal{I}^{(s)}_{\{\mu\},\{\nu\}}(x) = \sum_{k=0}^{[\frac{s}{2}]} b_k \delta^k_{\bar\mu\bar\mu}\delta^k_{\bar\nu\bar\nu}\mathcal{I}^{s-2k}_{\bar\mu\bar\nu}(x), \tag{25}$$

where the powers k and $s-2k$ denote the number of Kronecker tensors and the number of tensors $\mathcal{I}_{\mu\nu}(x)=\delta_{\mu\nu}-2x_\mu x_\nu/|x|^2$, respectively. The tensor $\mathcal{I}_{\mu\nu}(x)$ [11] is indeed the building block of all conformal tensors. We normalize (25) with $b_0=1$. The formal structure is the same as in (23), upon suitable symbolic replacements. Tracing, we get the same recursion relation as (24), but with $n\to n+1$. The solution is therefore

$$b_k = (-1)^k \frac{s!\ \ (s-k+\frac{n}{2}-2)!}{2^{2k}k!\ (s-2k)!\ \ (s+\frac{n}{2}-2)!}.$$

We have $\mathcal{I}^{(s)}_{\{\mu\},\{\nu\}}(x) = (-1)^s 2^s x_1\cdots x_{2s}/|x|^{2s} + \delta$-terms, and the complete conversion formula reads

$$\prod\nolimits^{(s)}_{\{\mu\},\{\nu\}}\left(\frac{1}{|x|^{2n-4}}\right) = \frac{(-1)^s 2^{2s}(2s+n-3)!(s+n-4)!\,(s+\frac{n}{2}-2)!}{(n-3)!\,(\frac{n}{2}-2)!(2s+n-4)!}\frac{\mathcal{I}^{(s)}_{\{\mu\},\{\nu\}}(x)}{|x|^{2s+2n-4}}.$$

The square of the conformal tensor $\mathcal{I}^{(s)}_{\{\mu\},\{\nu\}}(x)$ is equal to the identity, denoted by $\Im^{(s)}_{\{\mu\},\{\nu\}}$, in the space of symmetric, traceless, s-indexed tensors. Precisely,

$$\mathcal{I}^{(s)}_{\{\mu\},\{\rho\}}(x)\ \mathcal{I}^{(s)}_{\{\rho\},\{\nu\}}(x) = p_s^2\ \Im^{(s)}_{\{\mu\},\{\nu\}}, \tag{26}$$

for some factor p_s.

We write

$$\Im^{(s)}_{\{\mu\},\{\nu\}} = \sum_{k=0}^{\left[\frac{s}{2}\right]} c_k \delta^k_{\bar{\mu}\bar{\mu}} \delta^k_{\bar{\nu}\bar{\nu}} \delta^{s-2k}_{\bar{\mu}\bar{\nu}}.$$

The tracelessness condition shows immediately that the c_k are proportional to the b_k. The normalization of $\Im^{(s)}_{\{\mu\},\{\nu\}}$ is fixed by the condition

$$\Im^{(s)}_{\{\mu\},\{\rho\}} \ \Im^{(s)}_{\{\rho\},\{\nu\}} = \Im^{(s)}_{\{\mu\},\{\nu\}}.$$

This is easy to calculate. Indeed, only the first term of $\Im^{(s)}_{\{\mu\},\{\rho\}}$, namely $c_0 \delta^s_{\bar{\mu}\bar{\rho}}$, gives a non-vanishing result when acting on $\Im^{(s)}_{\{\rho\},\{\nu\}}$, since all the other terms contain at least one trace. Therefore $c_0 = 1$ and the c_k are precisely equal to the b_k of (1).

Now we study (26). Only the term $b_0 \mathcal{I}^s_{\bar{\mu}\bar{\rho}}(x)$ of $\mathcal{I}^{(s)}_{\{\mu\},\{\rho\}}(x)$ gives a non-vanishing contribution when acting on $\mathcal{I}^{(s)}_{\{\rho\},\{\nu\}}(x)$. On the other hand, the product $\mathcal{I}^s_{\bar{\mu}\bar{\rho}}(x)\mathcal{I}^{s-2k}_{\bar{\rho}\bar{\nu}}(x)$ with the $s-2k$ ρ-indices of $\mathcal{I}^{s-2k}_{\bar{\rho}\bar{\nu}}(x)$ contracted with corresponding indices in $\mathcal{I}^s_{\bar{\mu}\bar{\rho}}(x)$ is equal to $\delta^{s-2k}_{\bar{\mu}\bar{\nu}} \mathcal{I}^{2k}_{\bar{\mu}\bar{\rho}}(x)$. Further multiplication by $\delta^k_{\bar{\rho}\bar{\rho}}$ gives $\delta^{s-2k}_{\bar{\mu}\bar{\nu}} \delta^k_{\bar{\mu}\bar{\mu}}$. In conclusion, $p_s = 1$.

The conformal tensor $\mathcal{I}^{(s)}_{\{\mu\},\{\nu\}}(x)$ provides a universal way to normalize the two-point functions.

9 Trace anomaly

In this section we work out the contribution of the "generalized" vector field to the first central charge, called c, in the trace anomaly. The charge c provides the *integrated* trace anomaly at the quadratic level in the expansion around flat space. The topological invariants and the more important central charge a (see [9] for the definition and a quick introduction), instead, are not visible in this formula.

The stress tensor in arbitrary dimension reads

$$\begin{aligned} T_{\mu\nu} &= -\frac{1}{4}\varphi \overleftrightarrow{\partial_\mu} \overleftrightarrow{\partial_\nu} \varphi + \frac{1}{4}(\bar{\psi}\gamma_\mu \overleftrightarrow{\partial_\nu} \psi + \bar{\psi}\gamma_\nu \overleftrightarrow{\partial_\mu} \psi) + \frac{1}{4(n-1)}\pi_{\mu\nu}(\varphi^2) + F_{\mu\{\alpha\}}F_{\nu\{\alpha\}} - \frac{1}{n}\delta_{\mu\nu}F^2 \\ &= -\frac{1}{4}\mathcal{J}^S_2 + \frac{1}{2}\mathcal{J}^F_2 - 2\mathcal{J}^V_2, \end{aligned}$$

where the currents $\mathcal{J}^{V,F,S}$ are in the normalization of [4] and we assume that there are N_S real scalar fields φ, N_F Dirac fermions ψ and N_V $(n/2-1)$-forms $A_{\mu_1 \ldots \mu_{l-1}}$, $n = 2l$.

We get

$$\langle T_{\mu\nu}(x)\, T_{\rho\sigma}(0)\rangle = c_n \frac{\left(\frac{n}{2}\right)!\,\left(\frac{n}{2}-1\right)!}{(2\pi)^n (n+1)!} \prod^{(2)}_{\mu\nu,\rho\sigma} \Box^{\frac{n}{2}-2}\left(\frac{1}{|x|^n}\right),$$

where $\prod^{(2)}_{\mu\nu,\rho\sigma} = \frac{1}{2}(\pi_{\mu\rho}\pi_{\nu\sigma} + \pi_{\mu\sigma}\pi_{\nu\rho}) - \frac{1}{n-1}\pi_{\mu\nu}\pi_{\rho\sigma}$ is the spin-2 projector and

$$c_n = N_S + 2^{\frac{n}{2}-1}(n-1)N_F + \frac{n!}{2\left[\left(\frac{n}{2}-1\right)!\right]^2} N_V$$

is the desired central charge. The result matches with the well-known four-dimensional expression $c = N_S + 6N_F + 12N_V$. Using

$$\int \Theta = \mu \frac{\partial}{\partial \mu}, \qquad \mu \frac{\partial}{\partial \mu}\left(\frac{1}{|x|^n}\right) = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}\delta(x),$$

wehre Θ denotes the trace of the stress tensor, we have also

$$\int \Theta = c_n \frac{\left(\frac{n}{2}\right)!}{(4\pi)^{\frac{n}{2}}(n+1)!} \int h_{\mu\nu} \prod\nolimits^{(2)}_{\mu\nu,\rho\sigma} \Box^{\frac{n}{2}-2} h_{\rho\sigma},$$

the expression on the right-hand side being the first term of the expansion of the invariant $W_{\mu\nu\rho\sigma}\Box^{\frac{n}{2}-2}W^{\mu\nu\rho\sigma} + \cdots$ (W denoting the Weyl tensor - see [12] for other details) around the flat metric ($g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}$).

10 Complete form of the higher-spin currents

Finally, we give the complete expressions of the higher-spin currents, including the δ-terms. The derivation follows the strategy of the previous sections, which should be familiar by now, and therefore we just report the result.

We can write in compact notation, for a complex scalar field,

$$\mathcal{J}_s^S = \sum_{k=0}^{\left[\frac{s}{2}\right]} a_k \ \pi^k \ \left(\bar{\varphi} \overleftrightarrow{\partial}^{(s-2k)} \varphi\right),$$

π being the conserved projector $\partial\partial - \Box\delta$, as usual. Here we choose $a_0 = 1$. Conservation of $\mathcal{J}^{(s)}$ is implicit. Tracelessness gives the same recursion relation as in (24). The coefficients for the spinor and vector cases are easily obtained from the scalar ones with the replacements $s \to s-1$, $n \to n+2$ and $s \to s-2$, $n \to n+4$, respectively. In conclusion,

$$\mathcal{J}_s^S = \frac{s! \ (s+\frac{n}{2}-2)!}{(2s+n-4)!} \sum_{k=0}^{\left[\frac{s}{2}\right]} \frac{(-1)^k (2s-2k+n-4)!}{k! \ (s-2k)! \ (s-k+\frac{n}{2}-2)!} \ \pi^k \ \left(\bar{\varphi} \overleftrightarrow{\partial}^{(s-2k)} \varphi\right),$$

$$\mathcal{J}_s^F = \frac{(s-1)! \ (s+\frac{n}{2}-2)!}{(2s+n-4)!} \sum_{k=0}^{\left[\frac{s-1}{2}\right]} \frac{(-1)^k (2s-2k+n-4)!}{k! \ (s-2k-1)! \ (s-k+\frac{n}{2}-2)!} \ \pi^k \ \left(\overline{\psi}\gamma \overleftrightarrow{\partial}^{(s-2k-1)} \psi\right),$$

$$\mathcal{J}_s^V = \frac{(s-2)! \ (s+\frac{n}{2}-2)!}{(2s+n-4)!} \sum_{k=0}^{\left[\frac{s}{2}-1\right]} \frac{(-1)^k (2s-2k+n-4)!}{k! \ (s-2k-2)! \ (s-k+\frac{n}{2}-2)!} \ \pi^k \ \left(F^+_{\{\alpha\}} \overleftrightarrow{\partial}^{(s-2k-2)} F^-_{\{\alpha\}}\right),$$

in the normalization of ref. [4]. We have checked agreement with the explicit formulas of [4]. In [4] and the previous sections $\mathcal{J}_s^S$ was normalized with an additional factor 2.

Acknowledgements
I am grateful to M. Porrati and A. Zaffaroni for discussions.

11 Appendix: the $(n/2-1)$-form

The propagator of the "generalized" conformal vector field of section 5 is

$$\langle A_{\{\alpha\}}(x)\, A_{\{\beta\}}(0)\rangle = \frac{\Gamma\left(\frac{n}{2}\right)}{2(n-2)\pi^{\frac{n}{2}}}\frac{\mathcal{A}_{\{\alpha\},\{\beta\}}}{|x|^{n-2}}.$$

Here $\mathcal{A}_{\{\alpha\},\{\beta\}}$ is the identity in the space of antisymmetric tensors with $l-1$ indices in $2l$ dimensions, $\mathcal{A}_{\{\alpha\},\{\beta\}} = \frac{1}{(l-1)!}\delta^{\{\alpha\}}_{\{\beta\}}$ and $\delta^{\{\alpha\}}_{\{\beta\}}$ is the determinant of the matrix having Kronecher tensors as entries,

$$\mathcal{A}_{\{\alpha\},\{\beta\}} = \frac{1}{(l-1)!(l+1)!}\varepsilon_{\{\alpha\}\gamma_1\ldots\gamma_{l+1}}\varepsilon_{\{\beta\}\gamma_1\ldots\gamma_{l+1}}.$$

Note that the formula for the trace:

$$\mathcal{A}_{\{\alpha\},\{\alpha\}} = \frac{(2l)!}{(l-1)!(l+1)!}.$$

It is possible to define a dual field-strength,

$$\tilde{F}_{\mu_1\ldots\mu_l} = \frac{1}{l!}\varepsilon_{\mu_1\ldots\mu_l\nu_1\ldots\nu_l}F^{\nu_1\ldots\nu_l},$$

as well as self-dual and anti-self-dual tensors

$$F^{\pm}_{\mu_1\ldots\mu_l} = \frac{1}{2}(F_{\mu_1\ldots\mu_l} \pm \tilde{F}_{\mu_1\ldots\mu_l}),$$

so that the stress tensor reads

$$T_{\mu\nu} = F^{+}_{\mu\{\alpha\}}F^{-}_{\nu\{\alpha\}} + F^{-}_{\mu\{\alpha\}}F^{+}_{\nu\{\alpha\}}.$$

Observe that only in dimension multiple of 4, the two terms of this sum are equal. Indeed, we have

$$M^{\pm}_{\mu_1\ldots\mu_l} = \pm\frac{1}{l!}\varepsilon_{\mu_1\ldots\mu_l\nu_1\ldots\nu_l}M^{\pm}_{\nu_1\ldots\nu_l}, \qquad \text{for } l\text{= even}, \tag{27}$$

$$M^{\pm}_{\mu_1\ldots\mu_l} = \pm\frac{1}{l!}\varepsilon_{\mu_1\ldots\mu_l\nu_1\ldots\nu_l}M^{\mp}_{\nu_1\ldots\nu_l}, \qquad \text{for } l\text{= odd}, \tag{28}$$

where $M^{\pm}$ is any self-dual/anti-self-dual tensor in the sense specified above.

The field equations $\partial^\mu F_{\mu\alpha_1\ldots\alpha_{l-1}} = 0$ and Bianchi identities $\partial^\mu \tilde{F}_{\mu\alpha_1\ldots\alpha_{l-1}} = 0$ can be written in the form $\partial^\mu F^{\pm}_{\mu\alpha_1\ldots\alpha_{l-1}} = 0$. For l=even, we can write the stress tensor as

$$T_{\mu\nu} = 2F^{+}_{\mu\{\alpha\}}F^{-}_{\nu\{\alpha\}},$$

without care about the positions of the indices μ and ν. The same can be said of the currents with arbitrary spin. The proof of this statement follows from the equality

$$M^{+}_{\mu\{\alpha\}}N^{-}_{\nu\{\alpha\}} = M^{+}_{\nu\{\alpha\}}N^{-}_{\mu\{\alpha\}}, \qquad (l\text{= even}), \tag{29}$$

which can be shown using (27) for l=even, with the bonus

$$M^{+}_{\mu_1\ldots\mu_l}N^{-}_{\mu_1\ldots\nu_l}=0 \tag{30}$$

Here M and N are generic self-dual and anti-self-dual tensors, respectively.

In l=even conservation of the stress-tensor is obvious and tracelessness follows from (30). This means that the current-formalism of four dimensions generalizes straightforwardly to arbitrary dimension multiple of four.

If l is even the instanton equation

$$F^{+}_{\mu_1\ldots\mu_l}=0 \tag{31}$$

is meaningful. If l is odd, instead, (28) implies that self-dual and anti-self-dual tensors are not independent of each other, so that if (31) holds, then $F^{-}_{\mu_1\ldots\mu_l}$ also vanisehs and there are only trivial solutions. It is possible, nevertheless, to have a nontrivial notion of instanton in l=odd by considering complex fields.

So, let us assume that for l=odd the field is complex. In real notation we have two fields $A^i_{\mu_1\ldots\mu_{l-1}}$, $i=1,2$. The field strength $F^i_{\mu_1\ldots\mu_l}$ is defined as above, but now

$$F^{i\pm}_{\mu_1\ldots\mu_l}=\frac{1}{2}\left(F^i_{\mu_1\ldots\mu_l}\pm\frac{1}{l!}\varepsilon^{ij}\varepsilon_{\mu_1\ldots\mu_l\nu_1\ldots\nu_l}F^j_{\nu_1\ldots\nu_l}\right).$$

Now we have a good version of (27),

$$M^{\pm i}_{\mu_1\ldots\mu_l}=\pm\frac{1}{l!}\varepsilon^{ij}\varepsilon_{\mu_1\ldots\mu_l\nu_1\ldots\nu_l}M^{\pm j}_{\nu_1\ldots\nu_l}.$$

The stress tensor reads

$$T_{\mu\nu}=2F^{+i}_{\mu\{\alpha\}}F^{-i}_{\nu\{\alpha\}}$$

and everything else generalizes immediately, in particular formulas (29) and (30). The instanton equations are non-trivial.

All this is natural, because in two dimensions our "generalized vector field" reduces to the free (complex) scalar φ. Self-duality (anti-self-duality) of the "field strength" $F^{+i}_{\mu}=0$ ($F^{-i}_{\mu}=0$) is the holomorphicity (anti-holomorphicity) condition on φ. These observations might be useful to estabilish correspondences between instantons in different dimensions.

The instanton conditions (31) have non-trivial solutions on non-trivial manifolds, upon covariantization of the theory, with topological invariant

$$\int \mathrm{d}^n x\, F_{\mu_1\ldots\mu_l}\tilde{F}^{\mu_1\ldots\mu_l}.$$

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